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Existence and Optimal Controls for Fractional Stochastic Evolution Equations of Sobolev Type Via Fractional Resolvent Operators

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Abstract This paper is mainly concerned with controlled stochastic evolution equations of Sobolev type for the Caputo and Riemann–Liouville fractional derivatives. Some sufficient conditions are established for the existence of mild solutions and optimal state-control pairs of the limited Lagrange optimal systems. The main results are investigated by compactness of fractional resolvent operator family, and the optimal control results are derived without uniqueness of solutions for controlled evolution equations.

Keywords Fractional stochastic evolution equations of Sobolev type · Feasible pairs · Optimal controls · Resolvent operator

Mathematics Subject Classification 34A08 · 26A33 · 93E20

1 Introduction

Fractional calculus is an important generalization of ordinary differentiation and integration to arbitrary non-integer order, which has been regarded as one of the most

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powerful tools to describe long-memory processes in the last decades. Many phenomena from viscoelasticity, electrochemistry, nonlinear oscillation in mechanics, among others, can be modeled by differential equations involving fractional derivatives. We refer the reader to monographs [1,2] for more details. The optimal control is one of the fundamental topics in the field of mathematical control theory, which plays a key role in control systems [3,4]. In recent years, solvability and optimal control governed by fractional differential equations have been considerably studied. For instance, the existence and optimal control for semi-linear fractional finite time delay evolution systems of order $]0, 1[$ were investigated in [2]. Liu and Wang [5] considered the solvability and optimal controls for systems governed by semi-linear impulsive fractional differential equations of order $]0, 1]$. Since noise or stochastic perturbation are unavoidable in real world [6,7], it is of interest to consider stochastic effects in optimal control problems. Balasubramaniam and Tamilalagan [8] investigated the solvability and optimal controls for fractional stochastic systems of order in $]0, 1]$. Yan and Jia [9] discussed optimal controls for fractional stochastic functional differential equations of order $]1, 2]$.

On the other hand, Sobolev-type evolution equations find wide applications in mathematical models such as flow of fluid through fissured rocks, thermodynamics, propagation of long waves of small amplitude and so on; see for instance [10,11] and references therein. Sobolev-type fractional equations have attracted great interest recently. There are already some fundamental results on Sobolev-type fractional equations of order in $]0, 1]$. For example, Zhou gave some controllability results on Sobolev-type fractional systems in [2]. Debbouche et al established some sufficient conditions for the existence of optimal multi-controls governed by Sobolev-type fractional equations in [12]. Benchaabane and Sakthivel [13] investigated the existence and uniqueness of mild solutions for a semi-linear Sobolev-type fractional stochastic differential equations in Hilbert spaces. In above-mentioned topics, the solution operators of Sobolev-type fractional equations were obtained via the so-called subordination formulas of a strongly continuous semigroup. However, in order to ensure the existence of such operators, is necessary an order of fractional integration in $]0, 1[$ (for the Caputo fractional derivative) and the existence and compactness of certain inverse operator.

In this paper, we study the existence and optimal controls for Sobolev-type fractional stochastic evolution equations of order in $]1, 2[$ in Caputo and Riemann–Liouville fractional derivatives. We also note that there are few results available on existence and optimal controls for Sobolev-type fractional stochastic evolution equations of order in $]1, 2[$.

The rest of this paper is organized as follows. Section 2 presents the problem to study in this paper. Section 3 collects the Preliminaries. Sections 4 and 5 are devoted to the existence and optimal controls for addressed systems. Section 6 gives some applications and in Sect. 7 we present some Conclusions.

2 Problem Statement

The main objective of this paper is to investigate the following Sobolev-type fractional stochastic evolution equations of the order $\alpha \in]1, 2[$

$$\partial_t^\alpha (Ex)(t) = Ax(t) + f(t, x(t)) + \mathfrak{B}(t)u(t) + \Sigma(t, x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \ t \in I$$
$$Ex(0) = Ex_0, \ (Ex)'(0) = Ex_1, \tag{1}$$

and

$$\partial^\alpha (Ex)(t) = Ax(t) + f(t, x(t)) + \mathfrak{B}(t)u(t) + \Sigma(t, x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \ t \in I$$
$$E(g_{2-\alpha} * x)(0) = Ex_0, \ (E(g_{2-\alpha} * x))'(0) = Ex_1, \tag{2}$$

where $I := [0, b]$, ∂_t^α and ∂^α denote, respectively, the Caputo and Riemann–Liouville fractional derivatives, the state $x(t)$ takes values in a separable Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}$, A and E are two closed linear operators defined on $\mathcal{H}$ with domains $D(A)$ and $D(E)$, respectively. $\{W(t)\}_{t \geq 0}$ is a given $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, and $\mathcal{K}$ is another separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ and norm $\| \cdot \|_{\mathcal{K}}$. The control u takes values from a separable reflexive Hilbert Y, $\mathfrak{B} : I \to \mathcal{B}(Y, \mathcal{H})$, $f : I \times \mathcal{H} \to \mathcal{H}$, $\Sigma : I \times \mathcal{H} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$, which will satisfy some additional conditions. Here, $\mathcal{B}(\mathcal{K}, \mathcal{H})$ denotes the space of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$. If $\mathcal{K} = \mathcal{H}$, then we just write $\mathcal{B}(\mathcal{K})$. The function "$g_\star(\cdot)$" and its finite convolution "$*$" will be specified later.

Observe that the change of variable $y(t) = Ex(t)$ allows to write (1) as $\partial_t^\alpha y(t) = Ly(t) + g(t) + \mathfrak{B}(t)u(t) + h(t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}$, $y(0) = y_0$, $y'(0) = y_1$, where $g(t) = E^{-1} f(t, x(t))$, $L = AE^{-1}$, $h(t) = E^{-1}\Sigma(t, x(t))(t)$ and $y_0 = Ex_0$, $D(L) = E(D(A))$, $y_1 = Ex_1$. Then, formally the system (1) can be studied by above reduced system. However, this change of variable needs the existence of E^{-1} as a bounded operator, which in general is restrictive. On the other hand, a common assumption to deal with the problem (1) is: E, A *are closed linear operators;* $D(E) \subset D(A)$ *and* E *is bijective; and* E^{-1} *is a compact operators*. In this case, $-AE^{-1}$ is a bounded operator which generates a uniformly continuous C_0-semigroup. See for instance [13,14] and the references therein.

In this paper, we establish sufficient conditions for the existence of mild solutions to (1) and (2). And then we prove the existence of optimal state-control pairs of the limited Lagrange optimal systems governed by (1) and (2). The solution operators for (1) and (2) are directly expressed as fractional resolvent operator family generated by the pair (A, E) without assuming the existence or compactness of E^{-1}. The main results are investigated under the mixed Carathéodory and Lipschitz conditions via compactness of fractional resolvent operator family, and thus, the optimal control results are derived without uniqueness of solutions for controlled evolution equations.

