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Optimality Conditions and Duality for Robust Nonsmooth Multiobjective Optimization Problems with Constraints

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Abstract
In this paper, we investigate a robust nonsmooth multiobjective optimization problem related to a multiobjective optimization with data uncertainty. We firstly introduce two kinds of generalized convex functions, which are not necessary to be convex. Robust necessary optimality conditions for weakly robust efficient solutions and properly robust efficient solutions of the problem are established by a generalized alternative theorem and the robust constraint qualification. Further, robust sufficient optimality conditions for weakly robust efficient solutions and properly robust efficient solutions of the problem are also derived. The Mond–Weir-type dual problem and Wolfe-type dual problem are formulated. Finally, we obtain the weak, strong and converse robust duality results between the primal one and its dual problems under the generalized convexity assumptions.

Keywords Robust nonsmooth multiobjective optimization · Uncertain nonsmooth multiobjective optimization · Robust optimality conditions · Robust duality

Mathematics Subject Classification 65K10 · 90C25

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1 Introduction

Robust optimization, which is an important approach to study optimization problems with data uncertainty, has grown rapidly over the past two decades; see [1–8] and the references therein. An overview of robust approaches for uncertain vector-valued functions along with algorithms for obtaining robust solutions is presented in [9, Chapter 15.4]. An uncertain optimization problem can be studied through its robust counterpart. Robust optimization can be reviewed as a kind of sensitivity against perturbations in the decision space. In particular, robust optimization considers the case that no probability distribution information on the uncertain parameters is given. It is also common that the performance of the solutions is judged by multiple objectives that are in conflict, such as quality/benefits versus cost. This situation calls for a prioritization of the objectives, which can be difficult to articulate into a precise mathematical function prior to the procurement of information about the possible solutions to the problem. So, it is interesting to deeply study the theory and applications of robust multiobjective optimization.

In the optimization theory including multiobjective optimization, the main focus is placed on finding the global optimum or global efficient solutions, representing the best possible objective values. But, in practice, one may not always be interested in finding the so-called global best solutions, particularly when these solutions are quite sensitive to the variable perturbations, which cannot be avoided in practice. In such cases, one would be interested in finding robust solutions, which are less sensitive to small perturbations in variables. Gunawan and Azarm [10] studied the robust optimality for robust Pareto solution of nondifferentiable multiobjective robust optimization by using a sensitivity region concept, and obtained sensitivity results of its robust Pareto sets by a sensitivity measure, which did not require a presumed probability distribution of uncontrollable parameters. Deb and Gupta [11] presented two different robust multiobjective optimization procedures, where the emphasis was to find a robust solution, and showed the differences between global and robust multiobjective optimization principles. Kuroiwa and Lee [13] gave three kinds of robust efficient solutions for an uncertain multiobjective optimization problem (UMOP) where the objective functions and constraint functions were perturbed by different uncertain data, and established necessary optimality conditions for weakly and properly robust efficient solutions for (UMOP). Kim [14] discussed the Mond–Weir-type duality and weak saddle-point results for an uncertain multiobjective robust optimization problem. Fliege and Werner [15] studied general convex parametric multiobjective optimization problems with data uncertainty by using a robust approach, proved the relationship between the original multiobjective formulation and its scalarizations, and showed that standard techniques from multiobjective optimization can be applied to characterize this robust efficient frontier.

Recently, Goberna et al. [16] dealt with robust solutions of multiobjective linear semi-infinite programs under constraint data uncertainty, provided a radius of robust feasibility guaranteeing the feasibility of the robust counterpart under affine data parametrization, and established dual characterizations of robust solutions of the model that were immunized against data uncertainty by way of characterizing corresponding solutions of robust counterpart of the model. Ide and Köbis [17] introduced various concepts of efficiency for uncertain multiobjective optimization problems based on set less order relations, analyzed the resulting concepts of efficiency and presented numerical results on the occurrence of the various concepts. Goberna et al. [18] presented the numerically tractable optimality conditions for minmax robust weakly efficient solutions and highly robust weakly efficient solutions of multiobjective linear programming problems in the face of data uncertainty in both the objective function and the constraints, and derived a formula for the radius of robust feasibility guaranteeing constraint feasibility for all possible scenarios within a specified uncertainty set under affine data parametrization and the lower bounds for the radius of highly robust efficiency guaranteeing the existence of highly robust weakly efficient solutions under affine and rank-1 objective data uncertainty. Chuong [19] considered necessary/sufficient optimality conditions for robust (weakly) Pareto solutions of the a robust nonsmooth multiobjective optimization problem in terms of multipliers and limiting subdifferentials of the related functions, and explored weak/strong duality relations between the primal one and its dual robust problem under the (strictly) generalized convexity assumptions. Ide and Schöbel [20] compared and analyzed ten different concepts of robustness for (UMOP), and presented reduction results for the class of objective-wise uncertain multiobjective optimization problems. Lee and Lee [21] studied necessary/sufficient optimality conditions for a weakly robust efficient solution of a robust semi-infinite multiobjective optimization problem and gave its robust counterpart (RSIMP) of the problem in the worst case of the uncertain semi-infinite multiobjective optimization problem, and derived duality results on (RSIMP) and its Wolfe-type dual problem.

Very recently, Klamroth et al. [22] considered uncertain scalar optimization problems with infinite scenario sets, and developed a unified characterization of different concepts of robust optimization and stochastic programming by the existing methods arising from vector optimization in general spaces, set optimization as well as scalarization techniques. Bokrantz and Fredriksson [23] established necessary and sufficient conditions for robust efficiency to multiobjective optimization problems that depend on uncertain parameters by using scalarization method. Chen et al. [24] investigated the optimality conditions for a class of nonsmooth multiobjective optimization problems with cone constraints by a modified objective function method, and applied to multiobjective fractional programming problems. Suneja et al. [25] introduced some generalized convex function associated with a vector optimization problem, and obtained sufficient optimality conditions for the vector optimization problem and weak/strong duality results between the primal problem and its Mond–Weir-type dual problem using Clarke's generalized gradients. To the best of our knowledge, there are no results on the general weakly robust efficient solution and properly robust efficient solution of nonsmooth/nonconvex multiobjective optimization with constraint data uncertainty.

Motivated and inspired by the works [5,6,14,16,19,21,23], this paper is devoted to investigate a nonsmooth/nonconvex multiobjective optimization problems with constraint data uncertainty (shortly, (UCMOP)) by using a robust approach. The paper is organized as follows. In Sect. 2, we firstly present the robust counterpart of (UCMOP), which is called a robust nonsmooth multiobjective optimization problem. Together with the efficient solutions notions of [26–28], we introduce weakly robust efficient solution and properly robust efficient solution of (UCMOP) and several generalized convex functions, which are not necessary to be convex and smooth. Fritz–John/Kuhn–Tucker-type robust necessary optimality conditions for weakly robust efficient solution and properly robust efficient solution of (UCMOP) are established by the generalized alternative theorem of Illés and Kassay [29] in Sect. 3. Sufficient optimality conditions for weakly robust efficient solution and properly robust efficient solution of (UCMOP) are deduced under the generalized convexity assumptions. In Sect. 4, the Mond–Weir-type dual problem is formulated. Section 5 is concerned with the Wolfe-type dual problem. In these sections, we obtain the weak, strong and converse robust duality results between the primal one and its dual problems under the generalized convexity assumptions. Section 6 concludes this paper with a summary of the obtained results and an outlook to future research.

2 Preliminaries

Let $\mathbb{R}^n_+$ be the nonnegative orthant of the n-dimensional Euclidean space $\mathbb{R}^n$. A nonempty subset C of $\mathbb{R}^n$ is called a *convex cone* if $C + C \subseteq C$ and $tC \subseteq C$ for all $t>0$. Throughout this paper, without other specifications, let $Q \subseteq \mathbb{R}^k$ be proper, closed and convex cone with nonempty interior, V_i be a nonempty, convex and compact subset of $\mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m n_i = p$, and let $f = (f_1, \ldots, f_k)^\top : \mathbb{R}^n \to \mathbb{R}^k$ and $g = (g_1, \ldots, g_m)^\top : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ be vector-valued mappings and $g_i : \mathbb{R}^n \times \mathbb{R}^{n_i} \to \mathbb{R}$ for $i = 1, 2, \ldots, m$, where the superscript $\top$ denotes the transpose. We denote the closed hull, convex hull and interior of Q by $\mathrm{cl}Q$, $\mathrm{conv}Q$ and $\mathrm{int}Q$, respectively. The dual cone (positive polar cone) of Q is defined by

$$Q^* = \{u \in \mathbb{R}^m : u^\top x \geq 0,\ \forall x \in Q\}.$$

Consider the following *uncertain constrained multiobjective optimization problem*:

$$\min \quad f(x) \qquad \text{s.t.} \quad -g(x, v) \in S, \tag{UCMOP}$$

where $S \subseteq \mathbb{R}^m$ is a proper, closed and convex cone, $v := (v_1, v_2, \ldots, v_m)^\top \in \mathbb{R}^p$ is the vector of uncertain parameter with $v_i \in V_i, i = 1, 2, \ldots, m$, and $x \in \mathbb{R}^n$ is the vector of decision variable.

For the sake of brevity, we set $V := \prod_{i=1}^m V_i$. Since V_i are nonempty, convex and compact sets for all $i = 1, 2, \ldots, m$, then V is a nonempty, convex and compact subset of $\mathbb{R}^p$. One of the important approaches for studying (UCMOP) is the robust optimization approach, which can be reviewed as a kind of sensitivity against perturbations in the decision space. Particularly, robust optimization considers the case that no stochastic information on the uncertain parameters is given.

In this paper, we investigate (UCMOP) by a robust approach. The *robust counterpart* of (UCMOP) read as following:

$$\min \quad f(x) \qquad \text{s.t.} \quad -g(x, v) \in S,\ \forall\, v \in V. \tag{RMOP}$$

A vector x is called a robust feasible solution of (UCMOP) if it is a feasible solution of (RMOP). Denote by $F := \{x \in \mathbb{R}^n : -g(x, v) \in S, \ \forall v \in V\}$ the set of all robust feasible solutions of (UCMOP). It is noted that if $Q := \mathbb{R}^k_+$ and $S := \mathbb{R}^m_+$, then (RMOP) is reduced to the robust optimization problems of [14,19]; in particular, if $f : \mathbb{R}^n \to \mathbb{R}$, $Q := \mathbb{R}_+$ and $S := \mathbb{R}^m_+$, then (RMOP) is reduced to the scalar robust optimization problems of [5,6]. As we know, any system of inequalities and equalities can be characterized by the nonnegative orthant. Without loss of generality, we always assume that $S := \mathbb{R}^m_+$ throughout this paper.

