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Scalarization Functionals with Uniform Level Sets in Set Optimization

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Abstract
We use the original form of Gerstewitz's nonlinear scalarization functional to characterize upper and lower set-less minimizers of set-valued maps acting from a nonempty set into a real linear space with respect to the lower (resp. upper) set-less relation introduced by Kuroiwa. Our main results are as follows: An upper set-less minimizer to a set-valued map (with respect to the image space) is an upper set-less minimal solution to a scalarization of the set-valued map (with respect to the space of real numbers), where the *hypergraphical multifunction* is involved in the scalarization and vice versa, a lower set-less minimizer to a set-valued map (with respect to the image space) is an upper set-less minimal solution to an appropriate scalarization of the set-valued map (in the space of real numbers), where the *epigraphical multifunction* is involved in the scalarization and vice versa, and a lower set-less minimizer to a set-valued map becomes a (Pareto) minimizer to the same map provided that the map enjoys a domination property.

Keywords Nonlinear scalarization functional · Set optimization · Set-less relations · Scalarization characteristics

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1 Introduction

Set optimization means optimization of set-valued maps. It is an extension of vector optimization to the set-valued case. Recently, there has been an increasing interest in set optimization; see the books [1–4]. There are two main solution approaches: In the vector approach, a solution of a set-valued map is defined via the 'best' element of the image set of the map with respect to the usual ordering relation of vector optimization. In the set approach, a solution is defined via the 'best' set of the collection of all output sets with respect to set-less relations introduced by Kuroiwa [5] (compare [6] and references therein). We use the name *set optimization* for optimization with set-valued objective maps and Kuroiwa's set-less relations, and the name *set-valued optimization* when we work with the usual (Pareto) ordering relation in vector optimization.

Scalarization is a traditional approach to solve vector (single-valued) optimization problems. It converts a vector optimization problem into a scalar one via an extended real-valued scalarizing functional such that minimizers of a vector problem are minimal solutions of the scalarized problem and/or minimal solutions of scalarized problems are minimizers of the given vector problem. It is known that Gerstewitz' (nonlinear separation) scalarization functional introduced in [7,8] (see also Krasnoselskii [9] for assertions in the context of operator theory and compare the scalar optimization problem by Pascoletti and Serafini [10]) is an important tool in nonconvex vector optimization and it has had a number of Gerstewitz-type scalarization functionals in set optimization; see, for instance, [11–15]. They are defined from the space of sets in the image space into the real numbers, and the existing generalized differentiation theories could not be applied to compute subdifferentials of these Gerstewitz-type scalarization functionals. Therefore, we could not derive necessary conditions in terms of subdifferentials or coderivatives. It is important to stress that in [16] Bao and Tammer found an effective way to compute basic and singular subdifferentials of Gerstewitz' functionals and formulated several calculus rules. Motivated by promising necessary conditions for upper and lower set-less minimizers, we challenge ourselves to use the original form to characterize minimizers of set-valued maps. Doing this leads to dealing with scalarized set-valued optimization problems.

The main objective of this paper is a study on characterizations of lower and upper set-less minimizers of set-valued maps $F : X \rightrightarrows Y$ in the linear space setting. Our main results are as follows:

1. An upper set-less minimizer to a set-valued map F (with respect to the space Y) is an upper set-less minimal solution to a scalarization of the set-valued map F (with respect to the space of real numbers), where the *hypergraphical multifunction* is involved in the scalarization and vice versa.
2. A lower set-less minimizer to a set-valued map F (with respect to the space Y) is an upper set-less minimal solution to an appropriate scalarization of the set-valued map F (in the space of real numbers), where the *epigraphical multifunction* is involved in the scalarization and vice versa.
3. A lower set-less minimizer of a set-valued map F becomes a minimizer of F in the vector approach provided that the map F enjoys a domination property.

This paper is organized as follows: In Sect. 2, we recall the two main solution approaches in optimization with a set-valued objective map and present Gerstewitz' nonlinear scalarization functional as well as its basic properties in linear spaces. Section 3 provides characterizations of lower and upper set-less relations. Section 4 is devoted to characterizations of minimizers to set-valued maps with respect to lower and upper set-less relations.

2 Preliminaries

2.1 Nonlinear Scalarization

Let Y be a real linear space and A be a nonempty set in Y. The set A is said to be *pointed* if $A \cap (-A) \subseteq \{0\}$, and *a cone* if $\lambda a \in A$ for all $a \in A$ and $\lambda \geq 0$. A cone A is called nontrivial if $A \neq \{0\}$ and $A \neq Y$. A^c denotes the complement to A in Y.

Given a nonempty set A in Y, the notations core A and vcl A stand for the algebraic interior and the vector closure of A; i.e.,

$$\text{core}\, A := \{y \in Y \ : \ \forall v \in Y, \exists \lambda > 0 : \ y + [0, \lambda]v \subseteq A\},$$
$$\text{vcl}\, A := \{y \in Y \ : \ \exists v \in Y : \ \forall \lambda > 0, \exists t \in [0, \lambda], y + tv \in A\}.$$

For each vector $k \in Y$, we denote the vector closure of A in the direction k by

$$\text{vcl}_k A := \{y \in Y \ : \ \forall \lambda > 0, \exists t \in [0, \lambda], y + tk \in A\}$$

Compare [17–19] and references therein. For more results on directionally vector closedness and relationships with vector closedness and topological closedness, see [18] and the references therein.

When Y is a real linear topological space, notations $\text{int}(A)$, $\text{cl}(A)$, and $\text{bd}(A)$ stand for the topological interior, the topological closure, and the topological boundary of the set A, respectively. See [2,20,21] for basic definitions and concepts of vector optimization, and [8] for some scalarization methods in (convex and nonconvex) vector optimization with fixed ordering structure/ordering cone and important properties of these methods.

Furthermore, for a given number $t \in \mathbb{R}$, the t-level set of a functional $\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$ is denoted by $\text{Lev}\,(t; \varphi, \leq)$.

Let us recall a powerful nonlinear scalarization tool from [7,8] by Gerstewitz (Tammer) and Weidner (cf. [21] for the case of linear topological spaces) generated by (not necessarily closed or vectorially closed) sets in real linear spaces (cf. [18,19]).

Definition 2.1 (*Scalarization directions of sets*) Let A be a nonempty subset in a real linear space Y. A vector $k \in Y \setminus \{0\}$ is called a SCALARIZATION DIRECTION of A if the following conditions hold:

$$\forall t \geq 0, A + tk \subseteq A, \text{ and} \tag{1}$$
$$\forall y \in Y, \exists t \in \mathbb{R}, y + tk \notin A. \tag{2}$$

The set of all scalarization directions of A is denoted by $\operatorname{sd}(A)$.

Obviously, if $A = C$ is a convex cone, then $\operatorname{sd}(C) = C \setminus (-C)$.

Definition 2.2 (*Nonlinear scalarization functionals*) Let A be a nonempty subset of Y and $k \in \operatorname{sd}(A)$ be a scalarization direction of A. The functional $\varphi_{A,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi_{A,k}(y) := \inf\{t \in \mathbb{R} \ : \ y \in tk - A\} \tag{3}$$

with $\inf \emptyset = +\infty$, is called GERSTEWITZ' NONLINEAR (SEPARATING) SCALARIZATION FUNCTIONAL generated by the set A and the scalarization direction k.

Remark 2.1 (a) Condition (1) means that when we move $-A$ along the ray $[0, +\infty[\, k$, the set

$$A' := \{(y, t) \in Y \times \mathbb{R} \ : \ y \in tk - A\}$$

is of epigraphical type; i.e., if $(y, t) \in A'$ and $t' \geq t$, then $(y, t') \in A'$.

(b) Condition (2) means that A does not contain lines parallel to k and ensures that values of scalarization functionals are not $-\infty$. For example, let $Y = \mathbb{R}^2$, $A = \mathbb{R} \times \mathbb{R}_+$, and $k = (1, 0)$. Obviously, condition (2) is not satisfied and Gerstewitz' scalarization functional $\varphi_{A,k}$ takes values $-\infty$ for all $y \in \mathbb{R} \times \mathbb{R}_+$ and $+\infty$ otherwise.

The following lemma consists of most important properties of scalarization functionals; cf. [21, Theorem 2.3.1], [18, Theorem 4].

Lemma 2.1 (Properties of scalarization functionals) *Let Y be a real linear space, A be a nonempty set in Y, and $k \in \operatorname{sd}(A)$. Then, the following hold for the functional $\varphi_{A,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined in (3):*

(i) $\forall y \in Y,\ \varphi_{A,k}(y) = \varphi_{\operatorname{vcl}_k(A),k}(y)$.

(ii) $\operatorname{dom} \varphi_{A,k} = \mathbb{R}k - \operatorname{vcl}_k(A)$.

(iii) *$\varphi_{A,k}$ is finite-valued; i.e., $\operatorname{dom} \varphi_{A,k} = Y$ if and only if $\mathbb{R}k - \operatorname{vcl}_k(A) = Y$.*

(iv) *$\forall\, y \in Y,\ \forall\, t \in \mathbb{R}$, $\varphi_{A,k}(y + tk) = \varphi_{A,k}(y) + t$; the scalarization functional is linearly shifted along the scalarization direction k.*

(v) *The implication*

$$y \in -A \Longrightarrow \varphi_{A,k}(y) \leq 0$$

holds. The inverse implication ($\Longleftarrow$) holds under an additional condition that A is k-vectorially closed.

(vi) *The t-level set of $\varphi_{A,k}$ is given by*

$$\operatorname{Lev}(t; \varphi_{A,k}, \leq) = tk - \operatorname{vcl}_k(A).$$

(vii) *$\varphi_{A,k}$ is convex if and only if A is convex.*

(viii) *$\varphi_{A,k}$ is positively homogeneous, i.e., $\varphi_{A,k}(ty) = t\varphi_{A,k}(y)$ for all $t \geq 0$ and $y \in Y$ if and only if A is a cone.*

(ix) *Given* $B \subseteq Y$. $\varphi_{A,k}$ *is* ***B-monotone*** *if and only if* $A + B \subseteq A$, *where B-monotonicity of* $\varphi_{A,k}$ *means* $a \in b - B \implies \varphi_{A,k}(a) \leq \varphi_{A,k}(b)$.

(x) $\varphi_{A,k}$ *is subadditive if and only if* $A + A \subseteq A$.

The reader is referred to [21, Theorem 2.3.1] for stronger properties of the function $\varphi_{A,k}$ such as continuity or even Lipschitz continuity.

Remark 2.2 (a) $\varphi_{A,k}$ is called a scalarization functional with uniform level sets due to (vi).

(b) In [18], Gutiérrrez et al. proposed an extended form for the scalarization functional $\varphi^{\circ}_{B,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ for an arbitrary subset B of Y and an arbitrary element $k \in Y$ with

$$\varphi^{\circ}_{B,k}(y) := \begin{cases} \inf\{t \in \mathbb{R} \ : \ y \in tk - B\}, & \text{if } y \in \mathbb{R}k - B, \\ +\infty, & \text{if } y \notin \mathbb{R}k - B. \end{cases} \tag{4}$$

Obviously, $\operatorname{dom} \varphi^{\circ}_{B,k} = \mathbb{R}k - B$. By Lemma 2.1 (i), $\varphi^{\circ}_{B,k} = \varphi^{\circ}_{\operatorname{vcl}_k B,k}$, but their domains might be different due to the definition.

