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Constraint Qualifications and Proper Pareto Optimality Conditions for Multiobjective Problems with Equilibrium Constraints

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Abstract In this paper, we consider a class of multiobjective problems with equilibrium constraints. Our first task is to extend the existing constraint qualifications for mathematical problems with equilibrium constraints from the single-objective case to the multiobjective case, and our second task is to derive some stationarity conditions under the proper Pareto sense for the considered problem. After doing that, we devote ourselves to investigating the relationships among the extended constraint qualifications and the proper Pareto stationarity conditions.

Keywords Multiobjective problem with equilibrium constraints · Constraint qualification · Proper Pareto stationarity condition

Mathematics Subject Classification 90C29 · 90C33 · 90C46

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1 Introduction

Mathematical problems with equilibrium constraints (MPEC) play important roles in many fields such as engineering design, economic equilibria, multilevel game, transportation science, and mathematical programming itself. However, since the standard Mangasarian–Fromovitz constraint qualification (MFCQ) does not hold at any feasible point [1], this kind of problems is very difficult to deal with. See, e.g., [2], for more details about the basic theory, effective algorithms, and various applications of MPEC.

In this paper, we consider a multiobjective problem with equilibrium constraints (MOPEC), which has many practical applications in energy, environment, health and transportation, etc., see, e.g., [3–5]. Our main purpose is to extend various constraint qualifications and stationarity conditions from the single-objective case to the multiobjective case. Particularly, we mainly focus on the proper Pareto optimality conditions for MOPEC in this paper.

Note that the proper Pareto-type optimality conditions for MOPEC have been discussed in [3,6]. However, Lin et al. [3] mainly aim at locally proper Pareto optimal solutions, while Pandey and Mishra [6] only introduce a concept of proper Pareto M-stationarity in terms of the Clark subdifferentials for the nonsmooth multiobjective semi-infinite case. In this paper, we discuss the locally Pareto optimal solutions for the smooth MOPEC, and we especially present a number of generalized constraint qualifications and proper Pareto stationarity conditions. We show that the proper Pareto optimality conditions hold for the locally Pareto optimal solutions under appropriate constraint qualifications. Furthermore, we investigate the relationships among the constraint qualifications and the stationarity conditions.

The paper is organized as follows: In Sect. 2, we recall some useful concepts and properties. In Sect. 3, we first extend the existing MPEC-type constraint qualifications to MOPEC and then investigate the relationships among them. In Sect. 4, we discuss various stationarities in the proper Pareto sense for MOPEC. Finally, we make some conclusions in Sect. 5.

2 Preliminaries

In what follows, given a vector $d \in \mathbb{R}^n$ and an index set $I \subseteq \{1, \ldots, n\}$, we use d_I to denote the subvector composed from the components d_i $(i \in I)$. Moreover, we use $\mathbb{B}_\delta(x)$ to stand for the open ball centered at x with radius $\delta > 0$, and we denote by $\nabla F(x)$ its transposed Jacobian at x for a differentiable function F. We denote by e^i the vector whose i-th element is one and others are zero, and by e the vector whose elements are all one.

Consider the following MOPEC:

$$\begin{aligned} \min\ \ & f(x) \\ \text{s.t.}\ \ & g(x) \leq 0,\ \ h(x) = 0, \\ & G(x) \geq 0,\ \ H(x) \geq 0,\ \ \langle G(x), H(x) \rangle = 0, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^r$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^q$, and $G, H : \mathbb{R}^n \to \mathbb{R}^m$ are all continuously differentiable functions. We denote by $\mathcal{F}$ the feasible region of (1) and, for $x^* \in \mathcal{F}$, we let

$$\begin{aligned}
&\mathcal{I}_f := \{1, 2, \ldots, r\}, \quad \mathcal{I}_f^{-k} := \{i \in \mathcal{I}_f \ : \ i \neq k\}, \quad \mathcal{I}_h := \{1, 2, \ldots, q\},\\
&\mathcal{I}_g^* := \{i \ : \ g_i(x^*) = 0\}, \quad \mathcal{I}^* := \{i \ : \ G_i(x^*) = 0,\ H_i(x^*) > 0\},\\
&\mathcal{J}^* := \{i \ : \ G_i(x^*) = 0,\ H_i(x^*) = 0\}, \quad \mathcal{K}^* := \{i : G_i(x^*) > 0,\ H_i(x^*) = 0\}.
\end{aligned}$$

We next recall some basic concepts.

Definition 2.1 (a) The polar cone of a cone K is defined by

$$K^o := \left\{d \in \mathbb{R}^n : \langle d, x\rangle \le 0,\ \forall x \in K\right\}.$$

(b) The tangent cone of a set Q at $x^* \in \operatorname{cl} Q$ is defined by

$$\mathcal{T}(Q; x^*) := \left\{ d \in \mathbb{R}^n : \exists\{x^n\} \subseteq Q, t_n \downarrow 0 \text{ s.t. } x^n \to x^*,\ \frac{x^n - x^*}{t_n} \to d \right\}.$$

(c) The regular normal cone of a set Q at $x^* \in \operatorname{cl} Q$ is defined by

$$\widehat{\mathcal{N}}(Q; x^*) := \left\{ d \in \mathbb{R}^n : \langle d, x - x^*\rangle \le o(\|x - x^*\|), \forall x \in Q \right\}.$$

(d) The limiting normal cone of a set Q at $x^* \in \operatorname{cl} Q$ is defined by

$$\mathcal{N}(Q; x^*) := \left\{d \in \mathbb{R}^n : \exists\{x^k\} \in Q, d^k \in \widehat{\mathcal{N}}(Q; x^k) \text{ s.t. } x^k \to x^*, d^k \to d\right\}.$$

(e) For a set-valued mapping $\Gamma : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$, the sequential Painlevé–Kuratowski outer limit of $\Gamma(z)$ as $z \to z^*$ is defined by

$$\limsup_{z\to z^*} \Gamma(z) := \left\{ w^* \in \mathbb{R}^d \ : \ \exists (z^k, w^k) \to (z^*, w^*) \text{ with } w^k \in \Gamma(z^k) \right\}.$$

In addition, Γ is outer semicontinuous at z^* iff $\limsup_{z\to z^*} \Gamma(z) \subseteq \Gamma(z^*)$.

The following definition for (1) coincides with the standard definition of Pareto optimality in multiobjective optimization theory [7].

Definition 2.2 We call $x^* \in \mathcal{F}$ a Pareto optimal solution of (1) iff there does not exist another point $x \in \mathcal{F}$ such that $f(x) \le f(x^*)$ and $f_i(x) < f_i(x^*)$ for some i. We call $x^* \in \mathcal{F}$ a locally Pareto optimal solution of (1) iff there exists a neighborhood $U(x^*)$ such that it is Pareto optimal in $U(x^*) \cap \mathcal{F}$.

The following lemmas are useful in the subsequent sections.

Lemma 2.1 [8] *Let C and $\widetilde{C}$ be two nonempty cones. Then, we have*

$$C \subseteq \widetilde{C} \Rightarrow \widetilde{C}^o \subseteq C^o, \quad C \subseteq C^{oo}, \quad C^{oo} = \operatorname{conv} C,$$

where conv C *denotes the closed and convex hull of* C.

Lemma 2.2 [9] *Let $x^* \in C := C_1 \cap \cdots \cap C_m$ with $C_i \subseteq \mathbb{R}^n$. Then, we have*

$$\mathcal{T}(C; x^*) \subseteq \mathcal{T}(C_1; x^*) \cap \cdots \cap \mathcal{T}(C_m; x^*).$$

Lemma 2.3 [10] *Let $x := \sum_{i=1}^{m+p} \alpha_i v_i$, where $\{v_1, \ldots, v_m\}$ is linearly independent and $\alpha_i \neq 0$ for every $i \in \mathcal{I} := \{m+1, \cdots, m+p\}$. Then, there exist $\mathcal{J} \subseteq \mathcal{I}$ and $\bar{\alpha}_i$, $i \in \{1, \ldots, m\} \cup \mathcal{J}$, such that $x = \sum_{i \in \{1,\ldots,m\} \cup \mathcal{J}} \bar{\alpha}_i v_i$ with $\alpha_i \bar{\alpha}_i > 0$ for every $i \in \mathcal{J}$ and $\{v_i\}_{i \in \{1,\ldots,m\} \cup \mathcal{J}}$ to be linearly independent.*

Lemma 2.4 [11] *Let $f : \mathbb{R}^p \to \mathbb{R}$ be a continuously differentiable function. Assume that $x^l \to x^*$ with $f(x^l) \leq f(x^*)$ for each l and $d := \lim_{l \to \infty} t_l (x^l - x^*)$ with $t_l > 0$ for each l. Then, we have $\langle \nabla f(x^*), d \rangle \leq 0$.*

3 Constraint Qualifications for MOPEC

As is known to us, various constraint qualifications are very important in optimality theory and convergence analysis for numerical methods. However, as in the MPEC theory, because of the existence of the complementarity constraints, the Mangasarian–Fromovitz constraint qualification for multiobjective optimization problems does not hold for MOPEC (1) at any feasible point [12,13]. In this section, we first introduce a list of MOPEC-type constraint qualifications, which are generalizations of the ones for single-objective MPEC, and then, we discuss the relationships among them.

