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Global Uniqueness and Solvability of Tensor Variational Inequalities

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Abstract In this paper, we consider a class of variational inequalities, where the involved function is the sum of an arbitrary given vector and a homogeneous polynomial defined by a tensor; we call it the tensor variational inequality. The tensor variational inequality is a natural extension of the affine variational inequality and the tensor complementarity problem. We show that a class of multi-person noncooperative games can be formulated as a tensor variational inequality. In particular, we investigate the global uniqueness and solvability of the tensor variational inequality. To this end, we first introduce two classes of structured tensors and discuss some related properties, and then, we show that the tensor variational inequality has the property of global uniqueness and solvability under some assumptions, which is different from the existing result for the general variational inequality.

Keywords Tensor variational inequality · Global uniqueness and solvability · Noncooperative game · Strictly positive definite tensor · Exceptionally family of elements

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1 Introduction

The finite-dimensional variational inequality (VI) has been studied extensively due to its wide applications in many fields [1,2]. It is called an affine variational inequality if the involved function is linear. The existence and uniqueness of solution to the VI is a basic and important issue in the studies of the VI. It is well known that the VI has at most one solution when the involved function is strictly monotone [1–3] and a unique solution when the involved function is strongly monotone [1, 2].

It is well known that complementarity problem (CP) is an important subclass of the VIs, which has been studied extensively due to its wide applications [4,5]. Recently, a specific subclass of CPs, called the tensor complementarity problem (TCP) [6], has attracted much attention, and many theoretical results about the properties of the solution set of TCP have been developed, including existence of solution [7–11], global uniqueness of solution [11,12], boundedness of solution set [8,13–16], stability of solution [17], sparsity of solution [18], and so on. In addition, an application of the TCP was given in [19].

Inspired by the development of the TCP, we consider a subclass of the VIs, where the involved function is the sum of an arbitrary given vector and a homogeneous polynomial defined by a tensor, and we call it the tensor variational inequality (TVI). The concerned problem is a natural generalization of the TCP and the affine variational inequality. It is well known that the polynomial optimization problem is an important class of optimization problems, which has been studied extensively [20–22]. It is easy to see that the TVI is equivalent to a class of polynomial optimization problems. In addition, we show that a class of multi-person noncooperative games can be reformulated as a TVI. These are our motivations to consider the TVI.

In this paper, we mainly investigate the property of global uniqueness and solvability (GUS-property) of the TVI in the case that 0 belongs to the set involved in the TVI. In this case, we show that there is no strongly monotonously homogeneous polynomial whose degree is larger than 2. In order to investigate the GUS-property of the TVI, we first introduce two classes of structured tensors and discuss some related properties, and then, we show that the TVI has the GUS-property when the involved function is strictly monotone and the involved set contains 0, which is different from the existing result obtained in the case of the general variational inequality.

The rest of this paper is organized as follows: In Sect. 2, we recall some basic definitions and results. In Sect. 3, we introduce the TVI and reformulate a class of multi-person noncooperative games as a TVI. In Sect. 4, we define two classes of structured tensors and discuss some related properties. In particular, we show that the TVI has the GUS-property under some assumptions. In Sect. 5, we propose some open problems. The conclusions are given in Sect. 6.

2 Preliminaries

In this section, we recall some basic concepts and results, which are useful for our subsequent analysis.

Given a nonempty set $X \subseteq \mathbb{R}^n$ and a function $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$, then the variational inequality, denoted by the VI(X, F), is to find a point $x^* \in X$ such that

$$\langle y - x^*, F(x^*) \rangle \geq 0 \quad \text{for all} \quad y \in X. \tag{1}$$

It is called an affine variational inequality when the function F is linear. Moreover, if the set X is the nonnegative orthant $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$, then (1) reduces to

$$x \geq 0, \quad F(x) \geq 0, \quad x^\top F(x) = 0,$$

which is called the complementarity problem, denoted by the CP(F).

In the theoretical studies of the nonlinear variational inequality and complementarity problem, some special types of functions play important roles. The following two classes of functions will be used in this paper.

Definition 2.1 A mapping $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(i) *strictly monotone on X*, if and only if

$$\langle F(x) - F(y), x - y \rangle > 0 \quad \text{for all} \quad x, y \in X \quad \text{with} \quad x \neq y;$$

(ii) *strongly monotone on X*, if and only if there exists a constant $c > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq c \|x - y\|^2 \quad \text{for all} \quad x, y \in X. \tag{2}$$

Obviously, a strongly monotone function on $X \subseteq \mathbb{R}^n$ must be strictly monotone on X. Moreover, for $X = \mathbb{R}^n$ and an affine mapping, i.e., $F(x) = Ax + q$, where $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, F is strongly monotone if and only if it is strictly monotone, and if and only if A is positive definite [2]. However, such results do not hold for the general nonlinear function.