It is important to emphasize that

$$\varphi^{\circ}_{B,k}(y) = \varphi_{A,k}(y),$$

where $A = \mathcal{E}_k(B) := B + [0, +\infty[\, k$ is the epigraphical set of B with respect to k. This means that without loss of generality our setting is not less general than the one studied in the aforementioned paper and the references therein.

(c) In [14], Araya proposed four natural extensions for Gerstewitz' scalarization functionals to set optimization. Let Y be a linear topological space, $\mathcal{P}(Y)$ be the collection of all subsets in Y, C be a closed and convex cone of Y with a nonempty interior, and $k \in C \setminus (-C)$ be a scalarization direction. The functionals $h^l_{\inf}, h^u_{\inf}, h^l_{\sup}, h^u_{\sup} : \mathcal{P}(Y)^2 \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\begin{aligned} h^l_{\inf}(A, B) &:= \inf\{t \in \mathbb{R} \ : \ tk + B \subseteq A + C\}, \\ h^u_{\inf}(A, B) &:= \inf\{t \in \mathbb{R} \ : \ A \subseteq tk + B - C\}, \\ h^l_{\sup}(A, B) &:= \sup\{t \in \mathbb{R} \ : \ A \subseteq tk + B + C\}, \\ h^u_{\sup}(A, B) &:= \sup\{t \in \mathbb{R} \ : \ B + tk \subseteq A - C\} \end{aligned}$$

are called Gerstewitz-like scalarization functionals. Obviously, they are not subdifferentiable in the existing generalized differentiation theories. We strive to characterize set relations by using Gerstewitz' scalarization functional given by (3), which is subdifferential in terms of Mordukhovich/limiting differentiation, to derive necessary conditions in set optimization.

2.2 Solutions in Set Optimization

$F, G, H, \ldots$ stand for set-valued maps and $f, g, h, \ldots$ for single-valued functions. It is obvious that a single-valued function $f : X \to Y$ is a special case of a set-valued map $F : X \rightrightarrows Y$ in the sense that

$$F(x) := \begin{cases} \{f(x)\}, & \text{if } x \in \operatorname{dom} f, \\ \emptyset, & \text{otherwise} . \end{cases}$$

For the sake of simplicity, we write $F(x) = f(x)$ and $(F + g)(x) = F(x) + g(x)$ instead of $F(x) = \{f(x)\}$ and $(F + g)(x) = F(x) + \{g(x)\}$, respectively. Given a set-valued map $F : X \rightrightarrows Y$. The domain, the image set, and the graph of F are defined by

$$\operatorname{dom} F := \{x \in X \ : \ F(x) \neq \emptyset\}, F(X) := \cup\{F(x) \ : \ x \in \operatorname{dom} F\} \ \text{ and}$$
$$\operatorname{gph} F := \{(x, y) \in X \times Y \ : \ x \in \operatorname{dom} F, y \in F(x)\}.$$

Given $F : X \rightrightarrows \mathbb{R} \cup \{\pm\infty\}$ and $s : Y \rightrightarrows \mathbb{R} \cup \{\pm\infty\}$. The composition of F and s is a set-valued map $s \circ F : X \rightrightarrows Y$ defined by

$$(s \circ F)(x) := \{s(y) \ : \ y \in F(x)\}.$$

A set-valued optimization problem can be represented in the following way: Find a 'good' solution of a set-valued map $F : X \rightrightarrows Y$ from some nonempty set X to a real linear space Y, where a nontrivial and convex cone Θ in Y induces a preorder to elements in Y. In this paper, we consider two solution approaches: (1) vector approach: the 'best' element of the union of all outputs of F, and (2) set approach: the 'best' output set of the collection of all outputs of F.

Vector Approach

We first recall the definition of Pareto efficient points of sets with respect to a given nontrivial and convex cone, which is standard in vector/multi-objective optimization theory and various applications; see, e.g., the books [2,4,20] for more details.

Let Y be a real linear space and Θ be a nontrivial and convex cone in Y. Associate with this cone, we introduce a preorder in Y denoted by $\leq_\Theta$ and defined by

$$y_1 \leq_\Theta y_2 :\Longleftrightarrow y_1 \in y_2 - \Theta. \tag{5}$$

Assume in addition that Θ is pointed. Then, $\leq_\Theta$ becomes a partial order in Y.

Definition 2.3 (*Efficient points of sets*) Let A be a nonempty subset in Y and $\overline{a} \in A$. We say that:

(i) $\overline{a}$ is an EFFICIENT POINT of A with respect to $\leq_\Theta$ or a $\leq_\Theta$- MINIMAL POINT of A if $\forall a \in A, a \leq_\Theta \overline{a} \Longrightarrow \overline{a} \leq_\Theta a$, i.e.,

$$A \cap (\overline{a} - \Theta) \subseteq \overline{a} + (\Theta \setminus (\Theta \cap (-\Theta))). \tag{6}$$

(ii) $\overline{a}$ is a STRICT $\leq_\Theta$- MINIMAL POINT of A if $\forall a \in A \setminus \{\overline{a}\}, a \not\leq_\Theta \overline{a}$, i.e.,

$$A \cap (\overline{a} - \Theta) = \{\overline{a}\}. \tag{7}$$

Observe that the efficiency notion in (6) and the strict one in (7) are identical, when Θ is pointed, i.e., $\Theta \cap (-\Theta) = \{0\}$. Observe also that in the non-pointedness setting, every $\leq_\Theta$-minimal point of A is a strict $\leq_C$-minimal element of A with respect to the pointed cone

$$C := (\Theta \cap (Y \setminus (-\Theta))) \cup \{0\}.$$

Observe finally that most of publications dealing with nonpointed ordering cones and domination sets do not distinguish strict minimality and minimality.

When studying minimizers of a set-valued map based on the vector approach, we fix one element $\overline{y} \in F(\overline{x})$ so that we could use the definitions in vector optimization.

Definition 2.4 (*Solution concepts based on vector approach*) Given a set-valued map $F : X \rightrightarrows Y$ from a nonempty set to a real linear space and $(\overline{x}, \overline{y}) \in \operatorname{gph} F$. The pair $(\overline{x}, \overline{y})$ is a (resp. STRICT) $\leq_\Theta$- MINIMIZER of F if $\overline{y}$ is a (resp. strict) $\leq_\Theta$-minimal point of $F(X)$.

When $F = f : X \to Y$ is a single-valued function, $\overline{y} = f(\overline{x})$ is unique and thus it will not be mentioned; i.e., $\overline{x}$ is a (strict) minimizer of f.

Set Approach

In vector approach, $\overline{x}$ is a minimizer of F (see Definition 2.4) if $F(\overline{x})$ contains the best element of the image set $F(X)$. However, it might not be appropriate for some applications, for example, the problem of finding the best football team. A football team is a set of 11 players, and each player could be viewed as a vector of his ability, speed, power, stamina, skill, popularity, and other factors. Consider a set-valued map F corresponding the name of a team to the player set of that team. Obviously, the team with the best-of-all player is the best team according to the vector solution approach. However, a team consisting of the best-of-all player and worse-of-all 10 players is a very very bad team. In order to avoid this drawback of the vector approach, Kuroiwa [5] proposed new ways to choose a 'good' solution for a set-valued map by using different inclusions. Among them, the lower and upper set-less relations are promising to practical applications. See [22] for possibilities to uncertain optimization.

Definition 2.5 (*Set-less relations*) Let Y be a real linear space and Θ be a nontrivial and convex cone of Y. For any nonempty subsets A, B in Y, we say that

(i) A is LOWER SET- LESS than B, denoted by $A \preceq^l_\Theta B$ if $\forall b \in B, \exists a \in A : a \leq_\Theta b$, i.e.,

$$B \subseteq A + \Theta.$$

(ii) A is UPPER SET- LESS than B, denoted by $A \preceq^u_\Theta B$, if $\forall a \in A, \exists b \in B : a \leq_\Theta b$, i.e.,

$$A \subseteq B - \Theta.$$

Remark 2.3 **(Set-less relations in real numbers)** Set

$$\mathcal{M} := \{A \subseteq \mathbb{R} \ : \ A \text{ is a closed set in } \mathbb{R}\}.$$

Given $A, B \in \mathcal{M}$. The set-less relations given in Definition 2.5 become

$$A \preceq^l_{\mathbb{R}_+} B \iff \inf A \leq \inf B \quad \text{and} \quad A \preceq^u_{\mathbb{R}_+} B \iff \sup A \leq \sup B.$$

Definition 2.6 (*Solution concepts based on set approach*) Given a set-valued map $F : X \rightrightarrows Y$ from a nonempty set to a real linear space and $\bar{x} \in \operatorname{dom} F$. The relation $\preceq^\circ_\Theta$ stands for either $\preceq^l_\Theta$ or $\preceq^u_\Theta$ introduced in Definition 2.5. We say that

(i) $\bar{x}$ is a $\preceq^\circ_\Theta$-minimizer of F if

$$\forall x \in \operatorname{dom} F : F(x) \preceq^\circ_\Theta F(\bar{x}) \Longrightarrow F(\bar{x}) \preceq^\circ_\Theta F(x).$$

(ii) $\bar{x}$ is an x-partly strict $\preceq^\circ_\Theta$-minimizer of F if

$$\forall x \in \operatorname{dom} F : F(x) \preceq^\circ_\Theta F(\bar{x}) \Longrightarrow x = \bar{x}.$$

(iii) $\bar{x}$ is a y-partly strict $\preceq^\circ_\Theta$-minimizer of F if $F(\bar{x})$ is a strict $\preceq^\circ_\Theta$-minimal element of the image set $\{F(x) \ : \ x \in \operatorname{dom}(F)\}$; i.e.,

$$\forall x \in \operatorname{dom}(F), F(x) \preceq^\circ_\Theta F(\bar{x}) \Longrightarrow F(x) = F(\bar{x}).$$

(iv) $\bar{x}$ is a strong/ideal $\preceq^\circ_\Theta$-minimizer of F if

$$\forall x \in \operatorname{dom} F, F(\bar{x}) \preceq^\circ_\Theta F(x).$$

When $Y = \mathbb{R}$ and $\Theta = \mathbb{R}_+$, we use $\preceq^l_{\mathbb{R}_+}$- and $\preceq^u_{\mathbb{R}_+}$-minimal solutions.

Remark 2.4 **(On solution concepts in the set approach)**

(a) Observe that an x-partly strict minimizer of F w.r.t. $\preceq^l_\Theta$ was named as a minimizer in [23]. This adjective seems to be essential since the x-partly strict $\preceq^l_{\mathbb{R}_+}$-minimality to φ agrees with strict minimality of φ in scalar optimization, when $F = \varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ is an extended real-valued functional with

$\Theta = \mathbb{R}_+$. Note also that the y-partly strict minimality seems to be too restrictive in set optimization. However, it is weaker than the concept of strict minimality in scalar optimization. For example, consider a constant real-valued function $\psi(x) :\equiv c$ for some real number $c \in \mathbb{R}$. Then, ψ has no strict minimum, but every real number is a y-partly strict $\preceq^l_{\mathbb{R}_+}$-minimal solution of ψ.