In order to extend the MPEC-type constraint qualifications to MOPEC (1), we make use of the ε-constraint approach, which is often used to characterize Pareto optimal solutions for multiobjective optimization [7]. Recall that, given a multiobjective optimization problem

$$\min_{x \in X} f(x) \tag{2}$$

and an upper bound vector ε, the k-th ε-constraint problem for (2) is

$$\min \; f_k(x) \quad \text{s.t. } f_i(x) \leq \varepsilon_i, \quad i \in \mathcal{I}_f^{-k}, \; x \in X. \tag{3}$$

The following result reflects the relations between the Pareto optimal solutions of problem (2) and the optimal solutions of its ε-constraint problems.

Theorem 3.1 [14] *The vector $x^* \in X$ is a Pareto optimal solution of (2) if and only if x^* solves (3) for every $k \in \mathcal{I}_f$ with $\varepsilon_i := f_i(x^*)$ $(i \in \mathcal{I}_f^{-k})$.*

By means of Theorem 3.1, we can define some MOPEC-type constraint qualifications by means of the constraint qualifications for the MPEC

$$\begin{aligned} \min \quad & f_k(x) \\ \text{s.t.} \quad & f_i(x) \le f_i(x^*), \ \forall i \in \mathcal{I}_f^{-k}, \\ & g(x) \le 0, \ h(x) = 0, \ G(x) \ge 0, \ H(x) \ge 0, \ \langle G(x), H(x) \rangle = 0, \end{aligned} \tag{4}$$

where x^* is a locally Pareto optimal solution of (1). Roughly speaking, we say that some constraint qualification holds for MOPEC (1) at $x^* \in \mathcal{F}$ iff the corresponding MPEC-type constraint qualification holds for MPEC (4) at x^* for each k. To avoid disturbing the readability, here, we just list the MOPEC-type constraint qualifications used in this paper, which include

MOPEC-LCQ (or LICQ, WSCQ, SCQ, NNAMCQ, CRCQ, RCRCQ)

and

MOPEC-MFCQ (or CPLD, RCPLD, quasinormality, pseudonormality),

and we omit their mathematical descriptions. We refer readers to [15–17] for detailed definitions of the MPEC-type constraint qualifications involved above.

Remark 3.1 Note that the concepts defined for MOPEC (1) involve the objective functions. In the studies on multiobjective optimization problems, conditions to guarantee the positiveness of the multipliers associated with the objectives are usually called regularity conditions when the conditions involve the objective functions. Here, for consistency with the single-objective case discussed in [15–17], we still call them by constraint qualifications.

From [15–17], we can obtain the following relationships among the above MOPEC-type constraint qualifications:

- MOPEC-LICQ ⇒ MOPEC-CRCQ ⇒ MOPEC-CPLD ⇒ MOPEC quasinormality;
- MOPEC-LCQ ⇒ MOPEC-CRCQ ⇒ MOPEC-RCRCQ ⇒ MOPEC-RCPLD;
- MOPEC-LCQ ⇒ MOPEC pseudonormality;
- MOPEC-MFCQ ⇒ MOPEC-NNAMCQ ⇒ MOPEC pseudonormality ⇒ MOPEC quasinormality;
- MOPEC-LICQ ⇒ MOPEC-MFCQ ⇒ MOPEC-CPLD ⇒ MOPEC-RCPLD;
- MOPEC-SCQ ⇒ MOPEC-WSCQ.

We next extend the generalized Guignard constraint qualification (GGCQ) given in [18] for multiobjective optimization problems to MOPEC (1).

Definition 3.1 We say that the generalized Guignard constraint qualification (GGCQ) holds for (1) at $x^* \in \mathcal{F}$ iff $\mathcal{L}(Q; x^*) \subseteq \bigcap_{k=1}^{r} \text{conv}\mathcal{T}(Q^k; x^*)$, where

$$Q := \left\{ x \in \mathbb{R}^n : \begin{array}{l} f(x) \le f(x^*), g(x) \le 0, h(x) = 0 \\ G(x) \ge 0, H(x) \ge 0, \langle G(x), H(x) \rangle = 0 \end{array} \right\},$$

$$\mathcal{L}(Q; x^*) := \left\{ d \in \mathbb{R}^n \ : \ \begin{array}{l} \langle \nabla f(x^*), d \rangle \le 0 \\ \langle \nabla g_{\mathcal{I}_g^*}(x^*), d \rangle \le 0, \ \langle \nabla h(x^*), d \rangle = 0 \\ \langle \nabla G_{\mathcal{I}^*}(x^*), d \rangle = 0, \ \langle \nabla H_{\mathcal{K}^*}(x^*), d \rangle = 0 \\ \langle \nabla G_{\mathcal{J}^*}(x^*), d \rangle \ge 0, \ \langle \nabla H_{\mathcal{J}^*}(x^*), d \rangle \ge 0 \end{array} \right\},$$

and Q^k denotes the feasible region of (4).

We have the following result.

Theorem 3.2 *If the* MOPEC-WSCQ *holds at* x^*, *the* GGCQ *holds at* x^*.

Proof For each $k \in \mathcal{I}_f$, since the MOPEC-WSCQ holds at x^*, there exists x^k such that $f_{\mathcal{I}_f^{-k}}(x^k) < f_{\mathcal{I}_f^{-k}}(x^*)$, $g_{\mathcal{I}_g^*}(x^k) < 0$, $h(x^k) = 0$, $G_{\mathcal{I}^* \cup \mathcal{J}^*}(x^k) = 0$, and $H_{\mathcal{K}^* \cup \mathcal{J}^*}(x^k) = 0$. Denote by $d^k := x^k - x^*$. Since $\{f, g\}$ are convex and $\{h, G, H\}$ are affine, we have $\langle \nabla f_{\mathcal{I}_f^{-k}}(x^*), d^k \rangle < 0$, $\langle \nabla g_{\mathcal{I}_g^*}(x^*), d^k \rangle < 0$, $\langle \nabla h_{\mathcal{I}_h}(x^*), d^k \rangle = 0$, $\langle \nabla G_{\mathcal{I}^* \cup \mathcal{J}^*}(x^*), d^k \rangle = 0$, $\langle \nabla H_{\mathcal{K}^* \cup \mathcal{J}^*}(x^*), d^k \rangle = 0$. Let $d \in \mathcal{L}(Q; x^*)$, that is,

$$\begin{array}{l} \langle \nabla f(x^*), d \rangle \le 0, \ \langle \nabla g_{\mathcal{I}_g^*}(x^*), d \rangle \le 0, \ \langle \nabla h(x^*), d \rangle = 0, \\ \langle \nabla G_{\mathcal{I}^*}(x^*), d \rangle = 0, \langle \nabla H_{\mathcal{K}^*}(x^*), d \rangle = 0, \langle \nabla G_{\mathcal{J}^*}(x^*), d \rangle \ge 0, \langle \nabla H_{\mathcal{J}^*}(x^*), d \rangle \ge 0. \end{array}$$

For any positive sequence $\{\alpha_l\}$ converging to 0, by letting $d^l := d + \alpha_l d^k$ for each l, we have $\langle \nabla f_i(x^*), d^l \rangle = \langle \nabla f_i(x^*), d \rangle + \alpha_l \langle \nabla f_i(x^*), d^k \rangle < 0$ for any $i \in \mathcal{I}_f^{-k}$. For each l, taking a positive sequence $\{t_j\}$ converging to 0 and letting $x^j := x^* + t_j d^l$, we have that, for each j sufficiently large,

$$f_i(x^j) = f_i(x^* + t_j d^l) = f_i(x^*) + t_j \langle \nabla f_i(x^*), d^l \rangle + o(\|t_j d^l\|) \le f_i(x^*), \forall i \in \mathcal{I}_f^{-k}$$

and, similarly, we have

$$g_{\mathcal{I}_g^*}(x^j) \le 0, G_{\mathcal{J}^*}(x^j) \ge 0, H_{\mathcal{J}^*}(x^j) \ge 0, h(x^j) = 0, G_{\mathcal{I}^*}(x^j) = 0, H_{\mathcal{K}^*}(x^j) = 0.$$

For $i \notin \mathcal{I}_g^*$, since $g_i(x^*) < 0$, it follows from the continuity of g that $g_i(x^j) < 0$ for every j sufficiently large. Similarly, we have $G_{\mathcal{K}^*}(x^j) > 0$ and $H_{\mathcal{I}^*}(x^j) > 0$ for every j sufficiently large. As a result, without any loss of generality, we may assume $x^j \in Q^k$ for all j. Note that $x^j \in Q^k$ and $t_j \to 0$ imply $x^j \to x^*$ and $\lim_{j \to \infty} (x^j - x^*)/t_j = d^l$, which means $d^l \in \mathcal{T}(Q^k; x^*)$. Since $\mathcal{T}(Q^k; x^*)$ is closed, we have $d \in \mathcal{T}(Q^k; x^*)$.

