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FORUM

Some Perspectives on Set-Valued Optimization via Image Space Analysis

Letizia Pellegrini[1]

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Abstract This note aims at stimulating research on set-valued optimization through image space analysis.

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1 Introduction

The image space analysis (ISA) has shown to be instrumental in unifying several fields of the mathematical optimization theory, and to allow one to find new results, in particular, in the field of vector optimization problems (VOPs). This Forum paper aims at giving some perspectives on set-valued optimization by using the approach of ISA described in [1]. Some definitions and formulas of [1] are here reproduced for the reader's convenience.

2 Preliminaries

Let n, ℓ, m and k be positive integers, X a subset of $\mathbb{R}^n$, and $C \subset \mathbb{R}^\ell$ be a convex, closed and pointed cone with apex at the origin, which satisfies the condition $C + cl\ C = C$, where cl denotes topological closure. $\forall a, b \in \mathbb{R}^\ell$, we define the following inequalities:

✉ Letizia Pellegrini
letizia.pellegrini@univr.it

1 Department of Economics, University of Verona, Via Cantarane 24, 37129 Verona, Italy

$a \geq_C b \Leftrightarrow a - b \in C,\ a \not\geq_C b \Leftrightarrow a - b \notin C,\ a \leq_C b \Leftrightarrow b - a \in C,\ a \not\leq_C b \Leftrightarrow b - a \notin C.$

Assume we are given the functions $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^\ell$, and $g : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and consider the following constrained vector optimization problem:

$$\min_{C_0} f(x), \quad \text{s.t.} \quad x \in K := \{x \in X : g(x) \geq O\}, \tag{1}$$

where $\min_{C_0}$ marks vector minimum with respect to the cone $C_0 := C \backslash \{O\}$: $y \in K$ is a (global) vector minimum point (in short, *vmp*) of (1), iff

$$f(y) \not\geq_{C_0} f(x),\ \forall x \in K. \tag{2}$$

At $C = \mathbb{R}^\ell_+$, (1) becomes the classic *Pareto Vector Problem.*

Take an arbitrary $y \in \mathbb{R}^n$. Consider the vector-valued map $\mathscr{A}$ from $\mathbb{R}^n$ to $\mathbb{R}^{\ell+m}$ with $\mathscr{A}(x) := (f(y) - f(x), g(x))$, $x \in X \subseteq \mathbb{R}^n$, and the sets

$$\mathscr{K} := \{(u, v) \in \mathbb{R}^{\ell+m} : u = f(y) - f(x), v = g(x), x \in X\},$$

$$\mathscr{H} := \{(u, v) \in \mathbb{R}^{\ell+m} : u \geq_{C_0} O_\ell, v \geq O_m\},$$

where $\mathbb{R}^{\ell+m}$ denotes the product space $\mathbb{R}^\ell \times \mathbb{R}^m$. We observe that the sets $\mathscr{K}$ and $\mathscr{H}$ are subsets of $\mathbb{R}^{\ell+m}$, which is said to be the *image space* associated with (1) and $\mathscr{K}$ is said to be the *image* of (1) under the function $\mathscr{A}$. Take in particular $y \in K$. The ISA is based on the following immediate remark: it is easy to verify that y is a *vmp* to (1) if and only if $\mathscr{K} \cap \mathscr{H} = \emptyset$.

3 Some Perspectives

An extension of the fixed point approach of [2] to set-valued, in particular interval-valued, extremum problems is conceivable. A first proposal in this direction may consist in extending, to the set-valued case, the fixed point propositions of Sect. 4 of [2], which characterize the solutions of a VOP. The scalarization, contained in such a section, should produce an analogous vectorization. In this context, a special case is that, where the image of an objective function is an interval of $\mathbb{R}$. Let $\mathscr{I}$ denote the set of (compact, for the sake of simplicity, due to the compulsory shortness of this note) intervals of $\mathbb{R}$. Let F be an interval-valued function; i.e., $F : X \subseteq \mathbb{R}^n \to \mathscr{I}$, and assume that F^L (and F^U) are the inferior (respectively, superior) extremum of the interval $F(x)$. We consider the problem:

$$\min_{\preceq} \left[F(x) := \left(F^L(x), F^U(x)\right)\right], \quad \text{s.t.} \quad x \in K. \tag{3}$$

K is that of (1), [a,b] denotes an interval of $\mathbb{R}$, $\preceq$ denotes a partial order, $\min_{\preceq}$ denotes minimum with respect to that partial order. Let $A := \left[a^L, a^U\right]$ and $B := \left[b^L, b^U\right]$.