(b) The x-partly strict minimality is called strict minimality in [23, Definition 11 (c)]. The y-partly strict minimality is defined based on the strict minimality for sets w.r.t. set-less relations in [23, Definition 9 (c)]. Consider set-valued maps $F, G : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) :=]-\infty, |x|] \text{ and } G(x) :=]-\infty, 0].$$

It is easy to check that $\overline{x} = 0$ is both an x-partly and a y-partly strict $\preceq^u_{\mathbb{R}_+}$-minimal solution of F, and that $\overline{x} = 0$ is not an x-partly strict $\preceq^u_{\mathbb{R}_+}$-minimal solution of G, but a y-partly strict $\preceq^u_{\mathbb{R}_+}$-minimal solution of G.

Remark 2.5 **(On solution concepts in the vector approach)** In [23, Theorem 3.4 (i)], the authors dealt with set optimization problems, where the image space Y is equipped with a closed and convex cone being not necessarily pointed. They called a pair $(\overline{x}, \overline{y}) \in \text{gph } F$ satisfying the condition

$$\forall x \in \text{dom } F \setminus \{\overline{x}\},\ F(x) \cap (\overline{y} - \Theta) \subseteq \{\overline{y}\},$$

a minimizer of F; we name this type of minimizers *KTY-minimizers* in order to distinguish with $\leq_\Theta$-minimizers introduced in Definition 2.4 (i). Since the KTY-minimality is defined for $x \neq \overline{x}$, disregard of the image $F(\overline{x})$, the map $F : X \rightrightarrows \mathbb{R}$ with

$$F(x) := \begin{cases} \{|x|\}, & \text{if } x \neq 0, \\ \mathbb{R}, & \text{if } x = 0, \end{cases}$$

has infinitely many KTY-minimizers of the form $(0, \overline{y})$ for all $\overline{y} \in \mathbb{R}$, but it has no $\leq_\Theta$-minimizer. It is important to emphasize that both subdifferential and coderivative Fermat rules in terms of limiting normal cones and coderivatives (see [4] for definitions and calculus rules) do not hold at any KTY-minimizer of F. In fact, we have

$$N((0, \overline{y}); \text{epi } F) = \begin{cases} \{(0, 0)\}, & \text{if } \overline{y} > 0, \\ \text{gph } (-|\cdot|) \cup (\mathbb{R} \times \{0\}), & \text{if } \overline{y} = 0, \\ \mathbb{R} \times \{0\}, & \text{if } \overline{y} < 0 \end{cases}$$

and

$$N((0, \overline{y}); \text{gph } F) = \begin{cases} \mathbb{R} \times \{0\}, & \text{if } \overline{y} \neq 0, \\ \{(x, y) : |x| = |y|\} \cup (\mathbb{R} \times \{0\}), & \text{if } \overline{y} = 0. \end{cases}$$

Therefore, for any $\lambda \neq 0$, we have $\partial F(0, \overline{y})(\lambda) = D^*F(0, \overline{y})(\lambda) = \emptyset$ for all $\overline{y} \in \mathbb{R} \setminus \{0\}$ and $\partial F(0, 0)(\lambda) = D^*F(0, 0)(\lambda) = \{-\lambda, \lambda\}$.

Remark 2.6 Vector optimization may be viewed as a special case of set optimization. When $F = f : X \to Y$ is a single-valued function in the sense that $F(x) = \{f(x)\}$ for $x \in \operatorname{dom} f$ and $\emptyset$ otherwise, there is no difference between $\leq^u_\Theta$-optimality and $\leq^l_\Theta$-optimality; both of them reduce to $\leq_\Theta$-optimality.

3 Characterization of Set-Less Relations via Scalarization

In this section, we use Gerstewitz' scalarization functionals defined in (3) to characterize the upper and lower set-less relations introduced in Definition 2.5.

(H1) *Standing assumptions for sets* Y is a real linear space. Θ is a nontrivial and convex cone in Y. A and B are two nonempty subsets of Y.

Let us recall definitions of epigraphical sets and epigraphical multifunctions which play an important role in establishing necessary optimality conditions.

Definition 3.1 (*Epigraphical and hypergraphical properties*) Suppose that (H1) holds.

(i) The epigraphical (resp. hypergraphical) set of A with respect to Θ is defined by

$$\mathcal{E}^\Theta(A) := A + \Theta \text{ (resp. } \mathcal{H}^\Theta(A) := A - \Theta).$$

The set A is said to have the epigraphical (resp. hypergraphical) property if $A = \mathcal{E}^\Theta(A)$ (resp. $A = \mathcal{H}^\Theta(A)$).

(ii) The epigraphical (resp. hypergraphical) multifunction of a set-valued map $F : X \rightrightarrows Y$ with respect to Θ is defined by

$$\mathcal{E}^\Theta_F(x) := \mathcal{E}^\Theta(F(x)) = F(x) + \Theta \tag{8}$$

and

$$\mathcal{H}^\Theta_F(x) := \mathcal{H}^\Theta(F(x)) = F(x) - \Theta. \tag{9}$$

The map F is said to have the epigraphical (resp. hypergraphical) property if for every $x \in X$, $F(x)$ enjoys the corresponding property.

For the sake of simplicity, we drop the superscript Θ and/or subscript F in the epigraphical and hypergraphical notations.

Proposition 3.1 (Properties of epigraphical and hypergraphical sets) *Suppose that (H1) holds. Then,*

(i) $[\mathcal{E}(A)]^c$ *has the hypergraphical property, i.e.,* $\mathcal{H}([\mathcal{E}(A)]^c) = [\mathcal{E}(A)]^c - \Theta$.
(ii) $[\mathcal{H}(A)]^c$ *has the epigraphical property, i.e.,* $\mathcal{E}([\mathcal{H}(A)]^c) = [\mathcal{H}(A)]^c + \Theta$.
(iii) $A \leq^l_\Theta B \iff \mathcal{E}(A) \leq^l_\Theta \mathcal{E}(B) \iff [\mathcal{E}(A)]^c \leq^u_\Theta [\mathcal{E}(B)]^c$.
(iv) $A \leq^u_\Theta B \iff \mathcal{H}(A) \leq^u_\Theta \mathcal{H}(B) \iff [\mathcal{H}(A)]^c \leq^l_\Theta [\mathcal{H}(B)]^c$.

Proof Let us prove (i) and (iii); the others could be proved in a similar way.

(i) Since $0 \in \Theta$, we have $[\mathcal{H}(A)]^c \subseteq [\mathcal{H}(A)]^c + \Theta$. We prove the truth of the inverse inclusion by contradiction. Assume that there are $a \in [\mathcal{H}(A)]^c$ and $y \in \Theta$ such that $a + y \notin [\mathcal{H}(A)]^c$, i.e., $a + y \in \mathcal{H}(A) = A - \Theta$. Then, $a \in A - \Theta - y \subseteq A - \Theta = \mathcal{H}(A)$, clearly implying $a \notin [\mathcal{H}(A)]^c$. This contradiction justifies that $[\mathcal{H}(A)]^c + \Theta \subseteq [\mathcal{H}(A)]^c$ and thus $[\mathcal{H}(A)]^c + \Theta = [\mathcal{H}(A)]^c$.

(iii) We have

$$A \preceq^l_\Theta B \Longleftrightarrow B \subseteq A + \Theta \Longleftrightarrow B + \Theta \subseteq A + \Theta + \Theta \Longleftrightarrow \mathcal{E}(A) \preceq^l_\Theta \mathcal{E}(B)$$

and

$$\mathcal{E}(A) \preceq^l_\Theta \mathcal{E}(B) \Longleftrightarrow [\mathcal{E}(A)]^c \subseteq [\mathcal{E}(B)]^c \overset{(i)}{=} [\mathcal{E}(B)]^c - \Theta \Longleftrightarrow [\mathcal{E}(A)]^c \preceq^u_\Theta [\mathcal{E}(B)]^c.$$

The proof is complete. □

To characterize the upper set-less relation $\preceq^u_\Theta$ in Definition 2.5, we use the hypergraphical scalarization functional of a set B and a scalarization direction $k \in \mathrm{sd}\,(-\mathcal{H}(B))$ defined by

$$s_{B,k}(y) := \inf\{t \in \mathbb{R} \ : \ y \in tk + \mathcal{H}(B)\} = \varphi_{-\mathcal{H}(B),k}(y). \tag{10}$$

The next proposition provides several important properties of the hypergraphical scalarization functional $s_{B,k}(y)$ needed in the sequence.

Proposition 3.2 (Properties of hypergraphical scalarization functionals) *Assume that (H1) holds and $k \in \mathrm{sd}\,(-\mathcal{H}(B))$. Consider the hypergraphical scalarization functional $s_{B,k}$ defined in (10). Then,*

- (i) $y \in \mathcal{H}(B) \Longrightarrow s_{B,k}(y) \le 0$. *The inverse implication ($\Longleftarrow$) holds provided that $-\mathcal{H}(B)$ is k-vectorially closed.*
- (ii) *If $t \in s_{B,k}(\mathcal{H}(A))$, then $]-\infty, t] \subseteq s_{B,k}(\mathcal{H}(A))$.*
- (iii) *We always have $s_{B,k}(\mathcal{H}(B)) \subseteq \ ]-\infty, 0]$. When $-\mathcal{H}(B)$ is k-vectorially closed, $s_{B,k}(\mathcal{H}(B)) = \,]-\infty, 0]$.*
- (iv) $\mathcal{H}(A) \subseteq \mathcal{H}(B) \Longrightarrow s_{B,k}(\mathcal{H}(A)) \subseteq \ ]-\infty, 0]$, *where the inverse implication ($\Longleftarrow$) holds under an additional condition that $-\mathcal{H}(B)$ is k-vectorially closed.*
- (v) *Assume that both $-\mathcal{H}(B)$ and $-\mathcal{H}(A)$ are k-vectorially closed. Then, $s_{B,k}(\mathcal{H}(A)) = s_{A,k}(\mathcal{H}(B)) = \ ]-\infty, 0]$ if and only if $\mathcal{H}(A) = \mathcal{H}(B)$.*

Proof (i) By the definition of $s_{B,k} = \varphi_{-\mathcal{H}(B),k}$, we get from $y \in \mathcal{H}(B) = 0k - (-\mathcal{H}(B))$ that $s_{B,k}(y) = \inf\{t \in \mathbb{R} \ : \ y \in tk + \mathcal{H}(B)\} \le 0$. Assume now that $\mathcal{H}(B)$ is a k-vectorially closed set and there is some element y such that $s_{B,k}(y) = t \le 0$. By the definition of $s_{B,k} = \varphi_{-\mathcal{H}(B),k}$, we for every $\varepsilon \ge 0$ sufficiently small have

$$y \in (t + \varepsilon)k + \mathcal{H}(B) \Longleftrightarrow (-y + tk) + \varepsilon k \in -\mathcal{H}(B).$$

Taking to the account the k-vectorial closedness of $-\mathcal{H}(B)$ and the property of the scalarization direction, we obtain

$$-y + tk \in \text{vcl}_k(-\mathcal{H}(B)) = -\mathcal{H}(B) \Longleftrightarrow y \in tk + \mathcal{H}(B) \subseteq \mathcal{H}(B).$$

(ii) If $t \in s_{B,k}(\mathcal{H}(A))$, then we could find $a \in \mathcal{H}(A)$ such that $s_{B,k}(a) = t$. Since $a - \tau k \in \mathcal{H}(A)$ for any $\tau \in [0, +\infty[$, Lemma 2.1 (e) yields $s_{B,k}(a - \tau k) = s_{B,k}(b) - \tau = t - \tau$. Since τ was arbitrary in $[0, +\infty[$, we have $]-\infty, t] = t - [0, +\infty[\subseteq s_{B,k}(\mathcal{H}_A)$.

