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New Farkas-Type Results for Vector-Valued Functions: A Non-abstract Approach

Nguyen Dinh[1] · **Miguel A. Goberna**[2] · **Dang H. Long**[3,4] · **Marco A. López-Cerdá**[2,5]

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Abstract

This paper provides new Farkas-type results characterizing the inclusion of a given set, called contained set, into a second given set, called container set, both of them are subsets of some locally convex space, called decision space. The contained and the container sets are described here by means of vector functions from the decision space to other two locally convex spaces which are equipped with the partial ordering associated with given convex cones. These new Farkas lemmas are obtained via the complete characterization of the conic epigraphs of certain conjugate mappings which constitute the core of our approach. In contrast with a previous paper of three of the authors (Dinh et al. in J Optim Theory Appl 173:357–390, 2017), the aimed characterizations of the containment are expressed here in terms of the data.

Keywords Farkas-type results · Vector-valued functions · Qualification conditions

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✉ Marco A. López-Cerdá
marco.antonio@ua.es

Nguyen Dinh
ndinh@hcmiu.edu.vn

Miguel A. Goberna
mgoberna@ua.es

Dang H. Long
danghailong@tgu.edu.vn

1 International University, Vietnam National University-HCM, Thu Duc District, Ho Chi Minh City, Vietnam

2 Department of Mathematics, University of Alicante, 03080 Alicante, Spain

3 VNUHCM-University of Science, District 5, Ho Chi Minh City, Vietnam

4 Tien Giang University, Tien Giang Town, Vietnam

5 CIAO, Federation University, Ballarat, Australia

1 Introduction

Classical Farkas-type results deal with the characterization of the inclusion of a so-called *contained set* into a second set called *container set*, assuming that the contained and the container sets are subsets of some locally convex space called *decision space* and are described by means of inequality systems. More general Farkas-type results characterize the mentioned inclusions for pairs of sets which are described by means of functions, but not necessarily in terms of inequality systems. This family of results has been used to characterize solutions and strong duality of different classes of optimization, equilibrium problems, and variational inequalities.

Many papers (see, e.g., [1–5], etc.) provide Farkas lemmas focused on infinite programming, where the contained set (called *feasible set* in this setting) is a subset of an infinite-dimensional decision space and is described by means of some vector function while the container set is a sublevel set of some extended real-valued function. This type of scalar Farkas lemmas do not apply either to vector optimization and equilibrium problems or to vector variational inequalities. The limitation of the scalar nature of the function defining the container has been overcome in different ways in [6], where the decision space is Banach while the contained and the container sets are the images of two given convex cones by given convex set-valued functions, in [7], where the contained set is the solution set of a finite linear system while the container set is a given half-space of certain linearly ordered vector space, and also in [8], where the scalar function describing the container set was replaced by the restriction of some vector function to certain subset of the decision space, the so-called *constraint set*. This paper can be seen as a second part of [8], whose Farkas lemmas characterize the inclusion of the contained set into the container in terms of the constraint set, the vector function defining the container and the contained set, while here the characterization is expressed in terms of *the data*: the constraint set and the vector functions defining the contained and the container sets. So, the crucial difference between the Farkas-type results in this paper and the previous one [8] lies in the fact that the new characterizations of the inclusion are more easily checkable than the available ones. For this reason, the corresponding Farkas lemmas are called *abstract* and *non-abstract,* respectively. To do this, we extend, to the vector setting, the classical way of obtaining optimality conditions for scalar programs: reformulating the given constrained problem as an unconstrained one by summing up to the objective vector function the indicator function of the feasible set and, using the epigraph of the conjugate of this sum, to obtain non-abstract optimality conditions via Farkas-type results.

In contrast with scalar optimization, different kinds of optimal solutions can be considered in vector optimization, each one having its own set of advantages and disadvantages (see, e.g., [9, Section 3.2] and [10, Section 15.3]). In particular, regarding multiobjective optimization (when the involved spaces are finite-dimensional), it is usually admitted that weakly efficient solutions, efficient solutions, and super efficient solutions are preferable from the computational, practical, and stability perspectives, respectively (see, e.g., [9,11–13], and references therein). On the other hand, weak orders allow us to apply the elegant conjugate duality machinery. So, computability and mathematical elegance are the main reasons for having oriented our new Farkas-type results to the characterization of the weakly minimal elements of the image of the feasible set by the objective vector function.

Let us describe the structure of the paper. Section 2 introduces notations and the concepts of weak infimum (supremum) and weak minimum (maximum) in partially ordered spaces, together with the fundamentals of the theory of conjugate vector-valued functions. Section 3 provides some technical results to be used later. The main results in this paper are the representations of the epigraph of the conjugate of the sum of the objective vector function with the indicator of the feasible set obtained in Sect. 4, and whose consequences are the non-abstract Farkas-type results established in Sect. 5. The representations in Sect. 4 can be seen as extensions of scalar versions (as those in [14, Theorem 8.2], [15, p. 683], and [16, Section 4]), whose terminology we adopt, calling them *asymptotic*, when they involve a limiting process (typically, in the form of the closure of some set depending on the data), and *non-asymptotic* otherwise. Similarly, the vector Farkas-type results given in Sect. 5 are called *asymptotic* or *non-asymptotic* depending on the nature of the involved conditions. These results are called *Farkas lemmas*, when they characterize the inclusion of the contained set into the container one under certain assumption (typically the identity of two sets or the closedness of one of them) and *characterizations of Farkas lemma*, when they establish the equivalence of the mentioned assumption with some characterization of the inclusion. The final Sect. 6 provides the conclusions.

2 Preliminaries

Let Y be a locally convex Hausdorff topological vector space (lcHtvs in brief) with topological dual space denoted by Y^*. For a set $U \subset Y$, we denote by $\mathrm{cl}U$, $\mathrm{bd}\ U$, $\mathrm{conv}U$, $\mathrm{cl\ conv}U$, $\mathrm{lin}U$, $\mathrm{ri}U$, and $\mathrm{sqri}U$ the *closure*, the *boundary,* the *convex hull*, the *closed convex hull*, the *linear hull*, the *relative interior*, and the *strong quasi-relative interior* of U, respectively. Note that $\mathrm{cl\ conv}U = \mathrm{cl}\ (\mathrm{conv}U)$. The null vector in Y is denoted by 0_Y.

Let $K \subset Y$ be a closed, pointed, convex cone with nonempty interior, i.e., $\mathrm{int}K \neq \emptyset$. Then,

$$K + \mathrm{int}K = \mathrm{int}K, \tag{1}$$

or equivalently,

$$\left.\begin{array}{c} y \in K \\ y + y' \notin \mathrm{int}K \end{array}\right\} \Longrightarrow y' \notin \mathrm{int}K. \tag{2}$$

The cone K generates on Y an *ordering* "$\leqq_K$" and a *weak ordering* defined as $[y_1 \leqq_K y_2 \iff y_2 \in y_1 + K]$ and $[y_1 <_K y_2 \iff y_1 - y_2 \in -\mathrm{int}K]$, respectively. We enlarge Y by attaching to Y a *greatest element* $+\infty_Y$ and a *smallest element* $-\infty_Y$ with respect to $<_K$, which do not belong to Y, and we denote $Y^\bullet := Y \cup \{-\infty_Y, +\infty_Y\}$. We assume the usual convention rules (see, e.g., [8, (5)]). It is obvious that the order

$\leqq_K$ also can be extended to $Y^\bullet$ with the convention that $-\infty_Y \leqq_K y \leqq_K +\infty_Y$ for any $y \in Y^\bullet$ together with the mentioned rules.

We now recall the following basic definitions regarding a set M such that $\emptyset \neq M \subset Y^\bullet$ (see, e.g., [14], [9, Definition 7.4.1], [10,17–19], etc.). More details can be found in [8, Section 2].

- An element $\bar{v} \in Y^\bullet$ is said to be a *weakly infimal element* of M if for all $v \in M$ we have $v \not<_K \bar{v}$ and if for any $\tilde{v} \in Y^\bullet$ such that $\bar{v} <_K \tilde{v}$, there exists some $v \in M$ satisfying $v <_K \tilde{v}$. The set of all weakly infimal elements of M is denoted by $\mathrm{WInf}M$ and is called the *weak infimum* of M.
- An element $\bar{v} \in Y^\bullet$ is said to be a *weakly supremal element* of M if for all $v \in M$ we have $\bar{v} \not<_K v$ and if for any $\tilde{v} \in Y^\bullet$ such that $\tilde{v} <_K \bar{v}$, there exists some $v \in M$ satisfying $\tilde{v} <_K v$. The set of all weakly supremal elements of M is denoted by $\mathrm{WSup}M$ and is called the *weak supremum* of M.
- The *weak minimum* of M is the set $\mathrm{WMin}M = M \cap \mathrm{WInf}M$ and its elements are the *weakly minimal elements* of M. The definition of weak maximum of M is similar.
- An element $\overline{v} \in M$ is called a *strongly maximal element* of M if it holds $v \leqq_K \overline{v}$ for all $v \in M$. The set of all strongly maximal elements of M is denoted by $\mathrm{SMax}M$.

Concerning the weak supremum, as shown in [9, Remark 7.4.2] ,

$$\begin{aligned} +\infty_Y \in \mathrm{WSup}M &\iff \mathrm{WSup}M = \{+\infty_Y\} \\ &\iff \forall \tilde{v} \in Y,\ \exists v \in M : \tilde{v} <_K v. \end{aligned} \tag{3}$$

Additionally, if $\emptyset \neq M \subset Y$ and $\mathrm{WSup}M \neq \{+\infty_Y\}$, by [8, Proposition 2.1] and [19, Proposition 2.4], one has

$$\mathrm{WSup}M = \mathrm{cl}(M - \mathrm{int}K)\backslash(M - \mathrm{int}K), \tag{4}$$

$$\mathrm{WSup}M - \mathrm{int}K = M - \mathrm{int}K, \tag{5}$$

and

$$Y = (M - \mathrm{int}K) \cup (\mathrm{WSup}M) \cup (\mathrm{WSup}M + \mathrm{int}K). \tag{6}$$

Regarding the strong maximum, if $M \subset Y$, then

$$\mathrm{SMax}M = \{\bar{v} \in M : M \subset \bar{v} - K\}. \tag{7}$$

Moreover, in this case, if $\mathrm{SMax}M \neq \emptyset$ then $\mathrm{SMax}M$ is a singleton, i.e., the strongly maximum element of the set M in this case, if exists, will be unique. In such a case, we write $\overline{v} = \mathrm{SMax}M$ instead of $\mathrm{SMax}M = \{\overline{v}\}$.