3 Preliminaries

In this section, we list some basic definitions, notations and results. For more detailed facts on stochastic and fractional differential equations, see [1,2,15] and references therein.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathcal{F}_t, t \in I$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). For Q-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ with linear bounded covariance operator Q satisfying $tr Q < \infty$, we assume that there exists a complete orthonormal system $\{e_n\}_{n \geq 1}$ in $\mathcal{K}$, a bounded sequence of non-negative real numbers $\{\lambda_n\}_{n \geq 1}$ such that $Qe_n = \lambda_n e_n$, and a sequence $\{W_n\}_{n \geq 1}$ of independent Brownian motions such that $\langle W(t), e \rangle_{\mathcal{K}} = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle e_n, e \rangle_{\mathcal{K}} W_n(t)$, for $e \in \mathcal{K}, t \in I$ and $\mathcal{F}_t = \mathcal{F}_t^W$, where $\mathcal{F}_t^W$ is the σ-algebra generated by $\{W(s) : 0 \leq s \leq t\}$. Let $L^2(\Omega, \mathcal{H})$ be the space of all $\mathcal{F}_b$-measurable square mean integrable random variables with values in the Hilbert space $\mathcal{H}$. We denote by $C(I, L^2(\Omega, \mathcal{H}))$ the space of continuous maps from I into $L^2(\Omega, \mathcal{H})$ satisfying the expectation condition $\sup_{t \in I} \mathbb{E}\|x(t)\|^2 < \infty$. Let $\mathcal{C}(I, \mathcal{H})$ be the closed subspace of $C(I, L^2(\Omega, \mathcal{H}))$ consisting of measurable and $\mathcal{F}_t$-adapted $\mathcal{H}$-valued process $x \in C(I, L^2(\Omega, \mathcal{H}))$ endowed with the norm $\|x\|_{\mathcal{C}} = \left(\sup_{t \in I} \mathbb{E}\|x(t)\|^2 \right)^{\frac{1}{2}}$. Then $(\mathcal{C}, \| \cdot \|_{\mathcal{C}})$ is a Banach space.

In what follows, we introduce the admissible control set. Let Y be a separable reflexive Hilbert space from which the control u takes values. Let $L^2_{\mathcal{F}}(I, Y)$ be the closed subspace of $L^2_{\mathcal{F}}(I \times \Omega, Y)$, consisting of all measurable and $\mathcal{F}_t$-adapted, Y-valued stochastic processes satisfying the condition $\mathbb{E} \int_0^b \|u(t)\|_Y^2 \mathrm{d}t < \infty$, and furnished with the norm $\|u\| = \left(\mathbb{E} \int_0^b \|u(t)\|_Y^2 \mathrm{d}t \right)^{\frac{1}{2}}$. Let $\mathcal{P}(Y)$ be a class of nonempty closed and convex subsets of Y. We assume that the multivalued map $\mathcal{U} : I \to \mathcal{P}(Y)$ is graph measurable, $\mathcal{U}(\cdot) \subset \Xi$, where Ξ is a bounded set of Y. The set $U_{\mathrm{ad}} := \{u \in L^2_{\mathcal{F}}(I, \Xi) : u(t) \in \mathcal{U}(t) \text{ a.e. } t \in I\}$ is called the admissible control set, and from [16] it follows that $U_{\mathrm{ad}} \neq \emptyset$.

For $\alpha > 0$, we define $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, for $t > 0$ and $g_\alpha(t) = 0$ for $t \leq 0$, where $\Gamma(\cdot)$ is the Gamma function. We also define $g_0 \equiv \delta_0$, the Dirac delta. For $\alpha > 0$, $n = \lceil \alpha \rceil$ denotes the smallest integer n greater than or equal to α. The finite convolution of f and g is defined by $(f * g)(t) := \int_0^t f(t-s)g(s)\mathrm{d}s$.

Definition 3.1 Let $\alpha > 0$. The α-order Riemann–Liouville fractional integral of v is defined by $J^\alpha v(t) := (g_\alpha * v)(t)$. Also, we define J^0 as $J^0 v(t) = v(t)$.

Definition 3.2 Let $\alpha > 0$. The α-order Caputo and Riemann–Liouville fractional derivatives of v are defined, respectively by $\partial_t^\alpha v(t) := (g_{n-\alpha} * v^{(n)})(t)$ and $\partial^\alpha v(t) = \frac{d^n}{\mathrm{d}t^n}(g_{n-\alpha} * v)(t)$, where $n = \lceil \alpha \rceil$.

In the following, we recall some results on fractional resolvent operator family $\{S^E_{\alpha,\beta}(t)\}_{t \geq 0}$, which can be found in details in [17,18]. The E-modified resolvent set of A, $\rho_E(A)$, is defined by $\rho_E(A) := \{\lambda \in \mathbb{C} : (\lambda E - A) : D(A) \cap D(E) \to \mathcal{H}$ is invertible and $(\lambda E - A)^{-1} \in \mathcal{B}(\mathcal{H}, [D(A) \cap D(E)])\}$. The operator $(\lambda E - A)^{-1}$ is called the E-resolvent operator of A. A strongly continuous family $\{T(t)\}_{t \geq 0} \subseteq \mathcal{B}(\mathcal{H})$ is said to be of type (M, ω) or exponentially bounded if there exist $M > 0$ and $\omega \in \mathbb{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Note that, without loss of generality, we can assume $\omega > 0$.

Definition 3.3 Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$, $E : D(E) \subseteq \mathcal{H} \to \mathcal{H}$ be closed linear operators on a Hilbert space $\mathcal{H}$ satisfying $D(A) \cap D(E) \neq \{0\}$. Let $\alpha, \beta > 0$. The pair (A, E) is said to be the generator of an (α, β)-resolvent family, if there exist a constant $\mu \geq 0$ and a strongly continuous function $S^E_{\alpha,\beta} : [0, \infty[\to \mathcal{B}(\mathcal{H})$ such that $S^E_{\alpha,\beta}(t)$ is exponentially bounded, $\{\lambda^\alpha : \mathrm{Re}\lambda > \mu\} \subset \rho_E(A)$, and for all $x \in \mathcal{H}$, $\lambda^{\alpha-\beta} E\, (\lambda^\alpha E - A)^{-1}\, x = \int_0^\infty e^{-\lambda t} S^E_{\alpha,\beta}(t) x \mathrm{d}t$, $\mathrm{Re}\lambda > \mu$. In this case, $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α, β)-resolvent family generated by the pair (A, E).

Definition 3.4 The resolvent family $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{B}(\mathcal{H})$ is said to be compact, if for every $t > 0$, the operator $S^E_{\alpha,\beta}(t)$ is a compact operator.