The exceptionally family of elements is a powerful tool to investigate the solvability of the VI(X, F) [23–27]. There are several different definitions for the exceptionally family of elements. In this paper, we use the following definition.

Definition 2.2 [26, Definition 3.1] Let $\hat{x} \in \mathbb{R}^n$ be an arbitrary given point. A sequence $\{x^r\}_{r>0}$ is said to be an exceptionally family of elements for the VI(X, F) with respect to $\hat{x}$ if the following conditions are satisfied:

(i) $\|x^r\| \to \infty$ as $r \to \infty$;
(ii) $x^r - \hat{x} \in X$;
(iii) there exists $\alpha_r \in]0, 1[$ such that, for any $r \geq \|P_X(0) - \hat{x}\|$,

$$-[F(x^r - \hat{x}) + (1 - \alpha_r)(x^r - \hat{x})] \in \mathcal{N}_X(x^r - \hat{x}),$$

where $\mathscr{N}_X(x^r - \hat{x})$ denotes the normal cone of X at $x^r - \hat{x}$ and $P_X(\cdot)$ is the projection operator on X.

The normal cone of X at x is defined by

$$\mathscr{N}_X(x) = \begin{cases} \{z \in \mathbb{R}^n : z^\top(y - x) \leq 0, \forall y \in X\}, & \text{if } x \in X, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3}$$

About the relationship between the exceptionally family of elements and the solution of the VI(X, F), we will use the following lemma whose proof can be found in [26].

Lemma 2.1 [26, Theorem 3.1] *Let X be a nonempty, closed and convex set in $\mathbb{R}^n$ and $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Then, either the VI(X, F) has a solution or, for any point $\hat{x} \in \mathbb{R}^n$, there exists an exceptionally family of elements for the VI(X, F) with respect to $\hat{x}$.*

Throughout this paper, for any given positive integer n, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For any given positive integers $m, r_1, \ldots, r_{m-1}$ and r_m, an m-order $r_1 \times r_2 \times \cdots \times r_m$-dimensional real tensor can be denoted by $\mathscr{A} = (a_{i_1 i_2 \ldots i_m})$ with $a_{i_1 i_2 \ldots i_m} \in \mathbb{R}$ for any $i_j \in [r_j]$ and $j \in [m]$. Furthermore, if $r_j = n$ for all $j \in [m]$, then $\mathscr{A}$ is called an m-order n-dimensional real tensor, and we denote the set of all m-order n-dimensional real tensors by $\mathbb{T}_{m,n}$. In particular, $\mathscr{A} \in \mathbb{T}_{m,n}$ is called a symmetric tensor if the entries $a_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of their indices. For any $\mathscr{A} \in \mathbb{T}_{m,n}$ and $x \in \mathbb{R}^n$, $\mathscr{A}x^{m-1} \in \mathbb{R}^n$ is a vector defined by

$$(\mathscr{A}x^{m-1})_i := \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}, \quad \forall i \in [n].$$

3 The TVI and an Application

In this section, we first introduce the TVI and discuss the relationship between it and a class of polynomial optimization problems and then give an application of the TVI.

For any $\mathscr{A} \in \mathbb{T}_{m,n}$, $q \in \mathbb{R}^n$ and a nonempty set $X \subseteq \mathbb{R}^n$, the TVI we considered is given specifically in the following way: Find a vector $x^* \in X$ such that

$$\langle y - x^*, \mathscr{A}(x^*)^{m-1} + q \rangle \geq 0 \quad \text{for all } y \in X, \tag{4}$$

which is denoted by the TVI$(X, \mathscr{A}, q)$. From the relationship between the variational inequalities and the complementarity problems, which is also described at the beginning of Sect. 2, it is easy to see that, when $X = \mathbb{R}^n_+$, the TVI$(X, \mathscr{A}, q)$ is equivalent to the tensor complementarity problem: Find a vector $x^* \geq 0$ such that

$$\mathscr{A}(x^*)^{m-1} + q \geq 0, \quad \text{and} \quad (x^*)^T[\mathscr{A}(x^*)^{m-1} + q] = 0.$$

It should be noted that Song and Qi [6] proposed a TVI$(X, \mathscr{A}, q)$ with $q = 0$ in a question related to applications of structured tensors; but to the best of our knowledge, the TVI$(X, \mathscr{A}, q)$ has not been studied so far even in the case of $q = 0$.