Given a second lcHtvs X, we denote by $\mathcal{L}(X, Y)$ the space of linear continuous mappings from X to Y, and by $0_{\mathcal{L}} \in \mathcal{L}(X, Y)$ the zero mapping (i.e., $0_{\mathcal{L}}(x) = 0_Y$ for all $x \in X$). Obviously, $\mathcal{L}(X, Y) = X^*$ whenever $Y = \mathbb{R}$. We consider $\mathcal{L}(X, Y)$ equipped with the *weak topology*, that is, the one defined by the pointwise convergence.

In other words, given a net $(L_i)_{i\in I} \subset \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(X, Y)$, $L_i \to L$ means that $L_i(x) \to L(x)$ in Y for all $x \in X$.

Given a vector-valued mapping $F\colon X \to Y^\bullet$, the *domain* of F is defined by

$$\operatorname{dom} F := \{x \in X : F(x) \neq +\infty_Y\},$$

and F is *proper* when $\operatorname{dom} F \neq \emptyset$ and $-\infty_Y \notin F(X)$. The K*-epigraph* of F, denoted by $\operatorname{epi}_K F$, is defined by

$$\begin{aligned}\operatorname{epi}_K F &:= \{(x, y) \in X \times Y : F(x) \leqq_K y\}\\ &= \{(x, y) \in X \times Y : y \in F(x) + K\}.\end{aligned}$$

The *conjugate map* of F is the mapping $F^*\colon \mathcal{L}(X, Y) \rightrightarrows Y^\bullet$ such that

$$F^*(L) := \operatorname{WSup}\{L(x) - F(x) : x \in X\} = \operatorname{WSup}\{(L - F)(X)\}.$$

The *domain* and the (*strong*) *"max-domain"* of F^* are defined, respectively, as

$$\operatorname{dom} F^* := \big\{L \in \mathcal{L}(X, Y) \ : \ F^*(L) \neq \{+\infty_Y\}\big\},$$

and

$$\operatorname{dom}_M F^* := \big\{L \in \mathcal{L}(X, Y) \ : \ F^*(L) \subset Y \text{ and } \operatorname{SMax} F^*(L) \neq \emptyset\},$$

while the K*-epigraph* of F^* is

$$\operatorname{epi}_K F^* := \big\{(L, y) \in \mathcal{L}(X, Y) \times Y : y \in F^*(L) + K\big\}.$$

Let S be a nonempty convex cone in a third lcHtvs Z and $\leqq_S$ be the ordering on Z induced by the cone S. We also enlarge Z by attaching a greatest element $+\infty_Z$ and a smallest element $-\infty_Z$ (with respect to $\leqq_S$) which do not belong to Z, and define $Z^\bullet := Z \cup \{-\infty_Z, \ +\infty_Z\}$ with the usual convention rules.

The *cone of positive operators* (see [20,21]) is

$$\mathcal{L}_+(S, K) := \{T \in \mathcal{L}(Z, Y) : \ T(S) \subset K\}.$$

We also define the *cone of weakly positive operators* as

$$\mathcal{L}_+^w(S, K) := \{T \in \mathcal{L}(Z, Y) : T(S) \cap (-\operatorname{int} K) = \emptyset\}.$$

It is clear that $\mathcal{L}_+(S, K) \subset \mathcal{L}_+^w(S, K)$. Indeed, for any $T \in \mathcal{L}_+(S, K)$, one has $T(S) \subset K$ and so $T(S) \cap (-\operatorname{int} K) = \emptyset$ (as K is a pointed cone), and hence, $T \in \mathcal{L}_+^w(S, K)$. Examples 2.2 and 2.1 below, where X is finite-dimensional and infinite-dimensional, respectively, will be used for illustrative purposes along the paper. For instance, they show that the inclusion $\mathcal{L}_+(S, K) \subset \mathcal{L}_+^w(S, K)$ is generally strict. In Sect. 4, we interpret both cones in terms of domains of indicator functions.

It is worth noting that, when $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, the conjugate, the domain and the K-epigraph of a proper function $f: X \to \mathbb{R} \cup \{\pm\infty\}$ are nothing else but the ordinary conjugate, the domain, and the epigraph of the scalar function f. Moreover, since

$$T(S) \cap (-\mathrm{int}\mathbb{R}_+) = \emptyset \quad \Longleftrightarrow \quad T(S) \subset \mathbb{R}_+,$$

we have

$$\mathcal{L}_+^w(S, \mathbb{R}_+) = \mathcal{L}_+(S, \mathbb{R}_+) = S^* := \{z^* \in Z^* : \langle z^*, s\rangle \geq 0 \text{ for all } s \in S\},$$

which means that in the scalar case (when $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and f is proper), the cones $\mathcal{L}_+(S, K)$ and $\mathcal{L}_+^w(S, K)$ collapse to the usual (positive) *dual cone* S^* of S.

The K-epigraph of F^* might be not convex, as is shown in the next example, and it is so even in the case where Y is a finite-dimensional space as in Example 2.2.

Example 2.1 Take $X = Z = \mathbb{R}$, $Y = C[0, 1]$ equipped with the topology of the uniform convergence, $S = \mathbb{R}_+$, and $K := \{x \in C[0, 1] : x(t) \geq 0, \forall t \in [0, 1]\}$, the cone of all nonnegative functions on $C[0, 1]$. Then it is easy to check that:

- $\mathrm{int}K = \{x \in C[0, 1] : x(t) > 0, \forall t \in [0, 1]\}$,
- $\mathcal{L}(X, Y) = \mathcal{L}(Z, Y) \equiv C[0, 1]$, $\mathcal{L}_+(S, K) \equiv K$, and $\mathcal{L}_+^w(S, K) = \{z \in C[0, 1] : z(t) \geq 0 \text{ for some } t \in [0, 1]\}$.
- If $\emptyset \neq M \subset C[0, 1]$ and $\mathrm{WSup}M \subset C[0, 1]$ then

$$\mathrm{WSup}M = \left\{ v \in C[0, 1] : \begin{array}{l} \forall x \in M,\ \exists t_x \in [0, 1] : v(t_x) \geq x(t_x) \text{ and} \\ \forall \epsilon > 0,\ \exists x_\epsilon \in M : v(t) - \epsilon < x_\epsilon(t), \forall t \in [0, 1] \end{array} \right\},$$

- As a consequence of the last equality, we have, for $\emptyset \neq M \subset C[0, 1]$ with $\mathrm{WSup}M \subset C[0, 1]$,

$$\mathrm{WSup}M + K = \{y \in C[0, 1] : \forall x \in M,\ \exists t_x \in [0, 1] \text{ s.t. } y(t_x) \geq x(t_x)\}. \quad (8)$$

  We now take $F: \mathbb{R} \to C[0, 1]$ such that, for all $\alpha \in \mathbb{R}$, $F(\alpha): [0, 1] \to \mathbb{R}$ is the function $F(\alpha)(t) = t^2 + \alpha^2$. We will give a representation for $\mathrm{epi}_K F^*$ and show that this set is closed but not a convex one in $C[0, 1] \times C[0, 1]$.

- Take arbitrarily $x \in C[0, 1]$. By definition, $F^*(x) = \mathrm{WSup}\{\alpha x - F(\alpha) : \alpha \in \mathbb{R}\}$. Take $w_0: [0, 1] \to \mathbb{R}$ defined by $w_0(t) = \frac{[x(0)]^2}{4}$ for any $t \in [0, 1]$. Then for any $\alpha \in \mathbb{R}$ one has

$$w_0(0) = \frac{[x(0)]^2}{4} \geq \alpha x(0) - \alpha^2 = \alpha x(0) - F(\alpha)(0),$$

which means that $w_0 \nless_K \alpha x - F(\alpha)$ for any $\alpha \in \mathbb{R}$, and so, by (3), $F^*(x) \neq \{+\infty_Y\}$. It is also clear that $F^*(x) \neq \{-\infty_Y\}$. Applying (8), one gets

$$F^*(x) + K = \{y \in Y : \forall\alpha \in \mathbb{R},\ \exists t_\alpha \in [0,1] \text{ s.t. } y(t_\alpha) \geq \alpha x(t_\alpha) - (t_\alpha)^2 - \alpha^2\},$$

and hence,

$$\begin{aligned}\mathrm{epi}_K F^* = \{(x,y) \in (C[0,1])^2 : \forall\alpha \in \mathbb{R}, \exists t_\alpha \in [0,1] \text{ s.t. } y(t_\alpha) \geq \alpha x(t_\alpha) \\ -(t_\alpha)^2 - \alpha^2\}.\end{aligned} \tag{9}$$

- The K-epigraph of F^*, $\mathrm{epi}_K F^*$, is closed in $C[0,1] \times C[0,1]$. Indeed, let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{epi}_K F^*$ such that $(x_n, y_n) \to (x,y) \in C[0,1] \times C[0,1]$ (i.e., the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ uniformly converge to x and y, respectively). Fix an $\alpha \in \mathbb{R}$. Since $(x_n, y_n) \in \mathrm{epi}_K F^*$ we get, for each $n \in \mathbb{N}$, there is $t_\alpha^n \in [0,1]$ satisfying

$$y_n(t_\alpha^n) \geq \alpha x_n(t_\alpha^n) - (t_\alpha^n)^2 - \alpha^2.$$

As $(t_\alpha^n)_{n\in\mathbb{N}} \subset [0,1]$, without loss of generality, we can assume that $t_\alpha^n \to t_\alpha$ with some $t_\alpha \in [0,1]$. Now since x_n and y_n uniformly converge to x and y, respectively, one gets $y_n(t_\alpha^n) \to y(t_\alpha)$ and $x_n(t_\alpha^n) \to x(t_\alpha)$. It then follows that $y(t_\alpha) \geq \alpha x(t_\alpha) - (t_\alpha)^2 - \alpha^2$, showing that $(x,y) \in \mathrm{epi}_K F^*$.
- The K-epigraph of F^*, $\mathrm{epi}_K F^*$, is not a convex set. Indeed, consider the two pairs (θ, y_1) and (θ, y_2) where θ is the zero function (i.e., $\theta(t) = 0$ for all $t \in [0,1]$) in $C[0,1]$ and $y_1, y_2 \colon [0,1] \to \mathbb{R}$ defined by $y_1(t) = 2t^2 - 2$ and $y_2(t) = -4t^2$. It is easy to check that $(\theta, y_1), (\theta, y_2) \in \mathrm{epi}_K F^*$. However, $(\theta, \frac{1}{2}y_1 + \frac{1}{2}y_2) \notin \mathrm{epi}_K F^*$. Indeed, we have $y(t) := \frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = -t^2 - 1$, and with $\alpha = 0$, one gets $\alpha\theta(t) - (t)^2 - \alpha^2 = -t^2$ and hence, with $\alpha = 0$

$$y(t) := \frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = -t^2 - 1 < -t^2, \quad \forall t \in [0,1].$$

This shows that $(\theta, \frac{1}{2}y_1 + \frac{1}{2}y_2) \notin \mathrm{epi}_K F^*$ (see (9)), and $\mathrm{epi}_K F^*$ is not a convex subset of $C[0,1] \times C[0,1]$.

