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On a Sufficient Condition for Weak Sharp Efficiency in Multiobjective Optimization

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Abstract In this paper, we provide sufficient conditions entailing the existence of weak sharp efficient points of a multiobjective optimization problem. The approach uses variational analysis techniques, like regularity and subregularity of the diagonal subdifferential map related to a suitable scalar equilibrium problem naturally associated to the multiobjective optimization problem.

Keywords Weak sharp efficient point · Multiobjective optimization problem · Metric regularity · Metric subregularity · Equilibrium problem · Diagonal subdifferential operator

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1 Introduction

In the last decades, the theory and the methods of multiobjective optimization problems (MOP, in what follows) have attracted much attention from the researchers community

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(for a thorough presentation see, for instance [1–3]). Along the years, different solution concepts for MOP have been introduced and studied. Among them, without a doubt, one of the most investigated is the weak efficiency. Aiming at investigating the Hölder regularity of the solution set-valued mapping of perturbed multiobjective optimization problems, Bednarczuk considered in [4] (see also [5], and [6] for a slightly different definition) the more restrictive notion of weak sharp efficiency, where a growth condition on the vector-valued function is imposed. This approach was first proposed by Burke and Ferris [7] for scalar optimization problems, in order to overcome the strong condition of isolatedness of the local sharp minima. Later, it was applied in many other optimality frameworks, like variational inequalities and vector optimization problems, in order to analyse the finite convergence of approximation algorithms (see, for instance [8–10] and the references therein).

Inspired by the aforementioned ideas, in this paper we focus on the weak sharp efficiency proposed in [4]. Our purpose is to find sufficient conditions on the objective function entailing the existence of weak sharp efficient points, in their local version. The approach takes advantage of variational analysis techniques, already used by the authors, when studying sensitivity of equilibrium problems (see [11]), like regularity and subregularity of the diagonal subdifferential map.

In [4], the condition of weak sharp efficiency is investigated through the Hölder regularity of the ε-solution multifunction. Our goal is rather to find conditions on the components of a vector function, that guarantee the existence of weak sharp efficient points. The idea goes through a reformulation of the MOP via a suitable equilibrium problem, whose solution set is exactly the set of weak efficient points of the aforementioned vector function. In the same way, the set of its weak sharp efficient points can be identified with the solution set of a "stronger" equilibrium problem, whose solutions can be characterized by a suitable metric regularity of the so-called *diagonal subdifferential operator* already studied by the authors in [11].

The paper is organized as follows: in Sect. 2, we recall some regularity and subregularity notions of maps needed in the sequel, together with some lemmata. In Sect. 3, we investigate how the regularity properties of each function belonging to a finite family are inherited by the convex hull map of this family. In the last section, we use the previous results to give sufficient conditions for weak sharp efficiency.

2 Preliminaries and Notations

Let E, F be metric spaces, and $T : E \rightrightarrows F$ be a set-valued map. Denote by gph T the graph of T defined as $\operatorname{gph} T := \{(x, y) \in E \times F : y \in T(x)\}$, by dom T the domain of T defined as $\operatorname{dom} T := \{x \in E : T(x) \neq \emptyset\}$, and by $T^{-1} : F \rightrightarrows E$ the inverse map defined as $T^{-1}(y) := \{x \in E : y \in T(x)\}$. For any subsets A, B of a metric space E, the excess of A beyond B is defined as

$$e(A, B) := \sup_{a \in A} d(a, B) = \sup_{a \in A} \inf_{b \in B} d(a, b),$$

under the convention $e(\emptyset, B) := 0$, and $e(A, \emptyset) := +\infty$, for $A \neq \emptyset$.

Let us now recall some preliminary notions concerning the regularity of a map (see, for instance [12,13]). We will denote by $B(x,r)$ the open ball centered at x with radius r. Given a point $(\overline{x},\overline{y}) \in \operatorname{gph} T$, the set-valued map T is said to be:

(i) *Metrically regular* around $(\overline{x},\overline{y})$, iff there is a positive constant k, along with neighbourhoods $\mathcal{U}$ of $\overline{x}$ and $\mathcal{V}$ of $\overline{y}$, such that

$$d(x, T^{-1}(y)) \leq k d(y, T(x)), \quad \forall x \in \mathcal{U}, y \in \mathcal{V}; \tag{1}$$

(ii) *Metrically subregular* at $(\overline{x},\overline{y})$, iff there is a positive constant k, along with a neighbourhood $\mathcal{U}$ of $\overline{x}$, such that

$$d(x, T^{-1}(\overline{y})) \leq k d(\overline{y}, T(x)), \quad \forall x \in \mathcal{U}; \tag{2}$$

(iii) *Strongly metrically subregular* at $(\overline{x},\overline{y})$, iff there is a positive constant k, along with a neighbourhood $\mathcal{U}$ of $\overline{x}$, such that

$$d(x, \overline{x}) \leq k d(\overline{y}, T(x)), \quad \forall x \in \mathcal{U}; \tag{3}$$

(iv) *Calm* at $(\overline{x},\overline{y})$, iff there exist a positive constant k, and neighbourhoods $\mathcal{U}$ of $\overline{x}$ and $\mathcal{V}$ of $\overline{y}$, such that

$$e(T(x) \cap \mathcal{V}, T(\overline{x})) \leq k d(x, \overline{x}), \quad \forall x \in \mathcal{U}.$$

The metric regularity of T around $(\overline{x},\overline{y})$, with constant k, is equivalent to the *linear openness* of T around $(\overline{x},\overline{y})$; that is, there exist positive constants k and τ, along with neighbourhoods $\mathcal{U}$ of $\overline{x}$ and $\mathcal{V}$ of $\overline{y}$, such that

$$B(y,\rho) \subset T(B(x,k\rho)), \qquad \forall (x,y) \in \operatorname{gph} T \cap (\mathcal{U} \times \mathcal{V}), \quad 0 < \rho < \tau.$$

Similarly, the metric subregularity of T at $(\overline{x},\overline{y})$ is equivalent to the calmness of T^{-1} at $(\overline{y},\overline{x})$ (see, for instance, Propositions 2.7 and 2.2 in [14]). Moreover, it follows directly from the definition that, if T is strongly metrically subregular at $(\overline{x},\overline{y})$, then $T^{-1}(\overline{y}) \cap \mathcal{U} = \{\overline{x}\}$.