(iii) $s_{B,k}(\mathcal{H}(B)) = \;]-\infty, 0]$ is derived from (i) and (ii).

(iv) The $\Longleftarrow$ implication holds due to (i), and the $\Longrightarrow$ implication holds due to (iii).

(v) Assume that $-\mathcal{H}(B)$ and $-\mathcal{H}(A)$ are k-vectorially closed and

$$s_{B,k}(\mathcal{H}(A)) = s_{A,k}(\mathcal{H}(B)) = \;]-\infty, 0].$$

By (iv), $\mathcal{H}(A) \subseteq \mathcal{H}(B)$ and $\mathcal{H}(B) \subseteq \mathcal{H}(A)$, and thus $\mathcal{H}_A = \mathcal{H}_B$.

□

Remark 3.1 (a) Proposition 3.2 (iv) does not hold when the inclusion is replaced with an equality; i.e., $s_{B,k}(\mathcal{H}(A)) = \;]-\infty, 0]$ does not imply $\mathcal{H}(A) = \mathcal{H}(B)$. For example, let $\Theta = \mathbb{R}^2_+$, $k = (1, 1)$, $\mathcal{H}(B) = \mathbb{B}_{\mathbb{R}^2} - \mathbb{R}^2_+$ and $\mathcal{H}(A) =]-\infty, 1] \times]-\infty, -2]$. Obviously, $\mathcal{H}(A) \subseteq \mathcal{H}(B)$ and $\mathcal{H}(A) \neq \mathcal{H}(B)$. By Proposition 3.2 (ii), we have $s_{B,k}(\mathcal{H}(A)) = \;]-\infty, 0]$ since $s_{B,k}(\overline{y}) = 0$ and $\overline{y} = (1, -2) \in \mathcal{H}(A)$.

(b) The requirement $= \;]-\infty, 0]$ in Proposition 3.2 (v) is essential for $\mathcal{H}(A) = \mathcal{H}(B)$; in the other words, $s_{B,k}(\mathcal{H}(A)) = s_{A,k}(\mathcal{H}(B))$ does not imply $\mathcal{H}(A) = \mathcal{H}(B)$. Construct two sets $\mathcal{H}(A) := \;]-\infty, 0] \times \;]-\infty, 1]$ and $\mathcal{H}(B) := \;]-\infty, 1] \times]-\infty, 0]$ in $\mathbb{R}^2$ with $\Theta = \mathbb{R}^2_+$ and take $k = (1, 1)$. We have $s_{B,k}(\mathcal{H}(A)) = s_{A,k}(\mathcal{H}(B)) = \;]-\infty, 1]$, but $\mathcal{H}(A) \neq \mathcal{H}(B)$.

Proposition 3.3 (Characterization of the upper set-less relation $\preceq^u_\Theta$) *Assume that (H1) holds. Then,*

$$A \preceq^u_\Theta B \Longrightarrow \forall k \in \text{sd}\,(-\mathcal{H}(B)), s_{B,k}(\mathcal{H}(A)) \subseteq \;]-\infty, 0]. \tag{11}$$

The reverse implication holds provided that there is $k \in$ sd $(-H(B))$ such that $-\mathcal{H}(B)$ is k-vectorially closed.

Proof Assume that $A \preceq^u_\Theta B \Longleftrightarrow A \subseteq B - \Theta \Longleftrightarrow \mathcal{H}(A) \subseteq \mathcal{H}(B)$. For every $k \in \text{sd}\,(-\mathcal{H}(B))$ and for every $a \in \mathcal{H}(A) \subseteq \mathcal{H}(B)$, we have $a \in 0k + \mathcal{H}(B)$ and thus

$$s_{B,k}(a) = \inf\{t \in \mathbb{R} \;:\; a \in tk + \mathcal{H}(B)\} \leq 0.$$

Therefore, (11) holds.

Assume now that (11) holds for some $k \in \text{sd}\,(-\mathcal{H}(B))$ and $-\mathcal{H}(B)$ is a k-vectorially closed set. By Proposition 3.2, we have $\mathcal{H}(A) \subseteq \text{vcl}_k\mathcal{H}(B) = \mathcal{H}(B)$. By Proposition 3.1 (iv), the last inclusion is equivalent to $A \preceq^u_\Theta B$. The proof is complete. □

Remark 3.2 **(Comparisons with known results)** In [24, Theorem 3], Köbis and Tammer formulated the following characterization for the upper set-less relation $\preceq^u_C$ in a real linear topological space Y equipped with a nontrivial, closed, convex, and pointed cone C of Y. For any two sets $A, B \subseteq Y$, we have

$$A \preceq^u_C B \Longrightarrow \forall k \in C \setminus \{0\}, \sup_{a\in A} \inf_{b\in B} \varphi_{C,k}(a-b) \le 0. \tag{12}$$

Assume that there is $k \in C \setminus \{0\}$ such that for all $a \in A$ the infimum $\inf_{b\in B} \varphi_{C,k}(a-b)$ is attained, which is called the attainment property on $a - B$ for all $a \in A$. Then, the converse of (12) holds.

It is important to mention that $\mathrm{sd}\,(C) = C \setminus \{0\}$, when C is a nontrivial, closed, convex, and pointed cone. Observe that

$$\begin{aligned}\inf_{b\in B} \varphi_{C,k}(a-b) &= \inf_{b\in B} \inf\{t \in \mathbb{R} \ : \ a-b \in tk - C\} \\ &= \inf_{b\in B} \inf\{t \in \mathbb{R} \ : \ a \in tk + b - C\} \\ &= \inf\{t \in \mathbb{R} \ : \ a \in tk + B - C\} \\ &= \ \varphi_{-\mathcal{H}(B),k}(a) = s_{B,k}(a)\end{aligned}$$

and

$$\begin{aligned}&\sup_{a\in A} \inf_{b\in B} \varphi_{C,k}(a-b) \le 0 \Longleftrightarrow \sup_{a\in A} s_B(a) \le 0 \\ \Longleftrightarrow \quad &\sup_{a\in\mathcal{H}(A)} \ s_{B,k}(a) \le 0 \Longleftrightarrow s_{B,k}(\mathcal{H}(A)) \subseteq \,]-\infty, 0].\end{aligned}$$

The difference between these two scalar characterizations is that our result is established for the case of the k-vectorial closedness of $-\mathcal{H}(B)$ and the nonemptiness of $\mathrm{sd}\,(-\mathcal{H}(B))$, while the other needs the attainment assumption that seems to be not easy to verify.

Next, we will study scalar characterizations of the lower set-less relation $\preceq^l_\Theta$ in two ways. The first way is based on the relationship between the lower and upper set-less relations:

$$A \preceq^l_\Theta B \Longleftrightarrow B \subseteq A + \Theta = A - (-\Theta) \Longleftrightarrow B \preceq^u_{-\Theta} A.$$

Given $k \in \mathrm{sd}\,(-\mathcal{E}(A))$. Considering the epigraphical scalarization functional $e_{A,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ with

$$e_{A,k}(y) := \varphi_{-\mathcal{E}(A),k}(y), \tag{13}$$

we get the following characterization of the lower set-less relation $\preceq^l_\Theta$.

Corollary 3.1 (Characterization of the lower set-less relation $\preceq^l_\Theta$) *Assume that (H1) holds. Then,*

$$A \preceq^l_\Theta B \Longrightarrow \forall k \in (-\mathcal{E}(A)),\, e_{A,k}(\mathcal{E}(B)) \subseteq\,]-\infty, 0]. \tag{14}$$

Assume that there is $k \in \operatorname{sd}(-\mathcal{E}(A))$ such that $-\mathcal{E}(A)$ is k-vectorially closed. Then, $e_{A,k}(\mathcal{E}(B)) \subseteq]-\infty, 0]$ implies that $A \preceq^l_\Theta B$.

Proof Since $A \preceq^l_\Theta B \Longleftrightarrow B \preceq^u_{-\Theta} A$, we consider the space Y equipped with the nontrivial and convex cone $C := -\Theta$. Then, $\mathcal{H}_C(A) = A - C = A + \Theta = \mathcal{E}_\Theta(A)$, $\mathcal{H}_C(B) = \mathcal{E}_\Theta(B)$, and $e_{A,k}(y) = \varphi_{-\mathcal{H}_C(A),k}(y) = \varphi_{-\mathcal{E}_\Theta(A),k}(y)$. This corollary is derived from Proposition 3.3 applied to $B \preceq^u_C A$. □

The second way to derive a characterization for the lower set-less relation is based on the relationship

$$A \preceq^l_\Theta B \Longleftrightarrow [\mathcal{E}(A)]^c \preceq^u_\Theta [\mathcal{E}(B)]^c$$

[see, Proposition 3.1 (iii)]. For the sake of simplicity, we use $\mathcal{E}^c(A)$ for $[\mathcal{E}(A)]^c$. Given $k \in \operatorname{sd}(-\mathcal{E}^c(B))$. We define the functional $s^c_{B,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ by

$$s^c_{B,k}(y) := \varphi_{-\mathcal{E}^c(B),k}(y), \tag{15}$$

in order to obtain another scalar characterization for the lower set-less relation $\preceq^l_\Theta$.

Corollary 3.2 (Characterization of the lower set-less relation $\preceq^l_\Theta$) *Assume that (H1) holds. Then,*

$$A \preceq^l_\Theta B \Longrightarrow \forall k \in \operatorname{sd}(-\mathcal{E}^c(B)),\, s^c_{B,k}(\mathcal{E}^c(A)) \subseteq]-\infty, 0]. \tag{16}$$

Assume in addition that $\operatorname{vcl}(\mathcal{E}^c(B)) = [\operatorname{core}(\mathcal{E}(B))]^c$ *and* $\mathcal{E}(A) = \operatorname{vcl}\mathcal{E}(A)$. *Then,* $s^c_{B,k}(\mathcal{E}^c(A)) \subseteq]-\infty, 0]$ *implies* $A \preceq^l_\Theta B$.

Proof By Proposition 3.1 (iii), $A \preceq^l_\Theta B \Longleftrightarrow [\mathcal{E}(A)]^c \preceq^u_\Theta [\mathcal{E}(B)]^c$. By Proposition 3.3, (16) holds.