Example 2.2 Take $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, $F\colon \mathbb{R} \to \mathbb{R}^2$ the null mapping, and $G\colon \mathbb{R} \to \mathbb{R}$ such that $G(x) = -x$ for all $x \in \mathbb{R}$. In this case, it is easy to see that $\mathcal{L}(Z,Y) \equiv \mathbb{R}^2$, $\mathcal{L}_+(S,K) = \mathbb{R}^2_+$, and $\mathcal{L}^w_+(S,K) = \left\{(t_1, t_2) \in \mathbb{R}^2 : t_1 \geq 0 \vee t_2 \geq 0\right\}$. Moreover, given $(\alpha, \beta) \in \mathbb{R}^2$,

$$F^*(\alpha,\beta) = \mathrm{WSup}\{\mathbb{R}(\alpha,\beta)\} = \begin{cases} \left[(-\mathbb{R}_+) \times \{0\}\right] \cup \left[\{0\} \times (-\mathbb{R}_+)\right], & \text{if } \alpha = \beta = 0, \\ \left\{+\infty_{\mathbb{R}^2}\right\}, & \text{if } \alpha\beta > 0, \\ \mathbb{R}(\alpha,\beta), & \text{otherwise.} \end{cases}$$

Thus, $\mathrm{epi}_K F^* = \bigcup_{i=1}^{4} N_i$, where

$$
\begin{aligned}
N_1 &= \{(0, 0, y_1, y_2) : y_1 \geq 0 \vee y_2 \geq 0\}, \\
N_2 &= \left\{(\alpha, \beta, y_1, y_2) : \alpha\beta < 0 \wedge y_2 \geq \tfrac{\beta}{\alpha} y_1\right\}, \\
N_3 &= \{(\alpha, 0, y_1, y_2) : \alpha \neq 0 \wedge y_2 \geq 0\}, \\
N_4 &= \{(0, \beta, y_1, y_2) : \beta \neq 0 \wedge y_1 \geq 0\}.
\end{aligned}
$$

It is also easy to verify that $\mathrm{epi}_K F^*$ is closed. However, $\mathrm{epi}_K F^*$ is not convex as its image by the projection mapping $(\alpha, \beta, y_1, y_2) \mapsto (\alpha, \beta)$ is the domain of F^*, $\mathrm{dom}\, F^* = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha\beta \leq 0\}$, which is obviously non-convex.

3 Fundamental Tools

Let the spaces X, Y, Z and the cones K, S be as in Sect. 2. The results given below are used as fundamental tools in the next sections.

Lemma 3.1 *Given $\emptyset \neq M \subset Y^\bullet$, the following statements hold true:*
(i) *If $+\infty_Y \notin M$ and $M \cap \mathrm{int}K = \emptyset$, then $\mathrm{WSup}M \neq \{+\infty_Y\}$,*
(ii) *If there exists $v_0 \in \mathrm{int}K$ such that $\lambda v_0 \in M$ for all $\lambda > 0$, then $\mathrm{WSup}M = \{+\infty_Y\}$,*
(iii) *If $M \subset -K$ and $0_Y \in M$ then $\mathrm{WSup}M = \mathrm{WSup}(-K) = \mathrm{bd}(-K)$.*

Proof (i) Assume that $M \cap \mathrm{int}K = \emptyset$. Then, $0_Y \not<_K v$ for all $v \in M$ and it follows from (3) that $\mathrm{WSup}M \neq \{+\infty_Y\}$.

(ii) Assume that there is $v_0 \in \mathrm{int}K$ such that $\lambda v_0 \in M$ for all $\lambda > 0$. If $\mathrm{WSup}M \neq \{+\infty_Y\}$, then by (3), there exists $\tilde{v} \in Y$ such that $\tilde{v} \not<_K v$ for any $v \in M$. We get

$$
\begin{aligned}
\tilde{v} \not<_K \lambda v_0,\ \forall \lambda > 0 &\Longleftrightarrow \lambda v_0 - \tilde{v} \notin \mathrm{int}K,\ \forall \lambda > 0 \\
&\Longleftrightarrow v_0 - \frac{1}{\lambda}\tilde{v} \notin \mathrm{int}K,\ \forall \lambda > 0.
\end{aligned}
$$

Letting $\lambda \to +\infty$ we get $v_0 \notin \mathrm{int}K$, a contradiction.

(iii) Assume that $M \subset -K$ and $0_Y \in M$. Then $M - \mathrm{int}K = -\mathrm{int}K$. Indeed, $M - \mathrm{int}K \subset -K - \mathrm{int}K = -\mathrm{int}K$. Since $0_Y \in M$, we also have $-\mathrm{int}K \subset M - \mathrm{int}K$. On other hand, because K is a pointed cone, $M \subset -K$ yields $M \cap \mathrm{int}K = \emptyset$. So we get from (i) that $\mathrm{WSup}M \neq \{+\infty\}$. According to (4),

$$
\begin{aligned}
\mathrm{WSup}M &= \mathrm{cl}(M - \mathrm{int}K) \setminus (M - \mathrm{int}K) \\
&= \mathrm{cl}(-\mathrm{int}K) \setminus (-\mathrm{int}K) = \mathrm{WSup}(-K) = \mathrm{bd}(-K),
\end{aligned}
$$

and we are done. □

Lemma 3.2 *Assume $\emptyset \neq M \subset Y$, $\mathrm{WSup}M \subset Y$, and there exist $v_0 \in Y \setminus (-K)$ such that $\lambda v_0 \in M$ for all $\lambda > 0$. Then $\mathrm{SMax}(\mathrm{WSup}M) = \emptyset$.*

Proof Let us suppose by contradiction that $\mathrm{SMax}(\mathrm{WSup}M) \neq \emptyset$ and take $\bar{v} = \mathrm{SMax}(\mathrm{WSup}M)$ (note that $\mathrm{SMax}(\mathrm{WSup}M)$ is a singleton as $\mathrm{WSup}M \subset Y$). Since $\mathrm{WSup}M \subset Y$, one has $\mathrm{WSup}M \subset \bar{v} - K$ (see (7)). It follows from (4) and (5) that

$$\begin{aligned} M \subset \mathrm{cl}(M - \mathrm{int}K) &= [\mathrm{cl}(M - \mathrm{int}K) \setminus (M - \mathrm{int}K)] \cup (M - \mathrm{int}K) \\ &= \mathrm{WSup}M \cup (\mathrm{WSup}M - \mathrm{int}K) \\ &\subset \mathrm{WSup}M - K, \end{aligned}$$

and consequently, $M \subset \bar{v} - K - K = \bar{v} - K$. Thus, from the assumption $\lambda v_0 \in M$ for all $\lambda > 0$, one has

$$\lambda v_0 \in \bar{v} - K, \ \forall \lambda > 0 \quad \Longleftrightarrow \quad v_0 - \frac{1}{\lambda}\bar{v} \in -K, \ \forall \lambda > 0.$$

Letting $\lambda \to +\infty$ we get $v_0 \in -K$ which contradicts the assumption that $v_0 \in Y \setminus (-K)$ and the proof is complete. □

In order to obtain suitable interpretations of $\mathcal{L}_+(S, K)$ and $\mathcal{L}_+^w(S, K)$**,** we must extend the concept of indicator function from scalar to vector functions: the *indicator map* $I_D \colon X \to Y^\bullet$ of a set $D \subset X$ is defined by

$$I_D(x) = \begin{cases} 0_Y, & \text{if} \quad x \in D, \\ +\infty_Y, & \text{otherwise.} \end{cases}$$

In the case $Y = \mathbb{R}$, I_D is the usual indicator function i_D.

Proposition 3.1 *One has*

$$\mathcal{L}_+^w(S, K) = \mathrm{dom}\, I^*_{-S} \quad \textit{and} \quad \mathcal{L}_+(S, K) = \mathrm{dom}_M\, I^*_{-S}.$$

Proof • Taking an arbitrary $T \in \mathcal{L}(Z, Y)$, one has

$$I^*_{-S}(T) = \mathrm{WSup}\{T(z) : z \in -S\} = \mathrm{WSup}T(-S), \tag{10}$$

and so, $T \in \mathrm{dom}\, I^*_{-S}$ if and only if $\mathrm{WSup}T(-S) \neq \{+\infty_Y\}$. Two cases are possible.

(a) If $T \in \mathcal{L}_+^w(S, K)$, $T(S) \cap (-\mathrm{int}K) = \emptyset$, and consequently, $T(-S) \cap \mathrm{int}K = \emptyset$. So, it follows from Lemma 3.1(i) that $\mathrm{WSup}T(-S) \neq \{+\infty_Y\}$.

(b) If $T \in \mathcal{L}(Z, Y) \setminus \mathcal{L}_+^w(S, K)$, there exists $v_0 \in T(S) \cap (-\mathrm{int}K)$. Then, $-v_0 \in \mathrm{int}K$ and $\lambda(-v_0) \in T(-S)$ for all $\lambda > 0$ because S is a cone. So, by Lemma 3.1(ii), $\mathrm{WSup}T(-S) = \{+\infty_Y\}$. Consequently, $\mathrm{dom}\, I^*_{-S} = \mathcal{L}_+^w(S, K)$.

