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A Boundedness Result for Minimizers of Some Polyconvex Integrals

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Abstract We consider polyconvex functionals of the Calculus of Variations defined on maps from the three-dimensional Euclidean space into itself. Counterexamples show that minimizers need not to be bounded. We find conditions on the structure of the functional, which force minimizers to be locally bounded.

Keywords Local · Bounded · Minimizer · Polyconvex · Integral

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1 Introduction

Let us consider polyconvex integrals of the Calculus of Variations. Partial regularity results (that is, the regularity of minimizers up to a subset of the set of definition and the study of the properties of the singular set; see for example Section 4.2 in [1] and Section 1 in [2]) are contained in [3–10]. Only few everywhere regularity results are available: [11] where the everywhere continuity is proved in the two-dimensional case, [12] where Hölder continuity for extremals is dealt with in dimension two, [13] where local boundedness is proved in the three-dimensional case. Global pointwise bounds are in [14–19]. Interesting results are contained in [20–25]; see also [26,27]. Let us come back to [13]; in such a paper, the authors make an important step toward regularity: they prove boundedness of minimizers in the three-dimensional case; unfortunately, they make restrictions that rule out the most important polyconvex integral. In the present paper, we find a different set of assumptions, which allows us to deal with such a polyconvex integral. In the next section, we write assumptions and results; in Sect. 3 we collect some preliminaries and, in Sect. 4, we give the proof of the main theorem.

2 Assumptions and Results

In this paper we study the regularity of vectorial local minimizers of integral functionals

$$I(v,\Omega) = \int_\Omega f(x,\ Dv(x))\mathrm{d}x, \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is an open, bounded set, $v : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, $v = (v^1, v^2, v^3)$ and Dv is the Jacobian matrix of its partial derivatives

$$Dv = \left(v^\alpha_{x_i}\right)^{\alpha=1,2,3}_{i=1,2,3} = \begin{pmatrix} Dv^1 \\ Dv^2 \\ Dv^3 \end{pmatrix} = \begin{pmatrix} v^1_{x_1} & v^1_{x_2} & v^1_{x_3} \\ v^2_{x_1} & v^2_{x_2} & v^2_{x_3} \\ v^3_{x_1} & v^3_{x_2} & v^3_{x_3} \end{pmatrix},$$

moreover, $f : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty[$ is a Carathéodory function such that for fixed x

$$\xi \to f(x,\xi) \text{ is polyconvex}$$

that is

$$f(x,\xi) = g(x,\xi,\mathrm{adj}_2\xi,\det\xi) \quad \text{with} \quad (\xi,\lambda,t) \to g(x,\xi,\lambda,t) \quad \text{convex}, \tag{2}$$

see [28,29]. When dealing with models in nonlinear elasticity, f is the stored-energy function; moreover, ξ, $\mathrm{adj}_2\xi$, $\det\xi$ govern the deformation of line, surface and volume elements respectively. Our model is

$$f(x,Dv) = |Dv|^p + |\mathrm{adj}_2 Dv|^q + |\det Dv|^r, \tag{3}$$

where $\det Dv$ is the determinant of the matrix Dv, and $\mathrm{adj}_2 Dv$ denotes the adjugate matrix of order 2, whose components are

$$(\mathrm{adj}_2 Dv)_{ij} = (-1)^{i+j} \det \begin{pmatrix} v^\alpha_{x_k}, & v^\alpha_{x_\ell} \\ v^\beta_{x_k}, & v^\beta_{x_\ell} \end{pmatrix}, \quad i, j \in \{1, 2, 3\},$$

with $\alpha, \beta \in \{1, 2, 3\} \setminus \{i\}$, $\alpha < \beta$, and $k, \ell \in \{1, 2, 3\} \setminus \{j\}$, $k < \ell$. Moreover, $(\mathrm{adj}_2 Dv)^\alpha$ denotes the $\alpha-$row of $\mathrm{adj}_2 Dv$, that is

$$(\mathrm{adj}_2 Dv)^\alpha = \big((\mathrm{adj}_2 Dv)_{\alpha 1}, (\mathrm{adj}_2 Dv)_{\alpha 2}, (\mathrm{adj}_2 Dv)_{\alpha 3}\big).$$

In paper [13], the authors consider densities f for which the following splitting holds true

$$f(x, Dv) = \sum_{\alpha=1}^{3} F^\alpha(x, Dv^\alpha) + \sum_{\beta=1}^{3} G^\beta(x, (\mathrm{adj}_2 Dv)^\beta) + H(x, \det Dv) \tag{4}$$

for suitable nonnegative functions F^α, G^β, H. Note that model (3), with $p \neq 2$, cannot be written as (4); see Lemma A.1 in “Appendix A”. In this paper, we succeed in dealing with model (3) and we prove the following

Theorem 2.1 *Let Ω be a bounded and open subset of $\mathbb{R}^3$. Assume that $1 \leq r < q < p \leq 3$ with $2 < p$ and*

$$\begin{aligned} \frac{p}{p^*} &< \min\left\{1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)}\right\}, \qquad \text{if} \quad 1 < q \leq 2, \\ \frac{p}{p^*} &< \min\left\{1 - \frac{2p^*}{p(p^*-2)} - \frac{(q-2)p^*}{q(p^*-2)}, 1 - \frac{rp^*}{q(p^*-r)}\right\}, \qquad \text{if} \quad 2 < q; \end{aligned} \tag{5}$$

then all the local minimizers $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^3)$ of

$$\int_\Omega (|Du|^p + |\mathrm{adj}_2 Du|^q + |\det Du|^r) \tag{6}$$

are locally bounded in Ω.

We recall that p^* is the Sobolev exponent: $p^* = \frac{np}{n-p} = \frac{3p}{3-p}$ when $p < n = 3$; moreover, p^* is any number greater than p when $p = n = 3$, so it can be chosen large enough that (5) is satisfied by assuming only $1 \leq r < q < p$. We notice that we have restricted ourselves to the case $p \leq 3$ because, when $p > 3$, every function in $W^{1,p}_{loc}(\Omega)$ is trivially in $L^\infty_{loc}(\Omega)$ by the Sobolev theorem. Note that we have existence of minimizers for (6) when $2 \leq p$, $\frac{p}{p-1} \leq q$ and $1 < r$, provided a boundary datum $\overline{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$, with finite energy, has been fixed; see Remark 8.32 (iii) in [29] and Theorem 3.1 in [13]. Condition (5) is satisfied, for example, when $p = \frac{14}{5}$, $q = 2$, $r = \frac{3}{2}$ and this gives us the following.

Corollary 2.1 *Let Ω be a bounded and open subset of $\mathbb{R}^3$ and let $u \in W^{1,\frac{14}{5}}_{loc}(\Omega;\mathbb{R}^3)$ be a local minimizer of*

$$\int_\Omega (|Du|^{\frac{14}{5}} + |\text{adj}_2 Du|^2 + |\det Du|^{\frac{3}{2}}); \tag{7}$$

then u is locally bounded in Ω.

In the framework of Corollary 2.1, we have $\frac{p}{p-1} = \frac{14}{9} < 2 = q$, so the existence of minimizers is guaranteed as in the previous lines. Theorem 2.1 is a particular case of a more general result. Let us note that model (3) suggests we assume the following structure

$$f(x,\xi) = F(x,|\xi|^2) + G(x,|\text{adj}_2\xi|^2) + H(x,\det\xi), \tag{8}$$

where F, G and H are Carathéodory nonnegative functions. We assume p-growth with respect to ξ, q-growth with respect to $\text{adj}_2\xi$ and r-growth with respect to $\det\xi$

$$k_1 t^{p/2} - k_2 \le F(x,t) \le k_3 t^{p/2} + a(x) \tag{9}$$

$$k_1 t^{q/2} - k_2 \le G(x,t) \le k_3 t^{q/2} + b(x) \tag{10}$$

$$0 \le H(x,s) \le k_3|s|^r + c(x), \tag{11}$$

where k_1, k_2, k_3 are constants such that $k_1, k_3 \in]0,+\infty[$ and $k_2 \in [0,+\infty[$ and $a, b, c : \Omega \to [0,+\infty[$ are functions in $L^\sigma(\Omega)$, $\sigma > 1$; as far as exponents p, q, r are concerned, we assume that $2 < p \le 3$ and $1 \le r < q < p$. Now we need to control the behavior of F with respect to the sum from below

$$F(x,t_1) + F(x,t_2) - k_2 \le F(x,t_1+t_2). \tag{12}$$

A weaker condition is needed for G:

$$G(x,t_1) - k_2 \le G(x,t_1+t_2). \tag{13}$$

We also need to control the behavior of F with respect to the sum from above:

$$F(x,t_1+t_2) \le F(x,t_1) + F(x,t_2) + k_3 t_1 t_2^{\frac{p}{2}-1} + a(x). \tag{14}$$