We now recall some definitions and basic results in the literature.

Definition 2.1 A vector $\bar{x} \in F$ is called a *weakly robust efficient solution* of (UCMOP) iff

$$f(x) - f(\bar{x}) \notin -\text{int}Q, \ \forall x \in F.$$

Definition 2.2 ([27]) A vector $\bar{x} \in F$ is called a *properly robust efficient solution* of (UCMOP) iff there exists $\lambda \in \text{int}\, Q^*$ such that

$$\lambda^\top (f(x) - f(\bar{x})) \geq 0, \ \forall x \in F.$$

We denote the sets of all weakly robust efficient solutions and the properly robust efficient solutions of (UCMOP) by F^w and F^{pr}, respectively. It easily follows from Definitions 2.1 and 2.2 that $F^{pr} \subseteq F^w$. Besides, $\bar{x} \in F^{pr}$ if and only if there exists $\lambda \in \text{int}\, Q^*$ such that $\bar{x} \in \text{argmin}\left\{\lambda^\top f(x) : x \in F\right\}$. This shows that one can compute properly robust efficient solutions of (UCMOP) by a standard linear scalarization method.

The following result of properly robust efficient solutions of (UCMOP) is immediately obtained from Definition 2.2.

Lemma 2.1 $F^{pr} = \bigcup_{\lambda \in \text{int}\, Q^*} \text{argmin}\left\{\lambda^\top f(x) : x \in F\right\}$.

Definition 2.3 ([30]) A real-valued function $\ell : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally Lipschitz* iff, for any $z \in \mathbb{R}^n$, there exist a positive constant κ and a neighborhood U of z such that, for any $x, y \in U$,

$$|\ell(x) - \ell(y)| \leq \kappa \|x - y\|,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$.

In [30], the Clarke's generalized subgradient of ℓ at z is denoted by

$$\partial \ell(z) = \left\{\zeta \in \mathbb{R}^n : \ell^\circ(z; d) \geq \zeta^\top d, \ \forall d \in \mathbb{R}^n\right\},$$

where $\ell^\circ(z; d) := \limsup_{y \to z, t \to 0+} \frac{\ell(y+td) - \ell(y)}{t}$.

Clearly, $\ell^\circ(z; d) = \max\{\langle \zeta, d\rangle : \zeta \in \partial \ell(z)\}$. It is well known that if $\ell : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then the Clarke's generalized subgradient $\partial \ell : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is nonempty and compact-valued on $\mathbb{R}^n$ and also is upper semicontinuous on $\mathbb{R}^n$ (see [31, Proposition 2.5.5, p.54]), i.e., for any sequences (x_n) and (y_n) of $\mathbb{R}^n$ with $x_n \to x \in \mathbb{R}^n$ and $y_n \in \partial\ell(x_n)$ for all $n \in \mathbb{N}$, there exists a subsequence $(y_{n_k}) \to y \in \partial\ell(x)$.

A vector-valued function $f = (f_1, f_2, \ldots, f_k)^\top : \mathbb{R}^n \to \mathbb{R}^k$ is said to be locally Lipschitz on $\mathbb{R}^n$ iff f_i is locally Lipschitz on $\mathbb{R}^n$, $i = 1, \ldots, k$. In the following, we always assume that f is locally Lipschitz on $\mathbb{R}^n$ and that g is locally Lipschitz on $\mathbb{R}^n$ with respect to the first argument and its components are upper semicontinuous with respect to the second argument.

Definition 2.4 ([30]) Let $f = (f_1, f_2, \ldots, f_k)^\top : \mathbb{R}^n \to \mathbb{R}^k$ be locally Lipschitz on $\mathbb{R}^n$. The *generalized subgradient* of f at $z \in \mathbb{R}^n$ is the set

$$\partial f(z) = \{(\zeta_1, \ldots, \zeta_k) : \zeta_i \in \partial f_i(z),\ i = 1, \ldots, k\},$$

where $\partial f_i(z)$ is the Clarke's generalized subgradient of f_i $(i \in \{1, \ldots, k\})$ at $z \in \mathbb{R}^n$.

Definition 2.5 ([32]) Let X be a nonempty subset of $\mathbb{R}^n$. $f : X \to \mathbb{R}^k$ is said to be *Q-convexlike* iff the set $f(X) + Q$ is convex.

It is worth mentioning that a Q-convexlike function is usually not Q-convex function (see, e.g., [32, Example 2.1, p.322]): $f : X \to \mathbb{R}^k$ is said to be *Q-convex* iff the set,

$$\text{epi}_Q(f) := \{(x, y) : x \in X,\ y - f(x) \in Q\},$$

is convex. Besides, the Q-convexlike function contains Q-function as special case (see [33]): $f : X \to \mathbb{R}^k$ is said to be *Q-function* at $\bar{x} \in \mathbb{R}^n$ iff, for any $x, y \in \mathbb{R}^n$ and $t \in]0, 1[$,

$$tf(x) + (1-t)f(y) \in f(tx + (1-t)y) + Q.$$

A pair of functions (f, h) is said to be *$(Q, \mathbb{R}^m_+)$-convexlike* iff f is Q-convexlike and h is $\mathbb{R}^m_+$-convexlike, where $h : X \to \mathbb{R}^m$.

We next introduce two kinds of generalized convex functions corresponding to the robust feasible solutions set F.

Definition 2.6 (f, g) is said to be *type I $(Q, \mathbb{R}^m_+)$-generalized convex* at $\bar{x} \in \mathbb{R}^n$ iff, for each $x \in F$ and $A \in \partial f(\bar{x})$, $B \in \partial_x g(\bar{x}, v)$, $v \in V$, there exists $d \in \mathbb{R}^n$ such that

$$f(x) - f(\bar{x}) - Ad \in Q, \quad g(x, v) - g(\bar{x}, v) - Bd \in \mathbb{R}^m_+.$$

Definition 2.7 (f, g) is said to be *type II $(Q, \mathbb{R}^m_+)$-generalized convex* at $\bar{x} \in \mathbb{R}^n$ iff, for each $x \in F$ and $A \in \partial f(\bar{x})$, $B \in \partial_x g(\bar{x}, v)$, $v \in V$, there exists $d \in \mathbb{R}^n$ such that

$$f(x) - f(\bar{x}) - Ad \in Q, \quad -g(\bar{x}, v) - Bd \in \mathbb{R}^m_+.$$

Remark 2.1 (i) It follows from Definitions 2.5, 2.6 and 2.7 that the following relations hold: the type I $(Q, \mathbb{R}^m_+)$-generalized convexity of (f, g) at $\bar{x} \in \mathbb{R}^n$ $\Longrightarrow$ the type II $(Q, \mathbb{R}^m_+)$-generalized convexity of (f, g) at $\bar{x} \in \mathbb{R}^n$.

(ii) It is noted that the type II $(Q, \mathbb{R}^m_+)$-generalized convexity of (f, g) extends the $(Q, \mathbb{R}^m_+)$-generalized type I (see [25, Definition 5, p. 26] of (f, g) even if $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ are differentiable. Besides, if $Q := \mathbb{R}^k_+$, then the type I $(Q, \mathbb{R}^m_+)$-generalized convexity of (f, g) is reduced to the generalized convexity (see [19, Definition 3.9, p. 136]) of (f, g), which is not necessary convex (see [19, Example 3.10, p. 136]).

We need to point out that the converse of the assertion (i) in Remark 2.1 is not true (see Example 2.2). We next give two examples to show the type I (or, II) $(Q, \mathbb{R}^m_+)$-generalized convexity of a pair of functions (f, g).

Example 2.1 Let $\mathbb{R}^n := \mathbb{R}, \mathbb{R}^k = \mathbb{R}^m = \mathbb{R}^p := \mathbb{R}^2$, let $Q := \{(y, z)^\top \in \mathbb{R}^2 : y \leq 0, z \leq -y\}$, $V_1 := [0, 1]$, $V_2 := [-1, 1]$ and let

$$f(x) := \begin{cases} \begin{pmatrix} x \\ 1+x \end{pmatrix}, & \text{if } x < 0, \\ \begin{pmatrix} -x^2 \\ 1 \end{pmatrix}, & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad g(x, v) := \begin{cases} \begin{pmatrix} x^2 - v_1 - 3 \\ -v_2^2 \end{pmatrix}, & \text{if } x > 0, \\ \begin{pmatrix} -x + v_1 - 3 \\ -\frac{x}{2} - v_2^2 \end{pmatrix}, & \text{if } x \leq 0, \end{cases}$$

where $v = (v_1, v_2) \in V = V_1 \times V_2$. After calculation, we have

$$F = \left\{x \in \mathbb{R} : -g(x, v) \in \mathbb{R}^2_+, \ \forall v \in V\right\} = \left[0, \sqrt{3}\right].$$

Let $\bar{x} = 0 \in F$. Then, $\partial f(\bar{x})^\top = [0, 1] \times [0, 1]$ and $\partial_x g(\bar{x}, v)^\top = [-1, 0] \times [-1/2, 0]$ for all $v \in V$. For each $x \in F$, there exists $d = x^2 \in \mathbb{R}$ such that

$$f(x) - f(\bar{x}) - Ad = \begin{pmatrix} -(1 + a_1)\, x^2 \\ -a_2\, x^2 \end{pmatrix} \in Q$$

and

$$g(x, v) - g(\bar{x}, v) - Bd = \begin{cases} \begin{pmatrix} (1 - b_1)\, x^2 \\ -b_2\, x^2 \end{pmatrix} \in \mathbb{R}^2_+, & \text{if } x \in]0, \sqrt{3}], \\ \mathbf{0} \in \mathbb{R}^2_+, & \text{if } x = 0, \end{cases}$$

where $A = (a_1, a_2)^\top \in \partial f(\bar{x})^\top$ and $B = (b_1, b_2)^\top \in \partial_x g(\bar{x}, v)^\top$. Therefore, (f, g) is type I $(Q, \mathbb{R}^2_+)$-generalized convex at $\bar{x}$.