The TVI$(X, \mathscr{A}, q)$ arises in a natural way in the framework of polynomial optimization problems, which is given as follows:

Proposition 3.1 *For any given symmetric tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ and $q \in \mathbb{R}^n$, suppose that $f(x) = \frac{1}{m}\mathscr{A}x^m + q^\top x$ is a convex function and $X \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. Then, x^* solves the TVI$(X, \mathscr{A}, q)$ if and only if x^* is an optimal solution of the optimization problem $\min\{f(x) : x \in X\}$.*

Proof Since $\mathscr{A}$ is symmetric, it follows that $\nabla f(x) = \mathscr{A}x^{m-1} + q$. Then, the result is straightforward from [28, Page 10].   □

In the following, we consider an application of the TVI$(X, \mathscr{A}, q)$ related to the problem of m-person noncooperative game. For any $k \in [m]$, let $x^k \in \mathbb{R}^{r_k}$ and $X_k \subseteq \mathbb{R}^{r_k}$ be player k's strategy and strategy set, respectively. We denote

$$[m]_{-k} := [m] \backslash \{k\}, \quad n := \sum_{j \in [m]} r_j, \quad n_{-k} := \sum_{j \in [m]_{-k}} r_j,$$

$$x := (x^j)_{j \in [m]} \in \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_m} = \mathbb{R}^n,$$

$$x^{-k} := (x^j)_{j \in [m]_{-k}} \in \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_{k-1}} \times \mathbb{R}^{r_{k+1}} \times \cdots \times \mathbb{R}^{r_m} = \mathbb{R}^{n-k},$$

and

$$X := \prod_{j \in [m]} X_j \subseteq \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_m} = \mathbb{R}^n. \tag{5}$$

Then, for any $k \in [m]$, the kth player decides his own strategy by solving the following optimization problem with the opponents' strategy x^{-k} fixed:

$$\min f_k(y^k, x^{-k}) \quad \text{s.t.} \quad y^k \in X_k,$$

where $f_k : \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_m} \to \mathbb{R}$ denotes player k's cost function.

A tuple $x^* := ((x^1)^*, (x^2)^*, \dots, (x^m)^*)$ satisfying

$$(x^k)^* \in \arg\min_{y^k \in X_k} f_k(y^k, x^{-k}), \quad \forall k \in [m]$$

is called a Nash equilibrium of the m-person noncooperation game.

Proposition 3.2 [2, Proposition 1.4.2] *Let every $X_i \subseteq \mathbb{R}^{r_i}$ be closed and convex. Suppose that for each fixed $x := ((x^1), (x^2), \dots, (x^m)) \in X$, the function $f_k(y^k, x^{-k})$ is convex and continuously differentiable in y^k. Then $x^* := ((x^1)^*, (x^2)^*, \dots, (x^m)^*)$ is a Nash equilibrium of the m-person noncooperation game if and only if x^* is a solution of the VI(X, F) with $F(x) \equiv \left(\nabla_{x^k} f_k(x)\right)_{k \in [m]}$, where $\nabla_{x^k} f_k(x)$ is the gradient of the function $f_k(x)$ with respect to x^k.*

In this paper, for any $k \in [m]$, we use $\mathscr{A}^k = (a^k_{i_1 i_2 \dots i_m})$ to denote player k's payoff tensor and assume that player k's cost function f_k is given by

$$f_k(x^k, x^{-k}) = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_m=1}^{r_m} a^k_{i_1 i_2 \dots i_m} x^1_{i_1} x^2_{i_2} \dots x^{k-1}_{i_{k-1}} x^k_{i_k} x^{k+1}_{i_{k+1}} \dots x^m_{i_m}. \tag{6}$$

Then, the function $F(x) \equiv (\nabla_{x^k} f_k(x))_{k \in [m]}$ defined in Proposition 3.2 is a homogeneous polynomial function with the degree $m - 1$, which can be defined by a tensor. To this end, we first introduce the following symbols: for any tensor $\mathscr{B} \in \mathbb{T}_{m,n}$ and $u^k \in \mathbb{R}^{r_k}$ with $k \in [m]_{-1}$, we denote

$$
\mathscr{B}u^2 \ldots u^m =
\begin{pmatrix}
\sum_{i_2=1}^{r_2} \cdots \sum_{i_m=1}^{r_m} b_{1 i_2 \ldots i_n} u_{i_2}^2 \ldots u_{i_m}^m \\
\sum_{i_2=1}^{r_2} \cdots \sum_{i_m=1}^{r_m} b_{2 i_2 \ldots i_n} u_{i_2}^2 \ldots u_{i_m}^m \\
\vdots \\
\sum_{i_2=1}^{r_2} \cdots \sum_{i_m=1}^{r_m} b_{r_1 i_2 \ldots i_n} u_{i_2}^2 \ldots u_{i_m}^m
\end{pmatrix};
$$

