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On the R0-Tensors and the Solution Map of Tensor Complementarity Problems

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Abstract
Our purpose is to investigate several properties of the solution map of tensor complementarity problems. To do this, we focus on the R0-tensors and show some results on the local boundedness and the upper semicontinuity. Furthermore, by using a technique from semi-algebraic geometry, we obtain results on the finite-valuedness, the lower semicontinuity, and the local upper-Hölder stability of the map.

Keywords Tensor complementarity problem · R0-tensor · Semi-algebraic set · Solution map · Finite-valuedness · Local boundedness · Upper semicontinuity · Lower semicontinuity · Local upper-Hölder stability

Mathematics Subject Classification 90C33 · 90C31 · 14P10 · 54C60

1 Introduction

The tensor complementarity problem was firstly introduced by Song and Qi [1,2]. The problem has attracted a lot of attention from researchers [3–20]. In particular, Huang and Qi [3] have presented an explicit relationship between n-person noncooperative games and tensor complementarity problems.

The involved function in a tensor complementarity problem is the sum of a homogeneous polynomial and a vector. Thus, the tensor complementarity problem is a special case of the homogeneous complementarity problem, that was mentioned in the work [21] of Oettli and Yen, and of the polynomial complementarity problem, which has been recently introduced by Gowda [22]. All of them are natural extensions of the

Dedicated to Professor Boris Mordukhovich on the occasion of his 70th birthday.

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linear complementarity problems [23]. The local boundedness, the upper semicontinuity, the lower semicontinuity, and the local upper-Lipschitz stability of the solution map of linear complementarity problems have been deeply investigated [23–26].

In this paper, we firstly prove that the set of R0-tensors is open in the space of real tensors. As a result, the local boundedness of the solution map is shown. Secondly, using tools from semi-algebraic geometry, we show that the set of R0-tensors is semi-algebraic and generic. A lower bound for the dimension of the complement of R0-tensors is established. Furthermore, we prove that the solution map is generically finite-valued. Consequently, a necessary condition for the lower semicontinuity of the map is given. Thirdly, this paper shows a close relation between the upper semicontinuity of the solution map and the R0 property of the involved tensors. Finally, a result on the local upper-Hölder stability of the solution map is obtained.

The organization of the paper is as follows: Sect. 2 gives a brief introduction to tensor complementarity problems and semi-algebraic geometry. Section 3 investigates the local boundedness of the solution map, the semi-algebraicity, and the genericity of the set R0-tensors. The finite-valuedness and the lower semicontinuity are discussed in Sect. 4. The last two sections give results on the upper semicontinuity and the local upper-Hölder stability.

2 Preliminaries

In this section, we will recall some definitions, notations, and auxiliary results on tensor complementarity problems and from semi-algebraic geometry.

2.1 Tensor Complementarity Problems

The scalar product of two vectors x, y in the Euclidean space $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function. The *nonlinear complementarity problem* defined by F is the problem finding $x \in \mathbb{R}^n$ such that $x \geq 0$, $F(x) \geq 0$, and $\langle x, F(x)\rangle = 0$. We denote the problem and its solution set as $\mathrm{CP}(F)$ and $\mathrm{Sol}(F)$, respectively.

The following remark shows that a solution of a complementarity problem can be characterized by using some Lagrange multipliers.

Remark 2.1 A vector x solves $\mathrm{CP}(F)$ if and only if there exists a vector $\lambda \in \mathbb{R}^n$ such that next system is satisfied

$$F(x) - \lambda = 0,\ \langle \lambda, x\rangle = 0,\ \lambda \geq 0,\ x \geq 0.$$

To find the solution set of a complementarity problem, we will find the solutions on each pseudo-face of $\mathbb{R}^n_+$. For every index set $\alpha \subset [n] = \{1, \dots, n\}$, we associate that with the following *pseudo-face*

$$K_\alpha = \left\{x \in \mathbb{R}^n_+ : x_i = 0, \forall i \in \alpha;\ x_i > 0, \forall i \in [n]\backslash\alpha\right\}.$$

The pseudo-faces K_α, $\alpha \subset [n]$, establish a finite disjoint decomposition of $\mathbb{R}^n_+$. Therefore, we have

$$\mathrm{Sol}(F) = \bigcup_{\alpha \subset [n]} [\mathrm{Sol}(F) \cap K_\alpha]. \tag{1}$$

Throughout this paper, we assume that m and n are given integers, and $m, n \geq 2$. An mth-order n-dimensional *tensor* $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m} \in \mathbb{R}$, where $i_j \in [n]$ and $j \in [m]$. The set of all real mth-order n-dimensional tensors is denoted by $\mathbb{R}^{[m,n]}$. For any tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$, the Frobenius norm of $\mathcal{A}$ is defined and denoted as

$$\| \mathcal{A} \| := \sqrt{\sum_{i_1, i_2, \ldots, i_m = 1}^{n} a^2_{i_1 i_2 \cdots i_m}}.$$

This norm can be considered as a vector norm. So, the norm of $(\mathcal{A}, a)$ in $\mathbb{R}^{[m,n]} \times \mathbb{R}^n$ can be defined as follows

$$\|(\mathcal{A}, a)\| := \sqrt{\| \mathcal{A} \|^2 + \sum_{i=1}^{n} a_i^2}.$$

Clearly, $\mathbb{R}^{[m,n]}$ is a real vector space of dimension n^m, so each tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a real vector having n^m components. In particular, if $m = 2$ then $\mathbb{R}^{[2,n]}$ is the space of $n \times n$-matrices which is isomorphic to $\mathbb{R}^{n \times n}$. Note that if $\mathcal{A} = (a_{i_1 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 \cdots i_m})$ are tensors in $\mathbb{R}^{[m,n]}$, then the sum $\mathcal{A} + \mathcal{B}$ is $(a_{i_1 \cdots i_m} + b_{i_1 \cdots i_m})$.

For any $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $\mathcal{A}x^{m-1}$ is a vector whose ith component defined by

$$(\mathcal{A}x^{m-1})_i := \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \ \forall i \in [n], \tag{2}$$

and $\mathcal{A}x^m$ is a polynomial of degree m, defined by

$$\mathcal{A}x^m := \langle x, Ax^{m-1} \rangle = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

The polynomials $(Ax^{m-1})_i$ and $\mathcal{A}x^m$ are homogeneous of degree, respectively, $m-1$ and m, that is $\mathcal{A}(tx)^{m-1} = t^{m-1}(\mathcal{A}x^{m-1})$ and $\mathcal{A}(tx)^m = t^m(\mathcal{A}x^m)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$.

Remark 2.2 By the continuity of the polynomial function $\mathcal{A}x^{m-1}$, if U is a bounded set in $\mathbb{R}^n$, then there exists $\beta > 0$ such that $\| \mathcal{A}x^{m-1} \| \leq \beta \| \mathcal{A} \|$ for all $x \in U$.

Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $a \in \mathbb{R}^n$ be given. If $F(x) = \mathcal{A}x^{m-1} + a$, then one says that CP($F$) is a *tensor complementarity problem* defined by $\mathcal{A}$ and a. This problem and its solution set are denoted, respectively, by TCP($\mathcal{A}, a$) and Sol($\mathcal{A}, a$). By definition, x solves TCP($\mathcal{A}, a$) if and only if

$$x \geq 0,\ \mathcal{A}x^{m-1} + a \geq 0,\ \mathcal{A}x^m + \langle x, a\rangle = 0. \tag{3}$$

Clearly, the vector 0 solves TCP($\mathcal{A}, a$) for all $a \in \mathbb{R}^n_+$. The solution map of tensor complementarity problems is denoted and defined by

$$\text{Sol} : \mathbb{R}^{[m,n]} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n,\ (\mathcal{A}, a) \mapsto \text{Sol}(\mathcal{A}, a). \tag{4}$$

Remark 2.3 The graph of the map Sol, which is defined by

$$\text{gph(Sol)} = \left\{(\mathcal{A}, a, x) \in \mathbb{R}^{[m,n]} \times \mathbb{R}^n \times \mathbb{R}^n : x \in \text{Sol}(\mathcal{A}, a)\right\},$$

is closed. Indeed, take a sequence $\{(\mathcal{A}^k, a^k, x^k)\}$ in gph(Sol) such that

$$(\mathcal{A}^k, a^k, x^k) \to (\bar{\mathcal{A}}, \bar{a}, \bar{x}).$$

It follows that $\mathcal{A}^k \to \bar{\mathcal{A}}$, $a^k \to \bar{a}$ and $x^k \to \bar{x}$. From (3), one has

$$x^k \geq 0,\ \mathcal{A}^k(x^k)^{m-1} + a^k \geq 0,\ \mathcal{A}^k(x^k)^m + \left\langle x^k, a^k\right\rangle = 0.$$

Taking $k \to +\infty$, we can see that $\bar{x}$ solves TCP($\mathcal{A}, a$). So, $(\bar{\mathcal{A}}, \bar{a}, \bar{x})$ belongs to gph(Sol), and the graph is closed.