Note that in (14) there is an extra term with the product between t_1 and t_2. When $q > 2$ we assume

$$G(x,t_1+t_2) \le G(x,t_1) + G(x,t_2) + k_3 t_1 t_2^{\frac{q}{2}-1} + b(x). \tag{15}$$

When $q \le 2$ we do not need the product between t_1 and t_2 any longer; we require subadditivity

$$G(x,t_1+t_2) \le G(x,t_1) + G(x,t_2) + b(x). \tag{16}$$

Functions F verifying the previous assumptions are $F(x,t) = \gamma(x)t^{p/2}$ and $F(x,t) = \gamma(x)(1+t^2)^{p/4}$, provided $\gamma(x)$ is positive and away from both 0 and $+\infty$; similar examples for G and H: see Remarks 3.2, ..., 3.7. Our main result is the following

Theorem 2.2 *Let Ω be a bounded and open subset of $\mathbb{R}^3$ and let f be as in (8); assume that conditions (9)–(16) hold with $1 \le r < q < p \le 3$ such that $2 < p$ and*

$$\frac{p}{p^*} < \min\left\{1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)}, 1 - \frac{1}{\sigma}\right\}, \quad \text{if } 1 < q \le 2,$$
$$\frac{p}{p^*} < \min\left\{1 - \frac{2p^*}{p(p^*-2)} - \frac{(q-2)p^*}{q(p^*-2)}, 1 - \frac{rp^*}{q(p^*-r)}, 1 - \frac{1}{\sigma}\right\}, \quad \text{if } 2 < q. \tag{17}$$

Then, all the local minimizers $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^3)$ of I are locally bounded in Ω.

Note that $\frac{1}{\sigma} = 0$, if $\sigma = \infty$. In our Theorem 2.2, we assume (8); in [13] (4) was in force: in vectorial problems, some structure conditions are due to minimizers which can be unbounded: see De Giorgi's counterexample [30]; see also [31], Section 3 in [1] and [32]. As far as exponents p, q, r are concerned, (17) is the same as (2.5) in [13] when $1 < q \le 2$; if $2 < q$ then (17) seems to require a bit more than (2.5) in [13]: see comparison (72).

The integrals we consider show a $\tilde{p}$ growth from below and a $\tilde{q}$ growth from above, so we are in the class of functionals with $\tilde{p}, \tilde{q}$-growth. It is now well known, as in our result, that a restriction between $\tilde{p}$ and $\tilde{q}$ must be imposed due to counterexamples in [33–37]; see also [38,39]; we refer to [1] for a detailed survey on the subject.

3 Preliminaries

In this section, we recall some standard definitions and collect several lemmas useful in our proofs.

First of all, we recall the following

Definition 3.1 A function $u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^3)$ is a local minimizer of (1) if $f(Du) \in L^1_{loc}(\Omega)$ and

$$I(u, \operatorname{supp}\varphi) \le I(u+\varphi, \operatorname{supp}\varphi), \tag{18}$$

for all $\varphi \in W^{1,1}(\Omega, \mathbb{R}^3)$ with $\operatorname{supp}\varphi \subset\subset \Omega$.

All the norms we use on $\mathbb{R}^3$ and $\mathbb{R}^{3\times 3}$ will be the standard Euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices $\xi, \eta \in \mathbb{R}^{3\times 3}$ we write $\langle \xi, \eta \rangle := \operatorname{trace}(\xi^T \eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

Lemma 3.1 *For $a, b \ge 0$ we have that*

$$a^m + b^m \le (a+b)^m, \quad \text{if} \quad m \ge 1, \tag{19}$$
$$(a+b)^m \le a^m + b^m + mab^{m-1}, \quad \text{if} \quad 1 \le m \le 2. \tag{20}$$

Proof When $m = 1$, (19) and (20) are easy. We are left with the case $1 < m$. It is obvious that (19) and (20) hold true both for $b = 0$. We now assume $b > 0$ and we let $t = a/b$. It suffices to show that for $t \geq 0$,

$$t^m + 1 \leq (t+1)^m, \quad \text{if} \quad m > 1, \tag{21}$$

$$(t+1)^m \leq t^m + 1 + mt, \quad \text{if} \quad 1 < m \leq 2. \tag{22}$$

In order to prove (21), we let $h(t) = (t+1)^m - t^m - 1$. Since

$$h(0) = 0 \tag{23}$$

and, by $m > 1$,

$$h'(t) = m[(t+1)^{m-1} - t^{m-1}] \geq 0, \tag{24}$$

then $h(t) \geq 0$ and (21) follows.

Regarding (22), we let $g(t) = (t+1)^m - t^m - mt - 1$. Since

$$g(0) = 0, \tag{25}$$

$$g'(t) = m[(t+1)^{m-1} - t^{m-1} - 1] \leq m[t^{m-1} + 1 - t^{m-1} - 1] = 0, \tag{26}$$

where we used $1 < m \leq 2$ and Remark 3.1, then (22) follows. □

Remark 3.1 We recall the well-known inequality: for $a, b \geq 0$ we have

$$(a+b)^m \leq a^m + b^m, \qquad \text{if} \quad 0 < m \leq 1. \tag{27}$$

Lemma 3.2 *Fix* $m \in [-\frac{1}{2}, +\infty[$ *and consider* $V : \mathbb{R} \to \mathbb{R}$ *as follows*

$$V(s) = \left(1 + s^2\right)^m s; \tag{28}$$

then, $V : \mathbb{R} \to \mathbb{R}$ *is strictly increasing.*

Proof We compute the first derivative

$$V'(s) = \left(1 + s^2\right)^{m-1} [(2m+1)s^2 + 1]; \tag{29}$$

since $m \geq -\frac{1}{2}$, we have $V'(s) > 0$ for every $s \in \mathbb{R}$. This ends the proof. □

Lemma 3.3 *Fix* $p \in [2, +\infty[$*; consider* $w : \mathbb{R}^2 \to \mathbb{R}$ *as follows*

$$w(a, b) = [1 + (a+b)^2]^{p/4} - [1 + a^2]^{p/4} - [1 + b^2]^{p/4}. \tag{30}$$

Then,

$$a \geq 0, \quad b \geq 0 \Longrightarrow w(0, 0) \leq w(a, b). \tag{31}$$

Proof We compute the first partial derivatives:

$$\frac{\partial w}{\partial a}(a,b) = \frac{p}{4}[1+(a+b)^2]^{(p/4)-1}2(a+b) - \frac{p}{4}[1+a^2]^{(p/4)-1}2a$$
$$= \frac{p}{2}\{V(a+b) - V(a)\}$$

and

$$\frac{\partial w}{\partial b}(a,b) = \frac{p}{4}[1+(a+b)^2]^{(p/4)-1}2(a+b) - \frac{p}{4}[1+b^2]^{(p/4)-1}2b$$
$$= \frac{p}{2}\{V(a+b) - V(b)\}$$

where V is given by (28) with $m = (p/4) - 1$. Note that $m \geq -1/2$ since $p \geq 2$. Then, V is increasing so that, when $a \geq 0$ and $b \geq 0$, we have $V(a+b) - V(a) \geq 0$ and $V(a+b) - V(b) \geq 0$. This shows that

$$a \geq 0, \quad b \geq 0 \Longrightarrow \frac{\partial w}{\partial a}(a,b) \geq 0, \quad \frac{\partial w}{\partial b}(a,b) \geq 0. \tag{32}$$

Then, $a \to w(a,b)$ increases and $b \to w(a,b)$ increases too, if we restrict ourselves to $a \geq 0$ and $b \geq 0$; thus,

$$w(0,0) \leq w(0,b) \leq w(a,b), \tag{33}$$

provided $b \geq 0$ and $a \geq 0$. This ends the proof. □

Corollary 3.1 *Fix* $p \in [2, +\infty[$; *then,*

$$a \geq 0, \quad b \geq 0 \Longrightarrow [1+a^2]^{p/4} + [1+b^2]^{p/4} - 1 \leq [1+(a+b)^2]^{p/4}. \tag{34}$$