and, for any $k \in [m]$, by using the payoff tensor $\mathscr{A}^k = (a_{i_1 i_2 \ldots i_m}^k)$, we define a new tensor $\bar{\mathscr{A}}^k = (\bar{a}_{i_1 i_2 \ldots i_m}^k)$ with $\bar{a}_{i_1 i_2 \ldots i_m}^k = a_{i_k i_1 \ldots i_{k-1} i_{k+1} \ldots i_m}^k$ for any $i_j \in [r_j]$ and $j \in [m]$. Furthermore, we construct a new tensor

$$
\mathscr{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{T}_{m,n}, \tag{7}
$$

where for any $i_j \in [n]$ with $j \in [m]$,

$$
a_{i_1 i_2 \ldots i_m} =
\begin{cases}
a_{i_1 (i_2-r_1) \ldots (i_m - \sum_{j=1}^{m-1} r_j)}^1, \\
\quad \text{if } i_1 \in [r_1], i_2 \in [r_1 + r_2]\backslash[r_1], \ldots, i_m \in [\sum_{j=1}^{m} r_j]\backslash[\sum_{j=1}^{m-1} r_j], \\
a_{(i_1-r_1) i_2 (i_3-r_1-r_2) \ldots (i_m - \sum_{j=1}^{m-1} r_j)}^2, \\
\quad \text{if } i_1 \in [r_1 + r_2]\backslash[r_1], i_2 \in [r_1], \\
\qquad i_3 \in [\sum_{j=1}^{3} r_j]\backslash[r_1 + r_2], \ldots, i_m \in [\sum_{j=1}^{m} r_j]\backslash[\sum_{j=1}^{m-1} r_j], \\
a_{(i_1 - \sum_{j=1}^{k-1} r_j) i_2 (i_3-r_1) \ldots (i_{k-1} - \sum_{j=1}^{k-3} r_j) i_k (i_{k+1} - \sum_{j=1}^{k+1} r_j) \ldots (i_m - \sum_{j=1}^{m-1} r_j)}^k, \\
\quad \text{if } k \in [m]\backslash\{1, 2\}, \text{ and for any given } k, i_1 \in [\sum_{j=1}^{k} r_j]\backslash[\sum_{j=1}^{k-1} r_j], \\
\qquad i_2 \in [r_1], i_3 \in [r_1 + r_2]\backslash[r_1], \ldots, i_k \in [\sum_{j=1}^{k-1} r_j]\backslash[\sum_{j=1}^{k-2} r_j], \\
\qquad i_{k+1} \in [\sum_{j=1}^{k+1} r_j]\backslash[\sum_{j=1}^{k} r_j], \ldots, i_m \in [\sum_{j=1}^{m} r_j]\backslash[\sum_{j=1}^{m-1} r_j], \\
0, \quad \text{otherwise.}
\end{cases}
$$

Then, it is not difficult to get that

$$
\mathscr{A}x^{m-1} =
\begin{pmatrix}
\bar{\mathscr{A}}^1 x^2 \ldots x^m \\
\vdots \\
\bar{\mathscr{A}}^k x^1 \ldots x^{k-1} x^{k+1} \ldots x^m \\
\vdots \\
\bar{\mathscr{A}}^m x^1 x^2 \ldots x^{m-1}
\end{pmatrix}
=
\begin{pmatrix}
\nabla_{x^1} f_1(x^1, x^{-1}) \\
\vdots \\
\nabla_{x^k} f_k(x^k, x^{-k}) \\
\vdots \\
\nabla_{x^m} f_m(x^m, x^{-m})
\end{pmatrix}
= F(x). \tag{8}
$$

Therefore, from Propositions 3.2 and (8) we can obtain the following result.

Proposition 3.3 *For any $k \in [m]$, we assume the function f_k is defined by (6) and every set $X_i \subseteq \mathbb{R}^{r_i}$ is closed and convex, then a tuple $x^* := ((x^1)^*, (x^2)^*, \ldots, (x^m)^*)$ is a Nash equilibrium of the m-person noncooperation game if and only if x^* is a solution of the TVI$(X, \mathscr{A}, q)$ with $q = 0$ and $\mathscr{A}$ being defined by (7).*

4 GUS-Property of the TVI

The tensor variational inequality (4) is said to have the GUS-property if it has a unique solution for every $q \in \mathbb{R}^n$. Such an important property has been investigated for variational inequalities [1,2] and complementarity problems [29–32]. In this section, we discuss the GUS-property of the TVI$(X, \mathscr{A}, q)$.

For the general VI, the following results come from [1,2].

Lemma 4.1 *Let $X \subseteq \mathbb{R}^n$ be nonempty, closed and convex and $F : X \to \mathbb{R}^n$ be continuous.*

(i) *If F is strictly monotone on X, then the VI(X, F) has at most one solution;*
(ii) *If F is strongly monotone on X, then the VI(X, F) has a unique solution.*

Let $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$F(x) := \mathscr{A}x^{m-1} + q, \tag{9}$$

where $\mathscr{A} \in \mathbb{T}_{m,n}$ with $m > 2$ and $q \in \mathbb{R}^n$. Then, we have the following observation.