Example 2.2 Let $\mathbb{R}^n := \mathbb{R}, \mathbb{R}^k = \mathbb{R}^m = \mathbb{R}^p := \mathbb{R}^2$, let $Q := \{(y, z)^\top \in \mathbb{R}^2 : y \leq 0, z \leq -y\}$, $V_1 := [0, 1]$, $V_2 := [-1, 1]$ and let

$$f(x) := \begin{cases} \begin{pmatrix} x \\ 1+x \end{pmatrix}, & \text{if } x < 0, \\ \begin{pmatrix} -x^2 \\ 1 \end{pmatrix}, & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad g(x, v) := \begin{cases} \begin{pmatrix} x^2 - v_1 - 3 \\ -v_2^2 \end{pmatrix}, & \text{if } x < 0, \\ \begin{pmatrix} x^2 - v_1 - 3 \\ -\frac{x}{2} - v_2^2 \end{pmatrix}, & \text{if } x \geq 0, \end{cases}$$

where $v = (v_1, v_2) \in V = V_1 \times V_2$. After calculation, we have

$$F = \left\{x \in \mathbb{R} : -g(x, v) \in \mathbb{R}^2_+,\ \forall\, v \in V\right\} = \left[-\sqrt{3}, \sqrt{3}\right].$$

Let $\bar{x} = 0 \in F$. Then, $\partial f(\bar{x})^\top = [0, 1] \times [0, 1]$ and $\partial_x g(\bar{x}, v)^\top = [0, 1] \times [-1/2, 0]$ for all $v \in V$. For each $x \in F$, there exists $d = x^2 \in \mathbb{R}$ such that

$$f(x) - f(\bar{x}) - Ad = \begin{cases} \begin{pmatrix} (1 - a_1 x)x \\ (1 - a_2 x)x \end{pmatrix} \in Q, & \text{if } x < 0, \\ \begin{pmatrix} -(1 + a_1)x^2 \\ -a_2 x^2 \end{pmatrix} \in Q, & \text{if } x \geq 0, \end{cases}$$

and

$$-g(\bar{x}, v) - Bd = \begin{pmatrix} 3 + v_1 - b_1 x^2 \\ v_2^2 - b_2 x^2 \end{pmatrix} \in \mathbb{R}^2_+,$$

where $A = (a_1, a_2)^\top \in \partial f(\bar{x})^\top$ and $B = (b_1, b_2)^\top \in \partial_x g(\bar{x}, v)^\top$. Therefore, (f, g) is type II $(Q, \mathbb{R}^2_+)$-generalized convex at $\bar{x}$. However, for each $d \in \mathbb{R}$, there exist $x = 1 \in F$ and $B = (b_1, 0)^\top$ such that

$$g(x, v) - g(\bar{x}, v) - Bd = \begin{pmatrix} 1 - b_1 d \\ -\frac{1}{2} \end{pmatrix} \notin \mathbb{R}^2_+,$$

This thus implies that (f, g) is not type I $(Q, \mathbb{R}^2_+)$-generalized convex at $\bar{x}$.

We next present a generalized constraint qualification in the sense of robustness, which plays an important role in establishing necessary optimality conditions.

Definition 2.8 ([13]) The generalized robust Slater constraint qualification (GRSCQ) is satisfied if there exists $x_0 \in \mathbb{R}^n$ such that $-g(x_0, v) \in \text{int}\mathbb{R}^m_+$ for all $v \in V$.

Remark 2.2 (i) If $g(\cdot, v) := h(\cdot)$ for all $v \in V$, then (GRSCQ) reduces to the extended Slater constraint qualification: There exists $x_0 \in \mathbb{R}^n$ such that $-h(x_0) \in \text{int}\mathbb{R}^m_+$.

(ii) It is noted that the condition: "for each $v \in V$, there exists $x_0 \in \mathbb{R}^n$ such that $-g(x_0, v) \in \text{int}\mathbb{R}^m_+$"does not imply (GRSCQ). For instance, let $\mathbb{R}^n = \mathbb{R}^2$, $g(x, v) := \left(x_1^2 + 4x_1 - 5 - \ln(1 + v_1), x_2 - |v_2|\right)^\top$ and $V = V_1 \times V_2 =]0, 1] \times]0, 1]$. After verification, for each $v \in V$, there exists $x = (\frac{1}{2}, \frac{|v_2|}{2})^\top \in \mathbb{R}^2$ such that $-g(x, v) \in \text{int}\mathbb{R}^2_+$. In particular, set $\tilde{v} = (1, 1)^\top \in V$, $\tilde{x} = (\frac{1}{2}, \frac{1}{2})^\top \in \mathbb{R}^2$, then

$$-g(\tilde{x}, \tilde{v}) = \left(\frac{11}{4} + \ln 2, \frac{1}{2}\right)^\top \in \text{int}\mathbb{R}^2_+.$$

However, there exists $\hat{v} = (1, \frac{1}{2})^\top \in V$ such that $-g(\tilde{x}, \hat{v}) = \left(\frac{11}{4} + \ln 2, 0\right)^\top \notin \text{int}\mathbb{R}^2_+$, and consequently, $g_2(\tilde{x}, \hat{v}_2) = 0 \not< 0$.

Lemma 2.2 ([30]) *If $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in \{1, \ldots, k\})$ are locally Lipschitz, then*

$$\partial \left(\sum_{i=1}^{k} t_i f_i \right)(z) \subseteq \sum_{i=1}^{k} t_i \partial f_i(z), \ \forall z \in \mathbb{R}^n,$$

where $t = (t_1, \ldots, t_k)^\top \in \mathbb{R}^k$.

Lemma 2.3 ([34]) *Let $Q \subseteq \mathbb{R}^k$ be a closed convex cone with $\operatorname{int} Q \neq \emptyset$. Then,*

$$y \in \operatorname{int} Q \ \Leftrightarrow \ u^\top y > 0, \ \forall u \in Q^* \setminus \{\mathbf{0}\}.$$

It is worth noting that for any nonempty subset $\mathcal{U}$ of $\mathbb{R}^{k+m}$, $\mathcal{U} + (Q \times \mathbb{R}^m_+)$ has nonempty interior because of $\operatorname{int} Q \neq \emptyset$ and $\operatorname{int}\mathbb{R}^m_+ \neq \emptyset$. In view of this, the following theorem of the alternative can be derived from [29, Theorem 3.1, p.246].

Lemma 2.4 *Let X be a nonempty subset of $\mathbb{R}^n$. Assume that $(f, h) : X \to \mathbb{R}^{k+m}$ is $(Q, \mathbb{R}^m_+)$-convexlike. Then, the following assertions hold:*

(i) *If there is no $x \in X$ such that*

$$f(x) \in -intQ \ and \ h(x) \in -\mathbb{R}^m_+, \tag{1}$$

then there exist $\lambda \in Q^*$ and $\mu \in \mathbb{R}^m_+$ with $(\lambda, \mu) \neq \mathbf{0}$ such that

$$\lambda^\top f(x) + \mu^\top h(x) \geq 0, \ \forall x \in X. \tag{2}$$

(ii) *If there exist $\lambda \in Q^* \setminus \{\mathbf{0}\}$ and $\mu \in \mathbb{R}^m_+$ such that (2) holds, then there is no $x \in X$ such that (1) holds.*

The next important result is a generalized Fermat rule.

Lemma 2.5 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. If h attains a local minimum or maximum at $\bar{x}$, then $\mathbf{0} \in \partial h(\bar{x})$.*

Lemma 2.6 *Let I be a finite index set and $h_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in I)$ be locally Lipschitz. Set $h(x) := \max_{i \in I} h_i(x)$. Then, $\partial h(x) \subseteq conv\{\partial h_i(x) \ : \ i \in I(x)\}$ for all $x \in \mathbb{R}^n$, where $I(x) = \{i \in I \ : \ h_i(x) = h(x)\}$.*

In the rest of this paper, we always denote by **Assumption A** the following:

(A1) g is locally Lipschitz with respect to the first argument and uniformly on V with respect to the second argument, i.e., for each $x \in \mathbb{R}^n$, there exist an open neighborhood U of x and a positive constant L such that

$$\|g(y, v) - g(z, v)\| \leq L\|y - z\|, \ \forall y, z \in U, \ v \in V.$$

(A2) for each $j \in \{1, 2, \ldots, m\}$, the function $v_j \longmapsto g_j(\cdot, v_j)$ is concave on V_j.

For $j \in \{1, 2, \ldots, m\}$, we define a family of real-valued functions $\varphi, \varphi_j : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\varphi_j(x) := \max_{v_j \in V_j} g_j(x, v_j) \tag{3}$$

and

$$\varphi(x) := \max_{j \in \{1,2,\ldots,m\}} \varphi_j(x). \tag{4}$$

Since g_j is upper semicontinuous and V_j is nonempty, convex and compact for each $j \in \{1, 2, \ldots, m\}$, φ_j is well defined. By the auxiliary functions (3), the set of robust feasible solutions F can be equivalently characterized as follows:

$$F = \{x \in \mathbb{R}^n : \varphi_j(x) \leq 0,\ j = 1, 2, \ldots, m\} = \{x \in \mathbb{R}^n : \varphi(x) \leq 0\}. \tag{5}$$

Besides, for each $j \in \{1, 2, \ldots, m\}$, φ_j and φ are locally Lipschitz functions under the assumption (A1) (see, e.g., [19, (H1), p.131], and also see [35, p. 290] or [36, P.2043]).

3 Robust Optimality Conditions of (UCMOP)

In this section, we study Fritz–John/Kuhn–Tucker-type robust necessary optimality conditions for weakly robust efficient solutions and properly robust efficient solutions of (UCMOP) by using the generalized alternative theorem of Illés and Kassay [29] and (GRSCQ), and discuss sufficient robust optimality conditions for weakly robust efficient solutions and properly robust efficient solutions of (UCMOP) under the generalized convexity assumptions.