Assume now that $s^c_{B,k}(\mathcal{E}^c(A)) \subseteq]-\infty, 0]$. Then, we get

$$[\mathcal{E}(A)]^c \subseteq \operatorname{vcl}([\mathcal{E}(B)]^c) = [\operatorname{core}(\mathcal{E}(B))]^c$$

clearly implying that $\mathcal{E}(A) \supseteq \operatorname{core}(\mathcal{E}(B))$. Hence, we have

$$\mathcal{E}(A) = \operatorname{vcl}(\mathcal{E}(A)) \subseteq \operatorname{vcl}(\operatorname{core}(\mathcal{E}(B))) \subseteq \mathcal{E}(B)$$

and thus $A \preceq^l_\Theta B$. The proof is complete. □

4 Characterization of Set-Less Solutions

This section is devoted to establishing characterizations for $\leq_\Theta^\circ$-minimizers of a set-valued map F, where $\leq_\Theta^\circ$ stands for either lower or upper set-less relation $\leq_\Theta^l$ or $\leq_\Theta^u$ introduced in Definition 2.5. It is important to emphasize that the setting of this paper covers many important general situations. Let us provide two problems.

First, consider a constrained set optimization problem of the form

$$\leq_\Theta^\circ\text{–Min } F(x) \quad \text{subject to} \quad x \in \Omega,$$

where $F : X \rightrightarrows Y$ is a set-valued map, $\leq_\Theta^\circ$ is a set-less relation in Y defined via a nontrivial and convex cone Θ, and Ω is a subset of X. By considering a restricted set-valued map $F_\Omega : X \rightrightarrows Y$ with

$$F_\Omega(x) := \begin{cases} F(x), & \text{if } x \in \Omega, \\ \emptyset, & \text{if } x \in \Omega, \end{cases}$$

a point $\overline{x} \in \operatorname{dom} F \cap \Omega$ is a $\leq_\Theta^\circ$-minimizer of F over Ω if and only if it is a $\leq_\Theta^\circ$-minimizer of F_Ω.

The second example is to find a $\leq_\Theta^\circ$-minimal set from a family of sets. Given a family of nonempty sets of a real linear space Y denoted by $\mathcal{A} := \{A_i\}_{i\in I}$ and a set-less relation $\leq_\Theta^\circ$ in Y. By considering a set-valued map $F_\mathcal{A} : I \rightrightarrows Y$ denoted by

$$\forall i \in I, \ F_\mathcal{A}(i) := A_i,$$

A_{i_0} is a $\leq_\Theta^\circ$-minimal set of $\mathcal{A}$ if and only if it is a $\leq_\Theta^\circ$-minimizer of $F_\mathcal{A}$.

(H2) *Standing assumption for maps* X is a nonempty set. Y is a real linear space. $F : X \rightrightarrows Y$ is a set-valued map with $\operatorname{dom} F \neq \emptyset$. Θ is a nontrivial and convex cone in Y.

Next, we consider two important kinds of set-less relations in set optimization: the upper set-less relation $\leq_\Theta^u$ and the lower set-less relation $\leq_\Theta^l$ defined in Definition 2.5. We will show that by choosing an appropriate scalarization functional, an $\leq_\Theta^u$-minimizer of a set-valued map is also an $\leq_{\mathbb{R}_+}^u$-minimal solution of the scalar real-valued set-valued map.

Given a set-valued map $F : X \rightrightarrows Y$ and an element $\overline{x} \in \operatorname{dom} F$. Consider the hypergraphical scalarization functional $s_{\overline{x},k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ along the scalarization direction $k \in \operatorname{sd}(-\mathcal{H}(\overline{x}))$ from Y to the set of extended real-valued numbers defined by

$$s_{\overline{x},k}(y) := \varphi_{-\mathcal{H}(\overline{x}),k}(y) = \inf\{t \in \mathbb{R} \ : \ y \in tk + \mathcal{H}(\overline{x})\}. \tag{17}$$

It is well defined since $k \in \operatorname{sd}(-\mathcal{H}(\overline{x}))$ by Lemma 2.1.

The generalized composition of $\mathcal{H}$ and $s_{\overline{x},k}$ is a set-valued map with values

$$s_{\overline{x},k} \circ \mathcal{H}(x) := s_{\overline{x},k}(\mathcal{H}(x)) = \cup\{s_{\overline{x},k}(y) \ : \ \forall y \in F(x) - \Theta\}. \tag{18}$$

We call it the scalarized map of F via $s_{\overline{x},k}$. Furthermore, a minimizer of $s_{\overline{x},k} \circ \mathcal{H}$ with respect to the space of real numbers and the upper set-less relation $\preceq^u_{\mathbb{R}_+}$ is called an $\preceq^u_{\mathbb{R}_+}$-minimal solution for short.

Theorem 4.1 (Characterization of $\preceq^u_\Theta$-minimizers) *Assume that (H2) holds and $\overline{x} \in$ dom F.*

(i) *If $\overline{x}$ is an $\preceq^u_\Theta$-minimizer of F, then for every $k \in$ sd $(-\mathcal{H}(\overline{x}))$, $\overline{x}$ is an $\preceq^u_{\mathbb{R}_+}$-minimal solution of the scalarized map $s_{\overline{x},k} \circ \mathcal{H}$.*

(ii) *Assume that there is $k \in$ sd $(-\mathcal{H}(x))$ for all $x \in$ dom F and that $-\mathcal{H}$ is a k-vectorially closed valued map, i.e., $-\mathcal{H}(x)$ is k-vectorially closed for all $x \in$ dom $\mathcal{H}$. Assume also that*

$$\forall x \in \text{dom}\, F, s_{\overline{x},k}(\mathcal{H}(x)) = s_{\overline{x},k}(\mathcal{H}(\overline{x})) \Longrightarrow s_{x,k}\big(\mathcal{H}(\overline{x})\big) = s_{x,k}(\mathcal{H}(x)). \tag{19}$$

Then, $\overline{x}$ is an $\preceq^u_\Theta$-minimizer of F if $\overline{x}$ is an $\preceq^u_{\mathbb{R}_+}$-minimal solution of the scalarized map $s_{\overline{x},k} \circ \mathcal{H}$.

Proof (i) Assume that $\overline{x}$ is an $\preceq^u_\Theta$-minimizer of F, i.e.,

$$\forall x \in \text{dom}\, F, \mathcal{H}(x) \subseteq \mathcal{H}(\overline{x}) \Longrightarrow \mathcal{H}(x) = \mathcal{H}(\overline{x}),$$

where $\mathcal{H}(x) = \mathcal{H}(\overline{x})$ is used due to the hypergraphical property of $\mathcal{H}$. We need to show that $\overline{x}$ is an $\preceq^u_{\mathbb{R}_+}$-minimal solution of $s_{\overline{x},k} \circ \mathcal{H}$. By Proposition 3.2 (ii), $s_{\overline{x},k} \circ \mathcal{H}$ enjoys the hypergraphical property and thus it is sufficient to show that for every $k \in$ sd $(-\mathcal{H}(\overline{x}))$ and for every $x \in$ dom F, we have

$$s_{\overline{x},k}(\mathcal{H}(x)) \subseteq s_{\overline{x},k}(\mathcal{H}(\overline{x})) \Longrightarrow s_{\overline{x},k}(\mathcal{H}(x)) = s_{\overline{x},k}(\mathcal{H}(\overline{x})).$$

Fix an arbitrary scalarization direction $k \in$ sd $(-\mathcal{H}(\overline{x}))$ and an arbitrary element $x \in$ dom F such that $s_{\overline{x},k}(\mathcal{H}(x)) \subseteq s_{\overline{x},k}(\mathcal{H}(\overline{x}))$. By Proposition 3.2, $s_{\overline{x},k}(\mathcal{H}(x)) \subseteq]-\infty, 0]$, i.e., $\forall y \in \mathcal{H}(x), s_{\overline{x},k}(y) = \varphi_{-\mathcal{H}(\overline{x}),k}(y) \leq 0$. The latter implies that $y \in \mathcal{H}(\overline{x})$. Since y was arbitrary in $\mathcal{H}(x)$, $\mathcal{H}(x) \subseteq \mathcal{H}(\overline{x})$. The $\preceq^u_\Theta$-minimality of F implies that $\mathcal{H}(\overline{x}) = \mathcal{H}(x)$, and thus, $s_{\overline{x},k}(\mathcal{H}(x)) = s_{\overline{x},k}(\mathcal{H}(\overline{x}))$. This verifies the $\preceq^u_{\mathbb{R}}$ $_+$-minimality of $\overline{x}$ to $s_{\overline{x},k} \circ \mathcal{H}$.

(ii) Assume that $\overline{x}$ is an $\preceq^u_{\mathbb{R}}$ $_+$-minimal solution of $s_{\overline{x},k} \circ \mathcal{H}$ and condition (19) holds. Fix an arbitrary element $x \in$ dom F such that $\mathcal{H}(x) \subseteq \mathcal{H}(\overline{x})$. Obviously, we have $s_{\overline{x},k}(\mathcal{H}(x)) \subseteq s_{\overline{x},k}(\mathcal{H}(\overline{x}))$. The minimality of $s_{\overline{x},k} \circ \mathcal{H}$ forces $s_{\overline{x},k}(\mathcal{H}(x)) = s_{\overline{x},k}(\mathcal{H}(\overline{x}))$. Taking into account condition (19), we have $s_{x,k}(\mathcal{H}(x)) = s_{x,k}(\mathcal{H}(\overline{x}))$. Proposition 3.2 and the k-vectorial closedness of $-\mathcal{H}(x)$ imply that $\mathcal{H}(\overline{x}) \subseteq \mathcal{H}(x)$ and thus $\mathcal{H}(x) = \mathcal{H}(\overline{x})$. Since x was arbitrary, $\overline{x}$ is an $\preceq^u_\Theta$-minimizer of F. □

Let us illustrate Theorem 4.1 with an example.

Example 4.1 Consider a set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ with

$$F(x) := \begin{cases} [0,1] \times \{x\}, & \text{if } x \in [0,1[,\\ \{(a, \sqrt{1-a^2}) \ : \ a \in [0,1]\}, & \text{if } x = 1,\\ \emptyset, & \text{otherwise,} \end{cases}$$

and $\Theta = \mathbb{R}^2_+$. Then, the hypergraphical map of F is given by

$$\mathcal{H}(x) = \begin{cases}]-\infty, 1] \times]-\infty, x], & \text{if } x \in [0,1[,\\ \{(a, \sqrt{1-a^2}) \ : \ a \in [0,1]\} - \mathbb{R}^2_+, & \text{if } x = 1,\\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easy to check that $\overline{x} = 0$ is an $\preceq^u_{\mathbb{R}^2_+}$-minimizer of F and that $\hat{x} = 1$ is not an $\preceq^u_{\mathbb{R}^2_+}$-minimizer of F ($\mathcal{H}(0) \subseteq \mathcal{H}(1)$, but $\mathcal{H}(0) \neq \mathcal{H}(1)$).