In the sequel, our results will be stated in the framework of Euclidean spaces. The space $\mathbb{R}^k$ will be endowed with the ℓ_1-norm, i.e., $\|x\| = \sum_{i=1}^k |x_i|$, for every $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. Denote by Σ_{k-1} the simplex in $\mathbb{R}^k$ given by

$$\Sigma_{k-1} = \left\{ \lambda \in \mathbb{R}^k_+ : \|\lambda\| = 1 \right\}.$$

The usual inner product will be denoted by $\langle \cdot, \cdot \rangle$.

Let us consider a vector function $F = (f_1, f_2, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, and the associated multiobjective optimization problem

$$\min_{x \in \mathbb{R}^n} F(x), \tag{MOP}$$

where the ordering cone in $\mathbb{R}^m$ is given by the nonnegative orthant.

We will denote by WE_F the set of all global weakly efficient points of the MOP, i.e., $x \in \mathrm{WE}_F$ if and only if $F(z) \notin F(x) - \mathrm{int}(\mathbb{R}^m_+)$, for all $z \in \mathbb{R}^n$.

Under the assumption that f_i is differentiable for every $i = 1, 2, \ldots, m$, we associate to F the set-valued map $H_F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as

$$x \mapsto H_F(x) = \mathrm{conv}(\nabla f_1(x), \ldots, \nabla f_m(x)),$$

where $\mathrm{conv}(y_1, y_2, \ldots, y_m) = \{\sum_{i=1}^m \lambda_i y_i,\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i = 1\}$. It is well known that, under the additional assumption that f_i was convex on $\mathbb{R}^n$ for every $i = 1, 2, \ldots, m$, $0 \in H_F(\bar{x})$ if and only if $\overline{x} \in \mathrm{WE}_F$; that is,

$$\mathrm{WE}_F = H_F^{-1}(0).$$

Indeed, the convexity of the functions f_i entails that the set $F(\mathbb{R}^n) + \mathbb{R}^m_+$ is a convex subset of $\mathbb{R}^m$. From [1], Ch. 5, Corollary 5.29, $\overline{x} \in \mathrm{WE}_F$, if and only if there exists $\lambda \in \Sigma_{m-1}$ such that

$$\langle \lambda, F(\overline{x}) \rangle \leq \langle \lambda, F(x) \rangle, \quad \forall x \in \mathbb{R}^n,$$

i.e., $\overline{x}$ is a global minimum for the function $x \mapsto \langle \lambda, F(x) \rangle$. By the differentiability assumption on f_i, $i = 1, 2, \ldots, m$, this is equivalent to $\sum_{i=1}^m \lambda_i \nabla f_i(\overline{x}) = 0$, i.e., $0 \in H_F(\overline{x})$.

The more restrictive notion of weak sharp efficient points of order $\beta > 0$, WSE_F^β, of the function F, has been defined in [4] as

$$x \in \mathbb{R}^n : \quad F(z) \notin F(x) + \alpha d^\beta(z, \mathrm{WE}_F) B(0, 1) - \mathbb{R}^m_+, \tag{4}$$

for some positive α, and for all $z \in \mathbb{R}^n \setminus \mathrm{WE}_F$.

Let us now recall some monotonicity notions, that will play a significant role in the next results. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *locally monotone at* $\overline{x} \in \mathbb{R}^n$, iff there exists a neighbourhood $\mathcal{U}(\overline{x})$ of $\overline{x}$ such that

$$\langle f(x) - f(\overline{x}), x - \overline{x} \rangle \geq 0, \qquad \forall x \in \mathcal{U}(\overline{x}). \tag{5}$$

In particular, if

$$\langle f(x) - f(\overline{x}), x - \overline{x} \rangle \geq \alpha \|x - \overline{x}\|^2, \qquad \forall x \in \mathcal{U}(\overline{x}), \tag{6}$$

for some $\alpha > 0$, then we say that f is *locally strongly monotone at* $\overline{x}$. Note that, if f is locally strongly monotone at $\overline{x}$, then it is strongly metrically subregular at $(\overline{x}, f(\overline{x}))$, with constant $1/\alpha$. In addition, $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *locally monotone around* $\overline{x} \in \mathbb{R}^n$, iff there exists $\mathcal{U}(\overline{x})$ such that

$$\langle f(x) - f(x'), x - x' \rangle \geq 0, \qquad \forall x, x' \in \mathcal{U}(\overline{x}). \tag{7}$$

Moreover, if

$$\langle f(x) - f(x'), x - x' \rangle \geq \alpha \|x - x'\|^2, \qquad \forall x, x' \in \mathcal{U}(\overline{x}), \tag{8}$$

for some $\alpha > 0$, we say that f is *locally strongly monotone around* $\overline{x}$. In the sequel, when the neighbourhood is already given, the local monotonicity around a point will be simply denoted by monotonicity in the neighbourhood.

Let us now prove a relationship between strong monotonicity and metric regularity.