Theorem 3.1 (Fritz–John-type robust necessary optimality conditions) *Let g satisfy the* **Assumption A** *and that φ is $\mathbb{R}_+$-convexlike, and let f be Q-convexlike. If $\bar{x} \in F^w$, then there exist $\bar{\lambda} \in Q^*$, $\bar{\mu} \in \mathbb{R}^m_+$ not both zero and $\bar{v} \in V$ such that*

$$\begin{aligned} &\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \partial_x g(\bar{x}, \bar{v})^\top \bar{\mu}, \\ &\bar{\mu}^\top g(\bar{x}, \bar{v}) = 0. \end{aligned} \tag{6}$$

Proof Assume that $\bar{x} \in F^w$. Then, we have

$$f(x) - f(\bar{x}) \notin -\mathrm{int}Q,\ \ \forall x \in F. \tag{7}$$

Note that $x \in F \iff \varphi(x) \leq 0$. Therefore, (7) is equivalent to the infeasibility of the following system (in the unknown x):

$$(f(x) - f(\bar{x}), \varphi(x)) \in -(\mathrm{int}Q \times \mathbb{R}_+). \tag{8}$$

Since (f, φ) is $(Q, \mathbb{R}_+)$-convexlike, it follows from Lemma 2.4 and (8) that there exist $(\bar{\lambda}, \bar{\tau}) \in (Q^* \times \mathbb{R}_+) \setminus \{\mathbf{0}\}$ such that

$$\bar{\lambda}^\top (f(x) - f(\bar{x})) + \bar{\tau}\varphi(x) \geq 0, \ \forall x \in \mathbb{R}^n.$$

Taking $x = \bar{x}$ in the above inequality, we obtain that $\bar{\tau}\varphi(\bar{x}) \geq 0$. According to $\varphi(\bar{x}) \leq 0$, we have $\bar{\tau}\varphi(\bar{x}) = 0$. Consequently, $\bar{x}$ is a solution of the unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^n} \bar{\lambda}^\top f(x) + \bar{\tau}\varphi(x).$$

Since g satisfies the assumption (A1), φ is locally Lipschitz. From Lemmas 2.2 and 2.5, we deduce that

$$\mathbf{0} \in \partial \left(\bar{\lambda}^\top f(\cdot) + \bar{\tau}\varphi(\cdot)\right)(\bar{x}) \subseteq \partial f(\bar{x})^\top \bar{\lambda} + \partial\varphi(\bar{x})\bar{\tau}. \tag{9}$$

From Lemma 2.6, we deduce that

$$\partial\varphi(\bar{x}) \subseteq \mathrm{conv}\left\{\partial\varphi_i(\bar{x}) \ : \ i \in I(\bar{x})\right\}, \tag{10}$$

where $I(\bar{x}) = \{i \ : \ \varphi_i(\bar{x}) = \varphi(\bar{x}), \ i = 1, 2, \ldots, m\}$. According to [35, Theorem 2.4, p. 290], we have

$$\partial\varphi_i(\bar{x}) = \left\{\xi_i \ : \ \exists\, v_i \in V_i(\bar{x}) \text{ such that } \xi_i \in \partial_x g_i(\bar{x}, v_i)\right\}, \tag{11}$$

for all $i \in \{1, 2, \ldots, m\}$, where $V_i(\bar{x}) := \{v_i \in V_i \ : \ g_i(\bar{x}, v_i) = \varphi_i(\bar{x})\}$ is nonempty, convex and compact set. So, (10) and (11) imply that there exists $\bar{v} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m) \in V$ such that

$$g_i(\bar{x}, \bar{v}_i) = \varphi_i(\bar{x}), \ i \in \{1, 2, \ldots, m\}$$

and

$$\partial\varphi(\bar{x}) \subseteq \mathrm{conv}\left\{\partial_x g_i(\bar{x}, \bar{v}_i) \ : \ i \in I(\bar{x})\right\}.$$

This combined with (9) yields that

$$\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \bar{\tau}\mathrm{conv}\left\{\partial_x g_i(\bar{x}, \bar{v}_i) \ : \ i \in I(\bar{x})\right\}.$$

Hence, there exists $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_m)^\top \in \mathbb{R}^m_+$ with $\sum_{i \in I(\bar{x})} \tilde{\mu}_i = 1$ and $\tilde{\mu}_j = 0$ for all $j \in \{1, 2, \ldots, m\} \setminus I(\bar{x})$ such that

$$\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \bar{\tau}\partial_x g(\bar{x}, \bar{v})^\top \tilde{\mu}.$$

Set $\bar{\mu} = \bar{\tau}\tilde{\mu}$. Due to $\bar{\tau} \geq 0$, one has $\bar{\mu} \in \mathbb{R}^m_+$. Therefore, there exist $(\bar{\lambda}, \bar{\mu}) \in (Q^* \times \mathbb{R}^m_+) \setminus \{\mathbf{0}\}$ and $\bar{v} \in V$ such that

$$\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \partial_x g(\bar{x}, \bar{v})^\top \bar{\mu}. \tag{12}$$

Note that $\varphi_i(\bar{x}) = \varphi(\bar{x})$ for all $i \in I(\bar{x})$. Taking into account that $\bar{\tau}\varphi(\bar{x}) = 0$ and $\sum_{i \in I(\bar{x})} \tilde{\mu}_i = 1$, we have

$$\bar{\mu}^\top g(\bar{x}, \bar{v}) = \sum_{i=1}^{m} \bar{\tau}\tilde{\mu}_i \varphi_i(\bar{x}) = \sum_{i \in I(\bar{x})} \bar{\tau}\tilde{\mu}_i \varphi_i(\bar{x}) = \sum_{i \in I(\bar{x})} \tilde{\mu}_i[\bar{\tau}\varphi(\bar{x})] = \bar{\tau}\varphi(\bar{x}) = 0,$$

i.e., $\bar{\mu}^\top g(\bar{x}, \bar{v}) = 0$. This combined with (12) yields that (6) holds. □

Remark 3.1 Observe that many existing results on the robust necessary optimality conditions for uncertain optimization problems containing the scalar case and vector case are established under the convexity of every components of f and g with respect to the first argument by the subdifferential of the convex function; see, e.g., [6,7,35,36] and the references therein. It is not hard to verify that φ is $\mathbb{R}_+$-convexlike when every components of g with respect to the first argument are convex. So, the assumption: "φ is $\mathbb{R}_+$-convexlike"is reasonable.

On the other hand, since the robust necessary optimality conditions in Theorem 3.1 are established by the subdifferential in the sense of Clarke instead of the subdifferential in the sense of convex function, Theorem 3.1 is distinct with the corresponding results of [6,7,35,36].

Theorem 3.2 (Kuhn–Tucker-type robust necessary optimality conditions) *Assume that all conditions of Theorem* 3.1 *and (GRSCQ) hold. If* $\bar{x} \in F^w$, *then there exist* $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$, $\bar{\mu} \in \mathbb{R}^m_+$ *and* $\bar{v} \in V$ *such that* (6) *holds.*

Proof Assume that $\bar{x} \in F^w$. It thus follows from Theorem 3.1 that there exist $\bar{\lambda} \in Q^*$, $\bar{\mu} \in \mathbb{R}^m_+$ not both zero and $\bar{v} \in V$ such that (6) holds.

Let us prove that $\bar{\lambda} \neq \mathbf{0}$. Suppose that $\bar{\lambda} = \mathbf{0}$. Then, $\bar{\mu} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$. By the proof of Theorem 3.1, we have

$$\varphi(x) \geq 0, \ \forall x \in \mathbb{R}^n. \tag{13}$$

Since (GRSCQ) is satisfied, there exists $x_0 \in \mathbb{R}^n$ such that

$$-g(x_0, v) \in \text{int}\mathbb{R}^m_+, \ \forall v \in V,$$

i.e., for each $j = 1, 2, \ldots, m$, $g_j(x_0, v_j) < 0$ for all $v_j \in V_j$. Since g_j are upper semicontinuous with respect to the second argument and V_j are nonempty, convex and compact for all $j = 1, 2, \ldots, m$, then there exist $j \in \{1, 2, \ldots, m\}$ and $\tilde{v}_j \in V_j$ such that

$$\varphi(x_0) = \varphi_j(x_0) = g_j(x_0, \tilde{v}_j) < 0,$$

which contradicts (13). □

If $g(\cdot, v) := h(\cdot)$ for all $v \in V$, from Theorems 3.1 and 3.2, we can derive the following necessary optimality conditions for the weakly efficient solutions of the multiobjective optimization problem:

$$\text{(MOP)} \quad \min_{x \in \mathcal{F}} \ f(x),$$

where $\mathcal{F} := \{x \in \mathbb{R}^n \ : \ -h(x) \in S\}$.

Corollary 3.1 *Let (f, h) be $(Q, \mathbb{R}^m_+)$-convexlike. If $\bar{x}$ is a weakly efficient solution of (MOP), then there exist $\bar{\lambda} \in Q^*$, $\bar{\mu} \in \mathbb{R}^m_+$ not both zero such that $\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \partial h(\bar{x})^\top \bar{\mu}$ and $\bar{\mu}^\top h(\bar{x}) = 0$.*

Corollary 3.2 *Assume that (f, h) is $(Q, \mathbb{R}^m_+)$-convexlike and that the extended Slater constraint qualification is satisfied. If $\bar{x}$ is a weakly efficient solution of (MOP), then there exist $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$, $\bar{\mu} \in \mathbb{R}^m_+$ such that $\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \partial h(\bar{x})^\top \bar{\mu}$ and $\bar{\mu}^\top h(\bar{x}) = 0$.*

Remark 3.2 It is worth noting that the conditions of Corollaries 3.1 and 3.2 are distinct with that of [37, Theorems 3.1 and 3.2, pp.33-34], respectively, since Corollaries 3.1 and 3.2 do not involve the cone invexity of (f, h).

Similarly, we can get the necessary optimality conditions for properly robust efficient solutions of (UCMOP).

Theorem 3.3 (Fritz–John-type robust necessary optimality conditions) *Assume that all conditions of Theorem 3.1 are satisfied. If $\bar{x} \in F^{pr}$, then there exist $\bar{\lambda} \in Q^*$, $\bar{\mu} \in \mathbb{R}^m_+$ not both zero and $\bar{v} \in V$ such that*

$$\begin{aligned} &\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \partial_x g(\bar{x}, \bar{v})^\top \bar{\mu}, \\ &\bar{\mu}^\top g(\bar{x}, \bar{v}) = 0, \\ &\bar{\lambda}^\top f(\bar{x}) = \min_{x \in F} \bar{\lambda}^\top f(x). \end{aligned} \tag{14}$$

Proof Assume that $\bar{x} \in F^{pr}$. Then, there exists $\tilde{\lambda} \in \mathrm{int} Q^*$ such that

$$\tilde{\lambda}^\top (f(x) - f(\bar{x})) \geq 0, \ \ \forall x \in F. \tag{15}$$

Since f is Q-convexlike, we deduce that $\tilde{\lambda}^\top f(\cdot)$ is $\mathbb{R}_+$-convexlike. Similar to the proof of Theorem 3.1, there exist $(\bar{\alpha}, \bar{\mu}) \in (\mathbb{R}_+ \times \mathbb{R}^m_+) \setminus \{\mathbf{0}\}$ and $\bar{v} \in V$ such that

$$\mathbf{0} \in \partial f(\bar{x})^\top (\bar{\alpha}\tilde{\lambda}) + \partial_x g(\bar{x}, \bar{v})^\top \bar{\mu} \tag{16}$$

and $\bar{\mu}^\top g(\bar{x}, \bar{v}) = 0$. Due to $\tilde{\lambda} \in \mathrm{int} Q^*$, we have

$$\begin{aligned} &\bar{\alpha}\tilde{\lambda} \in \mathrm{int} Q^*, && \text{if } \bar{\alpha} > 0, \\ &\bar{\alpha}\tilde{\lambda} = \mathbf{0}, && \text{if } \bar{\alpha} = 0. \end{aligned}$$

Set $\bar{\lambda} = \bar{\alpha}\tilde{\lambda}$. So, $(\bar{\lambda}, \bar{\mu}) \in (Q^* \times \mathbb{R}^m_+) \setminus \{\mathbf{0}\}$ and

$$\bar{\lambda}^\top f(x) - \bar{\lambda}^\top f(\bar{x}) = \bar{\lambda}^\top (f(x) - f(\bar{x})) = \bar{\alpha}\tilde{\lambda}^\top (f(x) - f(\bar{x})) \overset{(15)}{\geq} 0, \quad \forall x \in F,$$

i.e., $\bar{\lambda}^\top f(\bar{x}) = \min_{x \in F} \bar{\lambda}^\top f(x)$. Therefore, using (16) implies

$$\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \partial_x g(\bar{x}, \bar{v})^\top \bar{\mu},$$

as required. □

Theorem 3.4 (Kuhn–Tucker-type robust necessary optimality conditions) *Assume that all conditions of Theorem 3.1 and (GRSCQ) hold. If $\bar{x} \in F^{pr}$, then there exist $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$, $\bar{\mu} \in \mathbb{R}^m_+$ and $\bar{v} \in V$ such that (14) holds.*

Proof It directly follows from Theorem 3.3 and the proof of Theorem 3.2. □

We next give two examples to illustrate Theorems 3.1, 3.2, 3.3 and 3.4.