Proof We write (31) explicitly and we get (34). □

Lemma 3.4 *Fix* $p \in]2, 3]$. *If* $a \geq 0$ *and* $b \geq 0$, *then*

$$[1+(a+b)^2]^{p/4} \leq [1+a^2]^{p/4} + [1+b^2]^{p/4} + \frac{p}{2}ab^{(p/2)-1} + 1. \tag{35}$$

Proof Since $p \in]2, 3]$ we have $\frac{p}{4} \in]\frac{1}{2}, \frac{3}{4}]$ and we can use (27) with $m = \frac{p}{4}$:

$$[1+(a+b)^2]^{p/4} \leq [1]^{p/4} + [(a+b)^2]^{p/4} = 1 + (a+b)^{p/2}; \tag{36}$$

now $\frac{p}{2} \in]1, \frac{3}{2}]$ and we can use (20) with $m = \frac{p}{2}$:

$$\begin{aligned} 1 + (a+b)^{p/2} &\leq 1 + a^{p/2} + b^{p/2} + \frac{p}{2} a b^{(p/2)-1} \\ &= 1 + (a^2)^{p/4} + (b^2)^{p/4} + \frac{p}{2} a b^{(p/2)-1} \\ &\leq 1 + [1+a^2]^{p/4} + [1+b^2]^{p/4} + \frac{p}{2} a b^{(p/2)-1}. \end{aligned} \tag{37}$$

This ends the proof. □

Lemma 3.5 *Fix $q \in]1, 2]$. Then,*

$$a \geq 0, \quad b \geq 0 \Longrightarrow [1 + (a+b)^2]^{q/4} \leq [1 + a^2]^{q/4} + [1 + b^2]^{q/4} + 1. \tag{38}$$

Proof Since $q \in]1, 2]$ we have $\frac{q}{4} \in]\frac{1}{4}, \frac{1}{2}]$ and we can use (27) with $m = \frac{q}{4}$:

$$[1 + (a+b)^2]^{q/4} \leq 1^{q/4} + [(a+b)^2]^{q/4} = 1 + (a+b)^{q/2}; \tag{39}$$

now $\frac{q}{2} \in]\frac{1}{2}, 1]$ and we can use (27) with $m = \frac{q}{2}$:

$$\begin{aligned} 1 + (a+b)^{q/2} &\leq 1 + a^{q/2} + b^{q/2} \\ &= 1 + (a^2)^{q/4} + (b^2)^{q/4} \\ &\leq 1 + [1+a^2]^{q/4} + [1+b^2]^{q/4}. \end{aligned} \tag{40}$$

This ends the proof. □

Now we are able to give examples of functions F, G, H verifying conditions required in Theorem 2.2.

Remark 3.2 Fix $p \in]2, 3]$ and define

$$F(x, t) = \gamma(x) t^{p/2} \tag{41}$$

for $t \in [0, +\infty[$, where $\gamma_1 \leq \gamma(x) \leq \gamma_2$ with $\gamma_1, \gamma_2 \in]0, +\infty[$. Then (9), (12), (14) hold true with $k_1 = \gamma_1$, $k_2 = 0$, $k_3 = \frac{p}{2}\gamma_2$, $a(x) = 0$. Indeed, we use (19) and (20) with $m = p/2$ in Lemma 3.1 and we are done.

Remark 3.3 Fix $q \in]1, 3[$ and define

$$G(x, t) = \gamma(x) t^{q/2} \tag{42}$$

for $t \in [0, +\infty[$, where $\gamma_1 \leq \gamma(x) \leq \gamma_2$ with $\gamma_1, \gamma_2 \in]0, +\infty[$. Then, when $q > 2$, (10), (13), (15) hold true with $k_1 = \gamma_1$, $k_2 = 0$, $k_3 = \frac{q}{2}\gamma_2$, $b(x) = 0$. Indeed, we use (20) with $m = q/2$ in Lemma 3.1 and we are done. Moreover, when $q \leq 2$, (10), (13), (16) hold true with $k_1 = \gamma_1$, $k_2 = 0$, $k_3 = \gamma_2$, $b(x) = 0$. Indeed, when $q \leq 2$, we use the well-known inequality (27) with $m = q/2$ and we are done.

Remark 3.4 Fix $r \in [1, 3[$ and define

$$H(x, s) = \gamma(x)|s|^r \tag{43}$$

for $s \in \mathbb{R}$, where $\gamma_1 \leq \gamma(x) \leq \gamma_2$ with $\gamma_1, \gamma_2 \in]0, +\infty[$. Then, (11) holds true with $k_3 = \gamma_2$, $c(x) = 0$.

Remark 3.5 Fix $p \in]2, 3]$ and define

$$F(x, t) = \gamma(x)[1 + t^2]^{p/4} \tag{44}$$

for $t \in [0, +\infty[$, where $\gamma_1 \leq \gamma(x) \leq \gamma_2$ with $\gamma_1, \gamma_2 \in]0, +\infty[$. Then (9), (12), (14) hold true with $k_1 = \gamma_1$, $k_2 = \gamma_2$, $k_3 = \frac{p}{2}\gamma_2$, $a(x) = \gamma_2$. Indeed, we use (27) with $m = p/4$, (34) and (35).

Remark 3.6 Fix $q \in]1, 3[$ and define

$$G(x, t) = \gamma(x)[1 + t^2]^{q/4} \tag{45}$$

for $t \in [0, +\infty[$, where $\gamma_1 \leq \gamma(x) \leq \gamma_2$ with $\gamma_1, \gamma_2 \in]0, +\infty[$. Then, when $q > 2$, (10), (13), (15) hold true with $k_1 = \gamma_1$, $k_2 = 0$, $k_3 = \frac{q}{2}\gamma_2$, $b(x) = \gamma_2$. Indeed, we use (27) with $m = q/4$, (35) and we are done. Moreover, when $q \leq 2$, (10), (13), (16) hold true with $k_1 = \gamma_1$, $k_2 = 0$, $k_3 = \gamma_2$, $b(x) = \gamma_2$. Indeed, when $q \leq 2$, we use (27) with $m = q/4$, (38) and we are done.

Remark 3.7 Fix $r \in [1, 3[$ and define

$$H(x, s) = \gamma(x)[1 + |s|^2]^{r/2} \tag{46}$$

for $s \in \mathbb{R}$, where $\gamma_1 \leq \gamma(x) \leq \gamma_2$ with $\gamma_1, \gamma_2 \in]0, +\infty[$. Then, (11) holds true with $k_3 = 2^{r/2}\gamma_2$, $c(x) = 2^{r/2}\gamma_2$.

The following lemma can be found in [13] as Lemma 4.1.

Lemma 3.6 *Consider the matrices $A, B \in \mathbb{R}^{3\times 3}$:*

$$A = \begin{pmatrix} A^1 \\ B^2 \\ B^3 \end{pmatrix}, \quad B = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}.$$

Then, the following estimates hold:

(a) $|A| \leq |A^1| + |B^2| + |B^3|$,
(b) $|\det A| \leq |A^1||(\mathrm{adj}_2 B)^1|$,
(c) $|(\mathrm{adj}_2 A)_{2j}| \leq |A^1||B^3|$ *and* $|(\mathrm{adj}_2 A)_{3j}| \leq |A^1||B^2|$*, for all* $j \in \{1, 2, 3\}$.

In order to get our main result, we have to prove a suitable Caccioppoli-type inequality for any component u^α of the local minimizer u of functional I (1) on every superlevel set $\{u^\alpha > k\}$. To this goal, we will use the following lemma (see [13] for a proof).

Lemma 3.7 *Let Ω be an open subset of $\mathbb{R}^3$. Consider a Carathéodory function $f : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty[$. Assume that there exist $c_1, c_3 > 0$ and $c_2 \geq 0$ such that, for every $\xi \in \mathbb{R}^{3\times 3}$,*

$$c_1(|\xi|^p + |\mathrm{adj}_2\xi|^q) - c_2 \leq f(x,\xi) \leq c_3(|\xi|^p + |(\mathrm{adj}_2\xi)|^q + |\det \xi|^r + 1 + \omega(x)),$$

with $1 \leq p$, $1 \leq q$, $1 \leq r$, $\omega(x) \geq 0$.