• Take an arbitrary $T \in \mathcal{L}_+(S, K)$. Then one has $T(S) \subset K$, or equivalently, $T(-S) \subset -K$. It is clear that $0_Y = T(0_Z) \in T(-S)$. According to Lemma 3.1(iii), $I^*_{-S}(T) = \mathrm{WSup}T(-S) = \mathrm{WSup}(-K)$. So, $\mathrm{SMax} I^*_{-S}(T) = \{0_Y\} \neq \emptyset$, and consequently, $T \in \mathrm{dom}_M\, I^*_{-S}$.

Now take an arbitrary $T \in \mathcal{L}(Z, Y) \setminus \mathcal{L}_+(S, K)$. One has $T(S) \not\subset K$, or equivalently, there exists $s_0 \in -S$ such that $T(s_0) \notin -K$. Thus, applying Lemma 3.2 with $M = T(-S)$ and $v_0 = T(s_0)$, we get that, if $\mathrm{WSup}T(-S) \subset Y$, then $\mathrm{SMax}\left[\mathrm{WSup}T(-S)\right] = \emptyset$. So, $T \notin \mathrm{dom}_M I^*_{-S}$ and we are done. □

Lemma 3.3 *Let $\emptyset \neq M \subset Y$, $\bar{y} \in Y$, and $y^* \in Y^*$ and assume that*

$$y^*(u) < y^*(\bar{y}), \quad \forall u \in M - \mathrm{int}K. \tag{11}$$

Then, the following statements hold:
(i) $y^*(v) \leq y^*(\bar{y})$, $\forall v \in M$,
(ii) $y^*(k) > 0$ *for all* $k \in \mathrm{int}K$ *and, consequently,* $y^* \in K^*$.

Proof (i) Take $k_0 \in \mathrm{int}K$. Then, for any $v \in M$, it follows from (11) that

$$y^*(v - \lambda k_0) < y^*(\bar{y}), \quad \forall \lambda > 0,$$

and by letting $\lambda \to 0$, we get $y^*(v) \leq y^*(\bar{y})$.

(ii) Take arbitrarily $k \in \mathrm{int}K$. We firstly show that there exists $\lambda > 0$ such that $\bar{y} - \lambda k \in M - \mathrm{int}K$. Indeed, take an $m_0 \in M$. Because of the continuity of the mapping $t \mapsto (m_0 - \bar{y})t + k$ at $t = 0$, there exists $\epsilon > 0$ such that $(m_0 - \bar{y})\epsilon + k \in \mathrm{int}K$. Taking $\lambda = \frac{1}{\epsilon}$, we obtain $m_0 - \bar{y} + \lambda k \in \lambda \mathrm{int}K$, and consequently,

$$\bar{y} - \lambda k \in m_0 - \lambda \mathrm{int}K \subset M - \mathrm{int}K$$

It now follows from (11) that $y^*(\bar{y} - \lambda k) < y^*(\bar{y})$, which yields $y^*(k) > 0$. Since $K = \mathrm{cl}(\mathrm{int}K)$, $y^*(k) \geq 0$ for all $k \in K$, showing that $y^* \in K^*$. □

We conclude this section with the two additional lemmas. The first onex gives a useful characterization of $\mathrm{epi}_K F^*$ and the second one shows the closedness of this set.

Lemma 3.4 *If $F : X \to Y \cup \{+\infty_Y\}$ is a proper mapping, then*

$$\mathrm{epi}_K F^* = \{(L, y) \in \mathcal{L}(X, Y) \times Y : y + F(x) - L(x) \notin -\mathrm{int}K, \ \forall x \in X\}.$$

Proof It follows from [8, Theorem 3.1] by taking $f = F$, $C = X$, and g such that $g(x) = 0_Z$ for all $x \in X$. □

Lemma 3.5 *If $F : X \to Y \cup \{+\infty_Y\}$ is a proper mapping, then $\mathrm{epi}_K F^*$ is a closed subset of $\mathcal{L}(X, Y) \times Y$.*

Proof Let $\{(L_i, y_i)\}_{i \in I} \subset \mathrm{epi}_K F^*$ be a net such that $(L_i, y_i) \to (L, y)$. We will show that $(L, y) \in \mathrm{epi}_K F^*$. Let us suppose the contrary, that is $(L, y) \notin \mathrm{epi}_K F^*$. Then, by Lemma 3.4, there exists $\bar{x} \in \mathrm{dom}\, F$ such that

$$y - L(\bar{x}) + F(\bar{x}) \in -\mathrm{int}K.$$

As $y_i - L_i(\bar{x}) + F(\bar{x}) \to y - L(\bar{x}) + F(\bar{x})$, there is a $i_0 \in I$ such that for all $i \in I$, $i \succsim i_0$ (where $\succsim$ is the net order),

$$y_i - L_i(\bar{x}) + F(\bar{x}) \in -\operatorname{int} K,$$

which, again by Lemma 3.4, $(L_i, y_i) \notin \operatorname{epi}_K F^*$ for all $i \succsim i_0$, a contradiction. $\square$

4 Representing $\operatorname{epi}_K(F + I_A)^*$

We are now in a position to obtain asymptotic and non-asymptotic representations of the set $\operatorname{epi}_K(F + I_A)^*$. The importance of these representations is twofold. From one side, they justify the qualification conditions introduced in the next section in order to establish non-asymptotic Farkas-type results for systems associated with vector functions (see also [8]). Secondly, they supply key tools for deriving refined asymptotic vector Farkas-type results which, to the best of the authors knowledge, are given for the first time in the next section of this paper.

Throughout this section X, Y, Z are as in Sect. 3, K is a closed, pointed, convex cone in Y with nonempty interior, and S is a convex cone in Z. Assume further that $F\colon X \to Y \cup \{+\infty_Y\}$ and $G\colon X \to Z^\bullet$ are proper mappings while C is a subset of X such that $A := C \cap G^{-1}(-S) \neq \emptyset$.

4.1 Asymptotic Representation of $\operatorname{epi}_K(F + I_A)^*$

We say that F is *K-convex* (*K-epi closed*) if $\operatorname{epi}_K F$ is a convex set (a closed set in $X \times Y$ equipped with the product topology, respectively). If F is K-convex, it is evident that $\operatorname{dom} F$ is a convex set in X. We also say that F is *positively K-lsc* if $y^* \circ F$ is lsc for all $y^* \in K^* \setminus \{0_{Y^*}\}$.

Remark 4.1 It is easy to prove that, if $F\colon X \to Y^\bullet$ is a K-convex mapping and A is a convex set satisfying $A \cap \operatorname{dom} F \neq \emptyset$, then $F(A \cap \operatorname{dom} F) + \operatorname{int} K$ is a convex subset of Y.

The next result, which is a natural extension of a similar scalar result (see Lemma 4.1), involves composite functions $T \circ G\colon X \to Y^\bullet$, with $T \in \mathcal{L}(Z, Y)$ and $G\colon X \to Z \cup \{+\infty_Z\}$, which are defined as follows:

$$(T \circ G)(x) := \begin{cases} T(G(x)), & \text{if} \quad G(x) \in Z, \\ +\infty_Y, & \text{if} \quad G(x) = +\infty_Z. \end{cases}$$

Theorem 4.1 (1st asymptotic representation of $\operatorname{epi}_K(F + I_A)^*$) *Let C be a convex and closed subset of X, F be a K-convex and positively K-lsc mapping, and G be an S-convex and S-epi closed mapping. Assume that $A \cap \operatorname{dom} F \neq \emptyset$. Then*

$$\operatorname{epi}_K(F + I_A)^* = \operatorname{cl}\left[\bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K(F + I_C + T \circ G)^*\right]. \tag{12}$$

Proof [“$\supset$”] According to [8, Lemma 4.1], we have

$$\operatorname{epi}_K(F+I_A)^* \supset \bigcup_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(F+I_C+T\circ G)^*, \tag{13}$$

and as $\operatorname{epi}_K(F+I_A)^*$ is closed (by Lemma 3.5),

$$\operatorname{epi}_K(F+I_A)^* \supset \operatorname{cl}\left[\bigcup_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(F+I_C+T\circ G)^*\right]. \tag{14}$$

[“$\subset$”] For this, take arbitrarily $(L,y)\in\operatorname{epi}_K(F+I_A)^*$ and let us show that

$$(L,y)\in \operatorname{cl}\left[\bigcup_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(F+I_C+T\circ G)^*\right]. \tag{15}$$

Observe that if $(L,y)\in\operatorname{epi}_K(F+I_A)^*$ then, by Lemma 3.4,

$$y\notin L(x)-F(x)-\operatorname{int}K,\quad \forall x\in A\cap\operatorname{dom}F,$$

or equivalently, $y\notin(L-F)(A\cap\operatorname{dom}F)-\operatorname{int}K$.

- Now, since G is S-convex, $G^{-1}(-S)$ is a convex set, and hence, $A=C\cap G^{-1}(-S)$ is convex, too. Moreover, $F-L$ is a K-convex mapping (as F is K-convex), and we get from Remark 4.1 that $(F-L)(A\cap\operatorname{dom}F)+\operatorname{int}K$ is convex, or equivalently, $(L-F)(A\cap\operatorname{dom}F)-\operatorname{int}K$ is convex.
- On the one hand, as $y\notin(L-F)(A\cap\operatorname{dom}F)-\operatorname{int}K$, the separation theorem [22, Theorem 3.4] ensures the existence of $y^*\in Y^*$ satisfying

  $$y^*(u)<y^*(y),\quad \forall u\in(L-F)(A\cap\operatorname{dom}F)-\operatorname{int}K.$$

  It then follows from Lemma 3.3 that

  $$y^*\circ(L-F)(x)\le y^*(y),\quad \forall x\in A\cap\operatorname{dom}F, \tag{16}$$

  $$y^*\in K^* \text{ and } y^*(k)>0 \quad \text{for all } k\in\operatorname{int}K. \tag{17}$$

- On the other hand, since $y^*\circ F$ is a proper convex lsc function, applying [14, Theorem 8.2] to the scalar function $y^*\circ F$, one gets

  $$\operatorname{epi}(y^*\circ F+i_A)^*=\operatorname{cl}\left[\bigcup_{z^*\in S^*}\operatorname{epi}(y^*\circ F+i_C+z^*\circ G)^*\right]. \tag{18}$$

  Note that (16) is equivalent to $y^*(y)\ge(y^*\circ F+i_A)^*(y^*\circ L)$ or, equivalently, $(y^*\circ L, y^*(y))\in\operatorname{epi}(y^*\circ F+i_A)^*$. Hence, by (18), there exist nets $\{z_i^*\}_{i\in I}\subset S^*$,