A nonempty subset $K \subset \mathbb{R}^n$ is called a cone [27, p. 89] if $\lambda > 0$ and $x \in K$ then $\lambda x \in K$. The cone K is bounded if and only if $K = \{0\}$.

Remark 2.4 The solution set of TCP($\mathcal{A}, 0$) is a nonempty and closed cone. Clearly, we have $0 \in \text{Sol}(\mathcal{A}, 0)$. Suppose that x is a solution of TCP($\mathcal{A}, 0$). For each $t > 0$, from (3), we have

$$tx \geq 0,\quad \mathcal{A}(tx)^{m-1} = t^{m-1}(\mathcal{A}x^{m-1}) \geq 0,\quad \mathcal{A}(tx)^m = t^m(\mathcal{A}x^m) = 0.$$

Hence, tx solves TCP($\mathcal{A}, 0$). This shows that Sol($\mathcal{A}, 0$) is a cone. The closedness of Sol($\mathcal{A}, 0$) is implied from the continuity of $F(x) = \mathcal{A}x^{m-1}$ and the closedness of $\mathbb{R}^n_+$.

Let us recall that $\mathcal{A}$ is an R0–tensor (sometimes, we say $\mathcal{A}$ is R0) if Sol($\mathcal{A}, 0$) $= \{0\}$. We denote $\mathcal{R}_0$ to be the set of all mth-order n-dimensional R0-tensors and $\mathcal{O} \in \mathbb{R}^{[m,n]}$ to be the zero tensor. The complement of $\mathcal{R}_0$ is denoted and defined by

$$C(\mathcal{R}_0) = \mathbb{R}^{[m,n]} \setminus \mathcal{R}_0\,.$$

Clearly, $\mathcal{O}$ belongs to $C(\mathcal{R}_0)$ since Sol($\mathcal{O}, 0$) $= \mathbb{R}^n_+$.

2.2 Semi-algebraic Geometry

Recall a subset in $\mathbb{R}^n$ is *semi-algebraic*, if it is the union of finitely many subsets of the form

$$\{x \in \mathbb{R}^n \ : \ f_1(x) = \cdots = f_\ell(x) = 0, \ g_{\ell+1}(x) < 0, \ldots, g_m(x) < 0\},$$

where ℓ, m are natural numbers, and $f_1, \ldots, f_\ell, g_{\ell+1}, \ldots, g_m$ are polynomials with real coefficients. The semi-algebraic property is preserved by taking finitely union, intersection, minus and taking closure of semi-algebraic sets. The well-known Tarski–Seidenberg theorem states that the image of a semi-algebraic set under a linear projection is a semi-algebraic set.

There are some ways to define the dimension of a semi-algebraic set. Here, we choose the geometric approach which is presented in [28, Corollary 2.8.9]. If $S \subset \mathbb{R}^n$ is a semi-algebraic set, then there exists a decomposition of S into a disjoint union of semi-algebraic subsets [28, Theorem 2.3.6]

$$S = \bigcup_{i=1}^{s} S_i,$$

where each S_i is semi-algebraically diffeomorphic to $]0,1[^{d_i}$. Let $]0,1[^0$ be a point, $]0,1[^{d_i} \subset \mathbb{R}^{d_i}$ be the set of points $x = (x_1, \ldots, x_{d_i})$ such that $x_j \in]0,1[$ for all $j = 1, \ldots, d_i$. The *dimension* of S is, by definition,

$$\dim(S) := \max\{d_1, \ldots, d_s\}.$$

This is well defined and not depend on the decomposition of S. Remind that if $S \neq \emptyset$ and $\dim(S) = 0$, then S has finitely many points.

Remark 2.5 For a semi-algebraic subset S of $\mathbb{R}^n$, the dimension of the complement $\mathbb{R}^n \backslash S$ is strictly less than n if and only if S is topologically generic in $\mathbb{R}^n$, i.e., S contains a countable intersection of dense and open sets (see, e.g., [29, Lemma 2.3]).

We will use the Tarski–Seidenberg theorem in the third form in the next section. To present the theorem, we have to describe semi-algebraic sets via the language of first-order formulas. A *first-order formula* (with parameters in $\mathbb{R}$) is obtained by the following induction rules [30]:

(i) If $p \in \mathbb{R}[X_1, \ldots, X_n]$, then $p > 0$ and $p = 0$ are first-order formulas;
(ii) If P, Q are first-order formulas, then "*P and Q*", "*P or Q*", and "*not Q*", which are denoted, respectively, by $P \wedge Q$, $P \vee Q$, and $\neg Q$, are first-order formulas;
(iii) If Q is a first-order formula, then $\exists X\ Q$ and $\forall X\ Q$, where X is a variable ranging over $\mathbb{R}$, are first-order formulas.

Formulas obtained by using only rules (i) and (ii) are called *quantifier-free formulas*. A subset $S \subset \mathbb{R}^n$ is semi-algebraic if and only if there is a quantifier-free formula $Q_S(X_1, \ldots, X_n)$ such that

$$(x_1, \ldots, x_n) \in S \text{ if and only if } Q_S(x_1, \ldots, x_n).$$

In this case, $Q_S(X_1, \ldots, X_n)$ is said to be a *quantifier-free formula defining* S.

Remark 2.6 The Tarski–Seidenberg theorem in the third form [30, Theorem 2.6] says that if $Q(X_1, \ldots, X_n)$ is a first-order formula, then the set

$$S = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : Q(x_1, \ldots, x_n)\right\}$$

is a semi-algebraic set.

3 The Set of R0-Tensors

We prove the openness of the set $\mathcal{R}_0$ in $\mathbb{R}^{[m,n]}$. Consequently, the local boundedness of the solution map is shown. Furthermore, we show that $\mathcal{R}_0$ is generic semi-algebraic, and give a lower bound for the dimension of the complement $C(\mathcal{R}_0)$.

3.1 Local Boundedness of the Solution Map

Proposition 3.1 *The set $\mathcal{R}_0$ of all R0-tensors is open in $\mathbb{R}^{[m,n]}$.*

Proof If the complement $C(\mathcal{R}_0)$ is closed, then $\mathcal{R}_0$ is open. So, we only need to prove the closedness of $C(\mathcal{R}_0)$. Let $\{\mathcal{A}^k\} \subset C(\mathcal{R}_0)$ be a convergent sequence with $\mathcal{A}^k \to A$. For each k, $\mathrm{Sol}(\mathcal{A}^k, 0)$ is unbounded. There exists an unbounded sequence $\{x^k\} \subset \mathbb{R}^n_+$ such that $x^k \in \mathrm{Sol}(\mathcal{A}^k, 0)$ and $x^k \neq 0$ for each k. Without loss of generality we can assume that $\|x^k\|^{-1}x^k \to \bar{x}$ and $\|\bar{x}\| = 1$. By definition, one has

$$\mathcal{A}^k(x^k)^{m-1} \geq 0, \; \mathcal{A}^k(x^k)^m = 0.$$

Dividing these ones by $\|x^k\|^{m-1}$ and $\|x^k\|^m$, respectively, and taking $k \to +\infty$, we obtain $\mathcal{A}(\bar{x})^{m-1} \geq 0$ and $\mathcal{A}(\bar{x})^m = 0$. It follows that $\bar{x} \in \mathrm{Sol}(\mathcal{A}, 0)$ and $\mathrm{Sol}(\mathcal{A}, 0) \neq \{0\}$. Hence, $\mathcal{A}$ must be in $C(\mathcal{R}_0)$, and $C(\mathcal{R}_0)$ is closed. The proof is completed. □

Remark 3.1 The set $\mathcal{R}_0$ is a cone in $\mathbb{R}^{[m,n]}$. Indeed, for any $t > 0$, one has

$$(t\,\mathcal{A})x^{m-1} = t^{m-1}(\mathcal{A}x^{m-1}), \; (t\,\mathcal{A})x^m = t^m(\mathcal{A}x^m).$$

It is easy to check that $\mathrm{Sol}(t\,\mathcal{A}, 0) = \mathrm{Sol}(\mathcal{A}, 0)$. Thus, $\mathcal{A} \in \mathcal{R}_0$ if and only if $t\,\mathcal{A} \in \mathcal{R}_0$. This implies that $\mathcal{R}_0$ is a cone.

The boundedness of solution sets of tensor complementarity problems and polynomial complementarity problems under the R0 condition is mentioned in [16] and [22]. Based on the openness of the set $\mathcal{R}_0$, we show that the solution map is locally bounded.

Here, $\mathbb{B}(\mathcal{O}, \varepsilon)$ stands for the closed ball in $\mathbb{R}^{[m,n]}$ centered at $\mathcal{O}$ with radius ε. Similarly, $B(0, \delta)$ is the closed ball in $\mathbb{R}^n$ centered at 0 with radius δ.