Lemma 3.1 *Let $\alpha > 0$ and $1 < \beta \leq 2$. Suppose that $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ is the (α, β)-resolvent family of type (M, ω) generated by (A, E). Then, the function $t \mapsto S^E_{\alpha,\beta}(t)$ is continuous in $\mathcal{B}(\mathcal{H})$.*

Lemma 3.2 *If (A, E) generates an (α, β)-resolvent family of type (M, ω) and $\gamma > 0$, then (A, E) also generates an $(\alpha, \beta + \gamma)$-resolvent family of type $\left(\frac{M}{\omega^\gamma}, \omega\right)$.*

Lemma 3.3 *Let $\alpha > 0$, $1 < \beta \leq 2$ and $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ be an (α, β)-resolvent family of type (M, ω) generated by (A, E). Then, the following assertions are equivalent: (i) $S^E_{\alpha,\beta}(t)$ is a compact operator for all $t > 0$; (ii) $E(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

Lemma 3.4 Let $1 < \alpha < 2$, and $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ be the $(\alpha, 1)$-resolvent family of type (M, ω) generated by (A, E). Suppose that $t \mapsto S^E_{\alpha,1}(t)$ is continuous in $\mathcal{B}(\mathcal{H})$. Then, the following assertions are equivalent: (i) $S^E_{\alpha,1}(t)$ is a compact operator for all $t > 0$; (ii) $E(\mu E - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.

Lemma 3.5 *Let $\frac{3}{2} < \alpha < 2$, and $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ be the $(\alpha, \alpha - 1)$-resolvent family of type (M, ω) generated by (A, E). Suppose that $t \mapsto S^E_{\alpha,\alpha-1}(t)$ is continuous in $\mathcal{B}(\mathcal{H})$. Then, the following assertions are equivalent: (i) $S^E_{\alpha,\alpha-1}(t)$ is compact for all $t > 0$; (ii) $E(\mu E - A)^{-1}$ is compact for all $\mu > \omega^{1/\alpha}$.*

Finally, we recall the following results, which can be found in [19].

Lemma 3.6 *If K is a compact subset of a Banach space X, then its convex closure is compact.*

Lemma 3.7 The closure and weak closure of a convex subset of a normed space are the same.

Lemma 3.8 *Let C be a closed, convex and nonempty subset of a Banach space X. Let operators N_1, N_2 satisfy that: (i) If $u, v \in C$, then $N_1 u + N_2 v \in C$; (ii) N_1 is a contraction; (iii) N_2 is compact and continuous. Then, there exists $z \in C$ such that $z = N_1 z + N_2 z$.*

4 Existence of Mild Solutions

In this section, we shall establish some existence results of mild solutions to systems (1) and (2) under the mixed Carathéodory and Lipschitz conditions. Let us list the following assumptions.

(A1) The pair (A, E) generates the $(\alpha, 1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ of type (M, ω), $E(\lambda^\alpha E - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho_E(A)$ with $\lambda > \omega^{\frac{1}{\alpha}}$.

(A2) $f : I \times \mathcal{H} \to \mathcal{H}$ satisfies the following conditions: (a) For a.e. $t \in I$, $f(t, \cdot)$ is continuous, and for each $x \in \mathcal{H}$, $f(\cdot, x)$ is measurable; (b) There exists a function $\phi \in L^1(I, \Re_+)$ such that $\mathbb{E}\|f(t, x)\|^2 \leq \phi(t)\mathbb{E}\|x\|^2$, $\forall t \in I, x \in \mathcal{H}$.

(A3) $\mathfrak{B} : I \to \mathcal{B}(Y, \mathcal{H})$ is essentially bounded, i.e. $\mathfrak{B} \in L^\infty(I, \mathcal{B}(Y, \mathcal{H}))$.

(A4) The function $\Sigma : I \times \mathcal{H} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is continuous and there exists a constant $L_\Sigma > 0$ such that $\mathbb{E}\|\Sigma(t, x) - \Sigma(t, y)\|^2 \leq L_\Sigma \mathbb{E}\|x - y\|^2$, $t \in I$, $x, y \in \mathcal{H}$.

(H1) Let $1 < \alpha < 2$, and the pair (A, E) generates the $(\alpha, \alpha - 1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type (M, ω) and the operator $E(\lambda^\alpha E - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho_E(A)$ with $\lambda > \omega^{\frac{1}{\alpha}}$.

Definition 4.1 An $\mathcal{F}_t$-adapted stochastic process $x : I \to \mathcal{H}$ is called a mild solution to (1), if for each $t \in I$ it verifies the following integral equation

$$x(t) = S^E_{\alpha,1}(t)x_0 + (g_1 * S^E_{\alpha,1})(t)x_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)\Sigma(s, x(s))\mathrm{d}W(s)$$
$$+ \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)\mathfrak{B}(s)u(s)\mathrm{d}s + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s) f(s, x(s))\mathrm{d}s.$$

Definition 4.2 An $\mathcal{F}_t$-adapted stochastic process $x : I \to \mathcal{H}$ is called a mild solution to (2), if for each $t \in I$ it verifies the following integral equation

$$x(t) = S^E_{\alpha,\alpha-1}(t)x_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)x_1 + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t - s)\Sigma(s, x(s))\mathrm{d}W(s)$$
$$+ \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t - s)\mathfrak{B}(s)u(s)\mathrm{d}s + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t - s) f(s, x(s))\mathrm{d}s.$$

Theorem 4.1 *If conditions (A1)–(A4) hold, then the system (1) has at least one mild solution on I provided that*

$$10\frac{M^2 e^{2\omega b} b}{\omega^{2(\alpha-1)}}\left[\frac{1}{2}\|\phi\|_{L^1} + L_\Sigma\right] < 1. \tag{3}$$

Proof For a constant $r > 0$, we define $B_r := \left\{x \in \mathcal{C}(I, \mathcal{H}) : \mathbb{E}\|x(t)\|^2 \leq r,\ t \in I\right\}$. We further define the map $N := N_1 + N_2 : \mathcal{C}(I, \mathcal{H}) \to \mathcal{C}(I, \mathcal{H})$ as

$$(N_1 x)(t) := S^E_{\alpha,1}(t)x_0 + (g_1 * S^E_{\alpha,1})(t)x_1$$
$$+ \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)\Sigma(s, x(s))\mathrm{d}W(s),$$

$$(N_2x)(t) := \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]\mathrm{d}s.$$

Obviously, the fixed points of N are mild solutions of the system (1). We shall show that N admits a fixed point. The proof will be given in several steps.
Step 1 There exists a constant $r > 0$ such that $N(B_r) \subseteq B_r$. Indeed, if it is not true, then for each $r > 0$, there exists $x \in B_r$, $\mathbb{E}\|(Nx)(t)\|^2 > r$ for some $t \in I$. From the definition of N, we have

$$r < \mathbb{E}\|(Nx)(t)\|^2 \leq 5M^2e^{2\omega b}\mathbb{E}\|x_0\|^2 + 5\frac{M^2e^{2\omega b}}{\omega^2}\mathbb{E}\|x_1\|^2 + 5\frac{M^2e^{2\omega b}b}{\omega^{2(\alpha-1)}}r\|\phi\|_{L^1}$$
$$+5\frac{M^2e^{2\omega b}}{\omega^{2(\alpha-1)}}\left[b\|\mathfrak{B}\|^2\|u\|^2_{L^2_{\mathcal{F}}(I,Y)} + 2L_\Sigma br + 2\int_0^b \mathbb{E}\|\Sigma(s,0)\|^2\mathrm{d}s\right].$$