Example 3.1 Let $\mathbb{R}^n = \mathbb{R}^k = \mathbb{R}^m = \mathbb{R}^p := \mathbb{R}^2$, and let $V_1 := [-0.5, 0]$, $V_2 := [-1, 1]$, $Q := \mathbb{R}^2_+$. Let the functions $f(x) := (2x_1, 3x_2^2)^\top$ and

$$g(x, v) := (x_1^2 + 4x_1 - 5 + \ln(1 + v_1), x_2 - |v_2|)^\top$$

for all $x, y \in \mathbb{R}^2$, where $v = (v_1, v_2) \in V = V_1 \times V_2$. It is easy to verify that $F = [-5, 1] \times (-\mathbb{R}_+)$, $Q^* = \mathbb{R}^2_+$, that g satisfies the assumptions (A1) and (A2) and f is Q-convexlike and that $\bar{x} = (-5, 0)^\top$ is a weakly robust efficient solution of (UCMOP). Besides, φ is convex on $\mathbb{R}^n$ and so it is $\mathbb{R}_+$-convexlike. Observe that there exist $\bar{\lambda} = (1, 1)^\top \in \text{int}Q^*$ and $\hat{x} = (-1, -1) \in \mathbb{R}^2$ such that

$$\begin{aligned} -g(\hat{x}, v) &= (-1 + 4 + 5 - \ln(1 + v_1), 1 + |v_2|)^\top \\ &= (8 - \ln(1 + v_1), 1 + |v_2|)^\top \in \text{int}\mathbb{R}^2_+, \ \forall v \in V \end{aligned}$$

and

$$\begin{aligned} &\bar{\lambda}^\top (f(x) - f(\bar{x})) = (1, 1)(2x_1 + 10, 3x_2^2)^\top = 2x_1 + 3x_2^2 + 10 \geq 0, \\ &\quad \forall x = (x_1, x_2)^\top \in F, \end{aligned}$$

i.e., (GRSCQ) is satisfied and $\bar{x}$ is a properly robust efficient solution of (UCMOP).

On the other hand, there exist $\bar{\mu} = (\frac{1}{3}, 0)^\top \in \mathbb{R}^2_+$ and $\bar{v} = \mathbf{0} \in V$ such that $\bar{\mu}^\top g(\bar{x}, \bar{v}) = 0$ and

$$\partial f(\bar{x})^\top \bar{\lambda} + \partial_x g(\bar{x}, \bar{v})^\top \bar{\mu} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} = \mathbf{0}.$$

As a matter of fact, we have $g(\bar{x}, \bar{v}) = \mathbf{0} \in \mathbb{R}^2_+$.

The next example shows that the condition “the closedness of V ”is indispensable to Theorems 3.1-3.4.

Example 3.2 Let $\mathbb{R}^n, \mathbb{R}^k, \mathbb{R}^m, \mathbb{R}^p, Q, g$ be the same as Example 3.1, let $V_2 := [-1,1] \setminus \{0\}$, $V_1 := [-0.5, 0[$, and let the function $f(x) := (2x_1 + x_2, 3x_2^2)^\top$. It is easy to check that g satisfies the assumptions (A1) and (A2), that (f, φ) is $(Q, \mathbb{R}_+)$-convexlike and that $\bar{x} = (-5, 0)^\top$ is a weakly robust efficient solution and a properly robust efficient solution of (UCMOP) and (GRSCQ) is satisfied. Taking into account that

$$\partial f(\bar{x})^\top \lambda = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 + \lambda_2 \\ 0 \end{pmatrix} \neq \mathbf{0}, \ \forall \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in Q^* \setminus \{\mathbf{0}\}.$$

Clearly, one has

$$g(\bar{x}, v) = (\ln(1 + v_1), -|v_2|)^\top \notin \mathbb{R}^2_+, \ \forall v \in V = V_1 \times V_2.$$

On the other hand, for each $v \in V$ and $\mu \in \mathbb{R}^2_+$, $\mu^\top g(\bar{x}, v) = 0$ if and only if $\mu = \mathbf{0}$. Altogether, there do not exist $(\lambda, \mu) \in \left(Q^* \times \mathbb{R}^2_+\right) \setminus \{\mathbf{0}\}$ and $\bar{v} \in V$ such that $\partial f(\bar{x})^\top \lambda + \partial_x g(\bar{x}, \bar{v})^\top \mu = \mathbf{0}$ and $\mu^\top g(\bar{x}, \bar{v}) = 0$.

It is easy to see that if $\bar{\lambda} \in \text{int} Q^*$ in (14), (14) implies $\bar{x} \in F^{pr}$. Now we further develop sufficient optimality conditions for weakly robust efficient solutions and properly robust efficient solutions of (UCMOP).

Theorem 3.5 *Let (f, g) be type II $(Q, \mathbb{R}^m_+)$-generalized convex at $\bar{x} \in F$. Assume that there exist $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$, $\bar{\mu} \in \mathbb{R}^m_+$ and $\bar{v} \in V$ such that (6) holds. Then, $\bar{x} \in F^w$.*

Proof Suppose that $\bar{x} \notin F^w$. Then, there exists $\hat{x} \in F$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\text{int} Q. \tag{17}$$

Since (f, g) is type II $(Q, \mathbb{R}^m_+)$-generalized convex at $\bar{x} \in \mathbb{R}^n$, then there exists $d \in \mathbb{R}^n$ such that

$$f(\hat{x}) - f(\bar{x}) - Ad \in Q, \ \forall A \in \partial f(\bar{x}) \tag{18}$$

and

$$-g(\bar{x}, \bar{v}) - Bd \in \mathbb{R}^m_+, \ \forall B \in \partial_x g(\bar{x}, \bar{v}). \tag{19}$$

Combining (17) with (18), one has

$$-Ad \in Q - (f(\hat{x}) - f(\bar{x})) \subseteq Q + \text{int} Q = \text{int} Q, \ \forall A \in \partial f(\bar{x}). \tag{20}$$

In view of $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$ and $\bar{\mu} \in \mathbb{R}^m_+$, it follows from Lemma 2.3 and (19), (20) that

$$\bar{\lambda}^\top Ad < 0, \ \forall A \in \partial f(\bar{x}) \tag{21}$$

and

$$-\bar{\mu}^\top(g(\bar{x},\bar{v})+Bd)=\bar{\mu}^\top(-g(\bar{x},\bar{v})-Bd)\geq 0,\ \forall B\in\partial_x g(\bar{x},\bar{v}). \tag{22}$$

Using (6) yields that $\bar{\mu}^\top g(\bar{x},\bar{v})=0$. Hence, we deduce from (22) that

$$\bar{\mu}^\top Bd\leq 0,\ \forall B\in\partial_x g(\bar{x},\bar{v}). \tag{23}$$

Adding both sides of (21) and (23), we obtain

$$\left(\bar{\lambda}^\top A+\bar{\mu}^\top B\right)d<0,\ \forall A\in\partial f(\bar{x}),\ B\in\partial_x g(\bar{x},\bar{v}). \tag{24}$$

Note that $\mathbf{0}\in\partial f(\bar{x})^\top\bar{\lambda}+\partial_x g(\bar{x},\bar{v})^\top\bar{\mu}$. Therefore, there exist $\bar{A}\in\partial f(\bar{x})$ and $\bar{B}\in\partial_x g(\bar{x},\bar{v})$ such that $\bar{A}^\top\bar{\lambda}+\bar{B}^\top\bar{\mu}=\mathbf{0}$. Consequently, $\left(\bar{\lambda}^\top\bar{A}+\bar{\mu}^\top\bar{B}\right)d=0$, which contradicts (24). □

The following sufficient optimality conditions are immediately derived from Remark 2.1(ii) and Theorem 3.5.

Corollary 3.3 *Let (f,g) be type I $(Q,\mathbb{R}^m_+)$-generalized convex at $\bar{x}\in F$. Assume that there exist $\bar{\lambda}\in Q^*\setminus\{\mathbf{0}\}$, $\bar{\mu}\in\mathbb{R}^m_+$ and $\bar{v}\in V$ such that (6) holds. Then, $\bar{x}\in F^w$.*

We now present sufficient optimality conditions for the properly robust efficient solutions of (UCMOP).

Theorem 3.6 *Let (f,g) be type II $(Q,\mathbb{R}^m_+)$-generalized convex at $\bar{x}\in F$. Assume that there exist $\bar{\lambda}\in intQ^*$, $\bar{\mu}\in\mathbb{R}^m_+$ and $\bar{v}\in V$ such that (6) holds. Then, $\bar{x}\in F^{pr}$.*

Proof Suppose that $\bar{x}\notin F^{pr}$. Then, there exists $\hat{x}\in F$ such that

$$\bar{\lambda}^\top\left(f(\hat{x})-f(\bar{x})\right)<0. \tag{25}$$

Since (f,g) is type II $(Q,\mathbb{R}^m_+)$-generalized convex at $\bar{x}\in\mathbb{R}^n$, then there exists $d\in\mathbb{R}^n$ such that

$$f(\hat{x})-f(\bar{x})-Ad\in Q,\ \forall A\in\partial f(\bar{x})$$

and

$$-g(\bar{x},\bar{v})-Bd\in\mathbb{R}^m_+,\ \forall B\in\partial_x g(\bar{x},\bar{v}),$$

which together with the definition of dual cone, (25) and $\bar{\mu}^\top g(\bar{x},\bar{v})=0$ imply that $\bar{\lambda}^\top Ad<0$ and $\bar{\mu}^\top Bd\leq 0$ for all $A\in\partial f(\bar{x})$ and $B\in\partial_x g(\bar{x},\bar{v})$. Consequently, one has

$$\left(\bar{\lambda}^\top A+\bar{\mu}^\top B\right)d<0,\ \forall A\in\partial f(\bar{x}),\ B\in\partial_x g(\bar{x},\bar{v}),$$

which contradicts the fact that $\mathbf{0}\in\partial f(\bar{x})^\top\bar{\lambda}+\partial_x g(\bar{x},\bar{v})^\top\bar{\mu}$. □

From Remark 2.1(ii) and Theorem 3.6, we can get the following sufficient optimality conditions for the properly robust efficient solutions of (UCMOP).