Proposition 4.1 *For any tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ with $m > 2$ and $q \in \mathbb{R}^n$, let the function F be defined by (9). Suppose that $0 \in X \subseteq \mathbb{R}^n$, then the function F is not strongly monotone on X.*

Proof Suppose that there exist a vector $q \in \mathbb{R}^n$ and a tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ with $m > 2$ such that the function F defined by (9) is strongly monotone on X, then there exists a positive constant c such that (2) holds for any $x, y \in X$. Let $y = 0 \in X$, then we get from (2) that

$$\mathscr{A}x^m \geq c\|x\|^2 \quad \text{for any} \quad x \in X. \tag{10}$$

For any $x \neq 0$, it follows from (10) that

$$\mathscr{A}\left(\frac{x}{\|x\|}\right)^m \geq c\left\|\left(\frac{x}{\|x\|}\right)\right\|^2 \|x\|^{2-m}. \tag{11}$$

Since $\left\|\frac{x}{\|x\|}\right\| = 1$, it follows that the left-hand side of inequality (11) is bounded; but when $\|x\| \to 0$, it is obvious that the right-hand side of inequality (11) tends to ∞, which leads to a contradiction. Therefore, there exists no strongly monotone function F in the form of $\mathscr{A}x^{m-1} + q$ for any $q \in \mathbb{R}^n$ and $\mathscr{A} \in \mathbb{T}_{m,n}$ with $m > 2$. $\square$

From Lemma 4.1(ii) and Proposition 4.1, a natural question is *whether or not the VI (X, F) has the GUS-property when $0 \in X$ and the function F is defined by* (9) *where $\mathscr{A} \in \mathbb{T}_{m,n}$ with $m > 2$ and $q \in \mathbb{R}^n$.* In this section, we answer this question. To this end, we first introduce two new classes of tensors in the next subsection and discuss the relationship between them.

4.1 Relationship of Two Classes of Tensors

In this subsection, we introduce two new classes of structured tensors and discuss the relationship between them.

Definition 4.1 Given a nonempty set $X \subseteq \mathbb{R}^n$. A tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ is said to be

 (i) *positive definite on X*, if and only if $\mathscr{A} x^m > 0$ for any $x \in X$ and $x \neq 0$;
(ii) *strictly positive definite on X*, if and only if

$$(x - y)^\top (\mathscr{A} x^{m-1} - \mathscr{A} y^{m-1}) > 0 \quad \text{for any} \quad x, y \in X \text{ with } x \neq y.$$

$\mathscr{A} \in \mathbb{T}_{m,n}$ is said to be a strictly positive definite tensor if it is strictly positive definite on $\mathbb{R}^n$.

When $X = \mathbb{R}^n$, the positive definite tensor on X defined by Definition 4.1(i) is just the positive definite tensor defined in [33]; when $X = \mathbb{R}^n_+$, the positive definite tensor on X defined by Definition 4.1(i) is just the strictly copositive tensor defined in [34]. From Definitions 2.1 and 4.1, it is easy to see that the function F defined by (9) is strictly monotone on X if and only if the tensor $\mathscr{A}$ is strictly positive definite on X.

A basic question is *whether or not there exists a strictly positive definite tensor on some subset of $\mathbb{R}^n$*. The following example gives a positive answer to this question.

Example 4.1 Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{T}_{4,2}$, where $a_{1111} = a_{2222} = 1$, and the others equal to zero. Then, $\mathscr{A}$ is a strictly positive definite tensor on any nonsingle-point subset X of $\mathbb{R}^2$.

It only needs to prove that $\mathscr{A}$ is strictly positive definite on $\mathbb{R}^2$.
Since

$$\mathscr{A} x^3 = \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix},$$

it follows that for any $x, y \in \mathbb{R}^2$,

$$\begin{aligned} (x_1 - y_1)[(\mathscr{A} x^3)_1 - (\mathscr{A} y^3)_1] &= (x_1 - y_1)(x_1^3 - y_1^3) \\ &= (x_1 - y_1)^2 (x_1^2 + x_1 y_1 + y_1^2); \end{aligned} \quad (12)$$

$$\begin{aligned} (x_2 - y_2)[(\mathscr{A} x^3)_2 - (\mathscr{A} y^3)_2] &= (x_2 - y_2)(x_2^3 - y_2^3) \\ &= (x_2 - y_2)^2 (x_2^2 + x_2 y_2 + y_2^2). \end{aligned} \quad (13)$$

For any $s, t \in \mathbb{R}$, we discuss the following three cases.