Dividing both sides by r and taking the lower limit as $r \to \infty$, we obtain $1 \leq 10\frac{M^2e^{2\omega b}b}{\omega^{2(\alpha-1)}}\left[\frac{1}{2}\|\phi\|_{L^1} + L_\Sigma\right]$, which is a contradiction to (3). Thus, there exists $r > 0$ such that $N(B_r) \subseteq B_r$.
Step 2 N_1 is a contraction in B_r. In fact, for $x, y \in B_r$ and $t \in I$, we have $\mathbb{E}\|(N_1x)(t) - (N_1y)(t)\|^2 \leq \frac{M^2e^{2\omega b}b}{\omega^{2(\alpha-1)}}L_\Sigma \sup_{s\in I}\mathbb{E}\|x(s) - y(s)\|^2$. Thus, we have $\mathbb{E}\|N_1x - N_1y\|^2 \leq \frac{M^2e^{2\omega b}b}{\omega^{2(\alpha-1)}}L_\Sigma\mathbb{E}\|x-y\|^2$. By (3), N_1 is a contraction in B_r.
Step 3 N_2 is continuous in B_r. In fact, let $x_n, x \in B_r$ be such that $x_n \to x$ in B_r as $n \to \infty$. Then $\mathbb{E}\|(N_2x^n)(t) - (N_2x)(t)\|^2 \leq \frac{M^2e^{2\omega b}b}{\omega^{2(\alpha-1)}}4r\int_0^t \phi(s)\mathrm{d}s$. Note that the function $s \mapsto \phi(s)$ is integrable on I. Thus, the Lebesgue-dominated convergence theorem implies that N_2 is continuous in B_r.
Step 4 N_2 is equicontinuous. Let $x \in B_r$, and take $0 \leq t_2 < t_1 \leq b$. Then

$$\begin{aligned}
&\mathbb{E}\|(N_2x)(t_1) - (N_2x)(t_2)\|^2\\
&\leq 4\,\mathbb{E}\left\|\int_{t_2}^{t_1}(g_{\alpha-1} * S^E_{\alpha,1})(t_1-s)f(s,x(s))\mathrm{d}s\right\|^2\\
&\quad+4\mathbb{E}\left\|\int_0^{t_1}(g_{\alpha-1} * S^E_{\alpha,1})(t_1-s)\mathfrak{B}(s)u(s)\mathrm{d}s\right\|^2\\
&\quad+4\mathbb{E}\left\|\int_0^{t_2}[(g_{\alpha-1} * S^E_{\alpha,1})(t_1-s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2-s)]f(s,x(s))\mathrm{d}s\right\|^2\\
&\quad+4\mathbb{E}\left\|\int_0^{t_2}[(g_{\alpha-1} * S^E_{\alpha,1})(t_1-s) - (g_{\alpha-1} * S^E_{\alpha,1})(t_2-s)]\mathfrak{B}(s)u(s)\mathrm{d}s\right\|^2\\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By (A2), for the term I_1, as $t_1 \to t_2$, we have $I_1 \leq 4\frac{M^2e^{2\omega b}}{\omega^{2(\alpha-1)}}\mathbb{E}\left\|\int_{t_2}^{t_1} f(s,x(s))\mathrm{d}s\right\|^2 \leq 4\frac{M^2e^{2\omega b}b}{\omega^{2(\alpha-1)}}\int_{t_2}^{t_1}\phi(s)\mathrm{d}s \to 0$. From (A3), for the term I_2, as $t_1 \to t_2$, we have $I_2 \leq$

$4\frac{M^2e^{2\omega b}b}{\omega^{2(\alpha-1)}}\|\mathfrak{B}\|^2\int_{t_2}^{t_1}\mathbb{E}\|u(s)\|^2\mathrm{d}s \leq 4\frac{M^2e^{2\omega b}b}{\omega^{2(\alpha-1)}}\|\mathfrak{B}\|^2\int_{t_2}^{t_1}\mathbb{E}\|u(s)\|^2\mathrm{d}s \to 0$. For terms I_3, I_4, we get by Hölder's inequality

$$I_3 \leq 4t_2 r\int_0^{t_2}\|(g_{\alpha-1}*S_{\alpha,1}^E)(t_1-s)-(g_{\alpha-1}*S_{\alpha,1}^E)(t_2-s)\|^2\phi(s)\mathrm{d}s,$$
$$I_4 \leq 4t_2\int_0^{t_2}\|(g_{\alpha-1}*S_{\alpha,1}^E)(t_1-s)-(g_{\alpha-1}*S_{\alpha,1}^E)(t_2-s)\|^2\mathbb{E}\|\mathfrak{B}(s)u(s)\|^2\mathrm{d}s.$$

Note that $\|(g_{\alpha-1}*S_{\alpha,1}^E)(t_1-\cdot)-(g_{\alpha-1}*S_{\alpha,1}^E)(t_2-\cdot)\|^2\phi(s) \leq 4\frac{M^2e^{2\omega b}}{\omega^{2(\alpha-1)}}\phi(s) \in L^1(I,\Re_+)$, $\|(g_{\alpha-1}*S_{\alpha,1}^E)(t_1-\cdot)-(g_{\alpha-1}*S_{\alpha,1}^E)(t_2-\cdot)\|^2\mathbb{E}\|\mathfrak{B}(s)u(s)\|^2 \leq 4\frac{M^2e^{2\omega b}}{\omega^{2(\alpha-1)}}\mathbb{E}\|\mathfrak{B}(s)u(s)\|^2 \in L^1(I,\Re_+)$, $(g_{\alpha-1}*S_{\alpha,1}^E)(t) = S_{\alpha,\alpha}^E(t)$ for all $t \geq 0$ (see Lemma 3.2) and $S_{\alpha,\alpha}^E(t)$ is norm continuous (see Lemma 3.1), we have $(g_{\alpha-1}*S_{\alpha,1}^E)(t_1-s)-(g_{\alpha-1}*S_{\alpha,1}^E)(t_2-s) \to 0$ in $\mathcal{B}(\mathcal{H})$ as $t_1 \to t_2$. By the dominated convergence theorem, we obtain $\lim_{t_1\to t_2} I_3 = 0$, $\lim_{t_1\to t_2} I_4 = 0$.