Fix $k = (1,1) \in \operatorname{sd}(\mathcal{H}(1)) = \operatorname{sd}(\mathcal{H}(0))$. Consider two hypergraphical scalarization functionals $s_{1,k} := \varphi_{-\mathcal{H}(1),k}$ and $s_{0,k} := \varphi_{-\mathcal{H}(0),k}$. We find

$$(s_{1,k} \circ \mathcal{H})(x) = \begin{cases}]-\infty, x^2/(2+2x)], & \text{if } x \in [0,1[\ ,\\ ]-\infty, 0], & \text{if } x = 1,\\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$(s_{0,k} \circ \mathcal{H})(x) = \begin{cases}]-\infty, x], & \text{if } x \in [0,1],\\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easy to check that $\hat{x} = 1$ is an $\preceq^u_{\mathbb{R}_+}$-minimal solution of the scalarized map $s_{1,k} \circ \mathcal{H}$, but not an $\preceq^u_{\mathbb{R}^2_+}$-minimizer of F. Since

$$s_{1,k}(\mathcal{H}(0)) = s_{1,k}(\mathcal{H}(1)) =]-\infty, 0] \ \text{ but}$$
$$s_{0,k}(\mathcal{H}(1)) =]-\infty, 1] \neq s_{0,k}(\mathcal{H}(0)) =]-\infty, 0],$$

condition (19) is not fulfilled for $\overline{x} = 0$ and thus Theorem 4.1 is not applicable to $\hat{x} = 1$.

We could easily check that $\overline{x} = 0$ is an $\preceq^u_{\mathbb{R}_+}$-minimal solution of $s_{0,k} \circ \mathcal{H}$ and condition (19) is satisfied; indeed, for all $x \in (0,1]$, we have $s_{0,k}(\mathcal{H}(x)) =]-\infty, x] \neq s_{0,k}(\mathcal{H}(0)) =]-\infty, 0]$. By Theorem 4.1, $\overline{x} = 0$ is an $\preceq^u_{\mathbb{R}^2_+}$-minimizer of F.

Theorem 4.2 (Characterization of strict/strong/ideal $\preceq^u_\Theta$-minimizers) *Assume that (H2) holds.*

(i) *If $\overline{x}$ is an x-partly strict $\preceq^u_{\mathbb{R}_+}$-minimizer of $s_{\overline{x},k} \circ F$ for some scalarization direction $k \in \operatorname{sd}(-\mathcal{H}(\overline{x}))$, then $\overline{x}$ is an x-partly strict $\preceq^u_\Theta$-minimizer of F. The reverse* *implication holds for every scalarization direction* $k \in \operatorname{sd}(-\mathcal{H}(\overline{x}))$ *such that* $-\mathcal{H}(\overline{x})$ *is* k*-vectorially closed.*

(ii) *Assume that* $\operatorname{sd}(-\mathcal{H}(x)) \neq \emptyset$ *for all* $x \in \operatorname{dom} F$. *If* $\overline{x}$ *is a strong/ideal* $\preceq^u_\Theta$*-minimizer of* F*, then*

$$\forall x \in \operatorname{dom} F, \forall k \in \operatorname{sd}(-\mathcal{H}(x)), s_{x,k}(\mathcal{H}(\overline{x})) \subseteq \mathbb{R}_-. \tag{20}$$

The reverse implication holds provided that for every $x \in \operatorname{dom} F$*, there exists* $k \in \operatorname{sd}(-\mathcal{H}(x))$ *such that* $-\mathcal{H}(x)$ *is* k*-vectorially closed.*

(iii) *If* $\overline{x}$ *is a strong/ideal* $\preceq^u_\Theta$*-minimizer of* F*, then* $\overline{x}$ *is a strong/ideal* $\preceq^u_{\mathbb{R}_+}$*-minimal solution of the scalarized map* $\varphi_{\mathcal{H}^c(\overline{x}),k} \circ \mathcal{H}^c$*, where*

$$\forall x \in \operatorname{dom} F, \mathcal{H}^c(x) := Y \setminus \mathcal{H}(x).$$

The reverse implication holds provided that for all $x \in \operatorname{dom} F$*, one has* $\mathcal{H}(x) = \operatorname{vcl} \mathcal{H}(x)$ *and* $\operatorname{core} \mathcal{H}(x) \neq \emptyset$*.*

Proof (i) Assume that $\overline{x}$ is an x-partly strict $\preceq^u_\Theta$-minimizer of F, i.e.,

$$\forall x \in \operatorname{dom} F, \mathcal{H}(x) \subseteq \mathcal{H}(\overline{x}) \Longrightarrow x = \overline{x}. \tag{21}$$

Fix an arbitrary scalarization direction $k \in \operatorname{sd}(\mathcal{H}(\overline{x}))$ and an arbitrary element $x \in \operatorname{dom} F$ such that

$$s_{\overline{x},k} \circ \mathcal{H}(x) \subseteq s_{\overline{x},k} \circ \mathcal{H}(\overline{x}) \subseteq]-\infty, 0]. \tag{22}$$

To verify the x-partly strict $\preceq^u_{\mathbb{R}_+}$-minimality of $\overline{x}$ to $s_{\overline{x},k} \circ F$, it is sufficient to show that $x = \overline{x}$. By the definition of the functional $s_{\overline{x},k}$, we get from (22) that

$$\forall y \in \mathcal{H}(x), s_{\overline{x},k}(y) \leq 0.$$

Since $-\mathcal{H}(\overline{x})$ is a k-vectorially closed set, we get $y \in \mathcal{H}(\overline{x})$. Since y was arbitrary in $\mathcal{H}(x)$, we have $\mathcal{H}(x) \subseteq \mathcal{H}(\overline{x})$. By (21), we obtain $x = \overline{x}$.

To prove the inverse implication, we assume that $\overline{x}$ is an x-partly strict $\preceq^u_{\mathbb{R}_+}$-minimal solution of $s_{\overline{x},k} \circ \mathcal{H}$ for some scalarization direction $k \in \operatorname{sd}(-H(\overline{x}))$, i.e.,

$$\forall x \in \operatorname{dom} F, s_{\overline{x},k} \circ \mathcal{H}(x) \subseteq s_{\overline{x},k} \circ \mathcal{H}(\overline{x}) \Longrightarrow x = \overline{x}.$$

Fix an arbitrary element $x \in \operatorname{dom} F$ such that $\mathcal{H}(x) \subseteq \mathcal{H}(\overline{x})$. Applying the functional $s_{\overline{x},k}$ to both sides of the inclusion, we have $s_{\overline{x},k} \circ \mathcal{H}(x) \subseteq s_{\overline{x},k} \circ \mathcal{H}(\overline{x})$. The x-partly strict $\preceq^u_{\mathbb{R}_+}$-minimality of $\overline{x}$ to $s_{\overline{x},k} \circ \mathcal{H}$ forces $x = \overline{x}$, and hence, $\overline{x}$ is an x-partly strict $\preceq^u_\Theta$-minimizer of F.

(ii) It is directly derived from Proposition 3.2.

$$\begin{aligned}
&\overline{x} \text{ is a strong } \preceq^u_\Theta \text{-minimizer of } F\\
&\Longleftrightarrow \forall x \in \operatorname{dom} F, \mathcal{H}(\overline{x}) \subseteq \mathcal{H}(x)\\
&\overset{\text{vcl}\,k}{\Longleftrightarrow} \forall x \in \operatorname{dom} F, s_{x,k}(\mathcal{H}(\overline{x})) \subseteq s_{x,k}(\mathcal{H}(x)) = \mathbb{R}_-.
\end{aligned}$$

(iii)

$$\begin{aligned}
&\overline{x} \text{ is a strong } \preceq^u_\Theta \text{-minimizer of } F\\
&\Longleftrightarrow \forall x \in \operatorname{dom} F, \mathcal{H}(\overline{x}) \subseteq \mathcal{H}(x)\\
&\Longleftrightarrow \forall x \in \operatorname{dom} F, \mathcal{H}^c(x) \subseteq \mathcal{H}^c(\overline{x})\\
&\Longrightarrow \forall x \in \operatorname{dom} F, \varphi_{-\mathcal{H}^c(\overline{x}),k}(\mathcal{H}^c(x)) \subseteq \varphi_{-\mathcal{H}^c(\overline{x}),k}(\mathcal{H}^c(\overline{x}))\\
&\Longleftrightarrow \overline{x} \text{ is a strong } \preceq^u_{\mathbb{R}_+} \text{-minimal solution of } \varphi_{-\mathcal{H}^c(\overline{x}),k} \circ \mathcal{H}^c.\\
&\overline{x} \text{ is a strong } \preceq^u_{\mathbb{R}_+} \text{-minimal solution of } \varphi_{-\mathcal{H}^c(\overline{x}),k} \circ \mathcal{H}^c\\
&\Longleftrightarrow \forall x \in \operatorname{dom} F, \varphi_{-\mathcal{H}^c(\overline{x}),k}(\mathcal{H}^c(x)) \subseteq \varphi_{-\mathcal{H}^c(\overline{x}),k}(\mathcal{H}^c(\overline{x}))\\
&\Longrightarrow \forall x \in \operatorname{dom} F, \operatorname{vcl} \mathcal{H}^c(x) \subseteq \operatorname{vcl} \mathcal{H}^c(\overline{x})\\
&\overset{\text{core}}{\Longleftrightarrow} \forall x \in \operatorname{dom} F, \operatorname{core} \mathcal{H}(\overline{x}) \subseteq \operatorname{core} \mathcal{H}(x)\\
&\overset{\text{vcl}}{\Longleftrightarrow} \forall x \in \operatorname{dom} F, \mathcal{H}(\overline{x}) \subseteq \mathcal{H}(x)\\
&\Longleftrightarrow \overline{x} \text{ is a strong } \preceq^u_\Theta \text{-minimizer of } F.
\end{aligned}$$

□

Next, we study a characterization for minimizers of a set-valued map F with respect to the lower set-less relation $\preceq^l_\Theta$. First, we derive a characterization for $\preceq^l_\Theta$-minimizers of F based on the known relationship between $\preceq^l_\Theta$-minimizers and $\preceq^u_\Theta$-minimizers by using the epigraphical scalarization functional $s^c_{\overline{x},k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ along the scalarization direction $k \in \text{sd}\ (-\mathcal{E}^c(\overline{x}))$ by

$$s^c_{\overline{x},k}(y) := \varphi_{-\mathcal{E}^c(\overline{x}),k}(y) = \inf\{t \in \mathbb{R} \ : \ y \in tk + \mathcal{E}^c(\overline{x})\}. \tag{23}$$

It is well defined due to Lemma 2.1.

The generalized composition of $\mathcal{E}^c$ and $s^c_{\overline{x},k}$ is a set-valued map with values

$$s^c_{\overline{x},k} \circ \mathcal{E}^c(x) = s^c_{\overline{x},k}(\mathcal{E}^c(x)) := \cup\{s^c_{\overline{x},k}(y) \ : \ \forall y \in \mathcal{E}^c(x) = F(x) - \Theta\}. \tag{24}$$

We call it the scalarized map of F via $s^c_{\overline{x},k}$. Furthermore, a minimal solution of the composition of $\mathcal{E}^c$ and $s^c_{\overline{x},k}$ is called upper set-less minimal solution of $s^c_{\overline{x},k} \circ \mathcal{E}^c$ (with respect to the space of real numbers), for short $\preceq^u_{\mathbb{R}_+}$-minimal solution.