(I) $|s| \neq |t|$. In this case, we have

$$s^2 + st + t^2 > 2|s||t| + st = \begin{cases} 3st \geq 0, & \text{if } st \geq 0, \\ -st > 0, & \text{if } st < 0, \end{cases}$$

which implies that $s^2 + st + t^2 > 0$.

(II) $s = t$. In this case, we have

$$(s - t)^2(s^2 + st + t^2) = 0.$$

(III) $s = -t \neq 0$. In this case, we have

$$(s - t)^2(s^2 + st + t^2) = 4s^4 > 0.$$

Now, for any $x, y \in \mathbb{R}^2$ and $x \neq y$, it follows that either $x_1 \neq y_1$ or $x_2 \neq y_2$. Therefore, by combining cases (I)–(III) with (12) and (13) we have

$$(x - y)^\top(\mathscr{A}x^3 - \mathscr{A}y^3) = \sum_{i=1}^{2}(x_i - y_i)^2(x_i^2 + x_i y_i + y_i^2) > 0,$$

which demonstrates that $\mathscr{A}$ is a strictly positive definite tensor on $\mathbb{R}^2$.

In what follows, we discuss the relationship between two classes of tensors defined by Definition 4.1.

Proposition 4.2 *Suppose that $0 \in X \subseteq \mathbb{R}^n$. Then, a strictly positive definite tensor on X must be positive definite on X.*

Proof Given a tensor $\mathscr{A} \in \mathbb{T}_{m,n}$. Take $y = 0 \in X$, it follows from Definition 4.1(ii) that for any $x \in X$ with $x \neq 0$,

$$\mathscr{A}x^m = (x - 0)^\top \left(\mathscr{A}x^{m-1} - \mathscr{A}0^{m-1}\right) > 0,$$

which, together with Definition 4.1(i), implies that $\mathscr{A}$ is positive definite on X. ☐

However, if $m > 2$, a positive definite tensor on X is not necessary a strictly positive definite tensor on X, which can be seen in the following example.

Example 4.2 Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{T}_{4,2}$, where $a_{1111} = a_{2222} = a_{2112} = 1, a_{1122} = -1$, and the others equal to zero. Denote $X := \mathbb{R}_+^2$. Then, $\mathscr{A}$ is positive definite on X but not strictly positive definite on X.

First, we show that $\mathscr{A}$ is positive definite on X. Since

$$\mathscr{A}x^3 = \begin{pmatrix} x_1^3 - x_1 x_2^2 \\ x_2^3 + x_1^2 x_2 \end{pmatrix},$$

it follows that for any $x \in \mathbb{R}^2 \backslash \{0\}$,

$$x^\top \mathscr{A} x^3 = x_1^4 - x_1^2 x_2^2 + x_2^4 + x_1^2 x_2^2 = x_1^4 + x_2^4 > 0.$$

Hence, $\mathscr{A}$ is positive definite on $\mathbb{R}^2$. Of course, $\mathscr{A}$ is positive definite on X.

Second, we show that $\mathscr{A}$ is not a strictly positive definite tensor on X. To this end, for any $\mu \in \mathbb{R}_+$ with $\mu \neq 0$, let $x = (2\mu, 3\mu)^\top$ and $y = (\mu, 3\mu)^\top$, then $x, y \in X$, $x \neq y$ and

$$
\begin{aligned}
(x - y)^\top (\mathscr{A} x^3 - \mathscr{A} y^3) &= (x_1 - y_1)[(\mathscr{A} x^3)_1 - (\mathscr{A} y^3)_1] \\
&\quad + (x_2 - y_2)[(\mathscr{A} x^3)_2 - (\mathscr{A} y^3)_2] \\
&= (2\mu - \mu)[(2\mu)^3 - 2\mu(3\mu)^2 - (\mu^3 - \mu(3\mu)^2)] + 0 \\
&= -2\mu^4 \\
&< 0.
\end{aligned}
$$

Therefore, $\mathscr{A}$ is not strictly positive definite on X.

4.2 Uniqueness of Solution to the TVI

In this subsection, we investigate the GUS-property of the TVI$(X, \mathscr{A}, q)$.

Theorem 4.1 *Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set and $\mathscr{A} \in \mathbb{T}_{m,n}$ be a strictly positive definite tensor on X. Then, for any given $q \in \mathbb{R}^n$, the TVI$(X, \mathscr{A}, q)$ has at most one solution.*

Proof Since $\mathscr{A}$ is a strictly positive definite tensor on X, it follows from Definition 4.1(ii) that the function $\mathscr{A} x^{m-1} + q$ is strictly monotone on X for any $q \in \mathbb{R}^n$. So, the desired result holds from Lemma 4.1 (i). □

Theorem 4.2 *Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set with $0 \in X$ and $\mathscr{A} \in \mathbb{T}_{m,n}$ be a positive definite tensor on X. Then, for any given $q \in \mathbb{R}^n$, the solution set of the TVI$(X, \mathscr{A}, q)$ is nonempty and compact.*

Proof If the set X is bounded, then the result is obvious from [1,35]. In what follows, we assume that the set X is unbounded.