Corollary 3.4 *Let (f, g) be type I $(Q, \mathbb{R}^m_+)$-generalized convex at $\bar{x} \in F$. Assume that there exist $\bar{\lambda} \in intQ^*$, $\bar{\mu} \in \mathbb{R}^m_+$ and $\bar{v} \in V$ such that (6) holds. Then, $\bar{x} \in F^{pr}$.*

Let us present the following example to show Theorems 3.3 and 3.6.

Example 3.3 Let $\mathbb{R}^n, \mathbb{R}^k, \mathbb{R}^m, \mathbb{R}^p, Q, V_1, V_2, f$ and g be the same as Example 2.2. After calculation, we have $Q^* = \{(y, z) \in \mathbb{R}^2 : z - y \geq 0 \geq z\}$. Let $\bar{x} = 0$. It then follows from Example 2.2 that $F = \left[-\sqrt{3}, \sqrt{3}\right]$, $\partial f(\bar{x})^\top = [0, 1] \times [0, 1]$ and $\partial_x g(\bar{x}, v)^\top = [0, 1] \times [-1/2, 0]$ for all $v \in V$ and that (f, g) is type II $(Q, \mathbb{R}^2_+)$-generalized convex at $\bar{x}$. Therefore, there exist $\bar{\lambda} = (-2, -1)^\top \in \text{int}Q^*$, $\bar{\mu} = (0, 1)^\top \in \mathbb{R}^2_+$ and $\bar{v} = \mathbf{0} \in V$ such that $\bar{\mu}^\top g(\bar{x}, \bar{v}) = 0$ and $\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \partial_x g(\bar{x}, \bar{v})^\top \bar{\mu}$. Then, for each $x \in F$, we have

$$f(x) - f(\bar{x}) = \begin{cases} (x, x)^\top \notin -\text{int}Q, & \text{if } x \in [-\sqrt{3}, 0[, \\ (-x^2, 0)^\top \notin -\text{int}Q, & \text{if } x \in \left[0, \sqrt{3}\right], \end{cases}$$

and

$$\bar{\lambda}^\top (f(x) - f(\bar{x})) = \begin{cases} -3x \geq 0, & \text{if } x \in [-\sqrt{3}, 0[, \\ 2x^2 \geq 0, & \text{if } x \in \left[0, \sqrt{3}\right]. \end{cases}$$

Hence, $\bar{x} = 0$ is a weakly robust efficient solution and properly robust efficient solution of (UCMOP).

4 Mond–Weir-Type Robust Dualities of (UCMOP)

In this section, we investigate the weak, strong and converse dualities between (RMOP) and the Mond–Weir-type robust dual problem of (UCMOP) in terms of the weakly robust efficient solutions and properly efficient solutions.

The Mond–Weir-type robust dual problem of (UCMOP) is formulated as follows:

$$\begin{aligned} \max f(y) \text{ s.t.} \mathbf{0} \in \partial f(y)^\top \lambda + \partial_x g(y, v)^\top \mu, \\ \mu^\top g(y, v) = 0, \ \ v \in V, \\ \lambda \in Q^* \setminus \{\mathbf{0}\}, \ \ \mu \in \mathbb{R}^m_+. \end{aligned} \tag{MWRD}$$

It is worth noting that (MWRD) can be understood as the optimistic counterpart of the Mond–Weir-type dual model of (UCMOP):

$$\begin{aligned} \max \ f(y) \ \text{ s.t. } \mathbf{0} \in \partial f(y)^\top \lambda + \partial_x g(y, v)^\top \mu, \\ \mu^\top g(y, v) = 0, \\ \lambda \in Q^* \setminus \{\mathbf{0}\}, \ \ \mu \in \mathbb{R}^m_+. \end{aligned} \tag{26}$$

where $v \in V$ is an uncertain parameter.

A vector (y, v, λ, μ) is said to be a feasible solution of (MWRD) if it satisfies (MWRD). We denote by F^D the set of the feasible solutions of (MWRD).

Definition 4.1 A vector $(\bar{y}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D$ is called a

(i) *weak robust solution* of (MWRD) iff

$$f(y) - f(\bar{y}) \notin \text{int}Q, \ \forall (y, v, \lambda, \mu) \in F^D;$$

(ii) *proper robust solution* of (MWRD) iff $\bar{\lambda} \in \text{int}Q^*$ and

$$\bar{\lambda}^\top (f(y) - f(\bar{y})) \leq 0, \ \forall (y, v, \lambda, \mu) \in F^D.$$

Theorem 4.1 (Weak duality) *Let $x \in F$ and $(y, v, \lambda, \mu) \in F^D$. If (f, g) is type II $(Q, \mathbb{R}^m_+)$-generalized convex at y, then $f(x) - f(y) \notin -\text{int}Q$.*

Proof Suppose that $f(x) - f(y) \in -\text{int}Q$. Since $(y, v, \lambda, \mu) \in F^D$, we obtain $\lambda \in Q^* \setminus \{\mathbf{0}\}$, $\mu \in \mathbb{R}^m_+$, $v \in V$, $\mu^\top g(y, v) = 0$ and

$$\mathbf{0} \in \partial f(y)^\top \lambda + \partial_x g(y, v)^\top \mu. \tag{27}$$

Since (f, g) is type II $(Q, \mathbb{R}^m_+)$-generalized convex at y, then there exists $d \in \mathbb{R}^n$ such that

$$f(x) - f(y) - Ad \in Q, \ \forall A \in \partial f(y)$$

and

$$-g(y, v) - Bd \in \mathbb{R}^m_+, \ \forall B \in \partial_x g(y, v).$$

Moreover, one has

$$\begin{aligned} -Ad = f(x) - f(y) - Ad - (f(x) - f(y)) \in Q + \text{int}Q = \text{int}Q, \\ \forall A \in \partial f(y) \end{aligned} \tag{28}$$

and

$$\mu^\top (-Bd) = \mu^\top (-g(y, v) - Bd) + \mu^\top g(y, v) \geq 0, \ \forall B \in \partial_x g(y, v). \tag{29}$$

Combined with (28) and (29), we deduce that $\lambda^\top Ad < 0$ for all $A \in \partial f(y)$ and $\mu^\top Bd \leq 0$ for all $B \in \partial_x g(y, v)$. Therefore, we obtain

$$\left(\lambda^\top A + \mu^\top B\right) d = \lambda^\top Ad + \mu^\top Bd < 0, \quad \forall A \in \partial f(y), \ B \in \partial_x g(y, v),$$

which contradicts (27). ◻

Theorem 4.2 (Strong duality) *Let $\bar{x} \in F^w$ and (GRSCQ) hold. Assume that all conditions of Theorem 3.1 are satisfied. Then, there exist $\bar{v} \in V$, $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$ and $\bar{\mu} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D$. Furthermore, if (f, g) is type II $(Q, \mathbb{R}^m_+)$-generalized convex at y, where $(y, v, \lambda, \mu) \in F^D$, then $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weak robust solution of (MWRD).*

Proof Since $\bar{x} \in F^w$ and (GRSCQ) is satisfied, then from Theorem 3.2, there exist $\bar{v} \in V$, $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$ and $\bar{\mu} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ satisfies (6), i.e., $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D$. By Theorem 4.1, we have

$$f(\bar{x}) - f(y) \notin -\text{int}Q, \ \forall (y, v, \lambda, \mu) \in F^D,$$

which implies that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weak robust solution of (MWRD). □

We next present a strong duality concerning the weakly robust efficient solution of (UCMOP) and properly robust solution of (MWRD).

Theorem 4.3 (Strong duality) *Let $\bar{x} \in F^w$ and (GRSCQ) hold. Assume that all conditions of Theorem 3.1 are satisfied. Then, there exist $\bar{v} \in V$, $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$ and $\bar{\mu} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D$. Furthermore, if $\bar{\lambda} \in \text{int}Q^*$ and $(-f, -g)$ is type II $(Q, \mathbb{R}^m_+)$-generalized convex at $\bar{x}$, then $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a proper robust solution of (MWRD).*

Proof By the same proof as Theorem 4.2, we deduce that there exist $\bar{v} \in V$, $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$ and $\bar{\mu} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D$. Let $\bar{\lambda} \in \text{int}Q^*$ and $(-f, -g)$ be type II $(Q, \mathbb{R}^m_+)$-generalized convex at $\bar{x}$.

Suppose that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is not a proper robust solution of (MWRD). Then, there exists $(\hat{y}, \hat{v}, \hat{\lambda}, \hat{\mu}) \in F^D$ such that

$$\bar{\lambda}^\top (f(\hat{y}) - f(\bar{x})) > 0. \tag{30}$$

Since $(-f, -g)$ is type II $(Q, \mathbb{R}^m_+)$-generalized convex at $\bar{x}$, then there exists $d \in \mathbb{R}^n$ such that

$$-f(\hat{y}) + f(\bar{x}) + Ad \in Q, \ \forall A \in \partial f(\bar{x})$$

and

$$g(\bar{x}, \bar{v}) + Bd \in \mathbb{R}^m_+, \ \forall B \in \partial_x g(\bar{x}, \bar{v}).$$

In view of $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D$, we have $\bar{\mu} \in \mathbb{R}^m_+$, $\bar{\mu}^\top g(\bar{x}, \bar{v}) = 0$ and

$$\mathbf{0} \in \partial f(\bar{x})^\top \bar{\lambda} + \partial_x g(\bar{x}, \bar{v})^\top \bar{\mu}. \tag{31}$$

Note that

$$-\bar{\lambda}^\top (f(\hat{y}) - f(\bar{x})) + \bar{\lambda}^\top Ad = \bar{\lambda}^\top (-f(\hat{y}) + f(\bar{x}) + Ad) \geq 0, \ \forall A \in \partial f(\bar{x}) \tag{32}$$

and

$$\bar{\mu}^\top Bd = \bar{\mu}^\top (g(\bar{x}, \bar{v}) + Bd) \geq 0, \ \forall B \in \partial_x g(\bar{x}, \bar{v}). \tag{33}$$

It therefore follows from (30) and (32) that

$$\bar{\lambda}^\top Ad \geq \bar{\lambda}^\top (f(\hat{y}) - f(\bar{x})) > 0, \ \forall A \in \partial f(\bar{x}).$$