$\{x_i^*\}_{i\in I} \subset X^*$ and $\{r_i\}_{i\in I} \subset \mathbb{R}$ such that $x_i^* \to y^* \circ L$, $r_i \to y^*(y)$ and

$$(x_i^*, r_i) \in \operatorname{epi}(y^* \circ F + i_C + z_i^* \circ G)^*, \quad \forall i \in I. \tag{19}$$

- Take an arbitrary $k_0 \in \operatorname{int} K$. Then $y^*(k_0) > 0$ (see (17)). Now for each $i \in I$, set

$$y_i := y + \frac{r_i - y^*(y)}{y^*(k_0)} k_0,$$

and define the mapping $L_i : X \to Y$ by

$$L_i(x) := L(x) + \frac{x_i^*(x) - (y^* \circ L)(x)}{y^*(k_0)} k_0, \ \forall x \in X.$$

It is easy to check that

$$y^*(y_i) = r_i, \ L_i \in \mathcal{L}(X, Y), \ y^* \circ L_i = x_i^*, \ \forall i \in I, \ \text{ and } (y_i, L_i) \to (y, L). \tag{20}$$

- We now claim that

$$(L_i, y_i) \in \bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K(F + I_C + T \circ G)^*, \quad \forall i \in I. \tag{21}$$

Indeed, for each $i \in I$, combining (19) and (20) we get

$$y^*(y_i) \geq (y^* \circ F + i_C + z_i^* \circ G)^*(y^* \circ L_i),$$

or equivalently,

$$y^*(y_i) \geq \big(y^* \circ L_i\big)(x) - \big(y^* \circ F\big)(x) - \big(z_i^* \circ G\big)(x), \quad \forall x \in C \cap \operatorname{dom} F. \tag{22}$$

For each $i \in I$, define $T_i : Z \to Y$ by

$$T_i(z) = \frac{z_i^*(z)}{y^*(k_0)} k_0, \ \forall z \in Z.$$

Then $T_i \in \mathcal{L}(Z, Y)$. Moreover, if $z \in S$, then $z_i^*(z) \geq 0$ (as $z_i^* \in S^*$) and so, $T_i(z) \in K$ (as $k_0 \in \operatorname{int} K$ and $y^*(k_0) > 0$). Consequently, $T_i \in \mathcal{L}_+(S, K)$.

- Since $y^* \circ T_i = z_i^*$, with the help of the mappings $T_i \in \mathcal{L}_+(S, K)$, $i \in I$, (22) can be rewritten as

$$y^*(y_i) \geq \big(y^* \circ L_i\big)(x) - \big(y^* \circ F\big)(x) - (y^* \circ T_i \circ G)(x), \quad \forall x \in C \cap \operatorname{dom} F,$$

or equivalently,

$$y^* \left(L_i(x) - F(x) - (T_i \circ G)(x) - y_i\right) \leq 0, \ \forall x \in C \cap \operatorname{dom} F.$$

The last inequality, together with (17), implies that

$$y_i \notin L_i(x) - F(x) - (T_i \circ G)(x) - \operatorname{int} K, \quad \forall x \in C \cap \operatorname{dom} F,$$

which together with Lemma 3.4 yields $(L_i, y_i) \in \operatorname{epi}_K (F + I_C + T_i \circ G)^*$, which is (21).

- Finally, (15) follows from (20) and (21). So, we are done.

□

We now introduce an alternative asymptotic representation of $\operatorname{epi}_K (F + I_A)^*$ where $\mathcal{L}_+^w(S, K)$ replaces $\mathcal{L}_+(S, K)$ as index set at the right-hand side union of sets.

Theorem 4.2 (2nd asymptotic representation of $\operatorname{epi}_K (F + I_A)^*$) *Assume that all the assumptions of Theorem 4.1 hold. Then,*

$$\operatorname{epi}_K (F + I_A)^* = \operatorname{cl}\left\{\bigcup_{T \in \mathcal{L}_+^w(S,K)} \left[\bigcap_{v \in I^*_{-S}(T)} [\operatorname{epi}_K (F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)]\right]\right\}. \tag{23}$$

Proof • We first show that

$$\operatorname{epi}_K (F + I_A)^* \supset \bigcap_{v \in I^*_{-S}(T)} \left[\operatorname{epi}_K (F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)\right], \quad \forall T \in \mathcal{L}(Z, Y). \tag{24}$$

Take an arbitrary $T \in \mathcal{L}(Z, Y)$ and

$$(L, y) \in \bigcap_{v \in I^*_{-S}(T)} \left[\operatorname{epi}_K (F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)\right].$$

Then,

$$(L, y - v) \in \operatorname{epi}_K (F + I_C + T \circ G)^*, \quad \forall v \in I^*_{-S}(T),$$

and, by Lemma 3.4, (10), and (5), the last inclusion is equivalent to

$$\begin{aligned}
& y - v - L(x) + F(x) + (T \circ G)(x) \notin -\operatorname{int} K, \quad \forall x \in C, \ \forall v \in I^*_{-S}(T) \\
\Longleftrightarrow\ & y - L(x) + F(x) + (T \circ G)(x) \notin I^*_{-S}(T) - \operatorname{int} K, \quad \forall x \in C \\
\Longleftrightarrow\ & y - L(x) + F(x) + (T \circ G)(x) \notin \operatorname{WSup} T(-S) - \operatorname{int} K, \quad \forall x \in C \\
\Longleftrightarrow\ & y - L(x) + F(x) + (T \circ G)(x) \notin T(-S) - \operatorname{int} K, \quad \forall x \in C \\
\Longleftrightarrow\ & y - L(x) + F(x) \notin u - (T \circ G)(x) - \operatorname{int} K, \quad \forall u \in T(-S), \ \forall x \in C.
\end{aligned} \tag{25}$$

Now, for any $x \in A$, taking $u = (T \circ G)(x)$ in (25) (note that $x \in A$ yields $G(x) \in -S$), we get $y - L(x) + F(x) \notin -\text{int}K$. Hence, again by Lemma 3.4, $(L, y) \in \text{epi}_K(F + I_A)^*$ and (24) follows.

- We now claim that, for each $T \in \mathcal{L}_+(S, K)$, one has

$$\text{epi}_K(F + I_C + T \circ G)^* = \bigcap_{v \in I^*_{-S}(T)} [\text{epi}_K(F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)]. \tag{26}$$

In fact, as $0_Y \in T(-S)$ (note that $0_Y = T(0_X)$), by Lemma 3.1(iii) and (10), $0_Y \in \text{bd}(-K) = I^*_{-S}(T)$. Hence, $\text{epi}_K(F + I_C + T \circ G)^*$ is a member of the collection in the right-hand side of (26) and we get

$$\bigcap_{v \in I^*_{-S}(T)} \left[\text{epi}_K(F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)\right] \subset \text{epi}_K(F + I_C + T \circ G)^*.$$

Conversely, take $(L, y) \in \text{epi}_K(F + I_C + T \circ G)^*$. Since $I^*_{-S}(T) = \text{bd}(-K) \subset -K$, we get

$$(L, y - v) \in \text{epi}_K(F + I_C + T \circ G)^*, \quad \forall v \in I^*_{-S}(T),$$

equivalently

$$(L, y) \in \text{epi}_K(F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v), \quad \forall v \in I^*_{-S}(T),$$

and we are done.

- Combining (24), the inclusion $\mathcal{L}_+(S, K) \subset \mathcal{L}^w_+(S, K)$ and (26), we get

$$\begin{aligned} \text{epi}_K\ (F + I_A)^* &\supset \bigcup_{T \in \mathcal{L}^w_+(S,K)} \left[\bigcap_{v \in I^*_{-S}(T)} [\text{epi}_K(F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)]\right] \\ &\supset \bigcup_{T \in \mathcal{L}_+(S,K)} \text{epi}_K(F + I_C + T \circ G)^*. \end{aligned} \tag{27}$$

- The conclusion follows from the closedness of $\text{epi}_K\ (F + I_A)^*$ (see Lemma 3.5), (27) and Theorem 4.1. □

4.2 Non-asymptotic Representation of $\text{epi}_K(F + I_A)^*$

In this subsection, we show that the closure of the sets in the right-hand side of the representations of $\text{epi}_K\ (F + I_A)^*$ in Theorems 4.1 and 4.2 can be removed under certain regularity conditions. The resulting expressions are then called non-asymptotic representations of $\text{epi}_K\ (F + I_A)^*$, and they will be used as key tools in establishing versions of non-asymptotic vector Farkas lemmas in the next section. We will need the following lemma on scalar functions, where we make explicit all the assumptions on the data (f, G, C).

Lemma 4.1 *Let C be a nonempty and convex subset of X, $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, and $G: X \to Z \cup \{+\infty_Z\}$ be a proper S-convex mapping. Let $D := G(C \cap \operatorname{dom} f \cap \operatorname{dom} G) + S$. Assume that $A \cap \operatorname{dom} f \neq \emptyset$ and that at least one of the following conditions is fulfilled:*

(C′1) *There exists $\widehat{x} \in C \cap \operatorname{dom} f$ such that $G(\widehat{x}) \in -\operatorname{int} S$;*
(C′2) *X, Z are Fréchet spaces, C is closed, f is lsc, G is S-epi closed and $0_Z \in \operatorname{sqri} D$;*
(C′3) $\dim \operatorname{lin} D < +\infty$ *and* $0_Z \in \operatorname{ri} D$.

Then

$$\operatorname{epi}(f + i_A)^* = \bigcup_{z^* \in S^*} \operatorname{epi}(f + i_C + z^* \circ G)^*.$$

Proof It is a direct consequence of [14, Theorem 3.4]. □

Theorem 4.3 (1st non-asymptotic representation of $\operatorname{epi}_K(F + I_A)^*$) *Let C be a nonempty and convex subset of X, $F: X \to Y \cup \{+\infty_Y\}$ be a proper K-convex mapping, and $G: X \to Z \cup \{+\infty_Z\}$ be a proper S-convex mapping. Consider the set $E := G(C \cap \operatorname{dom} F \cap \operatorname{dom} G) + S$. Assume that $A \cap \operatorname{dom} F \neq \emptyset$ and that at least one of the following conditions holds:*

(C1) *There exists $\widehat{x} \in C \cap \operatorname{dom} F$ such that $G(\widehat{x}) \in -\operatorname{int} S$;*
(C2) *X, Z are Fréchet spaces, C is closed, F is positively K-lsc, G is S-epi closed and $0_Z \in \operatorname{sqri} E$;*
(C3) $\dim \operatorname{lin} E < +\infty$ *and* $0_Z \in \operatorname{ri} E$.