Lemma 2.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and locally strongly monotone around $\overline{x}$, with constant $\alpha > 0$. Then, f is metrically regular around $(\overline{x}, f(\overline{x}))$ along with a suitable $\mathcal{U}'(\overline{x})$ and $\mathcal{V}(f(\overline{x})) = \mathbb{R}^n$, and constant $k = 1/\alpha$.*

Proof Denote by $\mathcal{U}(\overline{x})$ the neighbourhood of $\overline{x}$ where f is strongly monotone. Let $\overline{r} > 0$ be such that $B(\overline{x}, 2\overline{r}) \subset \mathcal{U}(\overline{x})$, and set $B' = \operatorname{cl} B(\overline{x}, 2\overline{r})$. We will prove that f is open at linear rate around $(\overline{x}, f(\overline{x}))$, i.e., there exists $\tau > 0$ such that, for every $\rho \in]0, \tau[$ and every $x \in B(\overline{x}, \overline{r}) = \mathcal{U}'(\overline{x})$,

$$B(f(x), \rho) \subset f(B(x, \rho/\alpha)).$$

Then, the result will follow by the well-known equivalence between metric regularity and openness at linear rate (see, for instance [14]). Consider the normal cone $N_{B'}(x)$ to B' at $x \in B'$, i.e.,

$$N_{B'}(x) := \{u \in \mathbb{R}^n : \langle u, x' - x \rangle \leq 0, \ \forall x' \in B'\},$$

and define the operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as follows:

$$T(x) = \begin{cases} f(x) + N_{B'}(x), & x \in B', \\ \emptyset, & x \notin B'. \end{cases}$$

By a classical result on monotone operators (see for instance [15]), T is maximal monotone. Since $\operatorname{dom} T$ is bounded, Theorem 2.17 in [16] entails that T is surjective. In addition,

$$\langle u - u', x - x' \rangle \geq \alpha \|x - x'\|^2, \quad \forall x, x' \in B', \forall u \in T(x), u' \in T(x'). \tag{9}$$

Fix now $\tau = \overline{r}\alpha$, let $\rho \in]0, \tau[$, and choose $x \in B(\overline{x}, \overline{r})$, and $y' \in B(f(x), \rho)$. By (9), T is one-to-one, therefore, taking into account the surjectivity of T, there exists a unique $x' \in B'$ such that $y' \in T(x')$. Since $T(x) = \{f(x)\}$, by (9), where $u = f(x)$, $u' = y'$, we get

$$\|x' - x\| \leq \frac{1}{\alpha} \|y' - f(x)\| < \frac{\rho}{\alpha},$$

hence $x' \in B(x, \rho/\alpha)$. Moreover, $\|x' - \overline{x}\| < \rho/\alpha + \overline{r} < 2\overline{r}$; this implies that $x' \in B(\overline{x}, 2\overline{r})$, and thus $T(x') = \{f(x')\}$, and $y' \in f(B(x, \rho/\alpha))$, thereby proving the openness of f around $(\overline{x}, f(\overline{x}))$, with constant $1/\alpha$. □

3 Regularity Properties of the Convex Hull Map

Let $\mathbb{R}^{nm}$ be the product space $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m \text{ times}}$, and define the convex hull map $\text{conv} : \mathbb{R}^{nm} \rightrightarrows \mathbb{R}^n$ as follows:

$$\text{conv}\left(x^1, x^2, \ldots, x^m\right) = \left\{\sum_{i=1}^{m} \lambda_i x^i,\ \lambda \in \Sigma_{m-1}\right\}.$$

Given the functions $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$, let us consider the closed and convex valued map $\text{conv}(g_1, g_2, \ldots, g_m) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as

$$\text{conv}(g_1, \ldots, g_m)(x) := \text{conv}(g_1(x), g_2(x), \ldots, g_m(x)).$$

This section is devoted to the investigation of metric regularity and calmness properties of the map $\text{conv}(g_1, g_2, \ldots, g_m)$.

Following [12], Section 1.3., let us first recall that a map $h : \mathbb{R}^n \to \mathbb{R}^l$ is said to be *calm* at $\overline{x} \in \mathbb{R}^n$ with constant k, iff there exists a neighbourhood $\mathcal{U}(\overline{x})$ of $\overline{x}$ such that

$$\|h(x) - h(\overline{x})\| \leq k\|x - \overline{x}\|, \quad \forall x \in \mathcal{U}(\overline{x}).$$

When the inequality is satisfied for every point $x' \in \mathcal{U}(\overline{x})$, and not only for $\overline{x}$, then h is said to be locally Lipschitz at $\overline{x}$, or Lipschitz on $\mathcal{U}(\overline{x})$. The following proposition holds:

Proposition 3.1 *Let $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$ and let $\overline{x} \in \mathbb{R}^n$. If we suppose that all the functions g_i are calm at $\overline{x}$ with constant k_i and common neighbourhood $\mathcal{U}(\overline{x})$, then the set-valued map $\text{conv}(g_1, g_2, \ldots, g_m)$ is calm at $(\overline{x}, \overline{y})$ for every $\overline{y} \in \text{conv}(g_1, g_2, \ldots, g_m)(\overline{x})$, with constant $k = m \cdot \max_i \{k_i\}$ and neighbourhood $\mathcal{U}(\overline{x})$.*

Proof Let us first prove that the map $\text{conv} : \mathbb{R}^{nm} \rightrightarrows \mathbb{R}^n$ satisfies

$$e\left(\text{conv}(x^1, x^2, \ldots, x^m), \text{conv}(y^1, y^2, \ldots, y^m)\right) \leq \sum_{i=1}^{m} \|x^i - y^i\| \tag{10}$$

for every $(x^1, x^2, \ldots, x^m), (y^1, y^2, \ldots, y^m) \in \mathbb{R}^{nm}$. Indeed, let us take two points $(x^1, x^2, \ldots, x^m),\ (y^1, y^2, \ldots, y^m) \in \mathbb{R}^{nm}$ and $\lambda \in \Sigma_{m-1}$. Hence,

$\sum_{i=1}^{m} \lambda_i x^i \in \operatorname{conv}(x^1, x^2, \ldots, x^m)$, $\sum_{i=1}^{m} \lambda_i y^i \in \operatorname{conv}(y^1, y^2, \ldots, y^m)$. Then,

$$\left\| \left(\sum_{i=1}^{m} \lambda_i x^i \right) - \left(\sum_{i=1}^{m} \lambda_i y^i \right) \right\| = \left\| \sum_{i=1}^{m} \lambda_i (x^i - y^i) \right\| \leq \sum_{i=1}^{m} \|x^i - y^i\|.$$

This implies that, for every $z \in \operatorname{conv}(x^1, x^2, \ldots, x^m)$,

$$d\left(z, \operatorname{conv}(y^1, y^2, \ldots, y^m)\right) \leq \sum_{i=1}^{m} \|x^i - y^i\|,$$

and therefore (10) holds.