Theorem 3.1 *The following two statements are equivalent:*

(a) *The tensor $\mathcal{A}$ is R0;*
(b) *There exists $\varepsilon > 0$ such that the following set*

$$S(\varepsilon, \delta) := \bigcup_{(\mathcal{B},b)\in\mathbb{B}(\mathcal{O},\varepsilon)\times B(0,\delta)} \mathrm{Sol}(\mathcal{A}+\mathcal{B}, a+b)$$

is bounded, for any $a \in \mathbb{R}^n$ and any $\delta > 0$.

Proof (a) $\Rightarrow$ (b) Let $\mathcal{A}$ be an R0–tensor. Since the set $\mathcal{R}_0$ is open in $\mathbb{R}^{[m,n]}$, due to Proposition 3.1, there exists $\varepsilon > 0$ such that $\mathcal{A}+\mathbb{B}(\mathcal{O}, \varepsilon) \subset \mathcal{R}_0$. On the contrary, we suppose that there exist $a \in \mathbb{R}^n$ and $\delta > 0$ such that the set $S(\varepsilon, \delta)$ is unbounded. Let $\{x^k\}$ be an unbounded sequence and $\{(\mathcal{B}^k, b^k)\}$ be a sequence in $\mathbb{B}(\mathcal{O}, \varepsilon) \times B(0, \delta)$ satisfying $x^k \in \mathrm{Sol}(\mathcal{A}+\mathcal{B}^k, a + b^k)$. We can assume that $x^k \neq 0$, $\|x^k\|^{-1}x^k \to \bar{x}$ and $\|\bar{x}\| = 1$. One has

$$(\mathcal{A}+\mathcal{B}^k)(x^k)^{m-1} + (a + b^k) \geq 0,\ (\mathcal{A}+\mathcal{B}^k)(x^k)^m + \left\langle a + b^k, x^k\right\rangle = 0. \tag{5}$$

By the compactness of the sets $\mathcal{A}+\mathbb{B}(\mathcal{O}, \varepsilon)$ and $a + B(0, \delta)$, we can assume that

$$\mathcal{A}+\mathcal{B}^k \to \bar{\mathcal{A}} \in \mathcal{A}+\mathbb{B}(\mathcal{O}, \varepsilon),\ a + b^k \to \bar{a} \in a + B(0, \delta). \tag{6}$$

From (5) and (6), it is easy to show that $\bar{x}$ solves $\mathrm{TCP}(\bar{\mathcal{A}}, 0)$. Since $\|\bar{x}\| = 1$, $\bar{\mathcal{A}}$ is not R0. This contradicts $\bar{\mathcal{A}} \in \mathcal{A}+\mathbb{B}(\mathcal{O}, \varepsilon) \subset \mathcal{R}_0$.

(b) $\Rightarrow$ (a) Suppose that there exists $\varepsilon > 0$ such that $S(\varepsilon, \delta)$ is bounded for any $a \in \mathbb{R}^n$ and $\delta > 0$. Take $a = 0$, one has $\mathrm{Sol}(\mathcal{A}, 0) \subset S(\varepsilon, \delta)$ and $\mathrm{Sol}(\mathcal{A}, 0)$ is bounded. Hence, $\mathcal{A}$ is an R0–tensor and the assertion is proved. □

Remark 3.2 The tensor $\mathcal{A}$ is R0 if and only if $\mathrm{Sol}(\mathcal{A}, a)$ is bounded for every $a \in \mathbb{R}^n$ (see [16, Theorem 3.2]). Moreover, $\mathcal{A}$ is an R0–tensor if and only if the set

$$\bigcup_{b\in B(0,\delta)} \mathrm{Sol}(\mathcal{A}, a + b)$$

is bounded, for any $a \in \mathbb{R}^n$ and $\delta > 0$ [22, Proposition 2.1]. Clearly, these assertions are corollaries of Theorem 3.1.

Recall that the set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is *locally bounded* at $\bar{x}$ if there exists an open neighborhood U of $\bar{x}$ such that the set $\cup_{x\in U}\Psi(x)$ is bounded.

Corollary 3.1 *The following three statements are equivalent:*

(a) *The tensor $\mathcal{A}$ is R0;*
(b) *The solution map $\mathrm{Sol}_{\mathcal{A}}$ is locally bounded at a, for any $a \in \mathbb{R}^n$;*
(c) *The solution map Sol is locally bounded at $(\mathcal{A}, a)$, for any $a \in \mathbb{R}^n$.*

Proof (a) $\Rightarrow$ (c) Suppose that $\mathcal{A}$ is an R0–tensor. By Theorem 3.1, there exists $\varepsilon > 0$ such that, for every $a \in \mathbb{R}^n$, the following set is bounded

$$S(\varepsilon, \varepsilon) \supset \bigcup_{(\mathcal{B},b)\in U} \mathrm{Sol}(\mathcal{B}, b),$$

where $U = (\mathcal{A}, a) + \mathrm{int}\,\mathbb{B}(\mathcal{O}, \varepsilon) \times \mathrm{int}\,B(0, \varepsilon)$ is an open neighborhood of $(\mathcal{A}, a)$. This means that the map Sol is locally bounded at $(\mathcal{A}, a)$.

(c) $\Rightarrow$ (b) Suppose that the assertion (c) is true. There exists an open neighborhood U of $(\mathcal{A}, a)$ such that

$$\bigcup_{b\in\varphi(U)} \mathrm{Sol}(\mathcal{A}, b) \subset \bigcup_{(\mathcal{B},b)\in U} \mathrm{Sol}(\mathcal{B}, b),$$

where $\varphi : \mathbb{R}^{[m,n]} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by $(\mathcal{B}, b) \mapsto b$, is bounded. Clearly, φ is surjective, continuous, and linear. According to the open mapping theorem [31, Theorem 2.11], $\varphi(U)$ is an open neighborhood of a. So, $\mathrm{Sol}_{\mathcal{A}}$ is locally bounded at a.

(b) $\Rightarrow$ (a) Suppose that (b) holds. Take $a = 0$, there exists an open neighborhood U of 0 such that

$$\mathrm{Sol}(\mathcal{A}, 0) \subset \bigcup_{b\in U} \mathrm{Sol}(\mathcal{A}, b)$$

is bounded. So, $\mathrm{Sol}(\mathcal{A}, 0)$ is also bounded and $\mathcal{A}$ is R0. □

3.2 Semi-algebraicity and Genericity of $\mathcal{R}_0$

Remind that the space $\mathbb{R}^{[m,n]}$ of real mth-order n-dimensional tensors can be considered as the Euclidean space $\mathbb{R}^{n^m}$. By abuse of terminology, we say that $\mathbb{S} \subset \mathbb{R}^{[m,n]}$ is semi-algebraic if $\varphi(\mathbb{S})$ is semi-algebraic in $\mathbb{R}^{n^m}$, where $\varphi : \mathbb{R}^{[m,n]} \to \mathbb{R}^{n^m}$ is an isomorphism.

Proposition 3.2 *The set $\mathcal{R}_0$ is semi-algebraic in $\mathbb{R}^{[m,n]}$.*

Proof Remind that $\mathcal{A} \in \mathcal{R}_0$ if and only if $\mathrm{Sol}(\mathcal{A}, 0) = \{0\}$. Since 0 always belongs to $\mathrm{Sol}(\mathcal{A}, 0)$, the set $\mathcal{R}_0$ can be described as follows:

$$\begin{aligned}\mathcal{R}_0 &= \left\{\mathcal{A} \in \mathbb{R}^{[m,n]} : \nexists x \in \mathbb{R}^n_+\backslash\{0\} \left(\left[\mathcal{A}x^{m-1} \geq 0\right] \wedge \left[\mathcal{A}x^m = 0\right]\right)\right\}\\ &= \left\{\mathcal{A} \in \mathbb{R}^{[m,n]} : \forall x \in \mathbb{R}^n_+\backslash\{0\} \left(\left[\mathcal{A}x^{m-1} \ngeq 0\right] \vee \left[\mathcal{A}x^m \neq 0\right]\right)\right\}.\end{aligned} \tag{7}$$

Because $\mathbb{R}^n_+$ and $\{0\}$ are semi-algebraic, the set $K = \mathbb{R}^n_+\backslash\{0\}$ is also a semi-algebraic set in $\mathbb{R}^n$. Let $Q_K(x)$ be the quantifier-free formula defining K. Since $\left(\mathcal{A}x^{m-1}\right)_i$, $i = 1, \ldots, n$, and $\mathcal{A}x^m$ are polynomials, the following formulas

$$Q_1(\mathcal{A}, x) := \bigvee_{i=1}^{m} \left[\left(\mathcal{A}x^{m-1}\right)_i < 0\right], \quad Q_2(\mathcal{A}, x) := \left[\mathcal{A}x^m > 0\right] \vee \left[\mathcal{A}x^m < 0\right],$$

are quantifier-free. From the last equation in (7), $\mathcal{A} \in \mathcal{R}_0$ if and only if $Q(\mathcal{A})$, where $Q(\mathcal{A})$ is the following first-order formula

$$Q(\mathcal{A}) := \forall x \left(Q_K(x) \wedge [Q_1(\mathcal{A}, x) \vee Q_2(\mathcal{A}, x)] \right).$$

According to the Tarski–Seidenberg theorem in the third form, $\mathcal{R}_0$ is a semi-algebraic set in $\mathbb{R}^{[m,n]}$. □

Let $\Phi : X \to Y$ be a differentiable map between manifolds, where $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$. A point $y \in Y$ is called a *regular value* for Φ if either the level set $\Phi^{-1}(y) = \emptyset$ or the derivative map

$$D\Phi(x) : T_x X \to T_y Y$$

is surjective at every point $x \in \Phi^{-1}(y)$, where $T_x X$ and $T_y Y$ denote, respectively, the tangent spaces of X at x and of Y at y. So y is a regular value of f if and only if rank $D\Phi(x) = n$ for all $x \in \Phi^{-1}(y)$.