Step 5 The set $\mathcal{V}(t) := \{(N_2x)(t) : x \in B_r\}$ is relatively compact for each $t \in I$. In fact, clearly the set $\mathcal{V}(0)$ is relatively in $\mathcal{H}$. Let $t \in]0,b]$ and $\varepsilon \in]0,t[$, we define $(N_2^\varepsilon x)(t) := \int_0^{t-\varepsilon}(g_{\alpha-1}*S_{\alpha,1}^E)(t-s)[f(s,x(s))+\mathfrak{B}(s)u(s)]\mathrm{d}s$. The condition (A1) and Lemma 3.3 imply that $(g_{\alpha-1}*S_{\alpha,1}^E)(t) = S_{\alpha,\alpha}^E(t)$ is compact for all $t > 0$. Considering $\mathfrak{B}u \in L^2(I,\mathcal{H})$ for all $u \in U_{ad}$, the set $\mathcal{O}_\varepsilon := \{(g_{\alpha-1}*S_{\alpha,1}^E)(t-s)[f(s,x(s))+\mathfrak{B}(s)u(s)] : x \in B_r, 0 \leq s \leq t-\varepsilon\}$ is compact for all $\varepsilon > 0$. Then, $\overline{\mathrm{conv}(\mathcal{O}_\varepsilon)}$ is also a compact set by Lemma 3.6. By the mean value theorem for Bochner integrals, we obtain that $(N_2^\varepsilon x)(t) \in (t-\varepsilon)\overline{\mathrm{conv}(\mathcal{O}_\varepsilon)}$ for all $t \in I$. Thus, the set $\mathcal{V}_\varepsilon(t) = \{(N_2^\varepsilon x)(t) : x \in B_r\}$ is relatively compact in $\mathcal{H}$ for every $\varepsilon \in]0,t[$. Furthermore, for $x \in B_r$, $\mathbb{E}\left\|(N_2x)(t)-(N_2^\varepsilon x)(t)\right\|^2 \leq 2\frac{M^2e^{2\omega b}}{\omega^{2(\alpha-1)}}\left[r\int_{t-\varepsilon}^t\phi(s)ds + b\|\mathfrak{B}\|^2\|u\|^2_{L^2_{\mathcal{F}}(I,Y)}\varepsilon^2\right]$. Since the map $s \mapsto \phi(s)$ belongs to $L^1([t-\varepsilon,t],\Re_+)$, by the dominated convergence theorem we have that $\lim_{\varepsilon\to 0}\mathbb{E}\|(N_2x)(t)-(N_2^\varepsilon x)(t)\|^2 = 0$. So, there are relatively compact sets arbitrarily close to the set $\mathcal{V}(t)$, and $\mathcal{V}(t)$ is relatively compact for every $t \in I$.

As a consequence of Steps 1–5, we deduce that N_1 is a contraction in B_r and N_2 is completely continuous in B_r. By Lemma 3.8, there exists a fixed point $x(\cdot)$ for $N_1 + N_2$ in B_r. Thus, the system (1) admits a mild solution. □

Theorem 4.2 *Assume that conditions (H1), (A2)–(A4) are satisfied. Then, the system (2) admits at least one mild solution on I provided that*

$$10\frac{M^2e^{2\omega b}b}{\omega^2}\left[\frac{1}{2}\|\phi\|_{L^1}+L_\Sigma\right] < 1. \tag{4}$$

Proof We define the operator $N := N_1 + N_2$ as

$$(N_1x)(t) := S_{\alpha,\alpha-1}^E(t)x_0 + (g_1*S_{\alpha,\alpha-1}^E)(t)x_1$$

$$+\int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)\Sigma(s,x(s))\mathrm{d}W(s),$$

$$(N_2x)(t) := \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)[f(s,x(s)) + \mathfrak{B}(s)u(s)]\mathrm{d}s.$$

As in the proof of Theorem 4.1, if for each $r > 0$ there exists $x \in B_r$ such that

$$r < \mathbb{E}\|(Nx)(t)\|^2 \leq 5M^2e^{2\omega b}\mathbb{E}\|x_0\|^2 + 5\frac{M^2e^{2\omega b}}{\omega}\mathbb{E}\|x_1\|^2 + 5\frac{M^2e^{2\omega b}b}{\omega^2}r\|\phi\|_{L^1}$$
$$+5\frac{M^2e^{2\omega b}}{\omega^2}\left[\|\mathfrak{B}\|^2\|u\|^2_{L^2_{\mathcal{F}}(I,Y)}b + 2L_\Sigma br + 2\int_0^b \mathbb{E}\|\Sigma(s,0)\|^2\mathrm{d}s\right],$$

then dividing by r and taking $r \to \infty$, we obtain $1 \leq 10\frac{M^2e^{2\omega b}b}{\omega^2}\left[\frac{1}{2}\|\phi\|_{L^1} + L_\Sigma\right]$. In view of (4), there exists a constant $r > 0$ such that $N(B_r) \subseteq B_r$. We can show N_1 is a contraction in B_r as Step 2 in Theorem 4.1. Since $S^E_{\alpha,\alpha-1}(t)$ is norm continuous (see (H1)) and $t \mapsto (g_1 * S^E_{\alpha,\alpha-1})(t)$ is also norm continuous by Lemma 3.1, we can similarly prove that $N_2(B_r)$ is equicontinuous. Lemma 3.3 implies the compactness of $(g_1 * S^E_{\alpha,\alpha-1})(t) = S^E_{\alpha,\alpha}(t)$ for all $t > 0$. Thus, the set $\mathcal{V}(t) := \{(N_2x)(t) : x \in B_r\}$ is relatively compact for each $t \in I$. By the Arzela–Ascoli theorem, we can deduce that N_2 is completely continuous. From Lemma 3.8, there exists a fixed point $x(\cdot)$ for $N_1 + N_2$ on B_r. Thus, the system (2) admits a mild solution. □

5 Existence of Optimal Controls

This section is concerned with the existence of optimal state-control pairs of the limited Lagrange optimal control problems governed by the systems (1) and (2), respectively. The main results are derived via some compactness results of corresponding operators, and thus the uniqueness of solutions to (1) and (2) is not necessarily needed.

For any $u \in U_{\mathrm{ad}}$, let B_r be defined as before and $\mathcal{S}(u)$ denotes the set of all mild solutions to the systems (1) or (2) in B_r. Let $x^u \in B_r$ denote the mild solution to the systems (1) or (2) corresponding to the control $u \in U_{\mathrm{ad}}$, and consider the following limited Lagrange problem (LP): Find $x^0 \in B_r \subseteq \mathcal{C}(I,\mathcal{H})$ and $u^0 \in U_{\mathrm{ad}}$ such that for all $u \in U_{\mathrm{ad}}$, $J(x^0,u^0) \leq J(x^u,u)$, where $x^0 \in B_r$ denotes the mild solution to the systems (1) or (2) related to the control $u^0 \in U_{\mathrm{ad}}$, and $J(x^u,u) := \mathbb{E}\left\{\int_0^b \mathfrak{L}(t,x^u(t),u(t))\mathrm{d}t\right\}$.

We remark that under the conditions of Theorems 4.1–4.2, a pair $(x(\cdot),u(\cdot))$ is feasible if it verifies the systems (1) or (2) for $x(\cdot) \in B_r$, and if $(x^u(\cdot),u(\cdot))$ is feasible, then $x^u \in \mathcal{S}(u) \subset B_r$.

In order to seek results of optimal control, we need the following conditions.