Now, for every $x \in \mathcal{U}(\overline{x})$, we obtain

$$\begin{aligned} e(\operatorname{conv}(g_1(x), \ldots, g_m(x)), \operatorname{conv}(g_1(\overline{x}), \ldots, g_m(\overline{x}))) &\leq \sum_{i=1}^{m} \|g_i(x) - g_i(\overline{x})\| \\ &\leq k\|x - \overline{x}\|, \end{aligned}$$

and the assertion easily follows with $\mathcal{V}(\overline{y}) = \mathbb{R}^n$. □

In order to investigate some metric regularity properties of the map $\operatorname{conv}(g_1, \ldots, g_m)$, we introduce the selections $g^\lambda : \mathbb{R}^n \to \mathbb{R}^n$, defined as

$$g^\lambda(x) = \sum_{i=1}^{m} \lambda_i g_i(x), \quad \lambda \in \Sigma_{m-1}.$$

Our first result provides a sufficient condition for the map g^λ to be strongly metrically subregular. The proof of this property is based on a result related to the inverse mapping theorem for strong metric subregularity (see [12], Section 3.9). As a matter of fact, according to [12], such an inverse function theorem cannot be stated if strong metric subregularity is relaxed to metric subregularity, since this last property is not stable under perturbations.

Lemma 3.1 (Theorem 3I.7 in [12]) *Let $\phi, \psi : \mathbb{R}^n \to \mathbb{R}^n$ be maps, and let $\overline{x} \in \mathbb{R}^n$. If ϕ is strongly metrically subregular at $\overline{x}$ with constant k, and ψ is calm at $\overline{x}$ with constant L, with a common neighbourhood $\mathcal{U}(\overline{x})$, and $Lk < 1$, then $\phi + \psi$ is strongly metrically subregular at $\overline{x}$ with constant $\frac{k}{1-Lk}$ within $\mathcal{U}(\overline{x})$.*

We can now prove the mentioned result about g^λ:

Theorem 3.1 *Let $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$, and let $\overline{x}$ be a point in $\mathbb{R}^n$. Suppose that the maps satisfy the following assumptions:*

(i) *There exists $i_0 \in \{1, 2, \ldots, m\}$ such that g_{i_0} is locally monotone and strongly metrically subregular at $\overline{x}$, with constant k;*
(ii) *For every $i \in \{1, 2, \ldots, m\} \setminus \{i_0\}$, g_i is calm at $\overline{x}$, with constant L_i, and locally strongly monotone at $\overline{x}$, with constant α_i.*

Then the function g^λ is strongly metrically subregular at $(\overline{x}, g^\lambda(\overline{x}))$ with constant

$$k' = k + \sum_{i=1,\, i\neq i_0}^{m} \frac{1 + kL_i}{\alpha_i},$$

for every $\lambda \in \Sigma_{m-1}$.

Proof Without loss of generality, let $i_0 = 1$. Fix $\varepsilon_i > 0$, $i = 2, \ldots, m$, such that

$$\sum_{i=2}^{m} (1 + kL_i)\varepsilon_i < 1. \tag{11}$$

Suppose first that $\lambda_i \in [0, \varepsilon_i]$ for every $i = 2, \ldots, m$. Then, by (11), we have $\sum_{i=2}^{m} \varepsilon_i < 1$, and hence $\lambda_1 > 0$.

Now, let us consider the map $g : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows

$$g(x) = \sum_{i=2}^{m} \lambda_i g_i(x).$$

Obviously, g is calm at $\overline{x}$ with constant $\sum_{i=2}^{m} \lambda_i L_i$. Furthermore, since g_1 is metrically subregular at $\overline{x}$ with constant k, then $\lambda_1 g_1$ is metrically subregular at $\overline{x}$ with constant k/λ_1. Since $k \sum_{i=2}^{m} \lambda_i L_i < \lambda_1$, then, by Lemma 3.1, the function $g^\lambda = \lambda_1 g_1 + g$ is strongly metrically subregular at $(\overline{x}, g^\lambda(\overline{x}))$ with constant

$$\frac{k}{\lambda_1 - k \sum_{i=2}^{m} \lambda_i L_i}. \tag{12}$$

Moreover, since $\lambda_i \in [0, \varepsilon_i]$ for every $i = 2, \ldots, m$, we have

$$\frac{k}{\lambda_1 - k \sum_{i=2}^{m} \lambda_i L_i} \le \frac{k}{(1 - \sum_{i=2}^{m} \varepsilon_i) - k \sum_{i=2}^{m} \varepsilon_i L_i}.$$

Therefore, the constant in (12) can be taken independent on λ, and is given by

$$\frac{k}{(1 - \sum_{i=2}^{m} \varepsilon_i) - k \sum_{i=2}^{m} \varepsilon_i L_i}.$$

It remains to consider the case where the condition: $\lambda_i \in [0, \varepsilon_i]$ for every $i = 2, \ldots, m$, is not satisfied. In this case, $\lambda = (\lambda_1, \ldots, \lambda_m) \in \cup_{i=2}^{m} S_i$, where

$$S_i = \{\lambda \in \Sigma_{m-1} : \ \lambda_i \in]\varepsilon_i, 1]\}.$$

If $\lambda \in S_i$, then the map $\lambda_i g_i$ is locally strongly monotone at $\overline{x}$ with constant $\lambda_i \alpha_i$; therefore, since $\lambda_j g_j$ is locally monotone at $\overline{x}$, for every $j \neq i$ (including $j = 1$), the map g^λ is locally strongly monotone with constant $\lambda_i \alpha_i$ at the same point. Since

$\lambda_i > \varepsilon_i$, the map g^λ is also locally strongly monotone at $\overline{x}$, with constant $\varepsilon_i \alpha_i$. Therefore, from the inequality

$$\varepsilon_i \alpha_i \|x - \overline{x}\|^2 \le \langle g^\lambda(x) - g^\lambda(\overline{x}), x - \overline{x} \rangle \le \|g^\lambda(x) - g^\lambda(\overline{x})\| \, \|x - \overline{x}\|,$$

it follows that g^λ is strongly metrically subregular at $\overline{x}$, with constant $1/\varepsilon_i \alpha_i$.