Let $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^3)$ be such that $x \to f(x, Du(x)) \in L^1_{loc}(\Omega)$. Fix $\eta \in C^1_0(\Omega)$, $\eta \geq 0$ and $k \in \mathbb{R}$, and denote, for almost every $x \in \{u^1 > k\} \cap \{\eta > 0\}$,

$$A = \begin{pmatrix} \mu\eta^{-1}(k-u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix}.$$

If

$$q < \frac{p^* p}{p^* + p} \quad \text{and} \quad r < \frac{p^* q}{p^* + q}$$

and $\omega \in L^1_{loc}(\Omega)$, then

$$\eta^\mu f(x, A) \in L^1(\{u^1 > k\} \cap \{\eta > 0\}), \ \ \forall \mu \geq p^*.$$

4 Proof of Theorem 2.2

We want to stress that the proof of our result follows the idea used in [13]: we provide the local boundedness of the minimizers by proving that each component is locally bounded. In the following lemma, we refer to the first component u^1: the core of the proof lies in the following Caccioppoli-type inequality, obtained on every superlevel set $\{u^1 > k\}$. We keep in mind that $p^* = \frac{np}{n-p}$ if $p < n = 3$ and p^* is any number $> p$ when $p = n = 3$.

Proposition 4.1 (Caccioppoli-type estimate) *Let f be as in (8) satisfying (9)–(16) with $1 \leq r < q < p \leq 3$ such that*

$$2 < p, \quad q < \frac{pp^*}{p + p^*}, \quad r < \frac{p^* q}{p^* + q}. \tag{47}$$

Let $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^3)$ be a local minimizer of I. Let $B_R(x_0) \subset\subset \Omega$ with $|B_R(x_0)| < 1$; fixed $k \in \mathbb{R}$, denote

$$A^1_{k,\tau} := \{x \in B_\tau(x_0) : u^1(x) > k\} \quad 0 < \tau \leq R.$$

Then, there exists $C = C(k_1, k_2, k_3, p, q, r, p^*) > 0$ such that, for every $0 < s < t \leq R$:

$$\int_{A^1_{k,s}} |Du^1|^p dx \leq C \int_{A^1_{k,t}} \left(\frac{u^1-k}{t-s}\right)^{p^*} \mathrm{d}x + C \left\{ 1 + \|a+b+c\|_{L^\sigma(B_R)} \right.$$
$$+ \left(\int_{B_R} (|Du^2|+|Du^3|)^p \mathrm{d}x\right)^{\frac{p^*(p-2)}{p(p^*-2)}} + \left(\int_{B_R} \left(|Du^2|+|Du^3|\right)^p \mathrm{d}x\right)^{\frac{qp^*}{p(p^*-q)}}$$
$$+ \left(\int_{B_R} |(\mathrm{adj}_2 Du)^1|^q \,\mathrm{d}x\right)^{\frac{rp^*}{q(p^*-r)}} + 1_{(2,+\infty)}(q) \left(\int_{B_R} (|Du^2|+|Du^3|)^p \mathrm{d}x\right)^{\frac{2p^*}{p(p^*-2)}}$$
$$\left. \times \left(\int_{B_R} |(adj_2 Du)^1|^q \mathrm{d}x\right)^{\frac{(q-2)p^*}{q(p^*-2)}} \right\} |A^1_{k,t}|^\theta, \tag{48}$$

where

$$\theta := \min\left\{1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)}, 1 - \frac{1}{\sigma}\right\}, \quad \text{if } 1 < q \leq 2,$$
$$\theta := \min\left\{1 - \frac{2p^*}{p(p^*-2)} - \frac{(q-2)p^*}{q(p^*-2)}, 1 - \frac{rp^*}{q(p^*-r)}, 1 - \frac{1}{\sigma}\right\}, \quad \text{if } 2 < q,$$

with $\frac{1}{\sigma} = 0$ *if* $\sigma = \infty$.

Proof The condition $|B_R(x_0)| = \frac{4\pi R^3}{3} < 1$ ensures $R < 1$. Let s, t be such that $0 < s < t \leq R$. Consider a cutoff function $\eta \in C^\infty_0(B_t(x_0))$ satisfying the following assumptions:

$$0 \leq \eta \leq 1, \ \eta \equiv 1 \text{ in } B_s(x_0), \ |D\eta| \leq \frac{2}{t-s}.$$

Fixing $k \in \mathbb{R}$, define $w \in W^{1,p}_{loc}(\Omega; \mathbb{R}^3)$,

$$w^1 := \max\{u^1 - k, 0\}, \ w^2 = 0, \ w^3 = 0,$$

and, for $\mu = p^*$,

$$\varphi := -\eta^\mu w.$$

For almost every $x \in \Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\})$ we have $\varphi = 0$, thus

$$f(x, Du + D\varphi) = f(x, Du) \tag{49}$$

almost everywhere in $\Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\})$.

For almost every $x \in \{\eta > 0\} \cap \{u^1 > k\}$ denote

$$A = \begin{pmatrix} \mu\eta^{-1}(k-u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix}. \tag{50}$$

We notice that

$$Du + D\varphi = \begin{pmatrix} (1-\eta^\mu)Du^1 + \mu\eta^{\mu-1}(k-u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix} = (1-\eta^\mu)Du + \eta^\mu A.$$

Moreover, since for almost every $x \in \{\eta > 0\} \cap \{u^1 > k\}$,

$$\det(Du + D\varphi) = (1-\eta^\mu)\det Du + \eta^\mu \det A$$

and

$$\mathrm{adj}_2(Du + D\varphi) = (1-\eta^\mu)\mathrm{adj}_2 Du + \eta^\mu \mathrm{adj}_2 A,$$

then, since f is polyconvex, we get that

$$f(x, Du + D\varphi) \le (1-\eta^\mu) f(x, Du) + \eta^\mu f(x, A) \tag{51}$$

almost everywhere in $\{\eta > 0\} \cap \{u^1 > k\}$.

By the minimality of u, $f(x, Du) \in L^1_{loc}(\Omega)$; note that in our case we can use Lemma 3.7, deducing that

$$\eta^\mu f(x, A) \in L^1(\{\eta > 0\} \cap \{u^1 > k\}).$$

Therefore, (49) and (51) imply $f(x, Du + D\varphi) \in L^1_{loc}(\Omega)$.

By the local minimality of u, (49) and (51), recalling that $A^1_{k,t}$ is the set $\{x \in B_t(x_0) : u^1(x) > k\}$, we have

$$\begin{aligned}\int_{A^1_{k,t}\cap\{\eta>0\}} f(x, Du)\mathrm{d}x &\le \int_{A^1_{k,t}\cap\{\eta>0\}} f(x, Du + D\varphi)\mathrm{d}x \\ &\le \int_{A^1_{k,t}\cap\{\eta>0\}} \{(1-\eta^\mu) f(x, Du) + \eta^\mu f(x, A)\}\mathrm{d}x.\end{aligned}$$

The inequality above implies

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu f(x, Du)\mathrm{d}x \le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu f(x, A)\mathrm{d}x.$$

Taking into account the expression of f (see (8)), we obtain from the above inequality that

$$\begin{aligned}&\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left[F(x, |Du|^2) + G(x, |\mathrm{adj}_2 Du|^2) + H(x, \det Du)\right]\mathrm{d}x \\ &\le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left[F(x, |A|^2) + G(x, |\mathrm{adj}_2 A|^2) + H(x, \det A)\right]\mathrm{d}x. \end{aligned} \tag{52}$$

Denote $\tilde{u} = (u^2, u^3)$ and

$$D\tilde{u} = \begin{pmatrix} Du^2 \\ Du^3 \end{pmatrix}.$$

We have

$$|Du|^2 = |Du^1|^2 + |D\widetilde{u}|^2;$$

we use (12) with $t_1 = |Du^1|^2$ and $t_2 = |D\widetilde{u}|^2$, so that

$$\begin{aligned} &\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left[F(x, |Du^1|^2) + F(x, |D\tilde{u}|^2) - k_2\right] \mathrm{d}x \\ &\quad \le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu F(x, |Du|^2)\mathrm{d}x. \end{aligned} \tag{53}$$

Note that

$$|A|^2 = |A^1|^2 + |D\widetilde{u}|^2;$$

by using (14) with $t_1 = |A^1|^2$ and $t_2 = |D\widetilde{u}|^2$, we obtain

$$\begin{aligned} &\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu F(x, |A|^2)\mathrm{d}x \\ &\quad \le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \Big[F(x, |A^1|^2) + F(x, |D\tilde{u}|^2) \\ &\qquad + k_3|A^1|^2(|D\tilde{u}|^2)^{\frac{p}{2}-1} + a(x)\Big] \mathrm{d}x. \end{aligned} \tag{54}$$