Consider the following partial order. *Pareto Partial Order* (PPO): in this case, y is a minimum point of (3), if and only if,

$$F(y) \preceq F(x), \quad \forall x \in K, \tag{4}$$

where the symbol $\preceq$ means:

$$A \preceq B \Leftrightarrow \begin{cases} a^L \leq b^L \\ a^U \leq b^U \end{cases} \text{or} \quad \begin{cases} a^L \leq b^L \\ a^U \geq b^U \end{cases} \text{or} \quad \begin{cases} a^L \geq b^L \\ a^U \leq b^U \end{cases}. \tag{5}$$

We associate (3) with the following VOP:

$$\min_C (F^L(x), F^U(x)), \quad \text{s.t.} \quad x \in K, \tag{6}$$

The meaning of $\min_C$ is as for (1). If we consider the case of PPO, Condition (5) suggests to set $C := \mathbb{R}^2_+ \backslash \{O\}$ in (6). We have that y *is a minimum point of* (3), *if and only if it is a vmp of* (6). Hence, a **first step** of this proposal consists in introducing the ISA for the interval-valued problem (3), passing through the VOP (6). Note that the case of (1) with $\ell = 2$ is not an elementary particular case, but an important one for applications. Hence, the interest of exploiting the ISA for (1) in the case of $\ell = 2$ follows.

Remark 3.1 Because the scope is to introduce ISA in this field, and not to discuss the partial orders, we consider only PPO. Obviously, there are infinite possible partial orders. For instance, we could consider, the *Classic Partial Order* (CPO). In this case, y is a minimum point of (3), if and only if, $\forall x \in K$, (4) holds, where the symbol $\preceq$ means: $A \preceq B \Leftrightarrow a^L \leq b^L, \; a^U \leq b^U$. Condition (4) suggests to set $C := \mathbb{R}^2 \backslash \mathbb{R}^2_-$ in (6). Unfortunately, in this case the cone C is not convex and hence problem (6) cannot be embedded in the format of problem (1). Anyway, in this case too, we have that y *is a minimum point of* (3), *if and only if it is a vmp of* (6), and hence, in this case too one can introduce the ISA for the interval-valued problem (3), passing through the VOP (6).

A **second step** consists in generalizing the above approach to the classic interval-valued optimization; more precisely, the previous family $\mathscr{I}$ can be considered as a set of intervals, which are compact subsets of a line of $\mathbb{R}^k$. Also in this more general case, we can reduce the interval-valued problem to a VOP. To this end, we suppose to be able to introduce a (total) order on $F(x)$, which leads to say what is $F^L(x)$ and what $F^U(x)$, where $F^L(x)$ and $F^U(x)$ are $k-$vectors. In the previous case (for $k = 1$), the order was not specified, because it was the usual one. Let $A := [a^L, a^U]$ and $B := [b^L, b^U]$ elements of $\mathscr{I}$, and hence, a^L, a^U, b^L, b^U are k-vectors. We introduce the following partial order for any pair of elements A and B of $\mathscr{I}$:

$$A \preceq B \Leftrightarrow \begin{cases} a^L \leq_{\mathbb{R}^k_+} b^L \\ a^U \leq_{\mathbb{R}^k_+} b^U \end{cases} \text{or} \quad \begin{cases} a^L \leq_{\mathbb{R}^k_+} b^L \\ a^U \geq_{\mathbb{R}^k_+} b^U \end{cases} \text{or} \quad \begin{cases} a^L \geq_{\mathbb{R}^k_+} b^L \\ a^U \leq_{\mathbb{R}^k_+} b^U \end{cases}. \tag{7}$$

Of course, instead of $\mathbb{R}^k_+$, we could have any cone of type C. We consider (3), where the objective functions are $2k$ and we associate with (3) problem (6), where $C := \mathbb{R}^k_+ \setminus \{O\}$. Henceforward, the development is the usual one for a VOP.