The last part of the proof is devoted to find a uniform constant of strong metric subregularity for every g^λ, when λ is any point in Σ_{m-1}. Let us consider the following linear system of $m-1$ equations with respect to $m-1$ unknowns $\varepsilon_2, \dots, \varepsilon_m$:

$$1 - \sum_{i=2}^{m} \varepsilon_i - k \sum_{i=2}^{m} \varepsilon_i L_i = k \varepsilon_j \alpha_j, \quad j = 2, \dots, m$$

It gives, as the unique solution, $\varepsilon_2, \dots, \varepsilon_m$ such that:

$$\frac{1}{\varepsilon_j} = \alpha_j \left(k + \sum_{i=2}^{m} \frac{1}{\alpha_i} + k \sum_{i=2}^{m} \frac{L_i}{\alpha_i} \right), \quad j = 2, \dots, m.$$

Note that $\sum_{i=2}^{m} (1 + kL_i) \varepsilon_i < 1$. With such a choice of ε_i, we have that, for every $\lambda \in \Sigma_{m-1}$, the function g^λ is strongly metrically subregular at $\overline{x}$ with constant k', where

$$k' = k + \sum_{i=2}^{m} \frac{1}{\alpha_i} + k \sum_{i=2}^{m} \frac{L_i}{\alpha_i} = k + \sum_{i=2}^{m} \frac{1 + kL_i}{\alpha_i}.$$

□

Remark 3.1 In Theorem 3.1, the monotonicity assumption plays a central role in proving the strong metric subregularity of the selections g^λ. Indeed, let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$g_1(x_1, x_2) = (x_1, x_2), \qquad g_2(x_1, x_2) = (3, 0) + (x_1, -x_2)$$

and let $\overline{x} = (0, 0)$. The function g_2 is not monotone. Since $g^\lambda(0, 0) = (3\lambda_2, 0)$, the inequality

$$\|x\| \le k \|g^\lambda(x_1, x_2) - g^\lambda(0, 0)\| = k \|(x_1, (\lambda_1 - \lambda_2) x_2)\|$$

cannot be fulfilled for any choice of k, if $\lambda_1 = \lambda_2 = 1/2$, and thus g^λ is not strongly metrically subregular. Moreover, for $\lambda_i \ne 1/2$, the function g^λ is strongly metrically subregular at $\overline{x}$, but the constant k cannot be chosen independent from λ.

Remark 3.2 The strong metric subregularity of the maps $\{g^\lambda\}_{\lambda \in \Sigma_{m-1}}$ does not imply the strong metric subregularity of the convex hull map. Take, for instance, $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g_1(x_1, x_1) = (x_1, x_2), \qquad g_2(x_1, x_2) = (3, 0) + (x_2, -x_1).$$

These functions satisfy the assumptions of Theorem 3.1 at (0, 0), with all constants equal to 1. From the final assertion of Theorem 3.1, the map g^λ is strongly metrically subregular at $((0, 0), g^\lambda(0, 0))$, for all $\lambda \in \Sigma_1$. We show that the set-valued map $T(x) := \text{conv}\{g_1(x_1, x_2), g_2(x_1, x_2)\}$ is not strongly metrically subregular at $((0, 0), (0, 0))$, i.e., it is not fulfilled that:

$$\|(x_1, x_2)\| \leq kd((0, 0), T(x_1, x_2)), \qquad (x_1, x_2) \in \mathcal{U}(0, 0),$$

for any $k > 0$. Indeed, standard computations show that $(0, 0) \in T(x_1, x_2)$ for every (x_1, x_2) such that $x_1^2 + x_2^2 + 3x_2 = 0$, and $x_1, x_2 \leq 0$. Thus the right-hand side evaluated at these points is 0, while the left-hand side is not.

For the reason explained in the remark above, we propose to study the metric regularity property of the map $\text{conv}(g_1, g_2, \ldots, g_m)$. To do this, a stronger version of Theorem 3.1 is required. The next result is the counterpart of Lemma 3.1 for metric regularity of the sum of two maps.

Lemma 3.2 (Theorem 3F.1 in [12]) *Let $\phi, \psi : \mathbb{R}^n \to \mathbb{R}^n$ be maps, and let $\overline{x} \in \mathbb{R}^n$. If ϕ is metrically regular at $\overline{x}$ with constant k, and ψ is Lipschitz with constant L, with common neighbourhood $\mathcal{U}(\overline{x})$, and $Lk < 1$, then $\phi + \psi$ is metrically regular at $\overline{x}$ with constant $\frac{k}{1-Lk}$ within $\mathcal{U}(\overline{x})$.*

Theorem 3.2 *Let $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$, and let $\overline{x}$ be a point in $\mathbb{R}^n$. Suppose that:*

(i) *There exists $i_0 \in \{1, 2, \ldots, m\}$ such that g_{i_0} is metrically regular around $(\overline{x}, g_{i_0}(\overline{x}))$ with constant k, and neighbourhoods $\mathcal{U}(\overline{x})$ and $\mathcal{V}(g_{i_0}(\overline{x}))$, monotone and continuous on $\mathcal{U}(\overline{x})$;*
(ii) *For every $i \in \{1, 2, \ldots, m\} \setminus \{i_0\}$, g_i is Lipschitz, with constant L_i, and strongly monotone, with constant α_i, on $\mathcal{U}(\overline{x})$.*

Then, for every $\lambda \in \Sigma_{m-1}$, the function g^λ is metrically regular around $(\overline{x}, g^\lambda(\overline{x}))$ with constant $k' = k + \sum_{i=1, i\neq i_0}^{m} \frac{1+kL_i}{\alpha_i}$, along with $\mathcal{U}(\overline{x})$ and $\mathbb{R}^n$.