Furthermore, setting

$$|\tilde{A}|^2 = |(\mathrm{adj}_2 Du)^2|^2 + |(\mathrm{adj}_2 Du)^3|^2, \qquad |\tilde{\tilde{A}}|^2 = |(\mathrm{adj}_2 A)^2|^2 + |(\mathrm{adj}_2 A)^3|^2 \tag{55}$$

and noticing

$$(\mathrm{adj}_2 A)^1 = (\mathrm{adj}_2 Du)^1,$$

we can write

$$\begin{aligned} |\mathrm{adj}_2 Du|^2 &= |(\mathrm{adj}_2 Du)^1|^2 + |(\mathrm{adj}_2 Du)^2|^2 + |(\mathrm{adj}_2 Du)^3|^2 = |(\mathrm{adj}_2 Du)^1|^2 + |\tilde{A}|^2, \\ |\mathrm{adj}_2 A|^2 &= |(\mathrm{adj}_2 A)^1|^2 + |(\mathrm{adj}_2 A)^2|^2 + |(\mathrm{adj}_2 A)^3|^2 = |(\mathrm{adj}_2 Du)^1|^2 + |\tilde{\tilde{A}}|^2. \end{aligned}$$

Applying (13) with $t_1 = |(\mathrm{adj}_2 Du)^1|^2$ and $t_2 = |\tilde{A}|^2$, we get

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left(G(x, |(\mathrm{adj}_2 Du)^1|^2) - k_2\right) \mathrm{d}x \\ \leq \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu G(x, |\mathrm{adj}_2 Du|^2)\mathrm{d}x. \tag{56}$$

Assumption (15) when $q > 2$ or (16) when $q \leq 2$, with $t_1 = |\tilde{\tilde{A}}|^2$ and $t_2 = |(\mathrm{adj}_2 Du)^1|^2$, yields

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu G(x, |\mathrm{adj}_2 A|^2)\mathrm{d}x \leq \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu [G(x, |\tilde{\tilde{A}}|^2) \\ + G(x, |(\mathrm{adj}_2 Du)^1|^2) + b(x) + 1_{(2,+\infty)}(q) k_3 |\tilde{\tilde{A}}|^2 \left(|(\mathrm{adj}_2 Du)^1|^2\right)^{\frac{q}{2}-1}]\mathrm{d}x. \tag{57}$$

By virtue of (53),(54), (56) and (57), from (52), we get

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu [F(x, |Du^1|^2) + F(x, |D\tilde{u}|^2) - 2k_2 \\ + G(x, |(\mathrm{adj}_2 Du)^1|^2) + H(x, \det Du)]\mathrm{d}x \\ \leq \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \Big[F(x, |A^1|^2) + F(x, |D\tilde{u}|^2) + k_3 |A^1|^2 (|D\tilde{u}|^2)^{\frac{p}{2}-1} \\ + a(x) + G(x, |\tilde{\tilde{A}}|^2) + G(x, |(\mathrm{adj}_2 Du)^1|^2) + b(x) \\ + 1_{(2,+\infty)}(q) k_3 |\tilde{\tilde{A}}|^2 \left(|(\mathrm{adj}_2 Du)^1|^2\right)^{\frac{q}{2}-1} + H(x, \det A)\Big]\mathrm{d}x$$

and then

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left[F(x, |Du^1|^2) - 2k_2 + H(x, \det Du)\right] \mathrm{d}x \\ \leq \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \Big[F(x, |A^1|^2) + k_3 |A^1|^2 (|D\tilde{u}|^2)^{\frac{p}{2}-1} + a(x) + G(x, |\tilde{\tilde{A}}|^2) \\ + b(x) + 1_{(2,+\infty)}(q) k_3 |\tilde{\tilde{A}}|^2 \left(|(\mathrm{adj}_2 Du)^1|^2\right)^{\frac{q}{2}-1} + H(x, \det A)\Big]\mathrm{d}x. \tag{58}$$

In order to estimate the first two terms on the right-hand side of (58), we recall that $\mu = p^* > p$ and

$$A^1 = \mu \eta^{-1}(k - u^1) D\eta.$$

By using the right-hand side of (9) and the fact $z^p \le 1 + z^{p^*}$ if $z \ge 0$, we obtain

$$\begin{aligned}\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu F(x,|A^1|^2) &\le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left[k_3|A^1|^p + a(x)\right]\mathrm{d}x\\ &\le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left[k_3\left\{1+|A^1|^{p^*}\right\} + a(x)\right]\mathrm{d}x\\ &\le \int_{A^1_{k,t}\cap\{\eta>0\}} \left\{\eta^\mu(k_3+a(x)) + k_3(2\mu)^{p^*}\eta^{\mu-p^*}\left(\frac{u^1-k}{t-s}\right)^{p^*}\right\}\mathrm{d}x\\ &\le \int_{A^1_{k,t}\cap\{\eta>0\}} \left\{k_3+a(x)+k_3(2\mu)^{p^*}\left(\frac{u^1-k}{t-s}\right)^{p^*}\right\}\mathrm{d}x. \end{aligned} \tag{59}$$

We will write d' to denote the Hölder conjugate of $d > 1$: $d' = \frac{d}{d-1}$. Regarding the second term on the right-hand side in (58), notice that

$$(p-2)\left(\frac{p^*}{2}\right)' < p;$$

we use Young inequality with $\frac{p^*}{2}$, $\left(\frac{p^*}{2}\right)'$ and Hölder inequality with $\frac{p}{(p-2)\left(\frac{p^*}{2}\right)'}$, $\frac{p}{p-(p-2)\left(\frac{p^*}{2}\right)'}$:

$$\begin{aligned}&\int_{A^1_{k,t}\cap\{\eta>0\}} k_3\eta^\mu|A^1|^2|D\tilde{u}|^{p-2}\mathrm{d}x\\ &\le \int_{A^1_{k,t}\cap\{\eta>0\}} k_3\eta^\mu|A^1|^{p^*}\mathrm{d}x + \int_{A^1_{k,t}\cap\{\eta>0\}} k_3\eta^\mu|D\tilde{u}|^{(p-2)\left(\frac{p^*}{2}\right)'}\mathrm{d}x\\ &\le k_3(2\mu)^{p^*}\int_{A^1_{k,t}\cap\{\eta>0\}}\left(\frac{u^1-k}{t-s}\right)^{p^*}\mathrm{d}x\\ &\quad + k_3\left(\int_{B_R}|D\tilde{u}|^p\mathrm{d}x\right)^{\left(1-\frac{2}{p}\right)\left(\frac{p^*}{2}\right)'}|A^1_{k,t}|^{1-\left(1-\frac{2}{p}\right)\left(\frac{p^*}{2}\right)'}. \end{aligned} \tag{60}$$

Now we estimate the fourth term in (58). By using (10), (55), Lemma 3.6-(c) and Young inequality with exponents $\frac{p^*}{q}$ and $(\frac{p^*}{q})'$, we estimate

$$\begin{aligned}\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu G(x,|\tilde{\tilde{A}}|^2) &\le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu k_3((|\tilde{\tilde{A}}|^2)^{\frac{q}{2}} + b(x))\mathrm{d}x\\ &= \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu (k_3\left(|(\mathrm{adj}_2 A)^2|^2 + |(\mathrm{adj}_2 A)^3|^2\right)^{\frac{q}{2}} + b(x))\mathrm{d}x\\ &\le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu (3^{\frac{q}{2}}k_3\left[|A^1|^2(|Du^2|^2+|Du^3|^2)\right]^{\frac{q}{2}} + b(x))\mathrm{d}x \end{aligned}$$

$$\leq \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu 3^{\frac{q}{2}} k_3 |A^1|^{p^*} \mathrm{d}x + \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu 3^{\frac{q}{2}} k_3 \left(|D\tilde{u}|\right)^{q(\frac{p^*}{q})'} \mathrm{d}x$$
$$+ \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu b(x) \mathrm{d}x.$$

Note that $q(\frac{p^*}{q})' < p$ when $q < \frac{pp^*}{p+p^*}$ and $\frac{pp^*}{p+p^*} > 1$ if $p > \frac{2n}{n+1}$.