Let us now outline the **third step**, that is a possible application to set-valued optimization. Given an ordering cone $D \subseteq \mathbb{R}^k$, we want now to define the minimum of the point-to-set map $\mathscr{F} : X \rightrightarrows \mathbb{R}^k$. Because the present comment aims at making a suggestion and not at giving any statement, for the sake of simplicity and shortness, we assume that each $\mathscr{F}(x)$ is compact with not empty interior, and that the following operations are possible. $\forall x \in X$, set:

$$\mathscr{F}_{\min}(x) := \{z \in \mathscr{F}(x) : (z - D) \cap \operatorname{int} \mathscr{F}(\mathrm{x}) = \varnothing\},$$

$$\mathscr{F}^{\max}(x) := \{z \in \mathscr{F}(x) : (z + D) \cap \operatorname{int} \mathscr{F}(\mathrm{x}) = \varnothing\}.$$

Now, $\forall z \in \mathscr{F}_{\min}(x)$, we consider an optimization problem for which we assume that the extremum exists and is unique; this means that $\forall z \in \mathscr{F}_{\min}(x)$ we can determine a unique solution $t(z) \in \mathscr{F}^{\max}(x)$. An example of such a problem is :

$$\min d(z, t), \quad \text{s.t.} \quad t \in \mathscr{F}^{\max}(x), \quad z, t \in \mathbb{R}^k, \tag{8}$$

where $d(z, t)$ denotes distance function. The solution of (8) defines, $\forall z \in \mathscr{F}_{\min}(x)$, the interval $[z, t(z)]$ of $\mathbb{R}^k$. Now, we consider the following interval-valued problem:

$$\min_{\preceq}[z, t(z)], \quad \text{s.t.} \quad z \in \mathscr{F}_{\min}(x), \tag{9}$$

where, again, $\min_{\preceq}$ is defined as for (7), and $[z, t(z)]$ is an interval $A = [a^L, a^U]$ of $\mathbb{R}^k$, where $a^L := z$ and $a^U := t(z)$. Again through a VOP, like described in the second step, we achieve a solution of (9), say:

$$F(x) = \left[F^L(x), F^U(x)\right] = \left[z^*(x), t(z^*(x))\right]. \tag{10}$$

Now, through the above approach, the minimum of $\mathscr{F}$ is reduced to a VOP of type (6).

Remark 3.2 Note that the above approach—described in the third step—takes into account both $\mathscr{F}_{\min}$ and $\mathscr{F}^{\max}$. If we consider, for a set-valued optimization problem, an approach only based on $\mathscr{F}_{\min}$ (as in some literature), then it could happen to have 2 sets, of the image of the set-valued map $\mathscr{F}$, like $\mathscr{F}(x')$ and $\mathscr{F}(\mathrm{x}'')$, for which $\mathscr{F}_{\min}(x') = \mathscr{F}_{\min}(x'')$, but $\mathscr{F}^{\max}(x')$ is much different from $\mathscr{F}^{\max}(x'')$. In such a case, we would not be sensitive to the difference between $\mathscr{F}(x')$ and $\mathscr{F}(x'')$. Instead, this does not happen with the above approach.
An example of application of set-valued optimization to a real problem allows us to better understand the difference between the two approaches. Consider a set of weather stations that—for each position x of a station and possibly also as a function of time t—take bearing of temperature, pressure and humidity. The classic approach takes into account only $\mathscr{F}_{\min}(x, t)$ and therefore the minimum values of the set-valued function.

The approach proposed here also takes into account $\mathscr{F}^{\max}(x, t)$; in the above example this means taking into consideration the excursions of the temperature, pressure and humidity. Clearly, this second approach is more complete and more meaningful than the classic one.

In the vector case, namely $\mathscr{F}(x) = f(x) : X \to \mathbb{R}^\ell$, (10) becomes:

$$\left[f_{\min}(x),\ f^{\max}(x)\right], \tag{11}$$

where $f_{\min}(x) := \min\{f_1(x), ..., f_\ell(x)\}$, $f^{\max}(x) := \max\{f_1(x), ..., f_\ell(x)\}$. If we adopt PPO, then the above approach shrinks to the classic Pareto VOP. In the scalar case, namely $\ell = 1$ and $F(x) = f(x) : X \to \mathbb{R}$, (10) and (11) become just $f(x)$, and the above approach shrinks to the classic scalar optimization.

4 Conclusions

In this note, some short and rough comments are presented on the possibility of applying the ISA on set-valued optimization. The presentation has been outlined in 3 steps. In the first one, the ISA has been introduced for an interval-valued problem with 2 objective functions. The second step consists in generalizing the first step to a classic interval-valued optimization problem in $\mathbb{R}^k$, where the objective functions are $2k$. Lastly, in the third step, a possible application to a set-valued optimization problem has been suggested.

References

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