(A5) The function $\mathfrak{L} : I \times \mathcal{H} \times Y \to \Re\bigcup\{\infty\}$ satisfies:

(A5.1) The function $\mathfrak{L} : I \times \mathcal{H} \times Y \to \Re\bigcup\{\infty\}$ is Borel measurable;

(A5.2) $\mathfrak{L}(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $\mathcal{H} \times Y$ for a.e. $t \in I$;

(A5.3) $\mathfrak{L}(t,x,\cdot)$ is convex on Y for each $x \in \mathcal{H}$ and a.e. $t \in I$;

(A5.4) There exist constants $c \geq 0$, $d > 0$, ψ is non-negative and $\psi \in L^1(I, \Re)$ such that $\mathfrak{L}(t, x, u) \geq \psi(t) + c\mathbb{E}\|x\|^2 + d\|u\|_Y^2$.

(A6) $t \mapsto S^E_{\alpha,1}(t)$ is continuous in $\mathcal{B}(\mathcal{H})$.

(H2) Let $\frac{3}{2} < \alpha < 2$, and the pair (A, E) generates the $(\alpha, \alpha - 1)$-resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type (M, ω). Assume that the operator $E(\lambda^\alpha E - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho_E(A)$ with $\lambda > \omega^{\frac{1}{\alpha}}$ and $t \mapsto S^E_{\alpha,\alpha-1}(t)$ is continuous in $\mathcal{B}(\mathcal{H})$.

Remark 5.1 Conditions (A1), (A6) and Lemma 3.4 imply that $S^E_{\alpha,1}(t)$ is compact for all $t > 0$. According to Lemma 3.5, $S^E_{\alpha,\alpha-1}(t)$ is also compact for all $t > 0$ under the condition (H2).

Taking into account the proofs of Lemma 3.2 and Corollary 3.3 of Chapter 3 in [3], we have the following similar results.

Lemma 5.1 *Assume that $S^E_{\alpha,\alpha}(t)$ is compact for all $t > 0$. Then, the operator Π by $(\Pi u)(\cdot) = \int_0^\cdot S^E_{\alpha,\alpha}(\cdot - s)\mathfrak{B}(s)u(s)\mathrm{d}s$, $\forall u(\cdot) \in U_{\mathrm{ad}} \subset L^2_{\mathcal{F}}(I, Y)$ is compact. Moreover, if $u_n \in U_{\mathrm{ad}}$ converges weakly to u as $n \to \infty$ in $L^2_{\mathcal{F}}(I, Y)$, then $\Pi u_n \to \Pi u$ as $n \to \infty$.*

Theorem 5.1 *Assume that conditions (A1)–(A6) and (3) hold. Then, the problem (LP) governed by (1) admits at least one optimal feasible pair.*

Proof We define $J(u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u)$, $\forall u \in U_{\mathrm{ad}}$. If the set $\mathcal{S}(u)$ admits finite elements, there exists some $\tilde{x}^u \in \mathcal{S}(u)$ such that $J(\tilde{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u) = J(u)$. If the set $\mathcal{S}(u)$ admits infinite elements and $\inf_{x^u \in \mathcal{S}(u)} J(x^u, u) = +\infty$, there is nothing to prove. Now, we assume that $J(u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u) < +\infty$. By (A5), we have $J(u) > -\infty$. We now divide the proof into several steps.

Step 1 By the definition of the infimum, there exists a sequence $\{x^u_n\} \subseteq \mathcal{S}(u)$ satisfying $J(x^u_n, u) \to J(u)$ as $n \to \infty$. Considering $\{x^u_n, u\}$ as a sequence of feasible pairs, we have

$$x^u_n(t) = S^E_{\alpha,1}(t)x_0 + (g_1 * S^E_{\alpha,1})(t)x_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)\Sigma(s, x^u_n(s))\mathrm{d}W(s) + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t - s)[f(s, x^u_n(s)) + \mathfrak{B}(s)u(s)]\mathrm{d}s, \quad t \in I. \tag{5}$$

Step 2 We now show that there exists some $\tilde{x}^u \in \mathcal{S}(u)$ such that $J(\tilde{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u) = J(u)$. To do this, we first prove that for each $u \in U_{\mathrm{ad}}$, the set $\{x^u_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}(I, \mathcal{H})$. Note that

$$\begin{aligned} x_n^u(t) &= S_{\alpha,1}^E(t)x_0 + (g_1 * S_{\alpha,1}^E)(t)x_1 + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)\Sigma(s, x_n^u(s))\mathrm{d}W(s) \\ &\quad + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)[f(s, x_n^u(s)) + \mathfrak{B}(s)u(s)]\mathrm{d}s \\ &:= I_1 x_n^u + I_2 x_n^u + I_3 x_n^u + I_4 x_n^u. \end{aligned}$$

From (A1), (A6), Lemmas 3.3–3.4 and Remark 5.1, and analogously to Steps 3–5 in Theorem 4.1, we can similarly obtain that $\{I_1 x_n^u\}_{n\in\mathbb{N}}$, $\{I_2 x_n^u\}_{n\in\mathbb{N}}$, $\{I_3 x_n^u\}_{n\in\mathbb{N}}$ and $\{I_4 x_n^u\}_{n\in\mathbb{N}}$ are all precompact subsets of $\mathcal{C}(I,\mathcal{H})$. Consequently, the set $\{x_n^u\}_{n\in\mathbb{N}}$ is precompact in $\mathcal{C}(I,\mathcal{H})$ for $u \in U_{ad}$. Without loss of generality, we may assume that $x_n^u \to \tilde{x}^u$ in $\mathcal{C}(I,\mathcal{H})$ for $u \in U_{ad}$ as $n \to \infty$. Let $n \to \infty$ in both sides of (5), by the Lebesgue-dominated convergence theorem, we obtain that

$$\begin{aligned} \tilde{x}^u(t) &= S_{\alpha,1}^E(t)x_0 + (g_1 * S_{\alpha,1}^E)(t)x_1 + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)\Sigma(s, \tilde{x}^u(s))\mathrm{d}W(s) \\ &\quad + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)[f(s, \tilde{x}^u(s)) + \mathfrak{B}(s)u(s)]\mathrm{d}s, \quad t \in I, \end{aligned}$$

which implies that $\tilde{x}^u \in \mathcal{S}(u)$. Now we claim that $J(\tilde{x}^u, u) = \inf_{x^u \in \mathcal{S}(u)} J(x^u, u) = J(u)$ for any $u \in U_{\mathrm{ad}}$. In fact, owing to $\mathcal{C}(I,\mathcal{H})$ is continuously embedded in $L^1(I,\mathcal{H})$, through the definition of a feasible pair, the assumption (A5) and Balder theorem ([20]), imply

$$J(u) = \lim_{n\to\infty} \int_0^b \mathfrak{L}(t, x_n^u(t), u(t))\mathrm{d}t \geq \int_0^b \mathfrak{L}(t, \tilde{x}^u(t), u(t))\mathrm{d}t = J(\tilde{x}^u, u) \geq J(u),$$

i.e. $J(\tilde{x}^u, u) = J(u)$. This shows that $J(u)$ attains its minimum at $\tilde{x}^u \in \mathcal{C}(I,\mathcal{H})$ for each $u \in U_{ad}$.