This together with (33) yields that

$$\left(\bar{\lambda}^\top A + \bar{\mu}^\top B\right) d = \bar{\lambda}^\top Ad + \bar{\mu}^\top Bd > 0, \ \forall A \in \partial f(\bar{x}), \ B \in \partial_x g(\bar{x}, \bar{v}),$$

which contradicts (31). □

Theorem 4.4 (Converse duality) *Let $(\bar{y}, \bar{v}, \bar{\lambda}, \bar{\mu})$ be a weak robust solution of (MWRD) with $\bar{y} \in F$. If (f, g) is type II $(Q, \mathbb{R}^m_+)$-generalized convex at $\bar{y}$, then $\bar{y} \in F^w$.*

Proof Since $(\bar{y}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weak robust solution of (MWRD) and $\bar{y} \in F$, it then follows from Theorem 4.1 that $f(x) - f(\bar{y}) \notin -\text{int}Q$ for all $x \in F$, i.e., $\bar{y} \in F^w$. □

Theorem 4.5 (Converse duality) *Let $(\bar{y}, \bar{v}, \bar{\lambda}, \bar{\mu})$ be a proper robust solution of (MWRD) with $\bar{y} \in F$. If (f, g) is type II $(Q, \mathbb{R}^m_+)$-generalized convex at $\bar{y}$, then $\bar{y} \in F^{pr}$.*

Proof Since $(\bar{y}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a proper robust solution of (MWRD), then $\bar{\lambda} \in \text{int}Q^*$, $\bar{\mu}^\top g(\bar{y}, \bar{v}) = 0$ and

$$\mathbf{0} \in \partial f(\bar{y})^\top \bar{\lambda} + \partial_x g(\bar{y}, \bar{v})^\top \bar{\mu}.$$

Consequently, $\bar{y} \in F^{pr}$ follows from Theorem 3.6. □

It is worth noting that the type II $(Q, \mathbb{R}^m_+)$-generalized convexity of $(-f, -g)$ cannot be dropped in Theorem 4.3. This fact is illustrated by the following example when (f, g) is type I $(Q, \mathbb{R}^m_+)$-generalized convex.

Example 4.1 Let $\mathbb{R}^n$, $\mathbb{R}^k$, $\mathbb{R}^m$, $\mathbb{R}^p$, Q, V_1, V_2, f and g be the same as Example 3.1. It follows from Example 3.1 that $\bar{x} = (-5, 0)^\top$ is a weakly robust efficient solution and properly robust efficient solution of (UCMOP) and (GRSCQ) is satisfied. Again, from Example 3.1, we have $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D$, where $\bar{\lambda} = (1, 1)^\top \in \text{int}Q^*$, $\bar{\mu} = (\frac{1}{3}, 0)^\top \in \mathbb{R}^2_+$ and $\bar{v} = \mathbf{0} \in V$. For any $(y, v, \lambda, \mu) \in F^D$, we conclude that $\lambda = (\lambda_1, \lambda_2)^\top \in Q^* \setminus \{\mathbf{0}\}$, $\mu = (\mu_1, \mu_2)^\top \in \mathbb{R}^2_+$, $\mu^\top g(y, v) = 0$ and

$$\mathbf{0} = \partial f(y)^\top \lambda + \partial_x g(y, v)^\top \mu = \begin{pmatrix} 2 & 0 \\ 0 & 6y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 2y_1 + 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

So, $\lambda_1 + (y_1 + 2)\mu_1 = 0$ and $6y_2\lambda_2 + \mu_2 = 0$. Combined with $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$, we have $y_1 \leq -2$ and $y_2 \leq 0$. Then, $\mu_2\,(y_2 - |v_2|) \leq 0$. Again, from $\mu^\top g(y, v) = 0$, one has

$$\mu^\top g(y, v) = \mu_1\left[(y_1 + 2)^2 - 9 - \ln(1 + v_1)\right] + \mu_2\,(y_2 - |v_2|) = 0.$$

Therefore, we get

$$(y_1 + 2)^2 - 9 - \ln(1 + v_1) \geq 0.$$

It implies that $y_1 \leq -2 - \sqrt{9 + \ln(1 + v_1)} \leq -5$. So, $y_1 + 5 \leq 0$.

For each $(y, v, \lambda, \mu), (\tilde{y}, \tilde{v}, \tilde{\lambda}, \tilde{\mu}) \in F^D$, there exists $d = y - \tilde{y}$ such that

$$\begin{aligned} f(y) - f(\tilde{y}) - \nabla f(\tilde{y})d &= \begin{pmatrix} 2y_1 \\ 3y_2^2 \end{pmatrix} - \begin{pmatrix} 2\tilde{y}_1 \\ 3\tilde{y}_2^2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 6\tilde{y}_2 \end{pmatrix}\begin{pmatrix} y_1 - \tilde{y}_1 \\ y_2 - \tilde{y}_2 \end{pmatrix} \\ &= \left(0, 3(y_2 - \tilde{y}_2)^2\right)^\top \in Q \end{aligned}$$

and

$$\begin{aligned} & g(y, v) - g(\tilde{y}, v) - \nabla_x g(\tilde{y}, v)d \\ &= \begin{pmatrix} y_1^2 + 4y_1 - (\tilde{y}_1^2 + 4\tilde{y}_1) \\ y_2 - \tilde{y}_2 \end{pmatrix} - \begin{pmatrix} 2\tilde{y}_1 + 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 - \tilde{y}_1 \\ y_2 - \tilde{y}_2 \end{pmatrix} \\ &= \left((y_1 - \tilde{y}_1)^2, 0\right)^\top \in \mathbb{R}^2_+; \end{aligned}$$

namely, (f, g) is type I $(Q, \mathbb{R}^2_+)$-generalized convex at $\tilde{y}$, where $(\tilde{y}, \tilde{v}, \tilde{\lambda}, \tilde{\mu}) \in F^D$. Taking into account that $y_1 + 5 \leq 0$ for all $(y, v, \lambda, \mu) \in F^D$, we obtain

$$f(y) - f(\bar{x}) = \left(2(y_1 + 5), 3y_2^2\right)^\top \notin \text{int}Q,\ \forall\,(y, v, \lambda, \mu) \in F^D,$$

i.e., $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weak robust solution of (MWRD).

Finally, we show that the type II $(Q, \mathbb{R}^2_+)$-generalized convexity of $(-f, -g)$ at $\bar{x}$ cannot be dropped in Theorem 4.3. Note that there exist $\hat{y} = (-6, -7)^\top$, $\hat{v} = \mathbf{0}$, $\hat{\lambda} = (4, 1/42)^\top$ and $\hat{\mu} = (1, 1)^\top$ such that $(\hat{y}, \hat{v}, \hat{\lambda}, \hat{\mu}) \in F^D$ and

$$-f(\hat{y}) + f(\bar{x}) + \nabla f(\bar{x})d = (2 + 2d_1, -147)^\top \notin Q$$

where $d = (d_1, d_2)^\top \in \mathbb{R}^2$. So, $(-f, -g)$ is not type II $(Q, \mathbb{R}^2_+)$-generalized convex at $\bar{x}$. Due to

$$\bar{\lambda}^\top \left(f(\hat{y}) - f(\bar{x})\right) = 2\hat{y}_1 + 10 + 3\hat{y}_2^2 = 145 \not\leq 0.$$

Therefore, $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is not a proper robust solution of (MWRD).

5 Wolfe-Type Robust Dualities of (UCMOP)

In this section, we investigate the weak, strong and converse dualities between (RMOP) and the Wolfe-type robust dual problem of (UCMOP) under some suitable conditions.

The Wolfe-type robust dual problem of (RMOP) is formulated as follows:

$$\begin{aligned}\text{(WRD)}\quad \max\ & f(y)+\mu^\top g(y,v)e\\ \text{s.t. } & \mathbf{0}\in\partial f(y)^\top\lambda+\partial_x g(y,v)^\top\mu,\\ & v\in V,\lambda^\top e=1,\lambda\in Q^*\setminus\{\mathbf{0}\},\ \ \mu\in\mathbb{R}^m_+,\end{aligned}$$

where $e\in\mathbb{R}^k$ and $\|e\|=1$.

We denote by F_w^D the set of the feasible solutions of (WRD).

Definition 5.1 A vector $(\bar{y},\bar{v},\bar{\lambda},\bar{\mu})\in F_w^D$ is called a

(i) *weak robust solution* of (WRD) iff

$$f(y)+\mu^\top g(y,v)e-\left(f(\bar{y})+\bar{\mu}^\top g(\bar{y},\bar{v})e\right)\notin \operatorname{int}Q,\ \ \forall(y,v,\lambda,\mu)\in F_w^D;$$

(ii) *proper robust solution* of (WRD) iff $\bar{\lambda}\in\operatorname{int}Q^*$ and

$$\bar{\lambda}^\top\left[f(y)+\mu^\top g(y,v)e-\left(f(\bar{y})+\bar{\mu}^\top g(\bar{y},\bar{v})e\right)\right]\leq 0,\ \ \forall(y,v,\lambda,\mu)\in F_w^D.$$

Theorem 5.1 (Weak duality) *Let $x\in F$ and $(y,v,\lambda,\mu)\in F_w^D$. If (f,g) is type II $(Q,\mathbb{R}^m_+)$-generalized convex at y, then*

$$f(x)-\left(f(y)+\mu^\top g(y,v)e\right)\notin -intQ.$$

Proof Suppose that

$$f(x)-\left(f(y)+\mu^\top g(y,v)e\right)\in -\operatorname{int}Q.\tag{34}$$

Since $(y,v,\lambda,\mu)\in F_w^D$ and $x\in F$, we obtain $\lambda\in Q^*\setminus\{\mathbf{0}\}$, $\mu\in\mathbb{R}^m_+$, $v\in V$, $\lambda^\top e=1$ and

$$\mathbf{0}\in\partial f(y)^\top\lambda+\partial_x g(y,v)^\top\mu.\tag{35}$$

From (34) and Lemma 2.3, one has

$$\lambda^\top(f(x)-f(y))-\mu^\top g(y,v)=\lambda^\top f(x)-\lambda^\top\left(f(y)+\mu^\top g(y,v)e\right)<0.\tag{36}$$