Proof We follow the proof of Theorem 3.1. The case $\lambda_i \in [0, \varepsilon_i], i = 2, \ldots, m$ can be similarly discussed via Lemma 3.2. When this condition is not satisfied, the assumptions on the maps $\{g_j\}_{j=2}^m$ entail that there exists $i \in \{2, 3, \ldots, m\}$ such that the map g^λ is strongly monotone on $\mathcal{U}(\overline{x})$ with constant $\varepsilon_i\alpha_i$. Furthermore, g^λ is continuous as a sum of continuous maps. By Lemma 2.1, g^λ is metrically regular with constant $\frac{1}{\varepsilon_i\alpha_i}$. As in Theorem 3.1, we can find a uniform constant for metric regularity of all the functions g^λ, where $\lambda \in \Sigma_{m-1}$. □

Taking into account the uniform metric regularity of the selection maps g^λ, we are now able to prove the following result:

Theorem 3.3 *Under the assumptions of Theorem 3.2, the set-valued map $\text{conv}(g_1, \ldots, g_m)$ is metrically regular around $(\overline{x}, \overline{y})$, for all $\overline{y} \in \text{conv}(g_1, \ldots, g_m)(\overline{x})$, with constant $k' = k + \sum_{i=1, i\neq i_0}^{m} \frac{1+kL_i}{\alpha_i}$.*

Proof Let $x \in \mathcal{U}(\overline{x})$, and $y \in \text{conv}(g_1, \ldots, g_m)(x)$. Then $y = g^\lambda(x)$ for some $\lambda \in \Sigma_{m-1}$. By Theorem 3.2, g^λ is metrically regular around $(\overline{x}, g^\lambda(\overline{x}))$. By the equivalence between metric regularity and linear openness, we have that

$$B(y, \rho) \subset g^\lambda(B(x, \rho k')) \subset \text{conv}(g_1, g_2, \ldots, g_m)(B(x, \rho k')), \quad 0 < \rho < \tau,$$

for some positive τ. This proves the linear openness of $\text{conv}(g_1, \ldots, g_m)$ around $(\overline{x}, \overline{y})$, and therefore, the assertion follows from the same equivalence. □

Note that the conditions of the theorem above are not necessary, as the next example highlights:

Example 3.1 Take, for instance, the functions

$$g_1(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases} \qquad g_2(x) = \begin{cases} -x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Both of them are not open; indeed, for any $x_0 \in \mathbb{R}$, and for any positive r,

$$g_1(B(x_0, r)) \subset \mathbb{Q}, \qquad g_2(B(x_0, r)) \subset \mathbb{Q},$$

therefore no open set can be included in the image of any ball. In particular, the functions are not open at linear rate for any point x_0. However, a trivial computation shows that the set-valued map $\text{co}(g_1, g_2)$, given by

$$\text{conv}(g_1, g_2)(x) = \begin{cases} [-x, x], & x \in \mathbb{Q}, \\ \{0\}, & x \notin \mathbb{Q}, \end{cases}$$

is open at linear rate $k = 1$ around any point $(x_0, y_0) \in \text{gph}\,\text{conv}(g_1, g_2)$.

4 Application: Weak Sharp Efficiency and Metric Subregularity

In this section, we deal with weak sharp efficiency, in its local version, for the multiobjective optimization problem. In particular, in our main result we will provide sufficient conditions on the functions f_i, that guarantee the existence of local weak sharp efficient points.

Let $F : \mathbb{R}^n \to \mathbb{R}^m$, $F = (f_1, f_2, \ldots, f_m)$, where f_i is convex, for every $i = 1, 2, \ldots, m$. First of all, we reformulate the MOP as a suitable equilibrium problem, which shares the solution set (see, for instance [17]). Denote by
$\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the bifunction defined as follows:

$$\varphi(x, y) := \max_{z \in \Sigma_{m-1}} \langle z, F(y) - F(x) \rangle; \tag{13}$$

it is easy to prove that $\overline{x} \in \text{WE}_F$ if and only if $\overline{x}$ is a solution of (EP), i.e.,

$$\varphi(\overline{x}, y) \geq 0, \quad \forall y \in \mathbb{R}^n.$$

Let us consider the diagonal subdifferential operator $A^{\varphi} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ associated to φ and given by

$$A^{\varphi}(x) := \{x^* \in \mathbb{R}^n : \ \varphi(x, y) \geq \langle x^*, y - x\rangle, \ \forall y \in \mathbb{R}^n\}.$$

Obviously, $\overline{x}$ is a solution of (EP), if and only if $0 \in A^{\varphi}(\overline{x})$, i.e., $\overline{x} \in (A^{\varphi})^{-1}(0)$. In the next proposition, we strengthen this property.

Let us first recall a well-known formula for the subdifferentials of supremum functions (see, for instance, Theorem 4.4.2 in [18]). Let S be a compact set in some metric space, and let $\{h_s\}_{s\in S}$ be a family of convex functions on $\mathbb{R}^n$. Suppose that the map $s \mapsto h_s(x)$ is upper semicontinuous for every $x \in \mathbb{R}^n$, and the function h defined as $h(x) = \sup_{s\in S} h_s(x)$ is finite everywhere. Then

$$\partial h(x) = \operatorname{cl}\operatorname{conv}(\cup_{s\in S}\{\partial h_s(x) \mid h_s(x) = h(x)\}).$$

Proposition 4.1 *Let $F : \mathbb{R}^n \to \mathbb{R}^m$, $F = (f_1, f_2, \ldots, f_m)$, where f_i is convex and differentiable, for every $i = 1, 2, \ldots, m$, and let φ given by* (13)*. Then*

$$A^{\varphi}(x) = H_F(x), \quad \forall x \in \mathbb{R}^n,$$

where $H_F(x) = \operatorname{conv}(\nabla f_1(x), \ldots, \nabla f_m(x))$.