Moreover, Hölder inequality, with $\frac{p}{q} \Big/ \left(\frac{p^*}{q}\right)'$ and $\left(\frac{p}{q} \Big/ \left(\frac{p^*}{q}\right)'\right)'$, yields

$$\begin{aligned}\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu |D\tilde{u}|^{q(\frac{p^*}{q})'} \mathrm{d}x &\leq \left(\int_{A^1_{k,t}} \left(|D\tilde{u}|\right)^p \mathrm{d}x\right)^{\frac{q}{p}(\frac{p^*}{q})'} |A^1_{k,t}|^{1-\frac{q}{p}(\frac{p^*}{q})'} \\ &\leq \left(\int_{B_R} \left(|D\tilde{u}|\right)^p \mathrm{d}x\right)^{\frac{q}{p}(\frac{p^*}{q})'} |A^1_{k,t}|^{1-\frac{q}{p}(\frac{p^*}{q})'};\end{aligned} \tag{61}$$

therefore, if we note that $(\frac{p^*}{q})' = \frac{p^*}{p^*-q}$, we have

$$\begin{aligned}\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu G(x, |\tilde{\tilde{A}}|^2) &\leq \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu 3^{\frac{q}{2}} k_3 |A^1|^{p^*} \mathrm{d}x \\ &+ 3^{\frac{q}{2}} k_3 \left(\int_{B_R} |D\tilde{u}|^p \mathrm{d}x\right)^{\frac{qp^*}{p(p^*-q)}} |A^1_{k,t}|^{1-\frac{qp^*}{p(p^*-q)}} + \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu b(x) \mathrm{d}x.\end{aligned} \tag{62}$$

Eventually, if $q > 2$, we have to estimate the sixth term in (58) and we use Lemma 3.6-(c) and Young inequality with exponents $\frac{p^*}{2}$ and $\left(\frac{p^*}{2}\right)'$, so having

$$\begin{aligned}&\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu |\tilde{\tilde{A}}|^2 \left(|(\mathrm{adj}_2 Du)^1|^2\right)^{\frac{q}{2}-1} \mathrm{d}x \\ &= \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left(|(\mathrm{adj}_2 A)^2|^2 + |(\mathrm{adj}_2 A)^3|^2\right) |(\mathrm{adj}_2 Du)^1|^{q-2} \mathrm{d}x \\ &\leq 3 \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu |A^1|^2 (|Du^3|^2 + |Du^2|^2) |(\mathrm{adj}_2 Du)^1|^{q-2} \mathrm{d}x \\ &\leq 3 \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu |A^1|^{p^*} \mathrm{d}x \\ &\quad + 3 \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu |D\tilde{u}|^{2\left(\frac{p^*}{2}\right)'} |(\mathrm{adj}_2 Du)^1|^{(q-2)\left(\frac{p^*}{2}\right)'} \mathrm{d}x.\end{aligned} \tag{63}$$

Observe that $2 < q < \frac{pp^*}{p+p^*}$ implies $p > \frac{12}{5}$; so we have $2\left(\frac{p^*}{2}\right)' < p$ and we apply Hölder inequality with exponents $\frac{p}{2\left(\frac{p^*}{2}\right)'}$ and $\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'$,

$$
\begin{aligned}
&\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^{\mu}|D\tilde{u}|^{2\left(\frac{p^*}{2}\right)'}|(\mathrm{adj}_2 Du)^1|^{(q-2)\left(\frac{p^*}{2}\right)'}\mathrm{d}x\\
&\quad\le \left(\int_{B_R}|D\tilde{u}|^p\right)^{\frac{2}{p}\left(\frac{p^*}{2}\right)'}\\
&\qquad\times\left(\int_{A^1_{k,t}\cap\{\eta>0\}}\eta^{\mu}|(\mathrm{adj}_2 Du)^1|^{(q-2)\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'}\right)^{1-\frac{2}{p}\left(\frac{p^*}{2}\right)'}
\end{aligned}
\tag{64}
$$

Furthermore, if $(q-2)\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)' < q$, we apply Hölder inequality again with exponents $\frac{q}{(q-2)\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'}$ and its conjugate:

$$
\begin{aligned}
&\left(\int_{B_R}|D\tilde{u}|^p\right)^{\frac{2}{p}\left(\frac{p^*}{2}\right)'}\left(\int_{A^1_{k,t}}|(\mathrm{adj}_2 Du)^1|^{(q-2)\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'}\right)^{1-\frac{2}{p}\left(\frac{p^*}{2}\right)'}\\
&\quad\le \left(\int_{B_R}|D\tilde{u}|^p\right)^{\frac{2}{p}\left(\frac{p^*}{2}\right)'}\left[\left(\int_{A^1_{k,t}}|(\mathrm{adj}_2 Du)^1|^q\right)^{\frac{(q-2)}{q}\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'}\right.\\
&\qquad\left.\times|A^1_{k,t}|^{1-\frac{(q-2)}{q}\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'}\right]^{1-\frac{2}{p}\left(\frac{p^*}{2}\right)'}.
\end{aligned}
\tag{65}
$$

Therefore, by (64) and (65), (63) becomes

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu |\tilde{\tilde{A}}|^2 \left(|(\mathrm{adj}_2 Du)^1|^2\right)^{\frac{q}{2}-1} \mathrm{d}x \le 3\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu |A^1|^{p^*} \mathrm{d}x$$
$$+3\left(\int_{B_R} |D\tilde{u}|^p \mathrm{d}x\right)^{\frac{2}{p}\left(\frac{p^*}{2}\right)'} \left\{\left(\int_{A^1_{k,t}} |(\mathrm{adj}_2 Du)^1|^q \mathrm{d}x\right)^{\frac{(q-2)}{q}\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'}\right.$$
$$\left.\times |A^1_{k,t}|^{1-\frac{(q-2)}{q}\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'}\right\}^{1-\frac{2}{p}\left(\frac{p^*}{2}\right)'}. \tag{66}$$

Please, note that the previous condition $(q-2)\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)' < q$ means $q < \frac{pp^*}{p+p^*}$. Finally, r-growth assumption (11) on $H(x,.)$ yields

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu H(x, \det A)\mathrm{d}x \le \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu (k_3|\det A|^r + c(x))\mathrm{d}x. \tag{67}$$

We compute $\det A$ with respect to the first row, see Lemma 3.6-(b),

$$\eta^\mu|\det A|^r \le \eta^\mu |A^1|^r |(\mathrm{adj}_2 Du)^1|^r \le (2\mu)^r \eta^{\mu-r}\left(\frac{u^1-k}{t-s}\right)^r |(\mathrm{adj}_2 Du)^1|^r$$
$$\le (2\mu)^r \left(\frac{u^1-k}{t-s}\right)^r |(\mathrm{adj}_2 Du)^1|^r.$$

Notice that $r < p < p^*$ and $\frac{rp^*}{p^*-r} < q$. By the Young inequality with exponents $\frac{p^*}{r}$ and $\frac{p^*}{p^*-r}$, one has

$$\left(\frac{u^1-k}{t-s}\right)^r |(\mathrm{adj}_2 Du)^1|^r \le \left(\frac{u^1-k}{t-s}\right)^{p^*} + |(\mathrm{adj}_2 Du)^1|^{\frac{rp^*}{p^*-r}}.$$

Hölder inequality with $\frac{q}{\frac{rp^*}{p^*-r}}$ and $\frac{q}{q-\frac{rp^*}{p^*-r}}$ leads to

$$\begin{aligned}\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^{\mu}|\det A|^r \mathrm{d}x &\le (2\mu)^r\Bigg[\int_{A^1_{k,t}\cap\{\eta>0\}} \left(\frac{u^1-k}{t-s}\right)^{p^*} \mathrm{d}x \\ &\quad + \int_{A^1_{k,t}\cap\{\eta>0\}} |(\mathrm{adj}_2 Du)^1|^{\frac{rp^*}{p^*-r}} \mathrm{d}x\Bigg] \\ &\le (2\mu)^r\Bigg[\int_{A^1_{k,t}\cap\{\eta>0\}} \left(\frac{u^1-k}{t-s}\right)^{p^*} \mathrm{d}x \\ &\quad + \left(\int_{B_R} |(\mathrm{adj}_2 Du)^1|^q \mathrm{d}x\right)^{\frac{rp^*}{q(p^*-r)}} |A^1_{k,t}|^{1-\frac{rp^*}{q(p^*-r)}}\Bigg].\end{aligned} \tag{68}$$