By the arbitrariness of k, we have

$$d \in \bigcap_{k=1}^{r} \mathcal{T}(Q^k; x^*) \subseteq \operatorname{conv} \bigcap_{k=1}^{r} \mathcal{T}(Q^k; x^*),$$

which means that the GGCQ holds at x^*. This completes the proof. □

We now extend the MPEC-ACQ, the MPEC-GCQ, the MPEC-KTCQ, and the MPEC-ZCQ in [1,8,19] for the single-objective MPEC to MOPEC (1).

Definition 3.2 Consider MOPEC (1), and let $x^* \in \mathcal{F}$.

(a) The MOPEC Abadie constraint qualification (MOPEC-ACQ) holds at x^* iff $\mathcal{L}_{\text{MOPEC}}(Q; x^*) = \mathcal{T}(Q; x^*)$, where

$$\mathcal{L}_{\text{MOPEC}}(Q; x^*) := \left\{ d : \begin{array}{l} \langle \nabla f(x^*), d \rangle \leq 0, \ \langle \nabla g_{\mathcal{I}_g^*}(x^*), d \rangle \leq 0, \ \langle \nabla h(x^*), d \rangle = 0 \\ \langle \nabla G_{\mathcal{I}^*}(x^*), d \rangle = 0, \ \langle \nabla H_{\mathcal{K}^*}(x^*), d \rangle = 0 \\ \langle \nabla G_{\mathcal{J}^*}(x^*), d \rangle \geq 0, \ \langle \nabla H_{\mathcal{J}^*}(x^*), d \rangle \geq 0 \\ \langle \nabla G_{\mathcal{J}^*}(x^*), d \rangle \cdot \langle \nabla H_{\mathcal{J}^*}(x^*), d \rangle = 0 \end{array} \right\}.$$

(b) The MOPEC generalized Abadie constraint qualification (MOPEC-GACQ) holds at x^* iff $\mathcal{L}_{\text{MOPEC}}(Q; x^*) = \bigcap_{k=1}^{r} \mathcal{T}(Q^k; x^*)$.

(c) The MOPEC Guignard constraint qualification (MOPEC-GCQ) holds at x^* iff $\mathcal{L}_{\text{MOPEC}}(Q; x^*)^o = \mathcal{T}(Q; x^*)^o$.

(d) The MOPEC generalized Guignard constraint qualification (MOPEC-GGCQ) holds at x^* iff $\mathcal{L}_{\text{MOPEC}}(Q; x^*) \subseteq \bigcap_{k=1}^{r} \text{conv}\mathcal{T}(Q^k; x^*)$.

(e) The MOPEC Zangwill constraint qualification (MOPEC-ZCQ) holds at x^* iff $\mathcal{L}_{\text{MOPEC}}(Q; x^*) = \text{cl } \mathcal{D}(Q; x^*)$, where $\mathcal{D}(Q; x^*)$ is the cone of the feasible directions of the set Q at x^* defined by

$$\mathcal{D}(Q; x^*) := \left\{ d \in \mathbb{R}^n \ : \ \exists \delta > 0 \text{ s.t. } x^* + td \in Q \ (\forall t \in]0, \delta[\,) \right\}.$$

(f) The MOPEC Kuhn–Tucker constraint qualification (MOPEC-KTCQ) holds at x^* iff $\mathcal{L}_{\text{MOPEC}}(Q; x^*) = \text{cl}\mathcal{A}(Q; x^*)$, where $\mathcal{A}(Q; x^*)$ denotes the cone of attainable directions of the set Q at x^* defined by

$$\mathcal{A}(Q; x^*) := \left\{ d \in \mathbb{R}^n : \begin{array}{l} \exists \delta > 0, \ \alpha : \mathbb{R} \to \mathbb{R}^n \text{ s.t. } \alpha(0) = x^* \\ \alpha(\tau) \in Q \ (\forall \tau \in]0, \delta[\,), \ \lim_{\tau \downarrow 0} \frac{\alpha(\tau) - \alpha(0)}{\tau} = d \end{array} \right\}.$$

Obviously, if the complementarity constraints in (1) vanish, the MOPEC-ACQ, the MOPEC-GACQ, and the MOPEC-GGCQ defined above coincide with the corresponding constraint qualifications defined for multiobjective optimization problems in [18].

From Lemmas 2.1–2.2 and Definition 3.2, we can easily obtain the following relationships:

- MOPEC-ACQ $\Rightarrow$ MOPEC-GACQ $\Rightarrow$ MOPEC-GGCQ;
- MOPEC-ACQ $\Rightarrow$ MOPEC-GCQ $\Rightarrow$ MOPEC-GGCQ;
- GGCQ $\Rightarrow$ MOPEC-GGCQ.

We next show that

- MOPEC-ZCQ $\Rightarrow$ MOPEC-KTCQ $\Rightarrow$ MOPEC-ACQ.

Theorem 3.3 *If the* MOPEC-ZCQ *holds at* $x^* \in \mathcal{F}$, *the* MOPEC-KTCQ *holds at* x^*. *If the* MOPEC-KTCQ *holds at* $x^* \in \mathcal{F}$, *the* MOPEC-ACQ *holds at* x^*.

Proof We first show $\mathcal{T}(Q; x^*) \subseteq \mathcal{L}_{\text{MOPEC}}(Q; x^*)$. In fact, since x^* is a feasible point of MPEC (4) for each $k \in \mathcal{I}_f$, it follows from [19] that

$$\mathcal{T}(Q^k; x^*) \subseteq \mathcal{L}_{\text{MPEC}}(Q^k; x^*) = \mathcal{L}_{\text{MOPEC}}(Q^k; x^*)$$

and so

$$\mathcal{T}(Q; x^*) \subseteq \bigcap_{k=1}^{r} \mathcal{T}(Q^k; x^*) \subseteq \bigcap_{k=1}^{r} \mathcal{L}_{\text{MOPEC}}(Q^k; x^*) = \mathcal{L}_{\text{MOPEC}}(Q; x^*). \tag{5}$$

Since $\mathcal{D}(Q; x^*) \subseteq \mathcal{A}(Q; x^*) \subseteq \mathcal{T}(Q; x^*)$ and $\mathcal{T}(Q^k; x^*)$ is closed, we have

$$\text{cl}\, \mathcal{D}(Q; x^*) \subseteq \text{cl}\, \mathcal{A}(Q; x^*) \subseteq \mathcal{T}(Q; x^*). \tag{6}$$

Then $\mathcal{L}_{\text{MOPEC}}(Q; x^*) = \text{cl}\, \mathcal{D}(Q; x^*)$ if the MOPEC-ZCQ holds at x^*. It follows from (5)–(6) that $\mathcal{L}_{\text{MOPEC}}(Q; x^*) = \text{cl}\mathcal{A}(Q; x^*)$, that is, the MOPEC-KTCQ holds at x^*. We can show the second half in a similar way. □

We further have the following result.

Theorem 3.4 *Let $x^* \in \mathcal{F}$.*

(i) *Suppose that one of the* MOPEC-WSCQ*, the* MOPEC *pseudonormality, the* MOPEC-RCRCQ *holds at x^*. Then, the* MOPEC-GACQ *holds at x^*.*
(ii) *Suppose that the* MOPEC-LCQ *holds at x^*. Then, the* MOPEC-ZCQ *holds at x^*.*

Proof (i) We only prove that the MOPEC-WSCQ implies the MOPEC-GACQ because the other two cases can be shown similarly. Let the MOPEC-WSCQ hold at x^*. It follows that, for each $k \in \mathcal{I}_f$, the MPEC-WSCQ for (4) holds at x^*. From Theorem 3.9 in [19], the MPEC-ACQ for (4) also holds at x^*, and hence, $\mathcal{L}_{\text{MPEC}}(Q^k; x^*) \subseteq \mathcal{T}(Q^k; x^*)$ for each $k \in \mathcal{I}_f$. Then, we have

$$\mathcal{L}_{\text{MOPEC}}(Q; x^*) = \bigcap_{k=1}^{r} \mathcal{L}_{\text{MPEC}}(Q^k; x^*) \subseteq \bigcap_{k=1}^{r} \mathcal{T}(Q^k; x^*),$$

that is, the MOPEC-GACQ holds at x^*.

(ii) Suppose that the MOPEC-LCQ holds at x^*, and $d \in \mathcal{L}_{\text{MOPEC}}(Q; x^*)$. Since all functions involved are affine and $x^* \in \mathcal{F}$, for any $\tau \in]0, 1]$ small enough, we have $f_i(x^* + \tau d) = \tau\langle \nabla f_i(x^*), d\rangle + f_i(x^*) \leq 0$ for each $i \in \mathcal{I}_f$ and

$$g_{\mathcal{I}_g^*}(x^* + \tau d) \leq 0,\ h_{\mathcal{I}_h}(x^* + \tau d) = 0,\ G_{\mathcal{I}^*}(x^* + \tau d) = 0,$$
$$H_{\mathcal{K}^*}(x^* + \tau d) = 0,\ G_{\mathcal{J}^*}(x^* + \tau d) \geq 0,\ H_{\mathcal{J}^*}(x^* + \tau d) \geq 0.$$

If $i \in \mathcal{J}^*$, we have $G_i(x^* + \tau d)H_i(x^* + \tau d) = \tau^2\langle \nabla G_i(x^*), d\rangle\langle \nabla H_i(x^*), d\rangle = 0$. Since $g_i(x^*) < 0$ when $i \notin \mathcal{I}_g^*$, it follows from the continuity of g that $g_i(x^* + \tau d) < 0$ for any $\tau \in]0, 1]$ small enough. Similarly, we have $G_{\mathcal{K}^*}(x^* + \tau d) > 0$

and $H_{\mathcal{I}^*}(x^* + \tau d) > 0$ when $\tau \in]0, 1]$ is small enough. Therefore, as long as $\tau \in]0, 1]$ is small enough, there must hold $x^* + \tau d \in Q$, and hence, $d \in \mathcal{D}(Q; x^*)$. This completes the proof. □

4 Proper Pareto Stationarities for MOPEC

In this section, we discuss the stationarities for MOPEC (1) under the proper Pareto sense. In what follows, the capitals "W, C, M, B, S" are, respectively, the abbreviations for "weak, Clarke, Mordukhovich, Boulligand, strong", which coincide with the terminologies for MPEC in the literature.