Then,

$$\operatorname{epi}_K(F + I_A)^* = \bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K(F + I_C + T \circ G)^*. \tag{28}$$

Comment before the proof Observe that the set D and the function f in Lemma 4.1 coincide with the set E and the vector-valued function F here whenever $Y = \mathbb{R}$.

Proof The proof goes in parallel with the one of Theorem 4.1, applying Lemma 4.1. For completeness, the proof is sketched below.

- As in the proof of Theorem 4.1, it follows from [8, Lemma 4.1] that

$$\operatorname{epi}_K(F + I_A)^* \supset \bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K(F + I_C + T \circ G)^*.$$

So, to prove (28), it suffices to show that

$$\operatorname{epi}_K(F + I_A)^* \subset \bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K(F + I_C + T \circ G)^*. \tag{29}$$

- Take $(L, y) \in \mathrm{epi}_K(F + I_A)^*$. Then, by the same argument as the one in the proof of Theorem 4.1, using Lemma 3.4, Remark 4.1, [22, Theorem 3.4], and Lemma 3.3 consecutively, there exists $y^* \in Y^*$ such that (16) and (17) hold. Observe also that (16) is equivalent to $y^*(y) \geq (y^* \circ F + i_A)^*(y^* \circ L)$, which accounts for

$$(y^* \circ L, y^*(y)) \in \mathrm{epi}(y^* \circ F + i_A)^*. \tag{30}$$

- Because $y^* \in K^*$ and F is a K-convex mapping, $y^* \circ F$ is a convex function. If one of the qualification conditions (C1), (C2) and (C3) holds then, by Lemma 4.1, one has

$$\mathrm{epi}(y^* \circ F + i_A)^* = \bigcup_{z^* \in S^*} \mathrm{epi}(y^* \circ F + i_C + z^* \circ G)^*. \tag{31}$$

  Statements (30) and (31) ensure the existence of $z^* \in S^*$ satisfying $(y^* \circ L, y^*(y)) \in \mathrm{epi}(y^* \circ F + i_C + z^* \circ G)^*$, which means that

$$y^*(y) \geq \big(y^* \circ L\big)(x) - \big(y^* \circ F\big)(x) - \big(z^* \circ G\big)(x), \quad \forall x \in C \cap \mathrm{dom}\, F. \tag{32}$$

- Now, pick $k_0 \in \mathrm{int}K$ and consider the linear operator $T\colon Z \to Y$ such that

$$T(z) = \frac{z^*(z)}{y^*(k_0)} k_0, \quad \forall z \in Z.$$

  Then $T \in \mathcal{L}_+(S, K)$ and $y^* \circ T = z^*$. Hence, (32) can be rewritten as

$$y^*(y) \geq \big(y^* \circ L\big)(x) - \big(y^* \circ F\big)(x) - (y^* \circ T \circ G)(x), \quad \forall x \in C \cap \mathrm{dom}\, F,$$

  or equivalently,

$$y^*\left(L(x) - F(x) - (T \circ G)(x) - y\right) \leq 0, \quad \forall x \in C \cap \mathrm{dom}\, F.$$

  So, by (17),

$$L(x) - F(x) - (T \circ G)(x) - y \notin \mathrm{int}K, \quad \forall x \in C \cap \mathrm{dom}\, F,$$

  which in turn yields, by Lemma 3.4, $(L, y) \in \mathrm{epi}_K(F + I_C + T \circ G)^*$. Hence, (29) has been proved and the proof is complete. □

We now show that the inclusion

$$\mathrm{epi}_K\, (F + I_A)^* \subset \bigcup_{T \in \mathcal{L}_+^w(S,K)} \mathrm{epi}_K\, (F + I_C + T \circ G)^* \tag{33}$$

might be strict under the assumptions of Theorem 4.3.

Example 4.1 (Example 2.2 revisited) Let X, Y, Z, F, and G be as in Example 2.2. Let $C = \mathbb{R}$. Due to the extreme simplicity of $A = C \cap G^{-1}(-S) = \mathbb{R}_+$ in this case, $\mathrm{epi}_K (F + I_A)^*$ can be calculated directly. In fact, since $(F + I_A)^* (\alpha, \beta) = \mathrm{WSup}\{\mathbb{R}_+(\alpha, \beta)\}$, one gets

$$(F + I_A)^* (\alpha, \beta) = \begin{cases} \left\{+\infty_{\mathbb{R}^2}\right\}, & \text{if} \quad \alpha > 0 \text{ and } \beta > 0, \\ \left[(-\mathbb{R}_+) \times \{0\}\right] \cup \left[\{0\} \times (-\mathbb{R}_+)\right], & \text{if} \quad \alpha \leq 0 \text{ and } \beta \leq 0, \\ \mathbb{R}\,(\alpha, \beta), & \text{if} \quad \alpha\beta = 0 \text{ and } \alpha + \beta > 0, \\ \mathbb{R}_+\,(\alpha, \beta) \cup \left[(-\mathbb{R}_+) \times \{0\}\right], & \text{if} \quad \alpha > 0 \text{ and } \beta < 0, \\ \mathbb{R}_+\,(\alpha, \beta) \cup \left[\{0\} \times (-\mathbb{R}_+)\right], & \text{if} \quad \alpha < 0 \text{ and } \beta > 0. \end{cases}$$

Thus, $\mathrm{epi}_K\ (F + I_A)^* = \bigcup_{i=1}^{5} P_i$, where

$$\begin{aligned} P_1 &= \{(\alpha, \beta, y_1, y_2) : \alpha \leq 0 \wedge \beta \leq 0 \wedge (y_1 \geq 0 \vee y_2 \geq 0)\}, \\ P_2 &= \{(0, \beta, y_1, y_2) : \beta > 0 \wedge y_1 \geq 0\}, \\ P_3 &= \{(\alpha, 0, y_1, y_2) : \alpha > 0 \wedge y_2 \geq 0\}, \\ P_4 &= \left\{(\alpha, \beta, y_1, y_2) : \alpha > 0 \wedge \beta < 0 \wedge y_2 \geq \min\left\{0, \tfrac{\beta}{\alpha} y_1\right\}\right\}, \\ P_5 &= \left\{(\alpha, \beta, y_1, y_2) : \alpha < 0 \wedge \beta > 0 \wedge y_1 \geq \min\left\{0, \tfrac{\alpha}{\beta} y_2\right\}\right\}. \end{aligned}$$

Since $\mathrm{dom}(F + I_A)^* = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \leq 0 \vee \beta \leq 0\right\}$ is not convex, $\mathrm{epi}_K (F + I_A)^*$ cannot be convex while its closedness follows from Lemma 3.5 applied to the proper vector function $F + I_A = I_A$.
According to Theorem 4.3, as the Slater condition $(C1)$ is satisfied by any positive number, we can also express

$$\mathrm{epi}_K (F + I_A)^* = \bigcup_{(t_1, t_2) \in \mathbb{R}^2_+} \mathrm{epi}_K ((t_1, t_2) \circ G)^*,$$

where

$$((t_1, t_2) \circ G)^* (\alpha, \beta) = \mathrm{WSup}\{\mathbb{R}\,(\alpha + t_1, \beta + t_2)\} = F^* (\alpha + t_1, \beta + t_2).$$

So, $\mathrm{epi}_K\ ((t_1, t_2) \circ G)^* = \bigcup_{i=1}^{4} Q_i\,(t_1, t_2)$, with

$$\begin{aligned} Q_1\,(t_1, t_2) &= \{(-t_1, -t_2, y_1, y_2) : y_1 \geq 0 \vee y_2 \geq 0\}, \\ Q_2\,(t_1, t_2) &= \left\{(\alpha, \beta, y_1, y_2) : (\alpha + t_1)\,(\beta + t_2) < 0 \wedge y_2 \geq \left(\tfrac{\beta + t_2}{\alpha + t_1}\right) y_1\right\}, \\ Q_3\,(t_1, t_2) &= \{(\alpha, -t_2, y_1, y_2) : \alpha \neq -t_1 \wedge y_2 \geq 0\}, \\ Q_4\,(t_1, t_2) &= \{(-t_1, \beta, y_1, y_2) : \beta \neq -t_2 \wedge y_1 \geq 0\}. \end{aligned}$$

From Theorem 4.3 and the inclusion $\mathcal{L}_+(S, K) \subset \mathcal{L}_+^w(S, K)$, one has

$$\operatorname{epi}_K (F + I_A)^* \subset \bigcup_{T \in \mathcal{L}_+^w(S,K)} \operatorname{epi}_K (F + I_C + T \circ G)^*.$$

Next we show that this inclusion might be strict under the assumptions of Theorem 4.3. Indeed,

$$\begin{aligned}(1, 0, 0, -1) &\in Q_1(-1, 0) \setminus \left(\bigcup_{i=1}^{5} P_i, \right) \\ &\subset \left[\bigcup_{T \in \mathcal{L}_+^w(S,K)} \operatorname{epi}_K\ (F + I_C + T \circ G)^* \right] \setminus \operatorname{epi}_K\ (F + I_A)^*.\end{aligned}$$

Example 4.2 (Example 2.1 revisited) Let $X,\ Y,\ Z,\ S,\ K$ and F be as in Example 2.1 (i.e., $X = Z = \mathbb{R}$, $Y = C[0, 1]$, $S = \mathbb{R}_+$, $K := \{x \in C[0, 1] : x(t) \geq 0, \forall t \in [0, 1]\}$ and $F(\alpha)(t) = t^2 + \alpha^2$ for all $t, \alpha \in \mathbb{R}$). Let $C = \mathbb{R}$ and take $G\colon \mathbb{R} \to \mathbb{R}$ defined by $G(\alpha) = -\alpha$ for all $\alpha \in \mathbb{R}$. Then $A = C \cap G^{-1}(-S) = \mathbb{R}_+$. Recall that from Example 2.1, we have $\mathcal{L}(X, Y) = \mathcal{L}(Z, Y) \equiv C[0, 1]$, $\mathcal{L}_+(S, K) \equiv K$ and

$$\mathcal{L}_+^w(S, K) = \{z \in C[0, 1] : z(t) \geq 0 \text{ for some } t \in [0, 1]\}.$$