Remark 3.3 Consider the differentiable semi-algebraic map $\Phi : X \to \mathbb{R}^n$ where $X \subset \mathbb{R}^n$. Assume that $y \in Y$ is a regular value of Φ and $\Phi^{-1}(y)$ is nonempty. According to the regular level set theorem [32, Theorem 9.9], one has dim $\Phi^{-1}(y) = 0$. It follows that the semi-algebraic set $\Phi^{-1}(y)$ has finite points.

Remark 3.4 Let $\Phi : \mathbb{R}^p \times X \to \mathbb{R}^n$ be a differentiable semi-algebraic map, where $X \subset \mathbb{R}^n$. Assume that $y \in \mathbb{R}^n$ is a regular value of Φ. According to the Sard theorem with parameter [29, Theorem 2.4], there exists a generic semi-algebraic set $\mathbb{S} \subset \mathbb{R}^p$ such that, for every $p \in \mathbb{S}$, y is a regular value of the map $\Phi_p : X \to Y$ with $\Phi_p(x) = \Phi(p, x)$.

Theorem 3.2 *The set $\mathcal{R}_0$ of all R0-tensors is generic in $\mathbb{R}^{[m,n]}$.*

Proof We will show that there exists a generic semi-algebraic set $\mathbb{S} \subset \mathbb{R}^{[m,n]}$ such that $\mathrm{Sol}(\mathcal{A}, 0) = \{0\}$ for all $\mathcal{A} \in \mathbb{S}$. Indeed, let $K_\alpha \neq \{0\}$ be a given pseudo-face of $\mathbb{R}^n_+$. To avoid confusion, we only consider the case $\alpha = \{1, \ldots, \ell\}$, where $\ell < n$, because other cases can be treated similarly. Then, if $x \in K_\alpha$ then $x_{\ell+1} \neq 0$. We consider the function

$$\Phi_\alpha : \mathbb{R}^{[m,n]} \times K_\alpha \times \mathbb{R}^\ell \ \to \ \mathbb{R}^{n+\ell},$$

which is defined by

$$\Phi_\alpha(\mathcal{A}, x, \lambda_\alpha) = \left(\mathcal{A} x^{m-1} - \lambda, x_\alpha \right)^T, \tag{8}$$

where $x_\alpha = (x_1, \ldots, x_\ell)$, $\lambda_\alpha = (\lambda_1, \ldots, \lambda_\ell)$, and $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0, \ldots, 0) \in \mathbb{R}^n$. The Jacobian matrix of Φ_α is determined as follows

$$D\Phi_\alpha = \left[\begin{array}{c|c|c} D_{\mathcal{A}}(\mathcal{A} x^{m-1} - \lambda) & D_x(\mathcal{A} x^{m-1} - \lambda) & D_{\lambda_\alpha}(\mathcal{A} x^{m-1} - \lambda) \\ \hline D_{\mathcal{A}}(x_\alpha) & D_x(x_\alpha) & D_{\lambda_\alpha}(x_\alpha) \end{array} \right].$$

We claim that the rank of $D\Phi_\alpha$ is $n+\ell$ for all $x \in K_\alpha$. Indeed, it is easy to check that the rank of $D_x(x_\alpha)$ is ℓ. Therefore, if we prove that the rank of $D_{\mathcal{A}}(\mathcal{A}x^{m-1}-\lambda)$ is n then the claim follows. Clearly, one has

$$D_{\mathcal{A}}(\mathcal{A}x^{m-1}-\lambda) = \begin{bmatrix} Q_1 & O & \cdots & O \\ O & Q_2 & \cdots & O \\ & & \ddots & \\ O & O & \cdots & Q_n \end{bmatrix},$$

where O is the zero $1 \times n^{m-1}$-matrix and Q_i is an $1 \times n^{m-1}$-matrix. From (2) and (8), for each $i \in [n]$, we conclude that Q_i is a nonzero matrix because

$$\frac{\partial(\mathcal{A}x^{m-1}-\lambda)_i}{\partial a_{i(\ell+1)\cdots(\ell+1)}} = x_{\ell+1}^{m-1} \neq 0.$$

This shows that rank $D_{\mathcal{A}}(\mathcal{A}x^{m-1}-\lambda) = n$.

Therefore, $0 \in \mathbb{R}^{n+\ell}$ is a regular value of Φ_α. According to Remark 3.4, there exists a generic semi-algebraic set $\mathbb{S}_\alpha \subset \mathbb{R}^{[m,n]}$ such that if $\mathcal{A} \in \mathbb{S}_\alpha$ then 0 is a regular value of the map

$$\Phi_{\alpha,\mathcal{A}} : K_\alpha \times \mathbb{R}^\ell \to \mathbb{R}^{n+\ell},\ \Phi_{\alpha,\mathcal{A}}(x,\lambda_\alpha) = \Phi_\alpha(\mathcal{A},x,\lambda_\alpha).$$

By Remark 3.3, if the set $\Omega(\alpha,\mathcal{A}) := \Phi_{\alpha,\mathcal{A}}^{-1}(0)$ is nonempty, then it is a finite set. Moreover, from (8) and Remark 2.1, one has

$$\mathrm{Sol}(\mathcal{A},0) \cap K_\alpha = \pi(\Omega(\alpha,\mathcal{A})),$$

where π is the projection $\mathbb{R}^{n+\ell} \to \mathbb{R}^n$, which is defined by $\pi(x,\lambda_\alpha) = x$. Thus, the cardinality of $\mathrm{Sol}(\mathcal{A},0) \cap K_\alpha$ is finite.

If $K_\alpha = \{0\}$, i.e., $\alpha = [n]$, then $\mathrm{Sol}(\mathcal{A},0) \cap K_\alpha = \{0\}$. By the finite decomposition in (1), $\mathrm{Sol}(\mathcal{A},0)$ is a finite set.

By setting $\mathbb{S} := \cap_{\alpha \subset [n]} \mathbb{S}_\alpha$, we see that $\mathbb{S}$ is generic in $\mathbb{R}^{[m,n]}$. For any $\mathcal{A}$ in $\mathbb{S}$, the cardinality of $\mathrm{Sol}(\mathcal{A},0)$ is finite. Since $\mathrm{Sol}(\mathcal{A},0)$ is a cone, one has $\mathrm{Sol}(\mathcal{A},0) = \{0\}$. This leads to $\mathbb{S} \subset \mathcal{R}_0$; consequently, $\mathcal{R}_0$ is generic in $\mathbb{R}^{[m,n]}$. The proof is completed. □

Remark 3.5 Theorem 6 in [21] asserts that the set of all R0–matrices is dense in $\mathbb{R}^{n\times n}$. This is a special case of Theorem 3.2 when $m = 2$.

3.3 The Dimension of $C(\mathcal{R}_0)$

From Remark 2.5 and Theorem 3.2, the complement $C(\mathcal{R}_0)$ is thin in the set of real mth-order n-dimensional tensors. A natural question is: *How thin is $C(\mathcal{R}_0)$ in $\mathbb{R}^{[m,n]}$?* The dimension of $C(\mathcal{R}_0)$ tells us about the thinness of this set. The following theorem gives a rough lower estimate for dim $C(\mathcal{R}_0)$.