Definition 4.1 Let $x^* \in \mathcal{F}$.

(a) x^* is called a proper Pareto W-stationary point of (1) iff there exist multipliers $\{\sigma, \lambda, \mu, u, v\}$ such that

$$\begin{aligned} &\nabla f(x^*)\sigma + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)u - \nabla H(x^*)v = 0, \\ &\sigma \geq e, \lambda \geq 0, \ \langle g(x^*), \lambda \rangle = 0, \ u_{\mathcal{K}^*} = 0, v_{\mathcal{I}^*} = 0. \end{aligned} \tag{7}$$

(b) x^* is called a proper Pareto C-stationary point of (1) iff there exist multipliers $\{\sigma, \lambda, \mu, u, v\}$ satisfying (7) and $u_i v_i \geq 0$ for each $i \in \mathcal{J}^*$.

(c) x^* is called a proper Pareto M-stationary point of (1) iff there exist multipliers $\{\sigma, \lambda, \mu, u, v\}$ satisfying (7) and either $u_i > 0, v_i > 0$ or $u_i v_i = 0$ for each $i \in \mathcal{J}^*$.

(d) x^* is called a proper Pareto B-stationary point of (1) iff, for any $k \in \mathcal{I}_f$,

$$\langle \nabla f_k(x^*), d \rangle \geq 0, \ \forall d \in \mathcal{L}_{\text{MOPEC}}(Q^k; x^*).$$

(e) x^* is called a proper Pareto S-stationary point of (1) iff there exist multipliers $\{\sigma, \lambda, \mu, u, v\}$ satisfying (7) and

$$u_i \geq 0, \ v_i \geq 0, \ \forall i \in \mathcal{J}^*. \tag{8}$$

Note that the above stationarity conditions are different from the ones given in [3,20], where the multiplier vector associated with the objective functions is only required to be nonzero and nonnegative. Obviously, the proper Pareto stationarities can ensure all objective functions to have effective influence in practice. Moreover, if all functions involved are differentiable and the number of inequality constraints is finite, the definition given in [6] coincides with the proper Pareto M-stationarity defined above.

Theorem 4.1 *Let* $x^* \in \mathcal{F}$.

(i) *If* x^* *is a proper Pareto S-stationary point of* (1)*,* x^* *is a proper Pareto B-stationary point of* (1)*.*
(ii) *If* x^* *is a proper Pareto B-stationary point of* (1)*,* x^* *is a proper Pareto M-stationary point of* (1)*.*

Proof (i) Let x^* be a proper Pareto S-stationary point of (1), that is, there exist multipliers $\{\sigma, \lambda, \mu, u, v\}$ satisfying (7)–(8). For any $d \in \mathbb{R}^n$, we have

$$\sum_{i\in\mathcal{I}_f} \sigma_i \langle \nabla f_i(x^*), d\rangle + \sum_{i\in\mathcal{I}_g^*} \lambda_i \langle \nabla g_i(x^*), d\rangle + \sum_{i\in\mathcal{I}_h} \mu_i \langle \nabla h_i(x^*), d\rangle \\ - \sum_{i\in\mathcal{I}^*\cup\mathcal{J}^*} u_i \langle \nabla G_i(x^*), d\rangle - \sum_{i\in\mathcal{K}^*\cup\mathcal{J}^*} v_i \langle \nabla H_i(x^*), d\rangle = 0.$$

Suppose by contradiction that x^* is not a proper Pareto B-stationary point of (1), which means that there exist $k \in \mathcal{I}_f$ and $d^k \in \mathcal{L}_{\mathrm{MOPEC}}(Q^k; x^*)$ such that $\langle \nabla f_k(x^*), d^k\rangle < 0$. Noting that $\sigma \geq e$, $\mathcal{I}_g^* \geq 0$, $u_{\mathcal{J}^*} \geq 0$, and $v_{\mathcal{J}^*} \geq 0$, we have from the definition of $\mathcal{L}_{\mathrm{MOPEC}}(Q^k; x^*)$ that

$$\sum_{i\in\mathcal{I}_f} \sigma_i \langle \nabla f_i(x^*), d^k\rangle + \sum_{i\in\mathcal{I}_g^*} \lambda_i \langle \nabla g_i(x^*), d^k\rangle + \sum_{i\in\mathcal{I}_h} \mu_i \langle \nabla h_i(x^*), d^k\rangle \\ - \sum_{i\in\mathcal{I}^*\cup\mathcal{J}^*} u_i \langle \nabla G_i(x^*), d^k\rangle - \sum_{i\in\mathcal{K}^*\cup\mathcal{J}^*} v_i \langle \nabla H_i(x^*), d^k\rangle < 0,$$

which is a contraction. This completes the proof of (i).

(ii) Let x^* be a proper Pareto B-stationary point. Then, for any $k \in \mathcal{I}_f$ and $d \in \mathcal{L}_{\mathrm{MOPEC}}(Q^k; x^*)$, we have $\langle \nabla f_k(x^*), d\rangle \geq 0$. Noting that

$$\mathcal{L}_{\mathrm{MOPEC}}(Q; x^*) = \bigcap_{k=1}^{r} \mathcal{L}_{\mathrm{MOPEC}}(Q^k; x^*),$$

we have $\sum_{k=1}^{r} \langle \nabla f_k(x^*), d\rangle \geq 0$ for every $d \in \mathcal{L}_{\mathrm{MOPEC}}(Q; x^*)$. This means that $d = 0$ is an optimal solution of

$$\min_{d\in\mathcal{L}_{\mathrm{MOPEC}}(Q;x^*)} \sum_{k=1}^{r} \langle \nabla f_k(x^*), d\rangle,$$

which is actually an MPEC. Since all constraint functions are affine, by Theorem 2.2 in [1], $d = 0$ is an M-stationary point of the above MPEC, that is, there exist multipliers $\{\sigma, \lambda, \mu, u, v\} \neq 0$ such that

$$\sum_{k=1}^{r} \nabla f_k(x^*) + \sum_{i \in \mathcal{I}_f} \sigma_i \nabla f_i(x^*) + \sum_{i \in \mathcal{I}_g^*} \lambda_i \nabla g_i(x^*) + \sum_{i \in \mathcal{I}_h} \lambda_i \nabla h_i(x^*)$$
$$- \sum_{i \in \mathcal{I}^* \cup \mathcal{J}^*} u_i \nabla G_i(x^*) - \sum_{i \in \mathcal{K}^* \cup \mathcal{J}^*} v_i \nabla H_i(x^*) = 0,$$
$$\sigma \geq 0,\ \lambda_{\mathcal{I}_g^*} \geq 0,\ u_{\mathcal{K}^*} = 0,\ v_{\mathcal{I}^*} = 0,$$
$$\text{either } u_i > 0, v_i > 0 \quad \text{or} \quad u_i v_i = 0,\ \forall i \in \mathcal{J}^*,$$

from which we know that x^* is a proper Pareto M-stationary point of (1). □

From the above theorem and Definition 4.1, the relationships among the proper Pareto stationarities can be summarized as follows:

- proper Pareto S-stationarity $\Rightarrow$ proper Pareto B-stationarity $\Rightarrow$ proper Pareto M-stationarity $\Rightarrow$ proper Pareto C-stationarity $\Rightarrow$ proper Pareto W-stationarity.

In order to show the relationship between the MOPEC quasinormality and the proper Pareto M-stationarity, we first give a lemma.