Proof Let $x^* \in H_F(x)$, i.e., $x^* = \sum_{i=1}^m \lambda_i \nabla f_i(x)$, for some $\lambda \in \Sigma_{m-1}$. Then, from the convexity of f_i, for every i, we get

$$\sum_{i=1}^m \lambda_i(f_i(y) - f_i(x)) \geq \sum_{i=1}^m \langle \lambda_i \nabla f_i(x), y - x\rangle = \langle x^*, y - x\rangle, \qquad \forall y \in \mathbb{R}^n.$$

This implies that

$$\varphi(x, y) = \max_{z\in\Sigma_{m-1}} \langle z, F(y) - F(x)\rangle \geq \langle x^*, y - x\rangle, \qquad \forall y \in \mathbb{R}^n, \tag{14}$$

thereby implying that $x^* \in A^{\varphi}(x)$.

Suppose now that $x^* \in A^{\varphi}(x)$, i.e., (14) holds. Then,

$$\begin{aligned} x^* &\in \partial(\max_{z\in\Sigma_{m-1}} \langle z, F(\cdot) - F(x)\rangle)|_{y=x} \\ &= \operatorname{cl}\operatorname{conv}\left(\bigcup_{z\in\Sigma_{m-1}} \sum_{i=1}^m z_i \nabla f_i(x)\right) = H_F(x), \end{aligned}$$

as required. □

As well as a relationship holds between the solutions of the equilibrium problem with bifunction φ and WE_F, a connection between the set WSE_F^{β} and the solutions of a "stronger" equilibrium problem defined by (13) can be highlighted.

Proposition 4.2 *Let $\overline{x}$ belong to* WSE_F^β *(with constant α); then,*

$$\varphi(\overline{x}, y) \geq \frac{\alpha}{m} d^\beta(y, (A^\varphi)^{-1}(0)), \qquad \forall y \in \mathbb{R}^n.$$

Conversely, if

$$\varphi(\overline{x}, y) \geq \alpha d^\beta(y, (A^\varphi)^{-1}(0)), \qquad \forall y \in \mathbb{R}^n,$$

then $\overline{x}$ belongs to WSE_F^β *(with constant α).*

Proof Note that the definition of WSE_F^β (4) can be equivalently restated as follows:

$$F(y) \notin F(x) + \alpha d^\beta(y, \text{WE}_F) B(0, 1) - \text{int}(\mathbb{R}^m_+), \quad \forall y \in \mathbb{R}^n. \tag{15}$$

By Proposition 4.1, we have that $\text{WE}_F = H_F^{-1}(0) = (A^\varphi)^{-1}(0)$. If $\overline{x} \in \text{WSE}_F^\beta$, then, for every $y \in \mathbb{R}^n$, there exists $i \in \{1, 2, \ldots, m\}$ such that

$$f_i(y) - f_i(\overline{x}) \geq \frac{\alpha}{m} d^\beta(y, \text{WE}_F).$$

This implies that $\varphi(\overline{x}, y) \geq \frac{\alpha}{m} d^\beta(y, \text{WE}_F)$.

The converse implication can be easily proved, arguing by contradiction. If $\overline{x} \notin \text{WSE}_F^\beta$, then, in particular, $f_i(y) - f_i(\overline{x}) < \alpha d^\beta(y, \text{WE}_F)$ for every $i \in \{1, 2, \ldots, m\}$ and for some $y \in \mathbb{R}^n$. This implies that

$$\langle z, F(y) - F(\overline{x}) \rangle < \alpha d^\beta \left(y, (A^\varphi)^{-1}(0) \right),$$

for every $z \in \Sigma_{m-1}$, and thus $\overline{x}$ cannot satisfy the assumption. □

The same result holds if we consider the local version: we say that $x \in \mathbb{R}^n$ is a *local weak sharp efficient minimum of order* $\beta > 0$ ($x \in \text{LWSE}_F^\beta$) iff

$$F(z) \notin F(x) + \alpha d^\beta(z, \text{WE}_F) B(0, 1) - \mathbb{R}^m_+, \tag{16}$$

for some positive α, and for all $z \in \mathcal{U}(x) \setminus \text{WE}_F$, where $\mathcal{U}(x)$ is a suitable neighbourhood of x. In particular, if

$$\varphi(\overline{x}, y) \geq \alpha d^\beta(y, (A^\varphi)^{-1}(0)), \qquad \forall y \in \mathcal{U}(\overline{x}), \tag{17}$$

then $\overline{x} \in \text{LWSE}_F^\beta$ with constant α.

In the following, we will focus on the case $\beta = 2$, and provide a sufficient condition for (17). To this purpose, we need the next result (see Theorem 2 in [11]):

Lemma 4.1 *Let $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction, and let $\overline{x} \in \mathbb{R}^n$ be such that*

(i) $\varphi(\overline{x}, \cdot)$ *is convex and lsc;*

(ii) $\varphi(\overline{x}, y) \leq \varphi(\overline{x}, z) + \varphi(z, y)$, *for all* $y, z \in \mathbb{R}^n$;
(iii) A^{φ} *is metrically subregular at* $(\overline{x}, 0)$, *with neighbourhood* $\mathcal{U} = B(\overline{x}, r)$, *and* $k > 0$.

Then, there exists $\mathcal{U}' = B(\overline{x}, 2r/3)$, *and* $0 < c < 1/4k$ *such that*

$$\varphi(\overline{x}, x) \geq c\, d^2(x, (A^{\varphi})^{-1}(0)) \quad \forall x \in \mathcal{U}'.$$

We can now apply Lemma 4.1 to the function φ defined in (13), and the next result follows:

Proposition 4.3 *Let* H_F *be metrically subregular at* $(\overline{x}, 0)$, *with neighbourhood* $B(\overline{x}, r)$ *and constant* k. *Then* $\overline{x} \in \mathrm{LWSE}_F^2$, *with* $\mathcal{U}(\overline{x}) = B(\overline{x}, r')$, *by taking* $\alpha < k/4$, *and* $r' = 2r/3$.