Step 3 It is shown that there exists $u^0 \in U_{ad}$ such that $J(u^0) \leq J(u)$ for all $u \in U_{\mathrm{ad}}$. If $\inf_{u\in U_{\mathrm{ad}}} J(u) = +\infty$, then there is nothing to prove. Assume that $\inf_{u\in U_{\mathrm{ad}}} J(u) < +\infty$. Similarly to Step 1, we can prove that $\inf_{u\in U_{\mathrm{ad}}} J(u) > -\infty$, and there exists a sequence $\{u_n\} \subseteq U_{\mathrm{ad}}$ such that $J(u_n) \to \inf_{u\in U_{\mathrm{ad}}} J(u)$ as $n \to \infty$. Since $\{u_n\} \subseteq U_{\mathrm{ad}}$, $\{u_n\}$ is bounded in $L^2_{\mathcal{F}}(I, Y)$ and $L^2_{\mathcal{F}}(I, Y)$ is a reflexive Banach space, there exists a subsequence still denoted by $\{u_n\}$ weakly convergent to some $u^0 \in L^2_{\mathcal{F}}(I, Y)$ as $n \to \infty$. Note that U_{ad} is closed and convex, and by Lemma 3.7 it follows that $u^0 \in U_{ad}$.

Suppose $\tilde{x}^{u_n}$ is the mild solution to Eq. (1) related to u_n, where $J(u_n)$ attains its minimum. Then, $(\tilde{x}^{u_n}, u_n)$ is a feasible pair and verifies the following integral equation for $t \in I$,

$$\tilde{x}^{u_n}(t) = S_{\alpha,1}^E(t)x_0 + (g_1 * S_{\alpha,1}^E)(t)x_1 + \int_0^t (g_{\alpha-1} * S_{\alpha,1}^E)(t-s)\Sigma(s, \tilde{x}^{u_n}(s))\mathrm{d}W(s)$$

$$
\begin{aligned}
&+\int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s,\tilde{x}^{u_n}(s))+\mathfrak{B}(s)u_n(s)]\mathrm{d}s \\
&:= \Lambda_1\tilde{x}^{u_n}(t)+\Lambda_2\tilde{x}^{u_n}(t)+\Lambda_3\tilde{x}^{u_n}(t)+\Lambda_4\tilde{x}^{u_n}(t)+\Lambda_5 u_n(t).
\end{aligned} \tag{6}
$$

By (A1), Lemmas 3.3–3.4, and analogously to Steps 3–5 in Theorem 4.1, we can similarly obtain that the sets $\{\Lambda_1\tilde{x}^{u_n}\}_{n\in\mathbb{N}}$, $\{\Lambda_2\tilde{x}^{u_n}\}_{n\in\mathbb{N}}$, $\{\Lambda_3\tilde{x}^{u_n}\}_{n\in\mathbb{N}}$, $\{\Lambda_4\tilde{x}^{u_n}\}_{n\in\mathbb{N}}$ are all relatively compact subsets of $\mathcal{C}(I,\mathcal{H})$. Furthermore, by Lemma 5.1, $\Lambda_5 u_n \to \Lambda_5 u^0$ in $\mathcal{C}(I,\mathcal{H})$ as $n\to\infty$ and Λ_5 is compact. Thus, the set $\{\tilde{x}^{u_n}\}\subset\mathcal{C}(I,\mathcal{H})$ is relatively compact, and there exists a subsequence still denoted by $\{\tilde{x}^{u_n}\}$, and $\tilde{x}^{u^0}\in\mathcal{C}(I,\mathcal{H})$ such that $\tilde{x}^{u_n}\to\tilde{x}^{u^0}$ in $\mathcal{C}(I,\mathcal{H})$ as $n\to\infty$. If $n\to\infty$ in both sides of (6), then we have

$$
\begin{aligned}
\tilde{x}^{u^0}(t) &= S^E_{\alpha,1}(t)x_0+(g_1 * S^E_{\alpha,1})(t)x_1+\int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)\Sigma(s,\tilde{x}^{u^0}(s))\mathrm{d}W(s) \\
&\quad +\int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)[f(s,\tilde{x}^{u^0}(s))+\mathfrak{B}(s)u^0(s)]\mathrm{d}s, \quad t\in I,
\end{aligned}
$$

which implies that $(\tilde{x}^{u^0},u^0)$ is a feasible pair. Since $\mathcal{C}(I,\mathcal{H})$ is continuously embedded in $L^1(I,\mathcal{H})$, by the assumption (A5) and Balder theorem ([20]), we have

$$
\begin{aligned}
\inf_{u\in U_{\mathrm{ad}}} J(u) &= \lim_{n\to\infty}\int_0^b \mathfrak{L}(t,\tilde{x}^{u_n}(t),u_n(t))\mathrm{d}t \geq \int_0^b \mathfrak{L}(t,\tilde{x}^{u^0}(t),u^0(t))\mathrm{d}t \\
&= J(\tilde{x}^{u^0},u^0) \geq \inf_{u\in U_{\mathrm{ad}}} J(u).
\end{aligned}
$$

Thus, $J(\tilde{x}^{u^0},u^0) = J(u^0) = \inf_{x^{u^0}\in\mathcal{S}(u^0)} J(x^{u^0},u^0)$. Moreover, $J(u^0)=\inf_{u\in U_{ad}} J(u)$, i.e., J attains its minimum at $u^0\in U_{ad}$. This finishes the proof. □

Theorem 5.2 Assume that conditions (H2), (A2)–(A5) and (4) hold. Then, the problem (LP) governed by (2) admits at least one optimal feasible pair.

Proof According to Remark 5.1, $S^E_{\alpha,\alpha-1}(t)$ is compact for all $t>0$ under the condition (H2). Since the operator $E(\lambda^\alpha E-A)^{-1}$ is compact for all $\lambda^\alpha\in\rho(A)$ with $\lambda>\omega^{\frac{1}{\alpha}}$ (see (H2)), the operator $(g_1 * S^E_{\alpha,\alpha-1})(t)=S^E_{\alpha,\alpha}(t)$ is compact for all $t>0$ by Lemma 3.3. Considering Lemma 5.1, we can proceed the remainder analogously to that of Theorem 5.1. We omit the details here. □

6 Some Applications

In this section, we present some applications. We consider the following fractional stochastic evolution equations of Sobolev type with $1<\alpha<2$, $t\in I$

$$
\partial_t^\alpha(Ex)(t) = Ax(t)+J^{2-\alpha}[f(t,x(t))+\mathfrak{B}(t)u(t)]+\Sigma(t,x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t},
$$

$$Ex(0)=Ex_0,\,(Ex)'(0)=Ex_1,\;u\in U_{\rm ad}, \tag{7}$$

and

$$\begin{aligned}&\partial^\alpha(Ex)(t)=Ax(t)+J^{2-\alpha}[f(t,x(t))+\mathfrak{B}(t)u(t)]+\Sigma(t,x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t},\\ &(E(g_{2-\alpha}*x))(0)=Ex_0,\,(E(g_{2-\alpha}*x))'(0)=Ex_1,u\in U_{\rm ad}.\end{aligned} \tag{8}$$