Lemma 4.1 *Consider the problem*

$$\min\ f(x) \quad \text{s.t. } F(x) \in \Lambda, \tag{9}$$

where $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable, and Λ is a closed subset in $\mathbb{R}^m$. If the quasinormality for (9) holds at a locally Pareto optimal solution x^*, that is, there is no nonzero vector $\eta^* \in \mathcal{N}(\Lambda; F(x^*))$ such that

- *$0 \in \nabla F(x^*)\eta^*$;*
- *there exists a sequence $\{(x^k, y^k, \eta^k)\}$ convergent to $(x^*, F(x^*), \eta^*)$ such that, for all k, $\eta^k \in \mathcal{N}(\Lambda; y^k)$ and $\eta_i^*(F_i(x^k) - y_i^k) > 0$ if $\eta_i^* \neq 0$,*

there exists $\sigma \geq 0$ with $\sigma \neq 0$ such that

$$0 \in \nabla f(x^*)\sigma + \nabla F(x^*)\mathcal{N}(\Lambda; F(x^*)).$$

Proof Note that (9) can be rewritten as

$$\min\ f(x) \quad \text{s.t. } F(x) - y = 0, \quad (x, y) \in \mathbb{R}^n \times \Lambda. \tag{10}$$

Obviously, problem (10) is a special case of the problem considered in [21] without inequality constraints and variational inequality constraints. Since x^* is a locally Pareto optimal solution of (9), it is easy to see that x^* is also a locally optimal solution of (9) under the weakly Pareto preference. Then, from Theorem 1.3 and Remark 2 in [21], there exist $\alpha \in \{0, 1\}$, $\mu \in \mathbb{R}^m$, and $\sigma \geq 0$ with $\sigma \neq 0$ such that

$$0 \in \alpha \sum_{i=1}^{n} \sigma_i \begin{pmatrix} \nabla f_i(x^*) \\ 0 \end{pmatrix} + \sum_{i=1}^{m} \mu_i \begin{pmatrix} \nabla F_i(x^*) \\ -e^i \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{N}(\Lambda; y^*) \end{pmatrix}.$$

Therefore, we have

$$0 \in \alpha \nabla f(x^*)\sigma + \nabla F(x^*)\mathcal{N}(\Lambda; F(x^*)). \tag{11}$$

We next prove that α in (11) can be taken as 1. By Theorem 4.8 in [21], we only need to prove that the error bound constraint qualification is satisfied at x^*, that is, there exist positive constants $\{\kappa, \delta, \epsilon\}$ such that

$$\text{dist}(x, \Sigma(0)) \leq \kappa\|p\|, \quad \forall p \in \epsilon\mathbb{B}, \ \forall x \in \Sigma(p) \cap \mathbb{B}_\delta(x^*),$$

where $\Sigma(p) := \{x : F(x) + p \in \Lambda\}$. In fact, since the quasinormality for problem (9) holds at x^*, by Theorem 5.2 in [22], there exist $\delta > 0$ and $\kappa > 0$ such that

$$\text{dist}(x, \Sigma(0)) \leq \kappa \text{dist}(F(x), \Lambda), \quad \forall x \in \mathbb{B}_\delta(x^*). \tag{12}$$

Note that $F(x) + p \in \Lambda$ for any $x \in \Sigma(p)$. Then, we have from (12) that, for any $x \in \mathbb{B}_\delta(x^*) \cap \Sigma(p)$,

$$\text{dist}(x, \Sigma(0)) \leq \kappa \text{dist}(F(x), \Lambda) \leq \kappa\|F(x) - (F(x) + p)\| = \kappa\|p\|.$$

Hence, the error bound constraint qualification for (9) holds at x^*. By Theorem 4.8 in [21], the parameter α in (11) can be taken as 1. This completes the proof. □

Theorem 4.2 *Let $x^* \in \mathcal{F}$ be a locally Pareto optimal solution of* MOPEC (1). *If the* MOPEC *quasinormality or the* MOPEC-RCPLD *holds at x^*, x^* is a proper Pareto M-stationary point of* (1).

Proof (i) Note that problem (1) can be rewritten as the form of (9), in which

$$F(x) := (g(x), h(x), G_1(x), H_1(x), \ldots, G_m(x), H_m(x)),$$
$$\Lambda :=]-\infty, 0]^p \times \{0\}^q \times C^m, \quad C := \{(a, b) \in \mathbb{R}^2_+ \ : \ ab = 0\}.$$

Suppose that the MOPEC quasinormality holds at x^*. Then, the MPEC quasinormality for the system

$$g(x) \geq 0, \ h(x) = 0, \ G(x) \geq 0, \ H(x) \geq 0, \ \langle G(x), H(x)\rangle = 0$$

holds at x^*. Let $\eta^* := (\lambda, \mu, u_i, v_i, \ldots, u_m, v_m)$, $\eta^k := \eta^*$, and $y^k := F(x^*)$ for all k. Then, the quasinormality for problem (9) holds at x^*. Thus, by Lemma 4.1, there exists $\sigma \geq 0$ with $\sigma \neq 0$ such that

$$0 \in \nabla f(x^*)\sigma + \nabla F(x^*)\mathcal{N}(\Lambda; F(x^*)),$$

which means

$$\nabla f(x^*)\sigma + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)u - \nabla H(x^*)v = 0,$$

$$\sigma \geq 0,\ \lambda \geq 0,\ \langle \lambda, g(x^*) \rangle = 0,\ u_{\mathcal{K}^*} = 0,\ v_{\mathcal{I}^*} = 0,$$
$$\text{either } u_i > 0, v_i > 0 \text{ or } u_i v_i = 0,\ \forall i \in \mathcal{J}^*.$$

We next show $\sigma > 0$. Suppose by contradiction that there exists $k \in \mathcal{I}_f$ such that $\sigma_k = 0$. In this case, the above conditions become

$$\sum_{i \neq k} \sigma_i \nabla f_i(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)u - \nabla H(x^*)v = 0,$$
$$\sigma \geq 0,\ \lambda \geq 0,\ \langle \lambda, g(x^*) \rangle = 0,\ u_{\mathcal{K}^*} = 0,\ v_{\mathcal{I}^*} = 0,$$
$$\text{either } u_i > 0, v_i > 0 \text{ or } u_i v_i = 0,\ \forall i \in \mathcal{J}^*.$$

Since x^* is a locally Pareto optimal solution of (1), it is also a local minimizer of MPEC (4). Then, in a similar way to the proof of Theorem 3.1 in [23], we can show that there exists a sequence $\{x^N\} \to x^*$ such that, for each N,

$$\begin{aligned}
&\sigma_i > 0 \Rightarrow \sigma_i \left(f_i\left(x^N\right) - f_i\left(x^*\right)\right) > 0,\\
&\lambda_j > 0 \Rightarrow \lambda_j g_j\left(x^N\right) > 0, \quad \mu_l \neq 0 \Rightarrow \mu_l h_l\left(x^N\right) > 0,\\
&u_\iota \neq 0 \Rightarrow -u_\iota G_\iota\left(x^N\right) > 0, \quad v_\jmath \neq 0 \Rightarrow -v_\jmath H_\jmath\left(x^N\right) > 0.
\end{aligned}$$

This contradicts the MOPEC quasinormality assumption at x^*, and hence, we must have $\sigma > 0$. That is, x^* is a proper Pareto M-stationary point of (1).

(ii) Suppose that the MOPEC-RCPLD holds at x^*. Since x^* is a locally Pareto optimal solution of (1), for any $k \in \mathcal{I}_f$, x^* is also a local minimizer of (4), and so $-\nabla f_k(x^*) \in \mathcal{N}(Q^k; x^*)$. Note that (4) can be rewritten as the following optimization problem with a geometric constraint:

$$\min f_k(x) \ \text{ s.t. } \ F^k(x) \in \Lambda,$$

where

$$F^k(x) := \begin{pmatrix} f_{\mathcal{I}_f^{-k}}(x) - f_{\mathcal{I}_f^{-k}}(x^*) \\ g(x) \\ h(x) \\ \Psi(x) \end{pmatrix}, \quad \Psi(x) := \begin{pmatrix} G_1(x) \\ H_1(x) \\ \vdots \\ G_m(x) \\ H_m(x) \end{pmatrix},$$
$$\Lambda :=]-\infty, 0]^{r+p-1} \times \{0\}^q \times C^m, \quad C := \{(a,b) \in \mathbb{R}^2_+ \ : \ ab = 0\}.$$

By Theorem 3.7 in [24], we have

$$\mathcal{N}(Q^k; x^*) \subseteq \limsup_{(x,z) \to (x^*, F^k(x^*))} \nabla F^k(x) \mathcal{N}(\Lambda; z),$$

and so

$$-\nabla f_k(x^*) \in \limsup_{(x,z)\to(x^*,F^k(x^*))} \nabla F^k(x)\mathcal{N}(\Lambda; z), \quad \forall k \in \mathcal{I}_f.$$

From the definition of outer limit, there are sequences $\{x^{k,t}\}$, $\{z^{k,t}\}$, $\{w^{k,t}\}$, and $\{\gamma^{k,t}\}$ such that $x^{k,t} \to x^*$, $z^{k,t} \to F^k(x^*)$, and $w^{k,t} \to -\nabla f_k(x^*)$ with $w^{k,t} := \nabla F^k(x^{k,t})\gamma^{k,t}$ and $\gamma^{k,t} \in \mathcal{N}(\Lambda; z^{k,t})$. By Lemma 3.4 in [24], we have

$$\limsup_{(x^{k,t},z^{k,t})\to(x^*,F^k(x^*))} \nabla F^k\left(x^{k,t}\right)\mathcal{N}\left(\Lambda; z^{k,t}\right) = \limsup_{(x^{k,t},z^{k,t})\to(x^*,F^k(x^*))} \nabla F^k\left(x^{k,t}\right)\widehat{\mathcal{N}}\left(\Lambda; z^{k,t}\right),$$

and hence, without any loss of generality, we may assume $\gamma^{k,t} \in \widehat{\mathcal{N}}(\Lambda; z^{k,t})$.