Then for any $x \in C[0, 1]$, $(F + I_A)^*(x) = \operatorname{WSup}\{\alpha x - F(\alpha) : \alpha \in \mathbb{R}_+\}$, and it follows from (8) that, for all $x \in \mathcal{L}(X, Y) = C[0, 1]$,

$$\begin{aligned}&(F + I_A)^*(x) + K \\ &= \{y \in C[0, 1] : \forall \alpha \in \mathbb{R}_+, \exists t_\alpha \in [0, 1] \text{ s.t. } y(t_\alpha) \geq \alpha x(t_\alpha) - (t_\alpha)^2 - \alpha^2\},\end{aligned}$$

and hence,

$$\begin{aligned}\operatorname{epi}_K(F + I_A)^* = \Big\{&(x, y) \in (C[0, 1])^2 : \forall \alpha \in \mathbb{R}_+, \exists t_\alpha \in [0, 1] \text{ s.t.} \\ &y(t_\alpha) \geq \alpha x(t_\alpha) - (t_\alpha)^2 - \alpha^2\Big\}.\end{aligned} \tag{34}$$

By Theorem 4.3, as in our setting the Slater condition $(C1)$ is satisfied (by any positive number), we can also express

$$\operatorname{epi}_K\ (F + I_A)^* = \bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K (F + I_C + T \circ G)^* = \bigcup_{z \in K} \operatorname{epi}_K (F + z \circ G)^*. \tag{35}$$

On the other hand, for all $x \in \mathcal{L}(X, Y) = C[0, 1]$ and $z \in \mathcal{L}_+^w(S, K)$,

$$\begin{aligned}(F + z \circ G)^*(x) &= \mathrm{WSup}\{\alpha x - F(\alpha) - z \circ G(\alpha) : \alpha \in \mathbb{R}\}\\ &= \mathrm{WSup}\{\alpha(x+z) - F(\alpha) : \alpha \in \mathbb{R}\}\\ &= F^*(x+z),\end{aligned}$$

and hence,

$$\mathrm{epi}_K(F+z\circ G)^* = \left\{(x,y) \in (C[0,1])^2 : \begin{array}{l}\forall\alpha \in \mathbb{R}, \exists t_\alpha \in [0,1] \text{ s.t.}\\ y(t_\alpha) \geq \alpha x(t_\alpha) + \alpha z(t_\alpha) - (t_\alpha)^2 - \alpha^2\end{array}\right\}.$$

From (35) and the inclusion $K = \mathcal{L}_+(S, K) \subset \mathcal{L}_+^w(S, K)$, one has

$$\mathrm{epi}_K\ (F + I_A)^* \subset \bigcup_{z \in \mathcal{L}_+^w(S,K)} \mathrm{epi}_K\ (F + z \circ G)^*.$$

We now show that the inclusion (33) is strict even though one of the assumptions of Theorem 4.3 holds. Indeed, take the functions $x_0, y_0, z_0 \colon [0,1] \to \mathbb{R}$ defined by $x_0(t) = 2t^2 - \frac{1}{2}$, $y_0(t) = t^2 - 2$, and $z_0(t) = \frac{1}{2} - 2t^2$. For any $\alpha \in \mathbb{R}$, we can take $t_\alpha = 1$ to get $y_0(t_\alpha) = -1 \geq -1 - \alpha^2 = \alpha x_0(t_\alpha) + \alpha z_0(t_\alpha) - (t_\alpha)^2 - \alpha^2$, which shows that $(x_0, y_0) \in \mathrm{epi}_K(F + z_0 \circ G)^*$. On the other hand, for $\alpha = 1$, one has

$$y_0(t) = t^2 - 2 < t^2 - \frac{3}{2} = \alpha(x_0(t)) - t^2 - \alpha^2,\ \ \forall t \in [0,1],$$

and so, $(x_0, y_0) \notin \mathrm{epi}_K(F + I_A)^*$ by (34).

Theorem 4.4 (2nd non-asymptotic representation of $\mathrm{epi}_K(F + I_A)^*$) *Assume that all the assumptions of Theorem 4.3 hold. Then one has*

$$\mathrm{epi}_K(F + I_A)^* = \bigcup_{T \in \mathcal{L}_+^w(S,K)} \left[\bigcap_{v \in I^*_{-S}(T)} [\mathrm{epi}_K(F + I_C + T \circ G)^* + (0_\mathcal{L}, v)]\right]. \tag{36}$$

Proof It follows from (27) and Theorem 4.3. □

It is worth noting that the conditions (28) and (36) can be seen as qualification conditions. The first one was introduced in [8] while the other (which is weaker) is new.

5 Farkas-Type Results for Vector-Valued Functions

Let the spaces X, Y, Z, and the cones K, S be as in Sect. 4. Regarding the data (C, F, G), we still assume that $F \colon X \to Y \cup \{+\infty_Y\}$ and $G \colon X \to Z^\bullet$ are proper mappings. As in Sect. 4, we assume that $A \cap \mathrm{dom} F \neq \emptyset$, where $A := C \cap G^{-1}(-S)$.

This section provides stable reverse Farkas-type results for systems involving vector-valued functions, as well as some Farkas lemma principles in asymptotic form (without any regularity condition) and in non-asymptotic form (under various regularity conditions). The results are established based on the key tools: the representations of epigraphs of conjugate mappings in Sect. 4. Concretely, on the one hand, the asymptotic Farkas-type results, Theorem 5.1 and Theorem 5.2, are new and significantly different from the main asymptotic versions of Farkas-type results (Theorem 3.1, Theorem 3.2) in [8]. The conditions (ii) or (iii) in these Theorems 5.1, 5.2 are expressed explicitly in terms of F , the constraint function G and the constraint set C instead of the relation of the form $(L, y) \in \operatorname{epi}_K(F+I_A)^*$ as in [9]. This justifies the specification " non-abstract approach" in the title of the present paper. On the other hand, non-asymptotic versions of vector Farkas lemmas presented in this section consist of Proposition 5.1 and Corollaries 5.2, 5.3, 5.4. The first conclusion of Proposition 5.1, i.e., $(a) \Longleftrightarrow (c)$, extends both Theorem 4.1 and Theorem 4.2 in [8] while the second conclusion $(b) \Longleftrightarrow (d)$ is new. Each of the Corollaries 5.2, 5.3, 5.4 is a specific version of vector Farkas lemmas for convex setting under various regularity conditions. These results are new and their proofs are based on Proposition 5.1 and the nice representations of $\operatorname{epi}_K(F + I_A)^*$ proved in Sect. 4.

In the case where $Y = \mathbb{R}$, the results extend or recover many known Farkas-type results in the literature which are stable in the sense that they are preserved by arbitrary linear perturbations of the function defining the set B (see e.g., [1–3,8,14,15,23], and the references therein).

Theorem 5.1 (1st asymptotic vector Farkas lemma) *Let C be a closed and convex subset of X, F be a K -convex and positively K-lsc mapping, and G be an S-convex and S-epi closed mapping. Then for any $y \in Y$ and any $L \in \mathcal{L}(X, Y)$, the following assertions are equivalent:*
(i) $G(x) \in -S,\ x \in C \implies F(x) - L(x) + y \notin -\operatorname{int} K$;
(ii) $\exists\{(L_i, y_i)\}_{i\in I} \subset \mathcal{L}(X, Y) \times Y$, $\exists\{T_i\}_{i\in I} \subset \mathcal{L}_+(S, K)$ *such that* $(L_i, y_i) \longrightarrow (L, y)$ *and*

$$F(x) + (T_i \circ G)(x) - L_i(x) + y_i \notin -\operatorname{int} K,\ \forall x \in C,\ \forall i \in I.$$

Proof Take $(L, y) \in \mathcal{L}(X, Y) \times Y$. Observing that $A = C \cap G^{-1}(-S)$ and applying Lemma 3.4 to $F + I_A$, we have

$$\text{(i)} \Longleftrightarrow (L, y) \in \operatorname{epi}_K(F + I_A)^*. \tag{37}$$

It now follows from (37) and Theorem 4.1 that

$$\begin{aligned}\text{(i)} &\Longleftrightarrow (L, y) \in \operatorname{cl}\left[\bigcup_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(F + I_C + T \circ G)^*\right]\\ &\Longleftrightarrow \exists\{(L_i, y_i)\}_{i\in I} \subset \bigcup_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(F + I_C + T \circ G)^* \text{ s.t. } (L_i, y_i) \to (L, y)\end{aligned}$$

$$\iff \exists\{(L_i, y_i)\}_{i\in I} \subset \mathcal{L}(X,Y)\times Y,\ \exists\{T_i\}_{i\in I} \subset \mathcal{L}_+(S,K)$$
$$\text{s.t. } (L_i, y_i) \to (L, y) \text{ and } (L_i, y_i) \in \operatorname{epi}_K(F + I_C + T_i \circ G)^* \ \forall i \in I,$$

which is (ii) and the proof is complete. □

Theorem 5.2 (2nd asymptotic vector Farkas lemma) *Assume that all the assumptions of Theorem 5.1 hold. Then for any $y \in Y$ and any $L \in \mathcal{L}(X,Y)$, the following assertions are equivalent:*
(i) $G(x) \in -S,\ x \in C \implies F(x) - L(x) + y \notin -\operatorname{int}K$;
(iii) $\exists\{(L_i, y_i)\}_{i\in I} \subset \mathcal{L}(X,Y)\times Y, \exists\{T_i\}_{i\in I} \subset \mathcal{L}_+^w(S,K)$ *such that* $(L_i, y_i) \to (L, y)$ *and*

$$F(x) + (T_i \circ G)(x) - L_i(x) + y_i \notin T_i(-S) - \operatorname{int}K,\ \forall x \in C,\ \forall i \in I.$$

Proof The conclusion follows by the same argument as in the proof of Theorem 5.1, using (25) and Theorem 4.2 (instead of Theorem 4.1). □

In the special case when $Y = \mathbb{R}$, both Theorems 5.1 and 5.2 collapse to the following scalar stable asymptotic Farkas lemma which extends [24, Corollary 3.4] (see also [2]).