Theorem 3.3 *The dimension of the semi-algebraic set $C(\mathcal{R}_0)$ satisfies the following inequalities*

$$(n-1)^m \le \dim C(\mathcal{R}_0) \le n^m - 1.$$

Proof The second inequality immediately follows from Theorem 3.2 and Remark 2.5. To prove the first inequality, let $\alpha \subset [n]$ be given with $\alpha \ne [n]$, and we consider the set

$$\mathbb{S}_\alpha = \left\{ \mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]} : a_{i_1 i_2 \cdots i_m} = 0,\ \forall i_j \in [n] \backslash \alpha \right\}.$$

It follows that $\mathbb{S}_\alpha$ is a subspace of $\mathbb{R}^{[m,n]}$ whose the dimension is $|\alpha|^m$. Hence, $\mathbb{S}_\alpha$ is semi-algebraic. Denote by $\bar{K}_\alpha$ the face

$$\bar{K}_\alpha = \left\{ x \in \mathbb{R}^n_+ : x_i = 0, \forall i \in \alpha;\ x_i \ge 0, \forall i \in [n] \backslash \alpha \right\}.$$

A trivial verification shows that $\bar{K}_\alpha \subset \mathrm{Sol}(\mathcal{A}, 0)$ for all $\mathcal{A} \in \mathbb{S}_\alpha$. We conclude that the subspace $\mathbb{S}_\alpha$ is a subset of $C(\mathcal{R}_0)$. Thus, one has

$$|\alpha|^m = \dim \mathbb{S}_\alpha \le \dim C(\mathcal{R}_0).$$

Taking $\alpha = \{2, \ldots, n\}$, one has $|\alpha| = n - 1$, and the first inequality is obtained. □

4 Lower Semicontinuity of the Solution Map

We will prove that the solution map of tensor complementarity problems is finite-valued on a generic semi-algebraic set in the parametric space. Consequently, a necessary condition for the lower semicontinuity of the solution map is given.

The set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is *finite-valued* on $S \subset \mathbb{R}^m$ if the cardinality of the image $\Psi(x)$ is finite, namely $|\Psi(x)| < +\infty$, for all $x \in S$. The map Ψ is *lower semicontinuous* at $\bar{x}$ if for every open set $V \subset \mathbb{R}^n$ such that $\Psi(\bar{x}) \cap V \ne \emptyset$, there exists a neighborhood , U of $\bar{x}$ such that $\Psi(x) \cap V \ne \emptyset$ for all $x \in U$. Remind that (see, e.g., [33, p.139]), if Ψ is lower semicontinuous at $\bar{x}$, then

$$\Psi(\bar{x}) \subset \liminf_{x \to \bar{x}} \Psi(x),$$

where

$$\liminf_{x \to \bar{x}} \Psi(x) = \left\{ u \in \mathbb{R}^n :\ \forall x^k \to \bar{x},\ \exists u^k \to u \ \text{ with } \ u^k \in \Psi(x^k) \right\}.$$

If Ψ is lower semicontinuous at every $x \in X$ then Ψ is said that to be lower semicontinuous on X.

Remark 4.1 The number of connected components of $\mathrm{Sol}(\mathcal{A}, a)$ does not exceed $\chi = d(2d-1)^{5n}$, where $d = \max\{2, m-1\}$. Indeed, let Ω be the set of all $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n$ such that the following conditions are satisfied

$$\mathcal{A}x^{m-1} + q - \lambda = 0,\ \langle \lambda, x\rangle = 0,\ \lambda \geq 0,\ x \geq 0.$$

Clearly, Ω is a semi-algebraic set determined by $3n+1$ polynomial equations and inequalities in $2n$ variables, whose degrees do not exceed the number d. According to [30, Proposition 4.13], the number of connected components of Ω does not exceed χ. By the definition of Ω, one has $\mathrm{Sol}(\mathcal{A}, a) = \pi(\Omega)$, where π is the projection

$$\mathbb{R}^{n+n} \to \mathbb{R}^n,\ \pi(x, \lambda) = x.$$

Since π is continuous, the number of connected components of $\mathrm{Sol}(\mathcal{A}, a)$ also does not exceed χ.

Let Sol be given by (4) and $\mathrm{Sol}_{\mathcal{A}}$ defined by

$$\mathrm{Sol}_{\mathcal{A}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n,\ a \mapsto \mathrm{Sol}_{\mathcal{A}}(a) = \mathrm{Sol}(\mathcal{A}, a), \tag{9}$$

where $\mathcal{A}$ is given.

Proposition 4.1 *There exists a generic semi-algebraic set $\mathbb{S} \subset \mathbb{R}^{[m,n]} \times \mathbb{R}^n$ such that the map* Sol *is finite-valued on $\mathbb{S}$.*

Proof To prove the assertion, we apply the argument in the proof of Theorem 3.2 again, the only difference being in the analysis of the function

$$\Phi_\alpha : \mathbb{R}^{[m,n]} \times \mathbb{R}^n \times K_\alpha \times \mathbb{R}^\ell \ \to\ \mathbb{R}^{n+\ell},$$

which is defined by

$$\Phi_\alpha(\mathcal{A}, a, x, \lambda_\alpha) = \left(\mathcal{A}x^{m-1} + a - \lambda, x_\alpha\right)^T.$$

Note that, since $D_{\mathcal{A}}\Phi_\alpha$ has rank n, the rank of $D\Phi_\alpha$ is $n+\ell$ for $x \in K_\alpha \neq \{0\}$. The proof is completed. □

Remark 4.2 Let $\mathcal{A}$ be given. There exists a generic semi-algebraic set $\mathbb{S}_{\mathcal{A}} \subset \mathbb{R}^n$ such that $\mathrm{Sol}_{\mathcal{A}}$ is finite-valued on $\mathbb{S}_{\mathcal{A}}$. This property is implied from [35, Theorem 3.2] with the note that $\mathbb{R}^n_+$ is a semi-algebraic set satisfying the linearly independent constraint qualification.

Theorem 4.1 *If the solution map* Sol *is lower semicontinuous at* $(\mathcal{A}, a)$*, then* $\mathrm{Sol}(\mathcal{A}, a)$ *has finite elements. Hence, if* $\dim \mathrm{Sol}(\mathcal{A}, a) \geq 1$*, then* Sol *is not lower semicontinuous at* $(\mathcal{A}, a)$.

Proof According to Proposition 4.1, there is a generic set $\mathbb{S}$ in $\mathbb{R}^{[m,n]} \times \mathbb{R}^n$ such that Sol is finite-valued on $\mathbb{S}$. By the density of $\mathbb{S}$, there exists a sequence $\{(\mathcal{A}^k, a^k)\} \subset \mathbb{S}$ such that $(\mathcal{A}^k, a^k) \to (\mathcal{A}, a)$. Remark 4.1 says that $\text{Sol}(\mathcal{A}^k, a^k)$ is finite and $|\,\text{Sol}(\mathcal{A}^k, a^k)| \le \chi$. Since Sol is lower semicontinuous, one has

$$\text{Sol}(\mathcal{A}, a) \subset \liminf_{k \to +\infty} \text{Sol}(\mathcal{A}^k, a^k).$$

This yields $|\,\text{Sol}(\mathcal{A}, a)| \le \chi$, and the first assertion is proved. The second assertion follows from the first one. □

Example 4.1 Consider the problem TCP$(\mathcal{A}, a)$ where $\mathcal{A} \in \mathbb{R}^{[3,2]}$ is given by setting $a_{111} = a_{122} = -1$, $a_{211} = a_{222} = -1$, and $a_{i_1 i_2 i_3} = 0$ for all other components. One has

$$\mathcal{A}x^{m-1} + q = \begin{bmatrix} -x_1^2 - x_2^2 \\ -x_1^2 - x_2^2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

where $a_1, a_2 \in \mathbb{R}$. Due to Remark 2.1, $x \in \text{Sol}(\mathcal{A}, a)$ if and only if there exists $\lambda \in \mathbb{R}^2$ such that

$$\begin{bmatrix} -x_1^2 - x_2^2 \\ -x_1^2 - x_2^2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \lambda_1 x_1 + \lambda_2 x_2 = 0, \ \lambda \ge 0, \ x \ge 0.$$

An easy computation shows that

$$\text{Sol}(\mathcal{A}, a) = \begin{cases} \left\{(0,0), (0, \sqrt{a_2})\right\}, & \text{if } 0 \le a_2 < a_1, \\ \left\{(0,0), (\sqrt{a_1}, 0)\right\}, & \text{if } 0 \le a_1 < a_2, \\ \{(0,0)\} \cup S_{a_1}, & \text{if } 0 \le a_1 = a_2, \\ \emptyset, & \text{if otherwise,} \end{cases}$$

where

$$S_{a_1} = \{(x_1, x_2) : x_1^2 + x_2^2 = a_1, x_1 \ge 0, x_2 \ge 0\}.$$

Clearly, $\text{Sol}_{\mathcal{A}}(a)$ is finite-valued for all $a \in \mathbb{S}$, where

$$\mathbb{S} = \mathbb{R}^2 \backslash \{a \in \mathbb{R}^2 : 0 < a_1 = a_2\}.$$

The set $\mathbb{S}$ is generic and semi-algebraic in $\mathbb{R}^2$. Moreover, since $\dim S_{a_1} = 1$ with $a_1 > 0$, according to Theorem 4.1, the map Sol is not lower semicontinuous at (A, a), where $a \in \mathbb{R}^2$ with $0 < a_1 = a_2$.