Denote by $\gamma^{k,t} := (\sigma_{-k}^{k,t}, \lambda^{k,t}, \mu^{k,t}, u_1^{k,t}, v_1^{k,t}, \ldots, u_m^{k,t}, v_m^{k,t})$. Then, when t is sufficiently large, we have

$$\begin{aligned} w^{k,t} = &\sum_{i\in\mathcal{I}_f^{-k}} \sigma_i^{k,t}\nabla f_i(x^{k,t}) + \sum_{i\in\mathcal{I}_g^{k,t}} \lambda_i^{k,t}\nabla g_i(x^{k,t}) + \sum_{i\in\mathcal{I}_h} \mu_i^{k,t}\nabla h_i(x^{k,t}) \\ &- \sum_{i\in\mathcal{I}^{k,t}\cup\mathcal{J}^{k,t}} u_i^{k,t}\nabla G_i(x^{k,t}) - \sum_{i\in\mathcal{K}^{k,t}\cup\mathcal{J}^{k,t}} v_i^{k,t}\nabla H_i(x^{k,t}), \end{aligned} \tag{13}$$

where $\mathcal{I}_g^{k,t} := \{i \,:\, g_i(x^{k,t}) = 0\}$, $\mathcal{I}^{k,t} := \{i \,:\, G_i(x^{k,t}) = 0,\ H_i(x^{k,t}) > 0\}$,

$$\mathcal{J}^{k,t} := \{i : G_i(x^{k,t})=0,\, H_i(x^{k,t})=0\},\, \mathcal{K}^{k,t} := \{i : G_i(x^{k,t})>0,\, H_i(x^{k,t})=0\}.$$

Since $\gamma^{k,t} \in \widehat{\mathcal{N}}(\Lambda; z^{k,t})$, we have $\sigma_i^{k,t} \geq 0$ for each $i \in \mathcal{I}_f^{-k}$, $\lambda_j^{k,t} \geq 0$ for each $j \in \mathcal{I}_g^{k,t}$, and $u_l^{k,t} \geq 0$, $v_l^{k,t} \geq 0$ for each $l \in \mathcal{J}^{k,t}$. Since

$$F^k(x^*) = \begin{pmatrix} f_{\mathcal{I}_f^{-k}}(x^*) - f_{\mathcal{I}_f^{-k}}(x^*) \\ g(x^*) \\ h(x^*) \\ \Psi(x^*) \end{pmatrix} = \begin{pmatrix} 0^{r-1} \\ g(x^*) \\ h(x^*) \\ \Psi(x^*) \end{pmatrix},$$

the values of $F^k(x^*)$ for $k = 1, \ldots, r$ are the same. For convenience, we denote it by $F^0(x^*)$. Then, when t is sufficiently large, there must exist $z^t \to F^0(x^*)$, $x^t \to x^*$, and $w^{k,t} \to -\nabla f_k(x^*)$ with $w^{k,t} \in \nabla F^k(x^t)\gamma^{k,t}$ and $\gamma^{k,t} \in \widehat{\mathcal{N}}^(\Lambda; z^t)$ for each $k \in \mathcal{I}_f$. It then follows from (13) that

$$w^{k,t} = \sum_{i\in\mathcal{I}_f^{-k}} \sigma_i^{k,t}\nabla f_i(x^t) + \sum_{i\in\mathcal{I}_g^t} \lambda_i^{k,t}\nabla g_i(x^t) + \sum_{i\in\mathcal{I}_h} \mu_i^{k,t}\nabla h_i(x^t)$$

$$- \sum_{i \in \mathcal{I}^t \cup \mathcal{J}^t} u_i^{k,t} \nabla G_i(x^t) - \sum_{i \in \mathcal{K}^t \cup \mathcal{J}^t} v_i^{k,t} \nabla H_i(x^t), \tag{14}$$

where $\mathcal{I}_g^t := \{i \ : \ g_i(x^t) = 0\}$, $\mathcal{I}^t := \{i \ : \ G_i(x^t) = 0, \ H_i(x^t) > 0\}$, $\mathcal{J}^t := \{i \ : \ G_i(x^t) = 0, \ H_i(x^t) = 0\}$, $\mathcal{K}^t := \{i \ : \ G_i(x^t) > 0, \ H_i(x^t) = 0\}$, and $\gamma^{k,t} \in \widehat{\mathcal{N}}(\Lambda; z^t)$. We next show that, for each $k \in \mathcal{I}_f$, $\{\gamma^{k,t}\}$ has a bounded subsequence.

In fact, since $\mathcal{I}^* \subseteq \mathcal{I}^t$ and $\mathcal{K}^* \subseteq \mathcal{K}^t$ when t is sufficiently large, from (14), we obtain

$$\begin{aligned} w^{k,t} = & \sum_{i \in \mathcal{I}_f^{-k} \cap \text{supp}(\sigma^{k,t})} \sigma_i^{k,t} \nabla f_i(x^t) + \sum_{i \in \mathcal{I}_g^t \cap \text{supp}(\lambda^{k,t})} \lambda_i^{k,t} \nabla g_i(x^t) \\ & - \sum_{i \in ((\mathcal{I}^t \setminus \mathcal{I}^*) \cup \mathcal{J}^t) \cap \text{supp}(u^{k,t})} u_i^{k,t} \nabla G_i(x^t) - \sum_{i \in ((\mathcal{K}^t \setminus \mathcal{K}^*) \cup \mathcal{J}^t) \cap \text{supp}(v^{k,t})} v_i^{k,t} \nabla H_i(x^t) \\ & + \sum_{i \in \mathcal{I}_h} \mu_i^{k,t} \nabla h_i(x^t) - \sum_{i \in \mathcal{I}^*} u_i^{k,t} \nabla G_i(x^t) - \sum_{i \in \mathcal{K}^*} v_i^{k,t} \nabla H_i(x^t), \end{aligned}$$

where $\text{supp}(a) := \{i \ : \ a_i \neq 0\}$. Let $\mathcal{I}_1 \subseteq \mathcal{I}_h$, $\mathcal{I}_2 \subseteq \mathcal{I}^*$, $\mathcal{I}_3 \subseteq \mathcal{K}^*$ be index sets such that $\{\nabla h_l(x^*), \nabla G_\iota(x^*), \nabla H_\jmath(x^*) \ : \ l \in \mathcal{I}_1, \ \iota \in \mathcal{I}_2, \ \jmath \in \mathcal{I}_3\}$ is a basis for $\text{span}\{\nabla h_l(x^*), \nabla G_\iota(x^*), \nabla H_\jmath(x^*) \ : \ l \in \mathcal{I}_h, \iota \in \mathcal{I}^*, \jmath \in \mathcal{K}^*\}$. Since the MOPEC-RCPLD holds at x^*, $\{\nabla h_l(x^t), \nabla G_\iota(x^t), \nabla H_\jmath(x^t) : l \in \mathcal{I}_1, \iota \in \mathcal{I}_2, \jmath \in \mathcal{I}_3\}$ is a basis for $\text{span}\{\nabla h_l(x^t), \nabla G_\iota(x^t), \nabla H_\jmath(x^t) : l \in \mathcal{I}_h, \iota \in \mathcal{I}^*, \jmath \in \mathcal{K}^*\}$ when t is sufficiently large. Thus, from Lemma 2.3, there exist index sets

$$\begin{aligned} &\mathcal{I}_4^t \subseteq \mathcal{I}_f^{-k} \bigcap \text{supp}\left(\sigma^{k,t}\right), \quad \mathcal{I}_6^t \subseteq ((\mathcal{I}^t \setminus \mathcal{I}^*) \bigcup \mathcal{J}^t) \bigcap \text{supp}\left(u^{k,t}\right), \\ &\mathcal{I}_5^t \subseteq \mathcal{I}_g^t \bigcap \text{supp}\left(\lambda^{k,t}\right), \quad \mathcal{I}_7^t \subseteq ((\mathcal{K}^t \setminus \mathcal{K}^*) \bigcup \mathcal{J}^t) \bigcap \text{supp}\left(v^{k,t}\right), \end{aligned}$$

and multipliers $\{\widetilde{\sigma}^{k,t}, \widetilde{\lambda}^{k,t}, \widetilde{\mu}^{k,t}, \widetilde{u}_1^{k,t}, \widetilde{v}_1^{k,t}, \ldots, \widetilde{u}_m^{k,t}, \widetilde{v}_m^{k,t}\}$ such that

$$\begin{aligned} w^{k,t} = & \sum_{i \in \mathcal{I}_4^t} \widetilde{\sigma}_i^{k,t} \nabla f_i(x^t) + \sum_{i \in \mathcal{I}_5^t} \widetilde{\lambda}_i^{k,t} \nabla g_i(x^t) + \sum_{i \in \mathcal{I}_1} \widetilde{\mu}_i^{k,t} \nabla h_i(x^t) \\ & - \sum_{i \in \mathcal{I}_2 \cup \mathcal{I}_6^t} \widetilde{u}_i^{k,t} \nabla G_i(x^t) - \sum_{i \in \mathcal{I}_3 \cup \mathcal{I}_7^t} \widetilde{v}_i^{k,t} \nabla H_i(x^t) \end{aligned} \tag{15}$$