Suppose that the TVI$(X, \mathscr{A}, q)$ has no solution, then for $\hat{x} = 0 \in \mathbb{R}^n$, it follows from Lemma 2.1 that there exists an exceptionally family of elements $\{x^r\}_{r>0}$ for the TVI$(X, \mathscr{A}, q)$ with respect to 0. That is, we have

(a) $\|x^r\| \to \infty$ as $r \to \infty$;
(b) $x^r \in X$ for any positive integer r;
(c) there exists $\alpha_r \in]0, 1[$ such that, for any $r \geq \|P_X(0)\|$,

$$-[\mathscr{A}(x^r)^{m-1} + (1 - \alpha_r)x^r] \in \mathscr{N}_X(x^r).$$

From the above (c) and the definition of the normal cone, we have

$$[\mathscr{A}(x^r)^{m-1} + (1 - \alpha_r)x^r]^\top (y - x^r) \geq 0 \quad \text{for any } y \in X,$$

which can be rewritten as

$$[\mathscr{A}(x^r)^{m-1}]^\top(y - x^r) \geq (\alpha_r - 1)(x^r)^\top(y - x^r) \quad \text{for any } y \in X. \tag{14}$$

From the above (a), it holds that $\|x^r\| > 0$ for sufficiently large r. So, by dividing $\|x^r\|^m$ in both sides of inequality (14), we get

$$\left[\mathscr{A}\frac{(x^r)^{m-1}}{\|x^r\|^{m-1}}\right]^\top\left(\frac{y}{\|x^r\|} - \frac{x^r}{\|x^r\|}\right) \geq \frac{\alpha_r - 1}{\|x^r\|^{m-2}}\left(\frac{x^r}{\|x^r\|}\right)^\top\left(\frac{y}{\|x^r\|} - \frac{x^r}{\|x^r\|}\right).$$

Let $z^r = \frac{x^r}{\|x^r\|}$, then the above inequality becomes

$$[\mathscr{A}(z^r)^{m-1}]^\top\left(\frac{y}{\|x^r\|} - z^r\right) \geq \frac{\alpha_r - 1}{\|x^r\|^{m-2}}(z^r)^\top\left(\frac{y}{\|x^r\|} - z^r\right). \tag{15}$$

Since the sequence $\{z^r\}$ is bounded, there exists a convergent subsequence. Without lose of generality, we denote this subsequence by $\{z^r\}$ and its limit point by z^*. Noting that $\alpha_r \in]0, 1[$ and $y \in X$ is an arbitrary given vector, by letting $r \to \infty$, it follows from (15) that $[\mathscr{A}(z^*)^{m-1}]^\top(-z^*) \geq 0$, i.e.,

$$\mathscr{A}(z^*)^m \leq 0. \tag{16}$$

Next, we show that $z^* \in X$. Since $\|x^r\| \to \infty$ as $r \to \infty$, it follows that $\frac{1}{\|x^r\|} < 1$ with sufficiently large r. Furthermore, since $0 \in X$ and X is convex, it follows from the above (b) that for sufficiently large r,

$$z^r = \frac{x^r}{\|x^r\|} = \left(1 - \frac{1}{\|x^r\|}\right)0 + \frac{1}{\|x^r\|}x^r \in X.$$

Thus, by the fact that the set X is closed, we get

$$z^* \in X.$$

This, together with (16), contradicts that $\mathscr{A}$ is a positive definite tensor on X. Therefore, the TVI$(X, \mathscr{A}, q)$ has at least one solution when $\mathscr{A}$ is a positive definite tensor on X.

Denote the solution set of the TVI$(X, \mathscr{A}, q)$ by SOL$(X, \mathscr{A}, q)$. Suppose that the sequence $\{x^k\} \subseteq$SOL$(X, \mathscr{A}, q)$ and $x^k \to x^*$ as $k \to \infty$, then it follows that

$$(y - x^k)^\top\left[\mathscr{A}(x^k)^{m-1} + q\right] \geq 0 \quad \text{for all } y \in X.$$

Thus, let $k \to \infty$, then we get

$$(y - x^*)^\top\left[\mathscr{A}(x^*)^{m-1} + q\right] \geq 0 \quad \text{for all } y \in X.$$

That is, $x^* \in$SOL$(X, \mathscr{A}, q)$. So, the solution set of the TVI$(X, \mathscr{A}, q)$ is closed.