5 Upper Semicontinuity of the Solution Map

This section establishes a closed relationship between the R0 property and the upper semicontinuity of the solution map of tensor complementarity problems. Furthermore, two results on the single-valued continuity of the solution map $\mathrm{Sol}_{\mathcal{A}}$ are obtained.

Now we recall that the set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is *upper semicontinuous* at $x \in \mathbb{R}^m$ iff for any open set $V \subset \mathbb{R}^n$ such that $\Psi(x) \subset V$ there exists a neighborhood U of x such that $\Psi(x') \subset V$ for all $x' \in U$. If Ψ is upper semicontinuous at every $x \in \mathbb{R}^m$ then Ψ is said to be upper semicontinuous on $\mathbb{R}^m$. Remind that if Ψ is closed and locally bounded at x then Ψ is upper semicontinuous at x (see, e.g., [33, p.139]).

5.1 Necessity and Sufficiency

Proposition 5.1 *If $\mathcal{A}$ is an R0–tensor and $a \in \mathbb{R}^n$ satisfying $\mathrm{Sol}(\mathcal{A}, a) \neq \emptyset$, then the following two statements are valid:*

(a) *The map* Sol *is upper semicontinuous at (A, a);*
(b) *The map* $\mathrm{Sol}_{\mathcal{A}}$ *is upper semicontinuous at a.*

Proof Suppose that $\mathcal{A}$ is an R0–tensor and $\mathrm{Sol}(\mathcal{A}, a)$ is nonempty. By Remark 2.3 and Corollary 3.1, the map Sol is closed and locally bounded at (A, a). Hence, Sol is upper semicontinuous at (A, a). The assertion (a) is proved. The closedness of $\mathrm{Sol}_{\mathcal{A}}$ follows that of Sol, according to Corollary 3.1; $\mathrm{Sol}_{\mathcal{A}}$ is locally bounded at a. Hence, $\mathrm{Sol}_{\mathcal{A}}$ is upper semicontinuous at a, and the proof of (b) is completed. □

Example 5.1 Consider the problem $\mathrm{TCP}(\mathcal{A}, a)$ given in Example 4.1. One has $\mathrm{Sol}_{\mathcal{A}}(0) = \{0\}$, so $\mathcal{A}$ is an R0–tensor. By Proposition 5.1, $\mathrm{Sol}_{\mathcal{A}}$ is upper semicontinuous on $\mathbb{R}^2_+$.

Remark 5.1 The inverse assertion of (b) in Proposition 5.1 is not true. Indeed, choose $\mathcal{A} = \mathcal{O} \in \mathbb{R}^{[3,2]}$; one has

$$\mathrm{Sol}_{\mathcal{O}}(a_1, a_2) = \begin{cases} \mathbb{R}^2_+, & \text{if } a_1 = 0, a_2 = 0, \\ \mathbb{R}_+ \times \{0\}, & \text{if } a_1 = 0, a_2 > 0, \\ \{0\} \times \mathbb{R}_+, & \text{if } a_1 > 0, a_2 = 0, \\ \{(0,0)\}, & \text{if } a_1 > 0, a_2 > 0, \\ \emptyset, & \text{if } \quad \text{otherwise.} \end{cases}$$

It is easy to check that $\mathrm{Sol}_{\mathcal{O}}$ is upper semicontinuous on $\mathrm{dom}\, \mathrm{Sol}_{\mathcal{O}} = \mathbb{R}^2_+$, but $\mathcal{O}$ does not have the R0 property.

Proposition 5.2 *Assume that* $\mathrm{Sol}(\mathcal{A}, a)$ *is nonempty and bounded. If the map* Sol *is upper semicontinuous at $(\mathcal{A}, a)$, then $\mathcal{A}$ is an R0–tensor.*

Proof Suppose that $\mathrm{Sol}(\mathcal{A}, 0) \neq \{0\}$ and $y \in \mathrm{Sol}(\mathcal{A}, 0)$ with $y \neq 0$. According to Remark 2.1, there exists $\lambda \in \mathbb{R}^n$ such that

$$\mathcal{A}\, y^{m-1} - \lambda = 0,\ \langle \lambda, y\rangle = 0,\ \lambda \geq 0,\ y \geq 0. \tag{10}$$

For each $t \in]0, 1[$, we take $y_t = t^{-1}y$ and $\lambda_t = t^{-(m-1)}\lambda$. We will show that for every t there exists $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$, with $\mathcal{A}_t \to \mathcal{A}$ when $t \to 0$, and the following system is satisfied

$$\mathcal{A}_t(y_t)^{m-1} + q - \lambda_t = 0,\ \langle \lambda_t, y_t\rangle = 0,\ \lambda_t \geq 0,\ y_t \geq 0. \tag{11}$$

Since $y = (y_1, \ldots, y_n) \neq 0$, there exists $y_\ell \neq 0$, so one has $y_\ell^{m-1} \neq 0$. Take $\mathcal{Q} \in \mathbb{R}^{[m,n]}$ such that

$$\mathcal{Q}\, x^{m-1} = \left(q_1 x_\ell^{m-1}, \ldots, q_n x_\ell^{m-1}\right),$$

where $q_j = -a_j / y_\ell^{m-1}$ for $j = 1, \ldots, n$. It is clear that $\mathcal{Q}\, y^{m-1} + a = 0$. We take $\mathcal{A}_t = \mathcal{A} + t\,\mathcal{Q}$ and claim that system (11) holds. Indeed, the last two inequalities in (11) are obvious. Consider the left-hand side of the first equation in (11); from (10), we have

$$\begin{aligned}\mathcal{A}_t(y_t)^{m-1} + a - \lambda_t &= (\mathcal{A} + t\,\mathcal{Q})(t^{-1}y)^{m-1} + a - t^{-(m-1)}\lambda\\ &= t^{-(m-1)}(\mathcal{A}\, y^{m-1} - \lambda) + (\mathcal{Q}\, y^{m-1} + a)\\ &= 0.\end{aligned}$$

The second equation in (11) is obtained by

$$\langle \lambda_t, y_t\rangle = \langle t^{-(m-1)}\lambda, t^{-1}y\rangle = t^{-m}\langle \lambda, y\rangle = 0.$$

According to Remark 2.1, system (11) leads to $y_t \in \mathrm{Sol}(\mathcal{A}_t, a)$. Remind that this argument holds for all $t \in]0, 1[$.

Since $\mathrm{Sol}(\mathcal{A}, a)$ is nonempty bounded, let V be a nonempty bounded open set containing $\mathrm{Sol}(\mathcal{A}, a)$. By the upper semicontinuity of Sol at $(\mathcal{A}, a)$, there exists $\delta > 0$ such that $\mathrm{Sol}(\mathcal{B}, b) \subset V$ for all $(\mathcal{B}, b) \in \mathbb{R}^{[m,n]} \times \mathbb{R}^n$ satisfying $\|(\mathcal{B}, b) - (\mathcal{A}, a)\| < \delta$. Taking t small enough such that $\|(\mathcal{A}_t, a) - (\mathcal{A}, a)\| < \delta$, we have $\mathrm{Sol}(\mathcal{A}_t, a) \subset V$. So, $y_t \in V$ for every $t > 0$ sufficiently small. This is impossible, because V is bounded and $y_t = t^{-1}y \to \infty$ as $t \to 0$. The assertion is proved. □

The main result of this section is the next theorem.

Theorem 5.1 *Let $\mathcal{A}$ be given. The following two statements are equivalent:*

(a) *The tensor $\mathcal{A}$ is R0;*
(b) *The map* Sol *is upper semicontinuous at $(\mathcal{A}, a)$, for every $a \in \mathbb{R}^n$ satisfying* $\mathrm{Sol}(\mathcal{A}, a) \neq \emptyset$.

Proof From Proposition 5.1, one has (a) ⇒ (b). Hence, we only need to prove (b) ⇒ (a). Clearly, $0 \in \mathrm{Sol}(\mathcal{A}, a) \neq \emptyset$ for every $a \in \mathbb{R}^n_+$. By Remark 4.2, there exists $q \in \mathbb{R}^n_+$ such that $\mathrm{Sol}(\mathcal{A}, a)$ is bounded. By assumptions, the map Sol is upper semicontinuous at $(\mathcal{A}, a)$. Proposition 5.2 says that $\mathcal{A}$ is R0. The proof is completed. □

5.2 Single-Valued Continuity

Recall that TCP($\mathcal{A}, a$) is said to have the *GUS-property* if TCP($\mathcal{A}, a$) has a unique solution for every $a \in \mathbb{R}^n$. Some special structured tensors which have GUS-property are shown in [4,9]. A new property of the GUS-property of tensor complementarity problems is given in the following theorem.

Theorem 5.2 *If* TCP($\mathcal{A}, a$) *has the GUS-property, then the map* $\mathrm{Sol}_{\mathcal{A}}$ *is single-valued continuous on* $\mathbb{R}^n$.