and the vectors $\{\nabla f_{\mathcal{I}_4^t}(x^t), \ \nabla g_{\mathcal{I}_5^t}(x^t), \ \nabla h_{\mathcal{I}_1}(x^t), \ \nabla G_{\mathcal{I}_2 \cup \mathcal{I}_6^t}(x^t), \ \nabla H_{\mathcal{I}_3 \cup \mathcal{I}_7^t}(x^t)\}$ are linearly independent for each t sufficiently large. Set $\widetilde{\sigma}_i^{k,t} := 0$ for $i \notin \mathcal{I}_4^t$, $\widetilde{\lambda}_i^{k,t} := 0$ for $i \notin \mathcal{I}_5^t$, $\widetilde{\mu}_i^{k,t} := 0$ for $i \notin \mathcal{I}_1$, $\widetilde{u}_i^{k,t} := 0$ for $i \notin \mathcal{I}_2 \cup \mathcal{I}_6^t$, and $\widetilde{v}_i^{k,t} := 0$ for $i \notin \mathcal{I}_3 \cup \mathcal{I}_7^t$. When t is sufficiently large, it is not difficult to see that

$$\widetilde{\gamma}^{k,t} := (\widetilde{\sigma}^{k,t}, \widetilde{\lambda}^{k,t}, \widetilde{\mu}^{k,t}, \widetilde{u}_1^{k,t}, \widetilde{v}_1^{k,t}, \ldots, \widetilde{u}_m^{k,t}, \widetilde{v}_m^{k,t}) \in \widehat{\mathcal{N}}(\Lambda; z^t) \subseteq \mathcal{N}(\Lambda; z^t).$$

Without any loss of generality, we may assume $\mathcal{I}_4^t \equiv \mathcal{I}_4$, $\mathcal{I}_5^t \equiv \mathcal{I}_5$, $\mathcal{I}_6^t \equiv \mathcal{I}_6$, and $\mathcal{I}_7^t \equiv \mathcal{I}_7$ for each t sufficiently large. It is easy to see that $\mathcal{I}_4 \subseteq \mathcal{I}_f^{-k}$, $\mathcal{I}_5 \subseteq \mathcal{I}_g^*$, $\mathcal{I}_6 \subseteq \mathcal{J}^*$, and $\mathcal{I}_7 \subseteq \mathcal{J}^*$ by $\mathcal{I}^t \cup \mathcal{J}^t \cup \mathcal{K}^t = \mathcal{I}^* \cup \mathcal{J}^* \cup \mathcal{K}^*$. Hence, the vectors

$$\left\{\nabla f_{\mathcal{I}_4}(x^t),\ \nabla g_{\mathcal{I}_5}(x^t),\ \nabla h_{\mathcal{I}_1}(x^t),\ \nabla G_{\mathcal{I}_2\cup\mathcal{I}_6}(x^t),\ \nabla H_{\mathcal{I}_3\cup\mathcal{I}_7}(x^t)\right\}$$

are linearly independent for each t sufficiently large.

We can claim that $\{(\widetilde{\sigma}^{k,t}, \widetilde{\lambda}^{k,t}, \widetilde{\mu}^{k,t}, \widetilde{u}^{k,t}, \widetilde{v}^{k,t})\}$ has a bounded subsequence. Otherwise, dividing (15) by $M^{k,t} := \|(\widetilde{\sigma}^{k,t}, \widetilde{\lambda}^{k,t}, \widetilde{\mu}^{k,t}, \widetilde{u}^{k,t}, \widetilde{v}^{k,t})\|$ and taking an adequate subsequence $\{(\widetilde{\sigma}^k, \widetilde{\lambda}^k, \widetilde{\mu}^k, \widetilde{u}^k, \widetilde{v}^k)\}$, for each $k \in \mathcal{I}_f$, we have

$$\begin{aligned} 0 = &\sum_{i\in\mathcal{I}_4} \widetilde{\sigma}_i^k \nabla f_i(x^*) + \sum_{i\in\mathcal{I}_5} \widetilde{\lambda}_i^k \nabla g_i(x^*) + \sum_{i\in\mathcal{I}_1} \widetilde{\mu}_i^k \nabla h_i(x^*) \\ &- \sum_{i\in\mathcal{I}_2\cup\mathcal{I}_6} \widetilde{u}_i^k \nabla G_i(x^*) - \sum_{i\in\mathcal{I}_3\cup\mathcal{I}_7} \widetilde{v}_i^k \nabla H_i(x^*), \end{aligned} \tag{16}$$

where $\{\widetilde{\sigma}^k, \widetilde{\lambda}^k, \widetilde{\mu}^k, \widetilde{u}^k, \widetilde{v}^k\} \neq 0$. Clearly, we have $\widetilde{\sigma}_i^k \geq 0$ for each $i \in \mathcal{I}_4$, $\widetilde{\lambda}_j^k \geq 0$ for each $j \in \mathcal{I}_5$, and either $\widetilde{u}_l^k > 0$, $\widetilde{v}_l^k > 0$ or $\widetilde{u}_l^k \widetilde{v}_l^k = 0$ for each $l \in \mathcal{J}^*$ by

$$\begin{aligned} &(\widetilde{\sigma}^k, \widetilde{\lambda}^k, \widetilde{\mu}^k, \widetilde{u}_1^k, \widetilde{v}_1^k, \ldots, \widetilde{u}_m^k, \widetilde{v}_m^k) \\ &\in \limsup_{t\to\infty} \mathcal{N}(\Lambda; z^t) \subseteq \limsup_{z\to F(x^*)} \mathcal{N}(\Lambda; z) \subseteq \mathcal{N}(\Lambda; F(x^*)), \end{aligned}$$

which follows from the outer semicontinuity of $\mathcal{N}(\Lambda; \cdot)$. From (16) and the MOPEC-RCPLD assumption at x^*, the vectors

$$\left\{\nabla f_{\mathcal{I}_4}(x^t),\ \nabla g_{\mathcal{I}_5}(x^t),\ \nabla h_{\mathcal{I}_1}(x^t),\ \nabla G_{\mathcal{I}_2\cup\mathcal{I}_6}(x^t),\ \nabla H_{\mathcal{I}_3\cup\mathcal{I}_7}(x^t)\right\}$$

are linearly dependent for each t sufficiently large, which is a contradiction.

Without any loss of generality, we assume that, for each $k \in \mathcal{I}_f$, there exists a bounded sequence $\{\gamma^{k,t}\} \in \widehat{\mathcal{N}}(\Lambda; z^t)$ such that (14) holds. Then, we have

$$\begin{aligned} \sum_{k=1}^r w^{k,t} = &\sum_{k=1}^r \sum_{i\in\mathcal{I}_f^{-k}} \sigma_i^{k,t} \nabla f_i(x^t) + \sum_{k=1}^r \sum_{i\in\mathcal{I}_g^t} \lambda_i^{k,t} \nabla g_i(x^t) + \sum_{k=1}^r \sum_{i\in\mathcal{I}_h} \mu_i^{k,t} \nabla h_i(x^t) \\ &- \sum_{k=1}^r \sum_{i\in\mathcal{I}^t\cup\mathcal{J}^t} u_i^{k,t} \nabla G_i(x^t) - \sum_{k=1}^r \sum_{i\in\mathcal{K}^t\cup\mathcal{J}^t} v_i^{k,t} \nabla H_i(x^t). \end{aligned} \tag{17}$$

Let

$$\overline{\sigma}_i^t := \sum_{k=1,k\neq i}^r \sigma_i^{k,t}, \quad \overline{\lambda}_i^t := \sum_{k=1}^r \lambda_i^{k,t}, \quad \overline{\mu}_i^t := \sum_{k=1}^r \mu_i^{k,t},$$

$$\overline{u}_i^t := \sum_{k=1}^{r} u_i^{k,t}, \quad \overline{v}_i^t := \sum_{k=1}^{r} v_i^{k,t}.$$

Then, (17) can be rewritten as

$$\begin{aligned}\sum_{k=1}^{r} w^{k,t} = \sum_{i\in\mathcal{I}_f} \overline{\sigma}_i^t \nabla f_i(x^t) + \sum_{i\in\mathcal{I}_g^t} \overline{\lambda}_i^t \nabla g_i(x^t) + \sum_{i\in\mathcal{I}_h} \overline{\mu}_i^t \nabla h_i(x^t) \\ - \sum_{i\in\mathcal{I}^t\cup\mathcal{J}^t} \overline{u}_i^t \nabla G_i(x^t) - \sum_{i\in\mathcal{K}^t\cup\mathcal{J}^t} \overline{v}_i^t \nabla H_i(x^t)\end{aligned} \tag{18}$$

with $\overline{\sigma}^t \geq 0$, $\overline{\lambda}_j^t \geq 0$ for each $j \in \mathcal{I}_g^t$, and $\overline{u}_l^t \geq 0, \overline{v}_l^t \geq 0$ for each $l \in \mathcal{J}^t$. Obviously, $\overline{\gamma}^t := (\overline{\sigma}^t, \overline{\lambda}^t, \overline{\mu}^t, \overline{u}_1^t, \overline{v}_1^t, \ldots, \overline{u}_m^t, \overline{v}_m^t) \in \widehat{\mathcal{N}}(\Lambda; z^t) \subseteq \mathcal{N}(\Lambda; z^t)$ and $\{\overline{\gamma}^t\}$ is a bounded sequence. Without any loss of generality, we assume

$$\lim_{t\to\infty} \overline{\gamma}^t = \overline{\gamma} := (\overline{\sigma}, \overline{\lambda}, \overline{\mu}, \overline{u}_1, \overline{v}_1, \cdots, \overline{u}_m, \overline{v}_m).$$