Corollary 5.1 (Scalar asymptotic convex Farkas lemma) *Let C be a closed and convex subset of X, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function, and G be an S-convex and S-epi closed mapping. Assume that $A \cap \operatorname{dom} f \neq \emptyset$. Then, for any $(x^*, \alpha) \in X^* \times \mathbb{R}$, the following statements are equivalent:*
(iv) $G(x) \in -S,\ x \in C \implies f(x) - \langle x^*, x\rangle \geq \alpha$,
(v) *There exists a net* $\{z_i^*\}_{i\in I} \subset S^*$ *such that*

$$f(x) - \langle x^*, x\rangle + \liminf_i \left(z_i^* \circ G\right)(x) \geq \alpha,\ \ \forall x \in C.$$

Proof In the case where $Y = \mathbb{R}$, Theorem 5.1 (applied to f and $y = -\alpha \in \mathbb{R}$) asserts that (iv) is equivalent to

$$\exists\{(x_i^*, \alpha_i)\}_{i\in I} \subset X^* \times \mathbb{R},\ \exists\{z_i^*\}_{i\in I} \subset S^* \text{ such that } (x_i^*, \alpha_i) \longrightarrow (x^*, \alpha),$$

and

$$f(x) + (z_i^* \circ G)(x) - \langle x_i^*, x\rangle \geq \alpha_i,\ \forall x \in C,\ \forall i \in I. \tag{38}$$

Thus, (v) follows by taking the liminf of the inequality (38), and considering that $(x_i^*, \alpha_i) \longrightarrow (x^*, \alpha)$. The implication [(v) $\Rightarrow$ (iv)] is obvious and the proof is complete. □

We are now in a position to obtain different versions of stable non-asymptotic vector Farkas lemmas (Corollaries 5.2, 5.4 , 5.3, and 5.5) based on the principles gathered in the following proposition, whose first statement, [(a) $\iff$ (c)] generalizes [8, Theorem 4.1].

Proposition 5.1 (1st non-asymptotic vector Farkas lemma principles) *Let $\mathcal{V} \subset \mathcal{L}(X, Y)$ and $\mathcal{W} \subset Y$. Consider the following statements:*

(a) $\operatorname{epi}_K (F + I_A)^* \cap (\mathcal{V} \times \mathcal{W}) = \bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K (F + I_C + T \circ G)^* \cap (\mathcal{V} \times \mathcal{W})$,

(b) $\operatorname{epi}_K (F + I_A)^* \cap (\mathcal{V} \times \mathcal{W})$

$$= \left\{ \bigcup_{T \in \mathcal{L}_+^w(S,K)} \left[\bigcap_{v \in I^*_{-S}(T)} [\operatorname{epi}_K (F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)] \right] \right\} \bigcap (\mathcal{V} \times \mathcal{W}),$$

(c) *For any $y \in \mathcal{W}$ and any $L \in \mathcal{V}$, the following assertions are equivalent:*

(c_1) $G(x) \in -S,\ x \in C \implies F(x) - L(x) + y \notin -\operatorname{int} K$,
(c_2) $\exists T \in \mathcal{L}_+(S, K) : F(x) + (T \circ G)(x) - L(x) + y \notin -\operatorname{int} K,\ \forall x \in C$,

(d) *For any $y \in \mathcal{W}$ and any $L \in \mathcal{V}$, the following assertions are equivalent:*

(c_1) $G(x) \in -S,\ x \in C \implies F(x) - L(x) + y \notin -\operatorname{int} K$,
(d_1) $\exists T \in \mathcal{L}_+^w(S, K)$: $F(x) + (T \circ G)(x) - L(x) + y \notin T(-S) - \operatorname{int} K,\ \forall x \in C$.

Then (a) $\iff$ (c) *and* (b) $\iff$ (d).

Proof [(b) $\iff$ (d)] For all $(L, y) \in \mathcal{V} \times \mathcal{W}$, observe that

$$\begin{aligned} &(L, y) \in \bigcup_{T \in \mathcal{L}_+^w(S,K)} \left[\bigcap_{v \in I^*_{-S}(T)} [\operatorname{epi}_K (F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)] \right] \\ &\iff \exists T \in \mathcal{L}_+^w(S, K) : (L, y) \in \bigcap_{v \in I^*_{-S}(T)} [\operatorname{epi}_K (F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v)] \\ &\iff (d_1) \qquad (\text{see } (25)). \end{aligned} \tag{39}$$

The conclusion follows from (37) and (39). □

We now combine Proposition 5.1 with the representation theorems of Sect. 4 to derive some useful Farkas lemmas and Farkas lemma principles for vector functions.

Corollary 5.2 (2nd non-asymptotic vector Farkas lemma principle) *Assume that all the assumptions of Theorem 5.1 hold. Then, the following statements are equivalent:*

(e) $\bigcup_{T \in \mathcal{L}_+(S,K)} \operatorname{epi}_K (F + I_C + T \circ G)^*$ *is closed,*

(c′) $\forall (L, y) \in \mathcal{L}(X, Y) \times Y,\ (c_1) \iff (c_2)$.

Proof It follows from Theorems 4.1 and 5.1 with $\mathcal{V} = \mathcal{L}(X, Y)$ and $\mathcal{W} = Y$. □

Corollary 5.3 (3rd non-asymptotic vector Farkas lemma principle) *Assume that all the assumptions of Theorem 5.1 hold. Then the following statements are equivalent:*

(f) $\bigcup_{T\in\mathcal{L}_+^w(S,K)} \left[\bigcap_{v\in I^*_{-S}(T)} \operatorname{epi}_K(F+I_C+T\circ G)^* + (0_{\mathcal{L}}, v) \right]$ *is closed,*
(d′) $\forall (L, y) \in \mathcal{L}(X, Y) \times Y$, $(\mathrm{c}_1) \Longleftrightarrow (\mathrm{d}_1)$.

Proof It follows from Theorems 4.1 and 5.1 with $\mathcal{V} = \mathcal{L}(X, Y)$ and $\mathcal{W} = Y$. □

Corollary 5.4 (Non-asymptotic vector Farkas lemma) *Let C be a convex subset of X, F be a K-convex mapping, and G be a proper S-convex mapping. Assume that at least one of the conditions* (C1), (C2) *and* (C3) *in Theorem 4.3 holds. Then the two statements* (c) *and* (d) *in Proposition 5.1 hold true for any $\mathcal{V} \subset \mathcal{L}(X, Y)$ and $\mathcal{W} \subset Y$.*

Proof It follows from Theorems 4.2 and 5.1. □

We have seen that in the special case where $Y = \mathbb{R}$, the asymptotic vector Farkas lemmas (i.e., Theorem 5.1 and Theorem 5.2) collapse to a scalar stable asymptotic Farkas lemma (Corollary 5.1), which extends some known results in the literature. It should be worth mentioning here that the Farkas-type results of the forms $[(c_1) \Longleftrightarrow (c_2)]$ or $[(c_1) \Longleftrightarrow (d_1)]$ in Corollaries 5.2-5.4 when they are specified to the case $Y = \mathbb{R}$, give scalar stable non-asymptotic Farkas lemmas which extend (or at least cover) many Farkas-type results and their stable forms in the literature, such as [1–3,14], and many others. As an illustration, we finally establish a scalar stable non-asymptotic Farkas-type result which is a direct consequence of Corollary 5.2.

Corollary 5.5 (Scalar convex Farkas lemma principle) [15, Theorem 3.1] *Assume that all the assumptions of Corollary 5.1 hold. Assume that $A \cap \operatorname{dom} f \neq \emptyset$. Then, for any $(x^*, \alpha) \in X^* \times \mathbb{R}$, the following statements are equivalent:*
(g) $\bigcup_{z^*\in S^*} \operatorname{epi}(f + i_C + z^* \circ G)^*$ *is a closed subset of* $X^* \times \mathbb{R}$,
(h) *For all* $(x^*, \alpha) \in X^* \times \mathbb{R}$, *the following statements are equivalent:*

$$
\begin{aligned}
&(h_1)\ G(x) \in -S,\ x \in C \implies f(x) - \langle x^*, x\rangle \geq \alpha,\\
&(h_2)\ \exists z^* \in S^* :\quad f(x) - \langle x^*, x\rangle + (z^* \circ G)(x) \geq \alpha,\ \forall x \in C.
\end{aligned}
$$

6 Conclusions

Let the spaces X, Y, Z, the cones K, S and the mappings $F : X \to Y \cup \{+\infty_Y\}$ and $G : X \to Z^\bullet$ be as in Sect. 4. This paper provides a variety of characterizations of the inclusion $A := C \cap G^{-1}(-S) \subset B$, where B depends on F, which are expressed in terms of the data (C, F, G).

- The key tools to obtain the characterizations of $A \subset B$ are the representations of $\operatorname{epi}_K(F + I_A)^*$ in Sect. 4, which can be either asymptotic (Theorems 4.1 and 4.2) or non-asymptotic (Theorems 4.3 and 4.4).
- Two different representations of $\operatorname{epi}_K(F + I_A)^*$ are given, a first one based on the known concept of cone of positive operators and a second one based on the new one of cone of weakly positive operators.

- These representations have been used to establish, in Sect. 5, asymptotic stable vector Farkas lemmas (Theorems 5.1 and 5.2, and Corollary 5.1) and non-asymptotic stable vector Farkas principles (Proposition 5.1 and Corollaries from 5.2 to 5.5).
- All Farkas-type results in Sect. 5 are *stable*, in the sense that they characterize the inclusion of A in perturbations of the container set B produced by continuous affine perturbations of the form $L - y$, with $y \in Y$ and $L \in \mathcal{L}(X, Y)$, of the vector function F defining B. By setting $L = 0_{\mathcal{L}}$ and $y = 0_Y$, one gets the corresponding characterization of $A \subset B$.
- Similar to [8], the non-abstract vector Farkas lemmas obtained in this paper can be used to establish optimality conditions and duality theorems (including stable ones) for vector optimization problems, and this will be published somewhere else.
- All the results in this paper are established under the condition that the interior of the cone K is nonempty. In the case when $\mathrm{int} K = \emptyset$ but its quasi(-relative) interior is nonempty, we can define weakly minimal/maximal elements by means of quasi(- relative) interior (see, e.g., [25,26]), or by means of any other set E satisfying $E + K = E$ (called *free disposal set* **,** by Debreu [27]) and it is possible that some of the main results in this paper can be extended to this case, and this will be done in another work.

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