Since (f,g) is type II $(Q,\mathbb{R}^m_+)$-generalized convex at y, then there exists $d\in\mathbb{R}^n$ such that

$$f(x)-f(y)-Ad\in Q,\ \ \forall A\in\partial f(y)$$

and

$$-g(y,v) - Bd \in \mathbb{R}^m_+, \ \forall B \in \partial_x g(y,v).$$

Moreover, one has

$$\lambda^\top (f(x) - f(y) - Ad) \geq 0, \ \forall A \in \partial f(y)$$

and

$$\mu^\top(-g(y,v) - Bd) \geq 0, \ \forall B \in \partial_x g(y,v).$$

Therefore, we obtain

$$\begin{aligned}
&\lambda^\top f(x) - \lambda^\top \left(f(y) + \mu^\top g(y,v)e\right) - \left(\lambda^\top A + \mu^\top B\right) d \\
&= \lambda^\top (f(x) - f(y) - Ad) + \mu^\top(-g(y,v) - Bd) \\
&\geq 0,
\end{aligned} \tag{37}$$

for all $A \in \partial f(y)$ and $B \in \partial_x g(y,v)$. Then, we have

$$\begin{aligned}
\left(\lambda^\top A + \mu^\top B\right) d &\leq \lambda^\top f(x) - \lambda^\top \left(f(y) + \mu^\top g(y,v)e\right) \\
&\overset{(36)}{<} 0, \ \forall A \in \partial f(y), \ B \in \partial_x g(y,v),
\end{aligned}$$

which contradicts (35). Accordingly, we derive that $f(x) - \left(f(y) + \mu^\top g(y,v)e\right) \notin -\mathrm{int}Q$. □

Theorem 5.2 (Strong duality) *Let $\bar{x} \in F^w$ and (GRSCQ) hold. Assume that all conditions of Theorem 3.1 are satisfied. Then, there exist $\bar{v} \in V$, $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$ and $\bar{\mu} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D_w$. Furthermore, if (f, g) is type II $(Q, \mathbb{R}^m_+)$-generalized convex at y, where $(y, v, \lambda, \mu) \in F^D_w$, then $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weak robust solution of (WRD).*

Proof It follows from Theorem 4.2 that there exist $\bar{v} \in V$, $\bar{\lambda} \in Q^* \setminus \{\mathbf{0}\}$ and $\bar{\mu} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu}) \in F^D_w$ and $\bar{\mu}^\top g(\bar{x}, \bar{v}) = 0$. By Theorem 5.1, we have

$$\begin{aligned}
&\left(f(\bar{x}) + \bar{\mu}^\top g(\bar{x}, \bar{v})e\right) - \left(f(y) + \mu^\top g(y,v)e\right) \\
&= f(\bar{x}) - \left(f(y) + \mu^\top g(y,v)e\right) \notin -\mathrm{int}Q, \quad \forall (y, v, \lambda, \mu) \in F^D_w,
\end{aligned}$$

i.e.,

$$\left(f(y) + \mu^\top g(y,v)e\right) - \left(f(\bar{x}) + \mu^\top g(\bar{x}, \bar{v})e\right) \notin \mathrm{int}Q, \ \forall (y, v, \lambda, \mu) \in F^D_w.$$

Consequently, $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weak robust solution of (WRD). □

Theorem 5.3 (Converse duality) *Let* $(\bar{y}, \bar{v}, \bar{\lambda}, \bar{\mu})$ *be a weak robust solution of (WRD) with* $\bar{y} \in F$. *If* (f, g) *is type II* $(Q, \mathbb{R}^m_+)$*-generalized convex at* $\bar{y}$*, then* $\bar{y} \in F^w$.

Proof Since $(\bar{y}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weak robust solution of (WRD) and $\bar{y} \in F$, it then follows from Theorem 5.1 that $f(x) - f(\bar{y}) \notin -\text{int}Q$ for all $x \in F$, i.e., $\bar{y} \in F^w$. □

Remark 5.1 It follows from Remark 2.1(ii) that if the type II $(Q, \mathbb{R}^m_+)$-generalized convexity of (f, g) is replaced by the type I $(Q, \mathbb{R}^m_+)$-generalized convexity of (f, g), we can get the corresponding weak duality, strong duality and converse duality results between (UCMOP) and (WRD).

6 Conclusions

This manuscript is concerned with a robust nonsmooth multiobjective optimization problem. We introduced two kinds of generalized convex functions, which are not necessarily convex. By means of a generalized alternative theorem and the obtained robust constraint qualification, robust necessary optimality conditions for weakly robust efficient solutions and properly robust efficient solutions of the problem are presented. We also establish robust sufficient optimality conditions for weakly robust efficient solutions and properly robust efficient solutions. Moreover, we formulated the Mond–Weir-type dual problem and Wolfe-type dual problem. Our paper concluded with some weak, strong and converse robust duality results between the primal one and its dual problems under the generalized convexity assumptions. Our research leaves various avenues for future study. For example, it would be interesting to study the problem (UCMOP) with uncertainties in the objective function. Additionally, more concepts of robustness, for example deviation robustness, light robustness, reliable robustness or ϵ-constraint robustness (see [12]), could be studied and examined in a similar way.

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References

1. Beck, A., Ben-Tal, A.: Duality in robust optimization: primal worst equals dual best. Oper. Res. Lett. **37**, 1–6 (2009)
2. Ben-Tal, A., Ghaoui, L.E., Nemirovski, A.: Robust Optimization. Princeton series in applied mathematics. Princeton University Press, Princeton (2009)
3. Ben-Tal, A., Nemirovski, A.: Robust optimization-methodology and applications. Math. Programm. Ser. B. **92**, 453–480 (2002)
4. Ben-Tal, A., Nemirovski, A.: A selected topics in robust convex optimization. Math. Programm. Ser. B. **112**, 125–158 (2008)
5. Jeyakumar, V., Li, G.: Strong duality in robust convex programming: complete characterizations. SIAM J. Optim. **20**, 3384–3407 (2010)
6. Jeyakumar, V., Li, G., Lee, G.M.: Robust duality for generalized convex programming problems under data uncertainty. Nonlinear Anal. **75**, 1362–1373 (2012)

7. Jeyakumar, V., Lee, G.M., Li, G.: Characterizing robust solutions sets convex programs under data uncertainty. J. Optim. Theory Appl. **64**, 407–435 (2015)
8. Lee, G.M., Yao, J.C.: On solution sets for robust optimization problems. J. Nonlinear Convex Anal. **17**, 957–966 (2016)
9. Khan, A.A., Tammer, Chr, Zălinescu, C.: Set-Valued Optimization: An Introduction with Applications. Springer, Berlin (2015)
10. Gunawan, S., Azarm, S.: Multiobjective robust optimization using a sensitivity region concept. Struct. Multidiscip. Optim. **29**, 50–60 (2005)
11. Deb, K., Gupta, H.: Introducing robustness in multiobjective optimization. Evol. Comput. **14**, 463–494 (2006)
12. Klamroth, K., Köbis, K., Schöbel, A., Tammer, Chr: A unified approach for different concepts of robustness and stochastic programming via nonlinear scalarizing functionals. Optimization **62**(5), 649–671 (2013)
13. Kuroiwa, D., Lee, G.M.: On robust multiobjective optimization. Vietnam J. Math. **40**, 305–317 (2012)
14. Kim, M.H.: Duality theorem and vector saddle point theorem for robust multiobjective optimization problems. Commun. Korean Math. Soc. **28**, 597–602 (2013)
15. Fliege, J., Werner, R.: Robust multiobjective optimization & applications in portfolio optimization. Eur. J. Oper. Res. **234**, 422–433 (2014)
16. Goberna, M.A., Jeyakumar, V., Li, G., Vicentre-Perez, J.: Robust solutions of multiobjective linear semi-infinite programs under constraint data uncertainty. SIAM J. Optim. **24**, 1402–1419 (2014)
17. Ide, J., Köbis, E.: Concepts of efficiency for uncertain multiobjective problems based on set order relations. Math. Meth. Oper. Res. **80**, 99–127 (2014)
18. Goberna, M.A., Jeyakumar, V., Li, G., Vicentre-Perez, J.: Robust solutions to multi-objective linear programs with uncertain data. Eur. J. Oper. Res. **242**, 730–743 (2015)
19. Chuong, T.D.: Optimality and duality for robust multiobjective optimization problems. Nonlinear Anal. **134**, 127–143 (2016)
20. Ide, J., Schöbel, A.: Robustness for uncertain multi-objective optimization: a survey and analysis of different concepts. OR Spectr. **38**, 235–271 (2016)
21. Lee, J.H., Lee, G.M.: On optimality conditions and duality theorems for robust semi-infinite multiobjective optimization problems. Ann. Oper. Res. **269**, 419–438 (2018)
22. Klamroth, K., Köbis, E., Schöbel, A., Tammer, C.: A unified approach to uncertain optimization. Eur. J. Oper. Res. **260**, 403–420 (2017)
23. Bokrantz, R., Fredriksson, A.: Necessary and sufficient conditions for Pareto efficiency in robust multiobjective optimization. Eur. J. Oper. Res. **262**, 682–692 (2017)
24. Chen, J.W., Cho, Y.J., Kim, J.K., Li, J.: Multiobjective optimization problems with modified objective functions and cone constraints and applications. J. Global Optim. **49**, 137–147 (2011)
25. Suneja, S.K., Khurana, S., Bhatia, M.: Optimality and duality in vector optimization involving generalized type I functions over cones. J. Global Optim. **49**, 23–35 (2011)
26. Luc, D.T.: Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems. Springer-Verlag, Berlin (1989)
27. Göpfert, A., Riahi, H., Tammer, Chr, Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
28. Chen, G.Y., Huang, X.X., Yang, X.O.: Vector Optimization: Set-valued and Variational Analysis. Lecture Notes in Economics and Mathematical Systems. Springer-Verlag, Berlin (2005)
29. Illés, T., Kassay, G.: Theorems of the alternative and optimality conditions for convexlike and general convexlike programming. J. Optim. Theory Appl. **101**, 243–257 (1999)
30. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley-Interscience, New York (1983)
31. Göpfert, A., Tammer, C., Riahi, H., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer-Verlag, New York (2003)
32. Frenk, J.B.G., Kassay, G.: On classes of generalized convex functions, Gordan–Farkas type theorems, and Lagrangian duality. J. Optim. Theory Appl. **102**, 315–343 (1999)
33. Giannessi, F.: Constrained Optimization and Image Space Analysis, vol. 1: Separation of Sets and Optimality Conditions. Springer, New York (2005)
34. Craven, B.D.: Control and Optimization. Chapman & Hall, London (1995)
35. Lee, G.M., Son, P.T.: On nonsmooth optimality theorems for robust optimization problems. Bull. Korean Math. Soc. **51**, 287–301 (2014)

36. Lee, G.M., Lee, J.H.: On nonsmooth optimality theorems for robust multiobjective optimization problems. J. Nonlinear Convex Anal. **16**, 2039–2052 (2015)
37. Suneja, S.K., Khurana, S.: Generalized nonsmooth invexity over cones in vector optimization. Eur. J. Oper. Res. **186**, 28–40 (2008)