Therefore, (67) and (68) imply

$$\begin{aligned}\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^{\mu} H(x,\det A)\mathrm{d}x &\le k_3(2\mu)^r\Bigg[\int_{A^1_{k,t}\cap\{\eta>0\}} \left(\frac{u^1-k}{t-s}\right)^{p^*} \mathrm{d}x \\ + \left(\int_{B_R} |(\mathrm{adj}_2 Du)^1|^q \mathrm{d}x\right)^{\frac{rp^*}{q(p^*-r)}} &|A^1_{k,t}|^{1-\frac{rp^*}{q(p^*-r)}}\Bigg] + \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^{\mu} c(x)\mathrm{d}x.\end{aligned} \tag{69}$$

By left-hand side inequalities in (9) and (11), using (58), (59), (60), (62), (66) and (69), we conclude

$$\begin{aligned}&\int_{A^1_{k,s}} |Du^1|^p \mathrm{d}x \le C\int_{A^1_{k,t}} \left(\frac{u^1-k}{t-s}\right)^{p^*} \mathrm{d}x + C\Bigg\{1+\|a+b+c\|_{L^{\sigma}(B_R)} \\ &+\left(\int_{B_R} (|Du^2|+|Du^3|)^p \mathrm{d}x\right)^{\left(1-\frac{2}{p}\right)\left(\frac{p^*}{2}\right)'} \\ &+\left(\int_{B_R} \left(|Du^2|+|Du^3|\right)^p \mathrm{d}x\right)^{\frac{qp^*}{p(p^*-q)}} + \left(\int_{B_R} |(\mathrm{adj}_2 Du)^1|^q \mathrm{d}x\right)^{\frac{rp^*}{q(p^*-r)}} \\ &+1_{(2,+\infty)}(q)\left(\int_{B_R} (|Du^2|+|Du^3|)^p \mathrm{d}x\right)^{\frac{2}{p}\left(\frac{p^*}{2}\right)'} \\ &\times\left(\int_{B_R} |(\mathrm{adj}_2 Du)^1|^q \mathrm{d}x\right)^{\frac{(q-2)}{q}\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'\left(1-\frac{2}{p}\left(\frac{p^*}{2}\right)'\right)}\Bigg\}|A^1_{k,t}|^{\theta},\end{aligned} \tag{70}$$

where

$$\theta := \min\left\{1-\left(1-\frac{2}{p}\right)\left(\frac{p^*}{2}\right)', \ 1-\frac{qp^*}{p(p^*-q)}, \ 1-\frac{rp^*}{q(p^*-r)}, \ 1-\frac{1}{\sigma}, \right.$$
$$\left. 1_{[1,2]}(q)+1_{(2,+\infty)}(q)\left(1-\frac{(q-2)}{q}\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'\right)\left(1-\frac{2}{p}\left(\frac{p^*}{2}\right)'\right)\right\}$$

and $C = C(k_1, k_2, k_3, p, q, r, p^*) > 0$; moreover, $1_E(q) = 1$ if $q \in E$ and $1_E(q) = 0$ if $q \notin E$. Now we note that

$$1-\left(1-\frac{2}{p}\right)\left(\frac{p^*}{2}\right)' \geq 1-\frac{p^*}{p(p^*-1)} \geq 1-\frac{qp^*}{p(p^*-q)},$$

where the last inequality is granted since $q \mapsto 1-\frac{qp^*}{p(p^*-q)}$ decreases. Then,

$$\theta := \min\left\{1-\frac{qp^*}{p(p^*-q)}, \quad 1-\frac{rp^*}{q(p^*-r)}, \quad 1-\frac{1}{\sigma}, \right.$$
$$\left. 1_{[1,2]}(q)+1_{(2,+\infty)}(q)\left(1-\frac{(q-2)}{q}\left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'\right)\left(1-\frac{2}{p}\left(\frac{p^*}{2}\right)'\right)\right\}.$$

Note that $\left(\frac{p^*}{2}\right)' = \frac{p^*}{p^*-2}$; then, the exponents in (70) can be written as follows

$$\left(1-\frac{2}{p}\right)\left(\frac{p^*}{2}\right)' = \frac{p^*(p-2)}{p(p^*-2)}, \quad \frac{2}{p}\left(\frac{p^*}{2}\right)'$$
$$= \frac{2p^*}{p(p^*-2)}, \left(\frac{p^*}{2}\right)'\left(\frac{p}{2\left(\frac{p^*}{2}\right)'}\right)'\left(1-\frac{2}{p}\left(\frac{p^*}{2}\right)'\right)$$
$$= \frac{p^*}{p^*-2},$$

so that (70) turns out to be

$$\int_{A^1_{k,s}} |Du^1|^p \mathrm{d}x \leq C\int_{A^1_{k,t}}\left(\frac{u^1-k}{t-s}\right)^{p^*}\mathrm{d}x + C\left\{ 1+\|a+b+c\|_{L^\sigma(B_R)}\right.$$
$$+\left(\int_{B_R}(|Du^2|+|Du^3|)^p \mathrm{d}x\right)^{\frac{p^*(p-2)}{p(p^*-2)}}$$
$$+\left(\int_{B_R}\left(|Du^2|+|Du^3|\right)^p \mathrm{d}x\right)^{\frac{qp^*}{p(p^*-q)}}$$

$$+\left(\int_{B_R}|(\mathrm{adj}_2 Du)^1|^q\,\mathrm{d}x\right)^{\frac{rp^*}{q(p^*-r)}}$$

$$+1_{(2,+\infty)}(q)\left(\int_{B_R}(|Du^2|+|Du^3|)^p\mathrm{d}x\right)^{\frac{2p^*}{p(p^*-2)}}$$

$$\times\left(\int_{B_R}|(\mathrm{adj}_2 Du)^1|^q\mathrm{d}x\right)^{\frac{(q-2)p^*}{q(p^*-2)}}\Bigg\}\,|A^1_{k,t}|^\theta, \tag{71}$$

where

$$\theta := \min\left\{1-\frac{qp^*}{p(p^*-q)}, 1-\frac{rp^*}{q(p^*-r)}, 1-\frac{1}{\sigma}\right\}, \quad \text{if} \quad 1<q\le 2,$$

$$\theta := \min\Bigg\{1-\frac{qp^*}{p(p^*-q)}, 1-\frac{rp^*}{q(p^*-r)}, 1-\frac{1}{\sigma},$$

$$\frac{p^*(p-2)-2p}{p(p^*-2)}-\frac{(q-2)p^*}{q(p^*-2)}\Bigg\}, \quad \text{if} \quad 2<q.$$

If $2<q<\frac{pp^*}{p+p^*}$, we have

$$1-\frac{qp^*}{p(p^*-q)} > \frac{p^*(p-2)-2p}{p(p^*-2)}-\frac{(q-2)p^*}{q(p^*-2)} = 1-\frac{2p^*}{p(p^*-2)}-\frac{(q-2)p^*}{q(p^*-2)} \tag{72}$$

and

$$\theta = \min\left\{1-\frac{2p^*}{p(p^*-2)}-\frac{(q-2)p^*}{q(p^*-2)}, 1-\frac{rp^*}{q(p^*-r)}, 1-\frac{1}{\sigma}\right\}.$$

This ends the proof of Proposition 4.1. □

We now proceed with the proof of Theorem 2.2. We fix $x_0 \in \Omega$ and $R_0 < \min\{\mathrm{dist}(x_0, \partial\Omega), \left(\frac{3}{4\pi}\right)^{1/3}\}$ such that

$$\int_{B_{R_0}}|u^1|^{p^*}\mathrm{d}x < 1, \tag{73}$$

where B_ρ is the ball centered at x_0 with radius ρ. Note that $R_0 < 1$, $|B_{R_0}| < 1$ and $B_{R_0} \subset\subset \Omega$. For every $R \in (0, R_0]$ we define the decreasing sequence of radii

$$\rho_h := \frac{R}{2}+\frac{R}{2^{h+1}}.$$

Fix a positive constant $d \ge 1$ and define the increasing sequence of positive levels

$$k_h := d\left(1-\frac{1}{2^{h+1}}\right).$$

We define the "excess"

$$J_h := \int_{A^1_{k_h,\rho_h}} (u^1 - k_h)^{p^*} \mathrm{d}x.$$

We use our Caccioppoli inequality (48) and Proposition 2.4 of [13]: we get

$$J_{h+1} \leq c \left(2^{\frac{p^* p^*}{p}} \right)^h (J_h)^{\theta \frac{p^*}{p}},$$

where the positive constant c is independent of h. See also [40,41]. Assumption (17) tells us that $\theta \frac{p^*}{p} > 1$; then, we can use Lemma 2.5 of [13] with $\gamma := \theta \frac{p^*}{p} - 1$, see also [42]:

$$J_h \leq \left(2^{\frac{p^* p^*}{p}} \right)^{-\frac{h}{\gamma}} J_0, \tag{74}$$

provided

$$J_0 \leq c^{-\frac{1}{\gamma}} \left(2^{\frac{p^* p^*}{p}} \right)^{-\frac{1}{\gamma^2}}. \tag{75}$$