Proof By assumptions, TCP($\mathcal{A}$, 0) has a unique solution. This implies that $\mathcal{A}$ is an R0–tensor. Proposition 5.1 shows that $\mathrm{Sol}_{\mathcal{A}}$ is upper semicontinuous on $\mathbb{R}^n$. Therefore, $\mathrm{Sol}_{\mathcal{A}}$ is single-valued continuous on $\mathbb{R}^n$. □

Example 5.2 Consider the problem TCP($\mathcal{A}, a$) where $\mathcal{A} \in \mathbb{R}^{[3,2]}$ given by setting $a_{111} = a_{222} = 1$ and $a_{i_1 i_2 i_3} = 0$ for all other components. Obviously, one has

$$\mathcal{A}x^{m-1} + q = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

where $a_1, a_2 \in \mathbb{R}$. An easy computation shows that

$$\mathrm{Sol}_{\mathcal{A}}(a_1, a_2) = \begin{cases} \left\{(\sqrt{-a_1}, \sqrt{-a_2})\right\}, & \text{if } a_1 < 0, a_2 < 0, \\ \left\{(0, \sqrt{-a_2})\right\}, & \text{if } a_1 \geq 0, a_2 < 0, \\ \left\{(\sqrt{-a_1}, 0)\right\}, & \text{if } a_1 < 0, a_2 \geq 0, \\ \{(0, 0)\}, & \text{if } a_1 \geq 0, a_2 \geq 0. \end{cases}$$

The problem TCP($\mathcal{A}, a$) has the GUS-property, the domain of $\mathrm{Sol}_{\mathcal{A}}$ is $\mathbb{R}^2$, and $\mathrm{Sol}_{\mathcal{A}}$ is single-valued continuous on $\mathbb{R}^2$.

Recall that a tensor $\mathcal{A}$ is *copositive*, if $\mathcal{A}x^m \geq 0$ for all $x \geq 0$. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* on $X \subset \mathbb{R}^n$, if for all $x, y \in X$ the following inequality is satisfied

$$\langle F(x) - F(y), x - y \rangle \geq 0. \tag{12}$$

If $F(x) = \mathcal{A}x^{m-1}$ is monotone on $\mathbb{R}^n_+$ then $\mathcal{A}$ is copositive. Indeed, if one takes $y = 0$ in (12), then $\mathcal{A}$ is copositive.

Remark 5.2 If the R0–tensor $\mathcal{A}$ is copositive, then $\mathrm{Sol}(\mathcal{A}, a)$ is nonempty for every $q \in \mathbb{R}^n$ [22, Corollary 7.2].

Theorem 5.3 *Assume that* $\mathcal{A}$ *is an R0–tensor. If* $F(x) = \mathcal{A}x^{m-1}$ *is monotone on* $\mathbb{R}^n_+$, *then the map* $\mathrm{Sol}_{\mathcal{A}}$ *is single-valued continuous on a generic semi-algebraic set in* $\mathbb{R}^n$.

Proof By the copositivity and the R0 property of $\mathcal{A}$, according to Corollary 7.2 in [22], one has $\mathrm{Sol}_{\mathcal{A}}(a) \neq \emptyset$ for all $a \in \mathbb{R}^n$. By Proposition 4.1, there exists a generic semi-algebraic set $\mathbb{S} \subset \mathbb{R}^n$ such that $\mathrm{Sol}_{\mathcal{A}}$ is finite-valued on $\mathbb{S}$.

For every $a \in \mathbb{R}^n$, by the monotonicity of F, $F + a$ is also monotone. It follows that $\mathrm{Sol}_{\mathcal{A}}(a)$ is convex [33, Theorem 2.3.5]. Since $\mathrm{Sol}_{\mathcal{A}}(a)$ is nonempty and has finite points, this set has a unique point. So, $\mathrm{Sol}_{\mathcal{A}}$ is single-valued on $\mathbb{S}$. Moreover, Proposition 5.1 says that $\mathrm{Sol}_{\mathcal{A}}$ is upper semicontinuous on $\mathbb{S}$. From what has already been shown, $\mathrm{Sol}_{\mathcal{A}}$ is single-valued continuous on $\mathbb{S}$. □

Example 5.3 Consider the problem TCP$(\mathcal{A}, a)$ where $\mathcal{A} \in \mathbb{R}^{[3,2]}$ is given by setting $a_{111} = a_{122} = 1$, $a_{211} = a_{222} = 1$, and $a_{i_1 i_2 i_3} = 0$ for all other components. Obviously, one has

$$F(x) = \mathcal{A}x^{m-1} + a = \begin{bmatrix} x_1^2 + x_2^2 \\ x_1^2 + x_2^2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

where the parameters $a_1, a_2 \in \mathbb{R}$. The Jacobian matrix of F is positive semidefinite on $\mathbb{R}^2_+$. Hence, F is monotone on $\mathbb{R}^2_+$. By Remark 2.1, an easy computation shows that

$$\mathrm{Sol}_{\mathcal{A}}(a_1, a_2) = \begin{cases} \left\{(0, \sqrt{-a_2})\right\}, & \text{if } a_2 < 0, a_2 \le a_1, \\ \left\{(\sqrt{-a_1}, 0)\right\}, & \text{if } a_1 < 0, a_1 \le a_2, \\ \{(0, 0)\}, & \text{if } 0 \le a_1, 0 \le a_2, \\ S_{-a_1}, & \text{if } a_1 = a_2 < 0, \end{cases}$$

where

$$S_{-a_1} = \{(x_1, x_2) : x_1^2 + x_2^2 = a_1, x_1 \ge 0, x_2 \ge 0\}, \; a_1 < 0.$$

The tensor $\mathcal{A}$ is R0 since $\mathrm{Sol}_{\mathcal{A}}(0, 0) = \{(0, 0)\}$. The map $\mathrm{Sol}_{\mathcal{A}}$ is single-valued continuous on the generic semi-algebraic set $\mathbb{S}$, where

$$\mathbb{S} = \mathbb{R}^2 \backslash \{(a_1, a_2) \in \mathbb{R}^2 : a_1 = a_2 < 0\}.$$

6 Stability of the Solution Map

We will show that the map $\mathrm{Sol}_{\mathcal{A}}$ is locally upper-Hölder when the involved tensor is R0. In addition, if the tensor is copositive then one obtains a result on the stability of the solution map.

The map $\mathrm{Sol}_{\mathcal{A}}$ defined in (9) is said to be *locally upper-Hölder* at a if there exist $\gamma > 0$, $c > 0$ and $\varepsilon > 0$ such that

$$\mathrm{Sol}_{\mathcal{A}}(b) \subset \mathrm{Sol}_{\mathcal{A}}(a) + \gamma \|b - a\|^c B(0, 1)$$

for all a satisfying $\|b - a\| < \varepsilon$, where $B(0, 1)$ is the closed unit ball in $\mathbb{R}^n$.

Proposition 6.1 *If $\mathcal{A}$ is R0 and $\mathrm{Sol}(\mathcal{A}, a) \neq \emptyset$, then the map $\mathrm{Sol}_{\mathcal{A}}$ is locally upper-Hölder at a.*

Proof By our assumptions and Proposition 5.1, it follows that $\mathrm{Sol}_{\mathcal{A}}$ is upper semicontinuous at a. According to [35, Theorem 4.1], the upper semicontinuity and the local upper-Hölder stability of $\mathrm{Sol}_{\mathcal{A}}$ at a are equivalent. Hence, the assertion is proved. □

Let C be a nonempty and closed cone. Here $\mathrm{int}\, C^+$ stands for the interior of the dual cone C^+ of C. Note that $q \in \mathrm{int}\, C^+$ if and only if $\langle v, q\rangle > 0$ for all $v \in C$ and $v \neq 0$ [34, Lemma 6.4].

Proposition 6.2 *If $\mathcal{A}$ is copositive and $a \in \mathrm{int}(\mathrm{Sol}(\mathcal{A}, 0)^+)$, then the map $\mathrm{Sol}_{\mathcal{A}}$ is locally upper-Hölder at a.*

Proof Suppose that $\mathcal{A}$ is copositive and $a \in \mathrm{int}(\mathrm{Sol}(\mathcal{A}, 0)^+)$. On account of [22, Corollary 7.3], $\mathrm{Sol}(\mathcal{A}, a)$ is nonempty and compact. Due to [35, Theorem 4.1], we only need to prove that $\mathrm{Sol}_{\mathcal{A}}$ is upper semicontinuous at a.