Under the condition (A1), the mild solution to Eq. (7) is given by

$$\begin{aligned}x(t)=&\,S^E_{\alpha,1}(t)x_0+(g_1*S^E_{\alpha,1})(t)x_1+\int_0^t(g_1*S^E_{\alpha,1})(t-s)f(s,x(s))\mathrm{d}s\\ &+\int_0^t(g_1*S^E_{\alpha,1})(t-s)\mathfrak{B}(s)u(s)\mathrm{d}s+\int_0^t(g_1*S^E_{\alpha,1})(t-s)\Sigma(s,x(s))\mathrm{d}W(s)\end{aligned}$$

and by condition (H1), the mild solution of Eq. (8) can be written as

$$\begin{aligned}x(t)=&\,S^E_{\alpha,\alpha-1}(t)x_0+(g_1*S^E_{\alpha,\alpha-1})(t)x_1+\int_0^t(g_{3-\alpha}*S^E_{\alpha,\alpha-1})(t-s)\Sigma(s,x(s))\mathrm{d}W(s)\\ &+\int_0^t(g_{3-\alpha}*S^E_{\alpha,\alpha-1})(t-s)[f(s,x(s))+\mathfrak{B}(s)u(s)]\mathrm{d}s.\end{aligned}$$

Since $\left\|(g_1*S^E_{\alpha,1})(t)\right\|\le\frac{Me^{\omega t}}{\omega}$, and $\left\|(g_{3-\alpha}*S^E_{\alpha,\alpha-1})(t)\right\|\le\frac{Me^{\omega t}}{\omega^{3-\alpha}}$, we can prove the following results similarly to Theorems 5.1–5.2.

Lemma 6.1 *Under conditions (A1)–(A6) and (4), the problem (LP) governed by Eq. (7) admits at least one optimal feasible pair.*

Lemma 6.2 *If* $10\frac{M^2e^{2\omega b}b}{\omega^{2(3-\alpha)}}\left[\frac{1}{2}\|\phi\|_{L^1}+L_\Sigma\right]<1$ *and (H2), (A2)–(A5) hold, then the problem (LP) governed by (8) has at least one optimal feasible pair.*

Example 6.1 Consider the following fractional system of order $\alpha\in]1,2[$

$$\begin{aligned}\partial_t^\alpha\left[x(t,\xi)-\frac{\partial^2x}{\partial\xi^2}(t,\xi)\right]&=\frac{\partial^4x}{\partial\xi^4}(t,\xi)+f(t,x(t,\xi))\\ &\quad+u(t,\xi)+\sigma(t,x(t,\xi))\frac{\mathrm{d}W(t)}{\mathrm{d}t},\\ x(t,0)&=x(t,\pi)=0,\quad t\in[0,1],\\ x(0,\xi)&=x_0(\xi),\quad \xi\in[0,\pi],\\ x_t(0,\xi)&=x_1(\xi),\quad \xi\in[0,\pi],\end{aligned} \tag{9}$$

where $f(t,x(t,\xi)):=\frac{e^{-t}x(t,\xi)}{(360+t)(1+|x(t,\xi)|)}$, $\sigma(t,x(t,\xi)):=\frac{e^{-t}x(t,\xi)}{(720+t)(1+|x(t,\xi)|)}$. Take $J(x,u):=\mathbb{E}\left\{\int_0^\pi\int_0^1|x(t,\xi)|^2\mathrm{d}t\mathrm{d}\xi+\int_0^\pi\int_0^1|u(t,\xi)|^2\mathrm{d}t\mathrm{d}\xi\right\}$, and $x(\cdot)(\xi):=x(\cdot,\xi)$,

$\mathfrak{B}(\cdot)u(\cdot)(\xi) := u(\cdot,\xi)$. Set $\mathcal{H} = Y := L^2[0,\pi]$, $W(t)$ is a standard Brownian motion in $\mathcal{H}$ defined on a stochastic space $(\Omega,\mathcal{F},\mathbb{P})$. Define $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ and $E : D(E) \subset \mathcal{H} \to \mathcal{H}$, respectively by $Ax := -\frac{\partial^4 x}{\partial \xi^4}$ and $Ex := x - \frac{\partial^2 x}{\partial \xi^2}$, with domain $D(E) = D(A) := \left\{x \in \mathcal{H} : x \in H^4([0,\pi]), x(t,0) = x(t,\pi) = 0\right\}$. It is known that A has discrete spectrum with eigenvalues $-n^4, n \in \mathbb{N}$, with corresponding eigenvectors $x_n(s) := \sqrt{\frac{2}{\pi}}\sin(ns)$. Furthermore, $\{x_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$, and operators A, E can be expressed by (see [10]) $Ax = -\sum_{n=1}^{\infty} n^4\langle x, x_n\rangle x_n)$, and $Ex = \sum_{n=1}^{\infty}(1+n^2)\langle x, x_n\rangle x_n$. According to [17, Example 6.3], we conclude that (A, E) generates the $(\alpha,1)$-resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ given by $S^E_{\alpha,1}(t)x = \sum_{n=1}^{\infty} h^n_{\alpha,1}(t)\langle x, x_n\rangle x_n$, for all $x \in \mathcal{H}$, where $h^n_{\alpha,1}(t) := e_{\alpha,1}\left(-\frac{n^4}{n^2+1}t^\alpha\right)$, and $e_{\alpha,1}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$. Moreover (see also [17, Example 6.3]), the operator $E(\lambda^\alpha E - A)^{-1}$ is compact on $\mathcal{H}$, the operator $S^E_{\alpha,1}(t)$ is norm continuous and satisfies $\left\|S^E_{\alpha,1}(t)x\right\| \leq 2\|x\|$ for each $x \in \mathcal{H}$. Thus, $S^E_{\alpha,1}(t)$ is of type $(2,1)$, i.e. $M = 2$ and $\omega = 1$. Let $I := [0,1]$ and $\Sigma(t,x) := \sigma(t,x)$. Then, Problem (9) can be written in the abstract form (1). Note that in this case $\|\phi\|_{L^1} \leq \frac{1}{360}$, $L_\Sigma \leq \frac{1}{720}$, $b = \omega = 1$, and thus $10\frac{M^2 e^{2\omega b} b}{\omega^{2(\alpha-1)}}\left[\frac{1}{2}\|\phi\|_{L^1} + L_\Sigma\right] \leq \frac{e^2}{9} < 1$. According to Theorems 4.1 and 5.1, there exists a mild solution to (9), and its corresponding limited Lagrange problem admits at least one optimal feasible pair.

7 Conclusions

In this paper, some sufficient conditions are established for the existence of mild solutions and optimal state-control pairs to Sobolev-type fractional stochastic evolution equations of order $\alpha \in]1,2[$. The main results are investigated under the mixed Carathéodory and Lipschitz conditions via compactness of fractional resolvent operator family, and thus the optimal control results are derived without uniqueness of solutions for addressed evolution equations. We propose to investigate the existence of solutions to Sobolev-type fractional stochastic equations by the uniform continuity or some decay properties of the resolvent family in future works.

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