Theorem 4.3 (Characterization of $\preceq^l_\Theta$-minimizers) *Assume that (H2) holds and* $\overline{x} \in \operatorname{dom} F$. *Assume also that* $\mathcal{E}(x) = \operatorname{vcl} \mathcal{E}(x)$ *and* $\operatorname{core} \mathcal{E}(x) \neq \emptyset$ *for all* $x \in \operatorname{dom} F$.

(i) *If* $\overline{x}$ *is a* $\preceq^l_\Theta$*-minimizer of* F*, then* $\overline{x}$ *is an* $\leq^u_{\mathbb{R}_+}$*-minimal solution of the scalarized map* $s^c_{\overline{x},k} \circ \mathcal{E}^c$ *for all* $k \in \operatorname{sd}(-\mathcal{E}^c(\overline{x}))$.

(ii) *Assume that there is* $k \in \operatorname{sd}(-\mathcal{E}^c(\overline{x}))$ *such that*

$$\forall x \in \operatorname{dom} F,\ s^c_{\overline{x},k}(\mathcal{E}^c(x)) = s^c_{\overline{x},k}(\mathcal{E}^c(\overline{x})) \Longrightarrow s^c_x(\mathcal{E}^c(\overline{x})) = s^c_x(\mathcal{E}^c(x)). \quad (25)$$

If $\overline{x}$ *is an* $\leq^u_{\mathbb{R}_+}$*-minimal solution of* $s^c_{\overline{x},k} \circ \mathcal{E}^c$*, then* $\overline{x}$ *is a* $\preceq^l_\Theta$*-minimizer of* F.

Proof To prove (i), first we observe that

$$\begin{aligned}
&\overline{x} \text{ is a } \preceq^l_\Theta \text{-minimizer of } F\\
\Longleftrightarrow\ &\forall x \in \operatorname{dom} F, F(x) \preceq^l_\Theta F(\overline{x}) \Longrightarrow F(\overline{x}) \preceq^l_\Theta F(x)\\
\Longleftrightarrow\ &\forall x \in \operatorname{dom} F, \mathcal{E}(\overline{x}) = F(\overline{x}) + \Theta \subseteq \mathcal{E}(x) = F(x) + \Theta \Longrightarrow \mathcal{E}(x) = \mathcal{E}(\overline{x})\\
\Longleftrightarrow\ &\forall x \in \operatorname{dom} F, \mathcal{E}^c(x) \subseteq \mathcal{E}^c(\overline{x}) \Longrightarrow \mathcal{E}^c(x) = \mathcal{E}^c(\overline{x})\\
\Longleftrightarrow\ &\forall x \in \operatorname{dom} F, \mathcal{E}^c(x) \preceq^u_\Theta \mathcal{E}^c(\overline{x}) \Longrightarrow \mathcal{E}^c(x) = \mathcal{E}^c(\overline{x})\\
\Longleftrightarrow\ &\overline{x} \text{ is an } \preceq^u_\Theta \text{-minimizer of } \mathcal{E}^c.
\end{aligned}$$

To show that $\overline{x}$ is an $\leq^u_{\mathbb{R}_+}$-minimal solution of $s^c_{\overline{x},k} \circ \mathcal{E}^c$, we fix an arbitrary scalarization direction $k \in \operatorname{sd} -\mathcal{E}^c$; it is sufficient to show that

$$\forall x \in \operatorname{dom} F, s^c_{\overline{x},k} \circ \mathcal{E}^c(x) \subseteq s^c_{\overline{x},k} \circ \mathcal{E}^c(\overline{x}) \Longrightarrow s^c_{\overline{x},k} \circ \mathcal{E}^c(x) = s^c_{\overline{x},k} \circ \mathcal{E}^c(\overline{x}).$$

Assume that $s^c_{\overline{x},k} \circ \mathcal{E}^c(x) \subseteq s^c_{\overline{x},k} \circ \mathcal{E}^c(\overline{x}) \subseteq]-\infty, 0]$. It yields $\mathcal{E}^c(x) \subseteq \operatorname{vcl} \mathcal{E}^c(\overline{x}) = X \setminus \{\operatorname{core}(\mathcal{E}(\overline{x}))\}$ and thus $\operatorname{core} \mathcal{E}(\overline{x}) \subseteq \mathcal{E}(x)$. Under the assumptions made, we have $\operatorname{vcl}(\operatorname{core} \mathcal{E}(\overline{x})) = \mathcal{E}(\overline{x}) \subseteq \operatorname{vcl} \mathcal{E}(x) = \mathcal{E}(x)$ and thus $\mathcal{E}^c(x) \subseteq \mathcal{E}^c(\overline{x})$. Since $\overline{x}$ is an $\preceq^u_\Theta$-minimizer of $\mathcal{E}^c$, the last inclusion implies $\mathcal{E}^c(x) = \mathcal{E}^c(\overline{x})$ and thus $s^c_{\overline{x},k} \circ \mathcal{E}^c(x) = s^c_{\overline{x},k} \circ \mathcal{E}^c(\overline{x})$, clearly verifying the $\leq^u_{\mathbb{R}_+}$-minimality of $\overline{x}$ to $s^c_{\overline{x},k} \circ \mathcal{E}^c$.

(ii) Assume that all assumptions are satisfied, to justify the $\preceq^l_\Theta$-minimalily of $\overline{x}$ to F, it is sufficient to show that $\overline{x}$ is an $\preceq^u_\Theta$-minimizer to $\mathcal{E}^c$, i.e.,

$$\mathcal{E}^c(x) \subseteq \mathcal{E}^c(\overline{x}) \Longrightarrow \mathcal{E}^c(x) = \mathcal{E}^c(\overline{x}).$$

Fix an arbitrary element $x \in \operatorname{dom} F$ such that $\mathcal{E}^c(x) \subseteq \mathcal{E}^c(\overline{x})$. We have $s^c_{\overline{x},k}(\mathcal{E}^c(x)) = s^c_{\overline{x},k}(\mathcal{E}^c(\overline{x}))$. By condition (25), we have

$$\begin{aligned}
&s^c_{x,k}(\mathcal{E}^c(\overline{x})) = s^c_{x,k}(\mathcal{E}^c(x)) \Longrightarrow \mathcal{E}^c(\overline{x}) \subseteq \operatorname{vcl} \mathcal{E}^c(x) = X \setminus \{\operatorname{core} \mathcal{E}(x)\}\\
\Longleftrightarrow\ &\operatorname{core} \mathcal{E}(x) \subseteq \mathcal{E}(\overline{x}) = \operatorname{vcl} \mathcal{E}(\overline{x}) \Longrightarrow \mathcal{E}(x) \subseteq \mathcal{E}(\overline{x}) \Longleftrightarrow \mathcal{E}^c(\overline{x}) \subseteq \mathcal{E}^c(x).
\end{aligned}$$

This together with the choice of x gives $\mathcal{E}^c(\overline{x}) = \mathcal{E}^c(x)$ and completes the proof. □

When $F = f : X \to Z$ is a vector-valued function; i.e., $F(x) = \{f(x)\}$ for all $x \in \operatorname{dom} f$, both minimality concepts with respect to $\preceq^u_\Theta$ and $\preceq^l_\Theta$ coincide with the efficiency concept with respect to $\leq_\Theta$. Therefore, we have two scalar characterizations for minimizers of vector-valued functions.

Corollary 4.1 (Version 1: Characterization for minimizers in vector optimization) *Let $F = f$ be a single-valued function with* $\operatorname{dom} f \neq \emptyset$ *and* $\overline{x} \in \operatorname{dom} f$. *Assume that (H2) holds.*

(i) *If $\overline{x}$ is a $\leq_\Theta$-minimizer of f, then for every $k \in \operatorname{sd}(\Theta)$, $\overline{x}$ is a minimal solution of the scalarized function $\varphi_{\Theta - f(\overline{x}),k} \circ f(x)$.*
(ii) *If $\overline{x}$ is a minimal solution of the scalarized function $\varphi_{\Theta - f(\overline{x}),k} \circ f$ and the condition*

$$\varphi_{\Theta - f(\overline{x}),k} \circ f(x) = 0 \Longrightarrow \varphi_{\Theta - f(x),k} \circ f(\overline{x}) = 0$$

holds, then $\overline{x}$ is a $\leq_\Theta$-minimizer of f.

Proof It is straightforward from Theorem 4.1 and the following observations:

(1) $\varphi_{\Theta - f(\overline{x}),k} f(x) = \sup_{\theta \in \Theta} \varphi_{\Theta - f(\overline{x}),k}(f(x) - \theta)$. Fix an element $\theta \in \Theta$ arbitrarily; we have $f(x) - \theta \leq_\Theta f(x)$. Since $\varphi_{\Theta - f(\overline{x}),k}$ is Θ-monotone, we get $\varphi_{\Theta - f(\overline{x}),k}(f(x) - \theta) \leq \varphi_{\Theta - f(\overline{x}),k}(f(x))$. Since θ was arbitrary, we have

$$\sup_{\theta \in \Theta} \varphi_{\Theta - f(\overline{x}),k}(f(x) - \theta) \leq \varphi_{\Theta - f(\overline{x}),k}(f(x)).$$

The equality holds since $0 \in \Theta$.
(2) $\overline{x}$ is an $\leq^u_{\mathbb{R}_+}$-minimal solution of the scalarized map $\varphi_{\Theta - f(\overline{x})} \circ \mathcal{H}_{f,\Theta}$ if and only if it is a minimal solution of the scalarized function of scalar optimization.

□

Corollary 4.2 (Version 2: Characterization for minimizers in vector optimization) *Let f, Θ as in Corollary 4.1. Assume that $\Theta = \operatorname{vcl} \Theta$ and $\operatorname{core} \Theta \neq \emptyset$.*

(i) *If $\overline{x}$ is a $\leq_\Theta$-minimizer of f, then $\overline{x}$ is an $\leq^u_{\mathbb{R}_+}$-minimal solution of the scalarized map $\varphi_{-(\mathcal{E}^c_{f,\Theta}(\overline{x})),k} \circ \mathcal{E}^c_{f,K}$.*
(ii) *If $\overline{x}$ is a minimal solution of the scalarized map $\varphi_{-\mathcal{E}^c_{f,\Theta}(\overline{x}),k} \circ \mathcal{E}^c_{f,K}$ and the condition*

$$\varphi_{-\mathcal{E}^c_{f,\Theta}(\overline{x}),k}(\mathcal{E}^c_{f,K}(x)) \subseteq \mathbb{R}_- \Longrightarrow \varphi_{-\mathcal{E}^c_{f,\Theta}(x),k}(\mathcal{E}^c_{f,\Theta}(\overline{x})) \subseteq \mathbb{R}_-$$

holds, then $\overline{x}$ is a minimizer of f.

Proof It is straightforward from Theorem 4.3. □

Remark 4.2 While Corollary 4.1 (i) is known in vector optimization, (ii) is new. It could be served as a sufficient condition for minimality. For example, consider a vector-valued function $f : \mathbb{R} \to \mathbb{R}^2$ with

$$f(x) := \begin{cases} (0, -x), & \text{if } x \geq 0, \\ (x, 0), & \text{if } x < 0. \end{cases}$$

Take $k = (1, 1)$ and $\overline{x} = 0$. Since

$$\forall x \in \mathbb{R}, \varphi_{\mathbb{R}^2_+,k} \circ f(x) = \varphi_{\mathbb{R}^2_+ - f(0),k}(f(x)) \equiv 0,$$

$\overline{x} = 0$ is a minimal solution of $\varphi_{\mathbb{R}^2_, k} \circ f$. Take $\hat{x} = 1$. We have $\varphi_{\mathbb{R}^2_+ - f(0),k}(f(1)) = 0$, but $\varphi_{\mathbb{R}^2_+ - f(1),k}(f(0)) = 1$. Corollary 4.1 (ii) ensures that $\overline{x} = 0$ is not a $\leq_{\mathbb{R}^2_+}$-minimizer of f.