We suppose that $\mathrm{Sol}_{\mathcal{A}}$ is not upper semicontinuous at a. There are a nonempty open set V containing $\mathrm{Sol}(\mathcal{A}, a)$, and two sequences $\{a^k\} \subset \mathbb{R}^n$, where $a^k \to a$, and $\{x^k\} \subset \mathbb{R}^n_+$ such that

$$x^k \in \mathrm{Sol}(\mathcal{A}, a^k)\backslash V. \tag{13}$$

The sequence $\{x^k\}$ is bounded. Indeed, if $\{x^k\}$ is unbounded, then we can assume that $\|x^k\|^{-1}x^k \to v$ and $\|v\| = 1$. Clearly, one has $v \in \mathrm{Sol}(\mathcal{A}, 0)$. From (3), (13), and the copositivity of $\mathcal{A}$, we have

$$-\left\langle x^k, a^k\right\rangle = \mathcal{A}(x^k)^m \geq 0.$$

It follows that $\langle v, a\rangle \leq 0$. This contradicts the fact that $a \in \mathrm{int}(\mathrm{Sol}(\mathcal{A}, 0)^+)$. So, $\{x^k\}$ is bounded, and we can suppose that $x^k \to \bar{x}$. It easy to check that $\bar{x} \in \mathrm{Sol}(\mathcal{A}, a)$. This leads to $\bar{x} \in V$. Besides, since V is an open nonempty set, the relation (13) implies that $\bar{x} \notin V$. We obtain a contradiction. Therefore, $\mathrm{Sol}_{\mathcal{A}}$ is upper semicontinuous at a. □

Theorem 7.5.1 in [23] is an interesting result on the upper-Lipschitz stability of the solution map of linear complementarity problems under the copositivity condition. Here, we obtain an analogous one for the solution map of tensor complementarity problems.

Theorem 6.1 *Assume that $\mathcal{A}$ is copositive and $a \in \mathrm{int}(\mathrm{Sol}(\mathcal{A}, 0)^+)$. Then there exist constants $\varepsilon > 0$, $\gamma > 0$, and $c > 0$ such that, if $\mathcal{B} \in \mathbb{R}^{[m,n]}$ and $b \in \mathbb{R}^n$ satisfy*

$$\max\{\| \mathcal{B} - \mathcal{A} \|, \|b - a\|\} < \varepsilon, \tag{14}$$

and $\mathcal{B}$ is copositive, then the following statements are true:

(a) *The set $\mathrm{Sol}(\mathcal{B}, b)$ is nonempty and bounded;*

(b) *One has*

$$\mathrm{Sol}(\mathcal{B}, b) \subset \mathrm{Sol}(\mathcal{A}, a) + \gamma(\| \mathcal{B} - \mathcal{A} \| + \|b - a\|)^c B(0, 1). \tag{15}$$

Proof (a) We prove that there exists $\varepsilon_1 > 0$ such that, if $\mathcal{B}$ is copositive and $b \in \mathbb{R}^n$ satisfy

$$\max\{\| \mathcal{B} - \mathcal{A} \|, \|b - a\|\} < \varepsilon_1, \tag{16}$$

then $\mathrm{Sol}(\mathcal{B}, b)$ is nonempty and bounded. Suppose that the assertion is false. Then there is a sequence $\{(\mathcal{B}^k, b^k)\}$, where $(\mathcal{B}^k, b^k) \to (\mathcal{A}, a)$ and $\mathcal{B}^k$ is copositive, such that $\mathrm{Sol}(\mathcal{B}^k, b^k)$ is empty or unbounded, for each $k \in \mathbb{N}$. From [22, Corollary 7.3], we conclude that $b^k \notin \mathrm{int}(\mathrm{Sol}(\mathcal{B}^k, 0)^+)$, and then there exists $x^k \in \mathrm{Sol}(\mathcal{B}^k, 0)$ such that $x^k \neq 0$ and $\langle x^k, b^k \rangle \leq 0$. We can suppose that $\|x^k\|^{-1}x^k \to \bar{x}$ with $\|\bar{x}\| = 1$. Clearly, one has $\langle \bar{x}, a \rangle \leq 0$.

If we prove that $\bar{x}$ solves $\mathrm{TCP}(\mathcal{A}, 0)$, then this contradicts the assumption that $a \in \mathrm{int}(\mathrm{Sol}(\mathcal{A}, 0)^+)$, and (a) is proved. Thus, we only need to show that $\bar{x} \in \mathrm{Sol}(\mathcal{A}, 0)$. Because x^k belongs to $\mathrm{Sol}(\mathcal{B}^k, 0)$, one has

$$\mathcal{B}^k(x^k)^{m-1} \geq 0, \;\; \mathcal{B}^k(x^k)^m = 0. \tag{17}$$

By dividing the inequality and the equation in (17) by $\|x^k\|^{m-1}$ and $\|x^k\|^m$, respectively, and taking $k \to +\infty$, we obtain

$$\mathcal{A}\bar{x}^{m-1} \geq 0, \;\; \mathcal{A}\bar{x}^m = 0.$$

This leads to $\bar{x} \in \mathrm{Sol}(\mathcal{A}, 0)$.

(b) We prove the inclusion (15). According to Proposition 6.2, there exist $\gamma_0 > 0$, $c > 0$ and ε such that

$$\mathrm{Sol}(\mathcal{A}, b) \subset \mathrm{Sol}(\mathcal{A}, a) + \gamma_0\|b - a\|^c B(0, 1) \tag{18}$$

for every b satisfying $\|b - a\| < \varepsilon$. Suppose that $\mathcal{B}$ is copositive and $b \in \mathbb{R}^n$ such that (16) holds. For each $z \in \mathrm{Sol}(\mathcal{B}, b)$, by setting

$$\bar{b} := b + (\mathcal{B} - \mathcal{A})\, z^{m-1}, \tag{19}$$

we have

$$Az^{m-1} + \bar{b} = \mathcal{B}\, z^{m-1} + b \geq 0, \;\; \left\langle z, \mathcal{A}\, z^{m-1} + \bar{b} \right\rangle = \left\langle z, \mathcal{B}\, z^{m-1} + b \right\rangle = 0.$$

These show that $z \in \mathrm{Sol}(\mathcal{A}, \bar{b})$. By the assertion (a), $\mathrm{Sol}(\mathcal{B}, b)$ is bounded and nonempty. Remark 2.2 states that there exists $\beta > 0$ such that

$$\|(\mathcal{B} - \mathcal{A})z^{m-1}\| \leq \beta\| \mathcal{B} - \mathcal{A} \|, \tag{20}$$

for any $z \in \mathrm{Sol}(\mathcal{B}, b)$. From (19) and (20), one has

$$\begin{aligned} \|\bar{b}-a\| &\leq \|b-a\| + \|(\mathcal{B}-\mathcal{A})z^{m-1}\| \\ &\leq \|b-a\| + \beta\|\mathcal{B}-\mathcal{A}\|. \end{aligned} \tag{21}$$

Hence, for ε_1 given in the proof of (a), on account of (16), we conclude that $\|\bar{b}-a\| \leq (1+\beta)\varepsilon_1$. Choose ε_1 as small as $\|\bar{b}-a\| < \varepsilon$. Since $z \in \mathrm{Sol}(\mathcal{A}, \bar{b})$, by (18) and (21), there exists $x \in \mathrm{Sol}(\mathcal{A}, a)$ such that

$$\begin{aligned} \|z-x\| &\leq \gamma_0\|\bar{b}-a\|^c \\ &\leq \gamma_0\left(\|b-a\| + \beta\|\mathcal{B}-\mathcal{A}\|\right)^c \\ &\leq \gamma\left(\|b-a\| + \|\mathcal{B}-\mathcal{A}\|\right)^c, \end{aligned}$$

where $\gamma = \max\left\{\gamma_0^c, \gamma_0^c\beta\right\}$. The inclusion (15) is obtained. □

Corollary 6.1 *Assume that $\mathcal{A}$ is an R0–tensor, $F(x) = \mathcal{A}x^{m-1}$ is monotone on $\mathbb{R}^n_+$. Then for any $a \in \mathbb{R}^n$, there exist constants $\varepsilon > 0, \gamma > 0$ and $c > 0$ such that, if $\mathcal{B} \in \mathbb{R}^{[m,n]}$ and $b \in \mathbb{R}^n$ satisfy (14) and $G(x) = \mathcal{B}x^{m-1}$ is monotone on $\mathbb{R}^n_+$, then the following statements are true:*

(a) *The set* $\mathrm{Sol}(\mathcal{B}, b)$ *is nonempty and bounded;*
(b) *The conclusion* (15) *is valid.*

Proof Since $\mathrm{Sol}(\mathcal{A}, 0) = \{0\}$, one has $\mathrm{int}(\mathrm{Sol}(\mathcal{A}, 0)^+) = \mathbb{R}^n$. By the monotonicity of F and G, both $\mathcal{A}, \mathcal{B}$ are copositive. Therefore, the assertions follow Theorem 6.1. □

7 Conclusions

In this paper, we have proved that the set $\mathcal{R}_0$ of all R0-tensors is an open generic semi-algebraic cone. Upper and lower estimates for the dimension of the complement $C(\mathcal{R}_0)$ are shown. Several results on local boundedness, upper semicontinuity, lower semicontinuity, finite-valuedness, and stability of the solution map have been obtained. In our further research, we intend to develop these results for polynomial variational inequalities.

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