Note that

$$J_0 = \int_{A^1_{\frac{d}{2},R}} \left(u^1 - \frac{d}{2} \right)^{p^*} \mathrm{d}x \to 0 \quad \text{as} \quad d \to +\infty;$$

then, we can choose $d \geq 1$ large enough so that (75) holds true. Thus, we have (74) with $\gamma > 0$, so that $J_h \to 0$ as $h \to +\infty$; since $\frac{R}{2} < \rho_h$ and $k_h < d$, we also have

$$0 \leq \int_{A^1_{d,\frac{R}{2}}} \left(u^1 - d \right)^{p^*} \mathrm{d}x \leq J_h;$$

then,

$$\int_{A^1_{d,\frac{R}{2}}} \left(u^1 - d \right)^{p^*} \mathrm{d}x = 0$$

so that $u^1 \leq d$ almost everywhere in $B_{\frac{R}{2}}$. We have proved that u^1 is locally bounded from above. In order to prove that u^1 is locally bounded from below, we note that $-u$ locally minimizes $\int_\Omega \tilde{f}(x, Dz(x))\mathrm{d}x$, where $\tilde{f}(x, \xi) = f(x, -\xi)$; then, we get that $-u^1$ is locally bounded from above, so u^1 is locally bounded from below. We have just shown that $u^1 \in L^\infty_{\mathrm{loc}}(\Omega)$.

Now we turn our attention to the second component u^2. We change the order of the two components u^1 and u^2: we get a new function v as follows:

$$v = \begin{pmatrix} u^2 \\ u^1 \\ u^3 \end{pmatrix};$$

then,

$$Dv = \begin{pmatrix} Du^2 \\ Du^1 \\ Du^3 \end{pmatrix}$$

and $\det Dv = -\det Du$; moreover $(\mathrm{adj}_2 Dv)^1 = -(\mathrm{adj}_2 Du)^2$, $(\mathrm{adj}_2 Dv)^2 = -(\mathrm{adj}_2 Du)^1$ and $(\mathrm{adj}_2 Dv)^3 = -(\mathrm{adj}_2 Du)^3$, so that

$$\mathrm{adj}_2 Dv = -\begin{pmatrix} (\mathrm{adj}_2 Du)^2 \\ (\mathrm{adj}_2 Du)^1 \\ (\mathrm{adj}_2 Du)^3 \end{pmatrix}.$$

If we write $C_{1,2}(\xi)$ to denote the matrix obtained from ξ by inverting line 1 and line 2, we have $Dv = C_{1,2}(Du)$ and $\mathrm{adj}_2 Dv = -C_{1,2}(\mathrm{adj}_2 Du)$. Then, v is a local minimizer of $\int_\Omega \tilde{\tilde{f}}(x, Dw(x))dx$, where $\tilde{\tilde{f}}(x, \xi) = f(x, C_{1,2}(\xi))$. Thus, the first component v^1 is locally bounded: $u^2 = v^1 \in L^\infty_{\mathrm{loc}}(\Omega)$. In a similar way we deal with the third component u^3: we change the order of the two components u^1 and u^3; we get a new function w as follows:

$$w = \begin{pmatrix} u^3 \\ u^2 \\ u^1 \end{pmatrix};$$

then,

$$Dw = \begin{pmatrix} Du^3 \\ Du^2 \\ Du^1 \end{pmatrix}$$

and $\det Dw = -\det Du$; moreover $(\mathrm{adj}_2 Dw)^1 = -(\mathrm{adj}_2 Du)^3$, $(\mathrm{adj}_2 Dw)^2 = -(\mathrm{adj}_2 Du)^2$ and $(\mathrm{adj}_2 Dw)^3 = -(\mathrm{adj}_2 Du)^1$, so that

$$\mathrm{adj}_2 Dw = -\begin{pmatrix} (\mathrm{adj}_2 Du)^3 \\ (\mathrm{adj}_2 Du)^2 \\ (\mathrm{adj}_2 Du)^1 \end{pmatrix}.$$

If we write $C_{1,3}(\xi)$ to denote the matrix obtained from ξ by inverting line 1 and line 3, we have $Dw = C_{1,3}(Du)$ and $\mathrm{adj}_2 Dw = -C_{1,3}(\mathrm{adj}_2 Du)$. Then, w is a local minimizer of $\int_\Omega \tilde{\tilde{\tilde{f}}}(x, Dz(x))dx$, where $\tilde{\tilde{\tilde{f}}}(x,\xi) = f(x, C_{1,3}(\xi))$. Thus, the first component w^1 is locally bounded: $u^3 = w^1 \in L^\infty_{\mathrm{loc}}(\Omega)$. This ends the proof of Theorem 2.2. ◻

5 Conclusions

We have been able to prove boundedness for minimizers of the most important three-dimensional polyconvex integral, provided the growth exponents verify some restrictions. It would be interesting to understand what happens when such restrictions are not in force.

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Appendix: Comparison Between Two Structures

Lemma A.1 *We assume that* $F^\alpha, G^\alpha : \mathbb{R}^3 \mapsto [0,+\infty[$ *and* $H : \mathbb{R} \mapsto [0,+\infty[$*; let* $p, q, r \in]0,+\infty[$ *with* $p \neq 2$*. Then, it is false that*

$$\sum_{\alpha=1}^{3} F^\alpha(\xi^\alpha) + \sum_{\alpha=1}^{3} G^\alpha((\mathrm{adj}_2\xi)^\alpha) + \mathrm{H}(\det\xi) = |\xi|^{\mathrm{p}} + |\mathrm{adj}_2\xi|^{\mathrm{q}} + |\det\xi|^{\mathrm{r}} \tag{76}$$

for every $\xi \in \mathbb{R}^{3\times 3}$.

Proof We argue by contradiction: if (76) holds true, then we can use (76) with

$$\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{77}$$

and we get

$$\mathrm{adj}_2\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{78}$$

with $\det\xi = 0$, so that

$$\sum_{\alpha=1}^{3} F^\alpha((0,0,0)) + \sum_{\alpha=1}^{3} G^\alpha((0,0,0)) + H(0) = 0; \tag{79}$$

we keep in mind that $F^\alpha, G^\alpha, H \geq 0$ and we get

$$F^\alpha((0,0,0)) = G^\alpha((0,0,0)) = H(0) = 0, \tag{80}$$

for every $\alpha = 1, 2, 3$. Now we use (76) with

$$\xi = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{81}$$

and we get

$$\mathrm{adj}_2 \xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{82}$$

with $\det \xi = 0$, so that

$$F^1((t,0,0)) + F^2((0,0,0)) + F^3((0,0,0)) + \sum_{\alpha=1}^{3} G^\alpha((0,0,0)) + H(0) = |t|^p; \tag{83}$$

we keep in mind (80) and we get

$$F^1((t,0,0)) = |t|^p, \tag{84}$$

for every $t \in \mathbb{R}$. In a similar manner, taking

$$\xi = \begin{pmatrix} 0 & 0 & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{85}$$

we get

$$F^2((t,0,0)) = |t|^p, \tag{86}$$

for every $t \in \mathbb{R}$. In the same way, taking

$$\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix}, \tag{87}$$

we get

$$F^3((t,0,0)) = |t|^p, \tag{88}$$

for every $t \in \mathbb{R}$. Eventually, we take

$$\xi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{89}$$

and (76) implies

$$\sum_{\alpha=1}^{3} F^{\alpha}((1,0,0)) + \sum_{\alpha=1}^{3} G^{\alpha}((0,0,0)) + H(0) = 3^{p/2}; \tag{90}$$

we use (80), (84), (86), (88) and we get

$$3 = 3^{p/2} : \tag{91}$$

such an equality is a contradiction, since $p \neq 2$. This ends the proof of Lemma A.1. □

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