Noting that $w^{k,t} \to -\nabla f_k(x^*)$ as $t \to \infty$ for each $k \in \mathcal{I}_f$ and taking a limit in (18), we get

$$\begin{aligned}-\sum_{k=1}^{r} \nabla f_k(x^*) = \sum_{i\in\mathcal{I}_f} \overline{\sigma}_i \nabla f_i(x^*) + \sum_{i\in\mathcal{I}_g^*} \overline{\lambda}_i \nabla g_i(x^*) + \sum_{i\in\mathcal{I}_h} \overline{\mu}_i \nabla h_i(x^*) \\ - \sum_{i\in\mathcal{I}^*\cup\mathcal{J}^*} \overline{u}_i \nabla G_i(x^*) - \sum_{i\in\mathcal{K}^*\cup\mathcal{J}^*} \overline{v}_i \nabla H_i(x^*).\end{aligned} \tag{19}$$

Clearly, we have $\overline{\sigma}_i \geq 0$ for each $i \in \mathcal{I}_f$, $\overline{\lambda}_j \geq 0$ for each $j \in \mathcal{I}_g^*$, and either $\overline{u}_l > 0, \overline{v}_l > 0$ or $\overline{u}_l\overline{v}_l = 0$ for each $l \in \mathcal{J}^*$ by

$$(\overline{\sigma}, \overline{\lambda}, \overline{\mu}, \overline{u}_1, \overline{v}_1, \ldots, \overline{u}_m, \overline{v}_m) \in \limsup_{t\to\infty} \mathcal{N}(\Lambda; z^t) \subseteq \limsup_{z\to F(x^*)} \mathcal{N}(\Lambda; z) \subseteq \mathcal{N}(\Lambda; F(x^*)).$$

Let $\overline{\sigma}_k^* := 1 + \overline{\sigma}_k$ for each $k \in \mathcal{I}_f$. It follows from (19) and the above discussion that x^* is a proper Pareto M-stationary point of (1). □

Theorem 4.3 *Let $x^* \in \mathcal{F}$ be a locally Pareto optimal solution of* (1)*. If the* MOPEC-GGCQ *holds at x^*, x^* is a proper Pareto B-stationary point of* (1)*.*

Proof We first prove that, for each $k \in \mathcal{I}_f$, $\langle \nabla f_k(x^*), d^k \rangle \geq 0$ for any $d^k \in \mathcal{T}(Q^k; x^*)$. In fact, since $d^k \in \mathcal{T}(Q^k; x^*)$, there exist $\{x^{k,\nu}\} \subseteq Q^k$ and $t_\nu \downarrow 0$ such that $x^{k,\nu} \to x^*$ and $\frac{x^{k,\nu}-x^*}{t_\nu} \to d^k$. Let $d^{k,\nu} := \frac{x^{k,\nu}-x^*}{t_\nu}$. Since $x^{k,\nu} \in Q^k$, we have

$$f_i(x^{k,\nu}) = f_i(x^* + t_\nu d^{k,\nu}) \leq f_i(x^*), \quad \forall i \in \mathcal{I}_f^{-k}.$$

It follows from Lemma 2.4 that $\langle \nabla f_{\mathcal{I}_f^{-k}}(x^*), d^k\rangle \le 0$. Similarly, we have

$$\langle \nabla g_{\mathcal{I}_g^*}(x^*), d^k\rangle \le 0,\ \langle \nabla h(x^*), d^k\rangle = 0,\ \langle \nabla G_{\mathcal{I}^*}(x^*), d^k\rangle = 0,$$
$$\langle \nabla H_{\mathcal{K}^*}(x^*), d^k\rangle = 0,\ \langle \nabla G_{\mathcal{I}^*\cup\mathcal{J}^*}(x^*), d^k\rangle \ge 0,\ \ \langle \nabla H_{\mathcal{J}^*\cup\mathcal{K}^*}(x^*), d^k\rangle\rangle \ge 0.$$

Since $G_i(x^k)H_i(x^k)=0=G_i(x^*+t_kd^k)H_i(x^*+t_kd^k)$ for each $i=1,\ldots,m$, we have $G_i(x^*)\langle\nabla H_i(x^*), d^k\rangle + H_i(x^*)\langle \nabla G_i(x^*), d^k\rangle = 0$, and hence, $\langle \nabla G_i(x^*), d^k\rangle = 0$ for each $i \in \mathcal{I}^*$ and $\langle \nabla H_i(x^*), d^k\rangle = 0$ for each $i \in \mathcal{K}^*$. By the assumptions, x^* is a locally Pareto optimal solution of (1), and so we have $x^* = \text{argmin}_{x\in Q^k} f_k(x)$. Hence, we have $f_k(x^{k,\nu}) = f_k(x^* + t_\nu d^{k,\nu}) \ge f_k(x^*)$, and so $\langle \nabla f_k(x^*), d^k\rangle \ge 0$. The above discussion indicates that, for each $k \in \mathcal{I}_f$,

$$\langle \nabla f_k(x^*), d^k\rangle \ge 0,\quad \forall d^k \in \text{conv}\mathcal{T}\left(Q^k; x^*\right).$$

If x^* is not a proper Pareto B-stationary point of (1), there must exist $k \in \mathcal{I}_f$ such that $\langle \nabla f_k(x^*), d^k\rangle < 0$ for some $d^k \in \mathcal{L}_{\text{MOPEC}}(Q^k; x^*)$. Since the MOPEC-GGCQ holds at x^*, we have $d^k \in \text{conv}\mathcal{T}(Q^k; x^*)$. Thus, we get a contradiction. This completes the proof. □

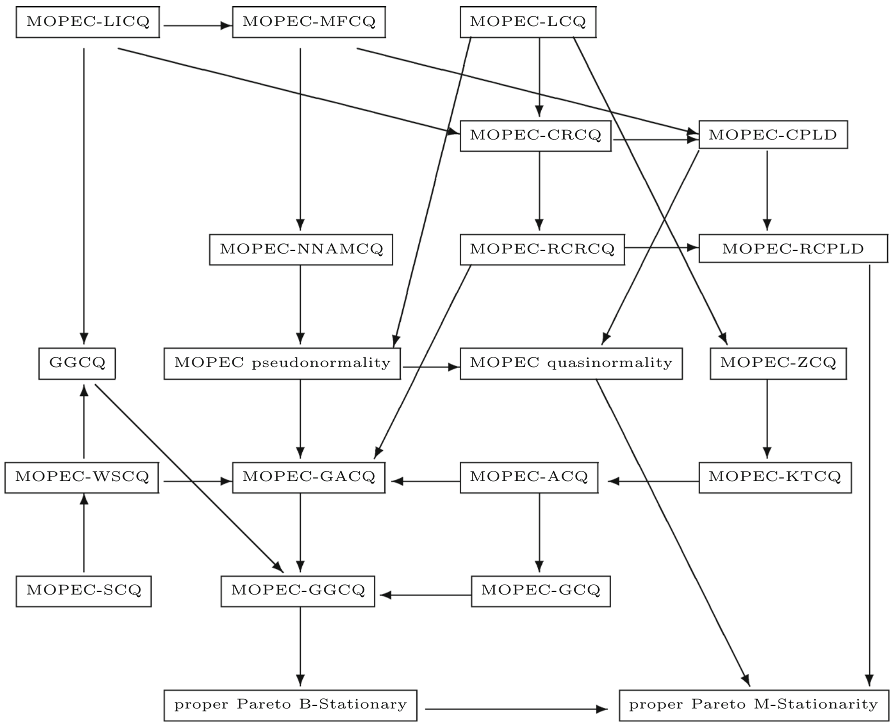

Fig. 1 Relations among various constraint qualifications

5 Conclusions

We have generalized the existing MPEC-type constraint qualifications from the single-objective case to the multiobjective case. Various relationships among these constraint qualifications have been investigated. We have further studied the MPEC-type stationarities in the proper Pareto sense for MOPEC (1). See Fig. 1 for a summary related to the relationships among the properties discussed in this paper.

Recall that Moldovan and Pellegrini [25,26] consider a particular theorem for linear separation between two sets in the image space associated with a single-objective constrained optimization problem. The two sets include a convex cone, which depends on the constraints, and the homogenization of its image set. They show that the regular linear separation between the above two sets is equivalent to the existence of Lagrangian multipliers with a positive multiplier associated with the objective function. When the constraint functions are all differentiable, they also show that the regularity condition is weaker than the Guignard constraint qualification. Although these results cannot be applied to our cases due to lack of convexity, it gives us some new ideas to establish similar weak regularity conditions to make sure the multiplier associate with the multiobjective function to be positive. We would like to leave it as a future topic.

In addition, in the next step, we are planning to extend the concepts given in this paper to the nonsmooth MOPEC, which has many applications in practice [6,27–30]. Another future work is to develop numerical algorithms by using the obtained theoretical results.

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