Proof By Proposition 4.1, $H_F = A^{\varphi}$. By taking into account Proposition 4.2, the assertion follows from Lemma 4.1. Indeed, condition (i) is satisfied from the convexity of the functions f_i. Moreover, for every $y, y' \in \mathbb{R}^n$,

$$\begin{aligned}\varphi(\overline{x}, y) &= \max_{z \in \Sigma_{m-1}} \langle z, F(y) - F(\overline{x}) \rangle \\ &\leq \max_{z \in \Sigma_{m-1}} \langle z, F(y') - F(\overline{x}) \rangle + \max_{z \in \Sigma_{m-1}} \langle z, F(y) - F(y') \rangle \\ &= \varphi(\overline{x}, y') + \varphi(y', y),\end{aligned}$$

thereby (ii) holds. □

Unfortunately, the example below shows that the local weak sharp efficiency cannot be characterized via the metric subregularity of the map H_F.

Example 4.1 Take $F : \mathbb{R} \to \mathbb{R}^2$, $F(t) = (|t|^{\gamma}, |t|^{2\gamma})$, with $\gamma > 1$ (in this case, both functions are differentiable everywhere). For every γ, the image is the same, and $t = 0$ is the only weak efficient point. Note that the map H_F given by

$$H_F(t) = \mathrm{conv}\left(\gamma\, \mathrm{sign}(t)|t|^{\gamma-1}, 2\gamma\, \mathrm{sign}(t)|t|^{2\gamma-1}\right)$$

is never metrically subregular at $(0, 0)$.

Indeed, $d(t, H_F^{-1}(0)) = |t|$, while $d(0, H_F(t)) = 2\gamma|t|^{2\gamma-1}$. For t small, the inequality $d(t, H_F^{-1}(0)) \leq kd(0, H_F(t))$ does not hold for any k. On the other hand, $t = 0$ belongs to LWSE_F^2 if and only if $\gamma \leq 2$, since $f_1(t) = |t|^{\gamma} \geq \alpha|t|^2$ for a suitable $\alpha > 0$, and for small values of t.

In order to get weak sharp efficient points of order 2, via Proposition 4.3, we are now interested in giving sufficient conditions for metric subregularity of the map H_F. In the next proposition, we prove a stronger result, i.e., the metric regularity of the map H_F around a point $(\overline{x}, 0)$, which in its turn will entail that $\overline{x}$ is a local weak sharp efficient point of order 2.

Theorem 4.1 *Let $F = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, with f_i convex and differentiable for every $i = 1, 2, \ldots, m$. Let $\overline{x} \in WE_F$, and suppose that there exists a neighbourhood $\mathcal{U}(\overline{x})$ of $\overline{x}$ such that:*

(i) *There exists $i_0 \in \{1, 2, \ldots, m\}$ such that ∇f_{i_0} is metrically regular around $(\overline{x}, \nabla f_{i_0}(\overline{x}))$ with constant k, along with $\mathcal{U}(\overline{x})$ and $\mathcal{V}(\nabla f_{i_0}(\overline{x}))$, and continuous on $\mathcal{U}(\overline{x})$;*
(ii) *For every $i \in \{1, 2, \ldots, m\} \setminus \{i_0\}$, ∇f_i is Lipschitz, with constant L_i, and strongly monotone, with constant α_i, within $\mathcal{U}(\overline{x})$.*

Then, H_F is metrically regular around $(\overline{x}, 0)$ with constant

$$k' = k + \sum_{i=1,\, i \neq i_0}^{m} \frac{1 + kL_i}{\alpha_i},$$

along with $\mathcal{U}(\overline{x})$ and $\mathbb{R}^n$. In particular, $\overline{x} \in \mathrm{LWSE}_F^2$.

Proof By applying Theorem 3.3 for $g_i = \nabla f_i$, $i = 1, 2, \ldots, m$, we obtain the first assertion. The second one is a consequence of Proposition 4.3. □

Via Lemma 2.1, we can provide sufficient conditions on the functions $\{f_i\}_{i=1}^m$, that entail the metric regularity of H_F and, as a by-product, the existence of a local weak sharp efficient solution of MOP.

Corollary 4.1 *Let $F = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, and $\overline{x} \in \mathrm{WE}_F$. Suppose that, for some positive r, and for every $i = 1, 2, \ldots, m$, $f_i \in \mathcal{C}^2(B(\overline{x}, r))$ and is strongly convex on $B(\overline{x}, r)$. Then, $\overline{x} \in \mathrm{LWSE}_F^2$.*

To conclude, note that the vector-valued function $F = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$, where

$$f_1(x) = \begin{cases} ax^2, & x < 0, \\ bx^2, & x \geq 0, \end{cases} \qquad f_2(x) = (x-1)^2,$$

satisfies the assumptions of the theorem above at $\overline{x} = 0$ for every $a, b > 0$, but not those of the corollary, if $a \neq b$.

5 Conclusions

In this paper, we study the existence of weak sharp efficient points for multiobjective optimization problems in their local version. To obtain our results, we use different techniques coming from variational analysis, like metric regularity and metric subregularity of the diagonal subdifferential operator associated to a suitable chosen equilibrium problem, strongly related to our multiobjective optimization problem.

The convex hull map built by the gradients of the objective functions plays an important role in investigating the existence of weak sharp efficient points. Namely, the metric regularity of this map around a certain point of its graph assures the local weak sharpness of the first component of that point. To this aim, we first investigate how the regularity properties of each function belonging to a finite family are inherited by the convex hull map of this family. We use these results to give sufficient conditions for weak sharp efficiency in terms of the objective functions of the investigated multiobjective optimization problem.

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