Again, $k = (1, 1)$ and $\overline{x} = 0$. Since

$$\forall x \in \mathbb{R}, \varphi_{-\overline{\mathcal{E}}^c_f(\overline{x}),k}\big(\overline{\mathcal{E}}^c_f(x)\big)) = \mathbb{R}_-,$$

$\overline{x} = 0$ is an $\leq^u_{\mathbb{R}_+}$-minimal solution of the scalarized map $\varphi_{-\overline{\mathcal{E}}^c_f(\overline{x}),k} \circ \overline{\mathcal{E}}^c_f$. Take $\hat{x} = 1$. We have $\varphi_{-\overline{\mathcal{E}}^c_f(0),k}(\overline{\mathcal{E}}^c_f(1)) = \mathbb{R}_-$, but $\varphi_{-\overline{\mathcal{E}}^c_f(1),k}(\overline{\mathcal{E}}^c_f(0)) = 1 + \mathbb{R}_-$. By Corollary 4.2 (ii), $\overline{x} = 0$ is not a $\leq_{\mathbb{R}^2_+}$-minimizer of f.

We close this section with a relation between $\leq^l_\Theta$- and $\leq_\Theta$-minimizers .

Theorem 4.4 (Characterization for $\leq^l_\Theta$-minimizers) *Let $\overline{x}$ be a $\leq^l_\Theta$-minimizer of a set-valued map F. Assume that Θ is pointed and $F(\overline{x})$ has a Θ-lower bound, i.e., there is $y_* \in Y$ such that $F(\overline{x}) \subseteq y_* + \Theta$.*

(i) *The pair $(\overline{x}, y_*)$ is a $\leq_\Theta$-minimizer of the adjusted map $F_{y_*} : X \rightrightarrows Y$ defined by*

$$F_{y_*}(x) := \begin{cases} y_* + \Theta, & \text{if } x = \overline{x} \\ F(x), & \text{if } x \neq \overline{x}. \end{cases} \tag{26}$$

(ii) *$\overline{x}$ is a $\leq^l_{\mathbb{R}_+}$-minimal solution of the scalarized map $\varphi_{y_*-\Theta,k} \circ F_{y_*}$; indeed, we have*

$$\forall x \in \operatorname{dom} F, \varphi_{y_*-\Theta,k}(F_{y_*}(x)) \subseteq [0, +\infty[$$

and $\varphi_{y_*-\Theta,k}(F_{y_*}(\overline{x})) = [0, +\infty[$.

Proof Assume that $\overline{x}$ is a $\leq^l_\Theta$-minimizer of F, i.e.,

$$\begin{aligned} &\forall x \in \operatorname{dom} F, F(x) \leq^l_\Theta F(\overline{x}) \Longrightarrow F(\overline{x}) \leq^l_\Theta F(x) \\ \Longleftrightarrow\ & x \in \operatorname{dom} F, F(\overline{x}) \subseteq F(x) + \Theta \Longrightarrow F(x) \subseteq F(\overline{x}) + \Theta. \end{aligned} \tag{27}$$

Arguing by contradiction, assume that $(\overline{x}, y_*)$ is not a Θ-minimizer of F_{y_*}, i.e., there are $x \in \operatorname{dom} F_{y_*} = \operatorname{dom} F$ and $y \in F_{y_*}(x)$ such that $y \leq_\Theta y_*$ and $y \neq y_*$. Observe that $x \neq \overline{x}$. Indeed, if $x = \overline{x}$, then $y \in F_{y_*}(\overline{x}) = y_* + \Theta$. This together with $y \leq_\Theta y_* \Longleftrightarrow y \in y_* - \Theta$ gives

$$y - y_* \in \Theta \cap (-\Theta).$$

Since Θ is a pointed cone, we have $y = y_*$ contradicting to $y \neq y_*$. Since $x \neq \overline{x}$, by the definition of F_{y_*}, we have $F_{y_*}(x) = F(x)$. We now get from $y \leq_\Theta y_*$ and the $\leq^l_\Theta$-minimality of $\overline{x}$ to F that

$$
\begin{aligned}
y \leq_\Theta y_* \Longleftrightarrow\ & y_* \in y + \Theta \\
\overset{\Theta+\Theta=\Theta}{\Longleftrightarrow}\ & y_* + \Theta \subseteq y + \Theta + \Theta = y + \Theta \\
\Longrightarrow\ & F(\overline{x}) \subseteq y_* + \Theta \subseteq y + \Theta \subseteq F(x) + \Theta \\
\overset{(27)}{\Longrightarrow}\ & F(x) \subseteq F(\overline{x}) + \Theta \\
\Longrightarrow\ & F(x) \subseteq y_* + \Theta \\
\Longrightarrow\ & y \in y_* + \Theta.
\end{aligned}
$$

We get from inclusions in the first line and the last line $y_* \in y + \Theta$ and $y \in y_* + \Theta$. Again, the pointedness of Θ yields $y = y_*$, contradicting $y \neq y_*$. The contradiction verifies that $(\overline{x}, y_*)$ is a $\leq_\Theta$-minimizer of F_{y_*} and completes the proof of (i).

(ii) It follows from (i) that

$$F(X) \cap (y_* - \Theta) \subseteq \{y_*\},$$

and thus for every scalarization direction $k \in \mathrm{sd}\,(\Theta)$, and for every $y \in F(X)$, we have $\varphi_{y_*-\Theta,k}(y) \geq 0$. Since $\varphi_{y_*-\Theta,k}(y_*) = 0$, the proof is complete. □

Let us illustrate the role of the Θ-minimality of y_* to $F(\overline{x})$ in Theorem 4.4.

Example 4.3 Consider a set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ with

$$
F(x) := \begin{cases} [(x, -\alpha x), (\sqrt{2}/2, \sqrt{2}/2)], & \text{if } x > 0, \\ I\!B((1,1), 1), & \text{if } x = 0, \\ [(x, -\beta x), (\sqrt{2}/2, \sqrt{2}/2)], & \text{if } x < 0, \end{cases}
$$

where $\alpha, \beta \geq 0$ $[(a, b), (c, d)]$ is the line segment from (a, b) to (c, d) and $I\!B((1, 1), 1)$ is the ball centered at $(1, 1)$ with radius 1. It is easy to check that for any $\alpha, \beta \geq 0$, $\overline{x} = 0$ is a $\leq^l_\Theta$-minimizer of F with $\Theta = \mathbb{R}^2_+$.

When $\alpha = +\infty$ and $\beta = +\infty$, i.e.,

$$
F_\infty(x) = \begin{cases} [(0, x), (\sqrt{2}/2, \sqrt{2}/2)], & \text{if } x > 0, \\ I\!B((1,1), 1), & \text{if } x = 0, \\ [(x, 0), (\sqrt{2}/2, \sqrt{2}/2)], & \text{if } x < 0, \end{cases}
$$

$\overline{x} = 0$ is no longer a $\leq^l_\Theta$-minimizer of F_∞.

When $\alpha < 0$ and $\beta < 0$, $\overline{x} = 0$ is not a $\leq^l_\Theta$-minimizer of F_∞.

The image of F could be any subset of $Y \setminus \{y_* - \Theta\}$ provided that it is not identical to $F(\overline{x})$.

Remark 4.4 **(Applications to necessary conditions)**

(a) Assume that $\overline{x}$ is an $\leq^u_{\mathbb{R}_+}$-minimal solution of a scalar set-valued map $\Psi : X \rightrightarrows \mathbb{R}$ and $\overline{y} = \max F(\overline{x}) \in F(\overline{x})$. Then, we have

$$\widehat{D}\mathcal{H}_\Psi(\overline{x}, \overline{y})(1) \subseteq \{0\},$$

where $\widehat{D}\mathcal{H}_\Psi(\overline{x}, \overline{y})$ is the regular/Fréchet coderivative of $\mathcal{H}_F$. Given a set-valued map $F : X \rightrightarrows Y$ between two Banach spaces and an $\leq^u_\Theta$-minimizer $\overline{x}$ of F. By Theorem 4.1, for every scalarization direction $k \in \text{sd}\,(-\mathcal{H}_F(\overline{x}))$, $\overline{x}$ is an $\leq^u_{\mathbb{R}_+}$-minimal solution of the scalarized map $s_{\overline{x},k} \circ \mathcal{H}$. Therefore, we have

$$\widehat{D}(s_{\overline{x},k} \circ \mathcal{H})(\overline{x}, 0)(1) \subseteq \{0\}.$$

(b) Theorem 4.4 provides a connection between efficiency of vector optimization and lower set-less minimality. This improves and corrects the results in [23]. Assume in addition that $y_* \in F(\overline{x})$. We have $F_{y_*} = F$. By [25], under appropriate closedness and sequential normal compactness assumptions, we obtain the following necessary condition for lower set-less minimizers: There is $y^* \in \Theta^+ \setminus \{0\}$ such that

$$0 \in D^* F(\overline{x}, y_*)(y^*),$$

i.e., there is a sequence $\{(x_k, y_k)\} \subseteq \text{gph}\, F$ and a sequence $\{(x_k^*, y_k^*)\}$ with

$$\limsup_{(u,v)\to(x_k,y_k)} \frac{\langle (x_k^*, y_k^*), (u, v) - (x_k, y_k)\rangle}{\|(u, v) - (x_k, y_k)\|} \leq 0$$

satisfying

$$(x_k, y_k) \to (\overline{x}, y_*) \text{ and } (x_k^*, y_k^*) \overset{w^*}{\to} (0, -y^*),$$

where $\Theta^+ := \{y^* \in Y^* \,:\, \forall y \in \Theta,\, y^*(y) \geq 0\}$ is the positive polar cone of Θ and $D^* F(\overline{x}, y_*)(y^*)$ is the limiting/Mordukhovich coderivative of F at $(\overline{x}, y_*)$. See [4] for the precise definitions and the full calculus of limiting differentiation.

Conclusions

In our paper, we established several important connections between upper/lower set-less minimizers of set-valued maps and upper/lower set-less minimal solutions of scalarized maps which are compositions of the map and Gerstewitz' nonlinear scalarization functional. This opens a new way to derive necessary conditions for lower and upper minimizers of set-valued maps; see Remark 4.4. In addition, we also provide a sufficient condition ensuring that a lower set-less minimizer of a set-valued map is a (Pareto) minimizer in the sense of vector optimization.

In a forthcoming paper, we will study characterizations via Gerstewitz' nonlinear scalarization functionals for all six types of minimizers of set-valued maps corresponding to six set relations introduced by Kuroiwa [5]. Furthermore, it is of interest to use our results for developing numerical procedures based on scalarization.

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