Suppose that the solution set of the TVI$(X, \mathscr{A}, q)$ is unbounded, then there exists a sequence $\{x^k\} \subseteq \mathrm{SOL}(X, \mathscr{A}, q)$ such that $\|x^k\| \to \infty$ as $k \to \infty$. Since

$$(y - x^k)^\top \left[\mathscr{A}(x^k)^{m-1} + q \right] \geq 0 \quad \text{for all } y \in X,$$

which leads to

$$\left(\frac{y}{\|x^k\|} - \frac{x^k}{\|x^k\|} \right)^\top \left[\mathscr{A} \left(\frac{x^k}{\|x^k\|} \right)^{m-1} + \frac{q}{\|x^k\|^{m-1}} \right] \geq 0.$$

Let $k \to \infty$ and denote $x^* = \lim_{k \to \infty} \frac{x^k}{\|x^k\|}$, then we have that

$$x^* \in X \quad \text{and} \quad -\mathscr{A}(x^*)^m \geq 0,$$

which contradicts the condition that $\mathscr{A}$ is a positive definite tensor on X. So, the solution set of the TVI$(X, \mathscr{A}, q)$ is bounded.

The proof is complete. □

Corollary 4.1 *Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set with $0 \in X$ and $\mathscr{A} \in \mathbb{T}_{m,n}$ be a strictly positive definite tensor on X. Then, for any given $q \in \mathbb{R}^n$, the solution set of the TVI$(X, \mathscr{A}, q)$ is nonempty and compact.*

Proof Since $0 \in X$, it follows from Proposition 4.2 that a strictly positive definite tensor on X is necessary a positive definite tensor on X. Thus, the result is obvious from Theorem 4.2. □

Theorem 4.3 *Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set with $0 \in X$ and $\mathscr{A} \in \mathbb{T}_{m,n}$ be a strictly positive definite tensor on X. Then, for any given $q \in \mathbb{R}^n$, the TVI$(X, \mathscr{A}, q)$ has a unique solution.*

Proof By virtue of Theorem 4.1 and Corollary 4.1, the result is straightforward. □

Equivalently, we have the following result.

Corollary 4.2 *Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set with $0 \in X$ and $\mathscr{A} \in \mathbb{T}_{m,n}$. Suppose that the function $F(x) := \mathscr{A}x^{m-1} + q$ is strictly monotone on X, then the VI(X, F) has a unique solution for any $q \in \mathbb{R}^n$.*

Remark 4.1 Suppose that 0 belongs to the strategy set X defined by (5) and the set X is closed and convex. Then, we can know from Theorem 4.3 that when the tensor $\mathscr{A}$ defined by (7) is strictly positive definite, the m-person noncooperative game has a unique Nash equilibrium.

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set and the function F be given by $F(x) = \mathscr{A}x^{m-1} + q$, where $\mathscr{A} \in \mathbb{T}_{m,n}$ and $q \in \mathbb{R}^n$. We have shown that, in the case of $0 \in X$, the VI(X, F) has the GUS-property if the function F is strictly monotone on X. What would happen if $0 \notin X$? From Lemma 4.1, we know that the VI(X, F)

has the GUS-property if the function F is strongly monotone on X. A natural question is *whether or not there exists a strongly monotone function $F(x) = \mathscr{A}x^{m-1} + q$ (with $m > 2$) on X with $0 \notin X$*. The following example gives a positive answer to this question.

Example 4.3 Let

$$X := \{(u, 1)^\top : u \in \mathbb{R}, u \geq 1\}, \tag{17}$$

and $\mathscr{A} \in \mathbb{T}_{m,n}$ be defined in Example 4.1, then $F(x) := \mathscr{A}x^{m-1} + q$ with any $q \in \mathbb{R}^2$ is strongly monotone on X.

For any $x, y \in X$, it follows that there exist $u \geq 1$ and $v \geq 1$ such that $x = (u, 1)^\top$ and $y = (v, 1)^\top$. Furthermore, for any $q \in \mathbb{R}^2$, we have

$$(x - y)^\top [F(x) - F(y)] = (x - y)^\top \left(\mathscr{A}x^3 - \mathscr{A}y^3 \right) = (u - v)^2 (u^2 + uv + v^2);$$

but for $\mu = 1$, we have

$$\mu \|x - y\|^2 = (u - v)^2.$$

Obviously,

$$(u - v)^2 (u^2 + uv + v^2) \geq 3uv(u - v)^2 \geq 3(u - v)^2 \geq (u - v)^2.$$

Thus, for any $x, y \in X$ and $q \in \mathbb{R}^2$, there exists a constant $\mu = 1$ such that

$$(x - y)^\top \left(\mathscr{A}x^3 - \mathscr{A}y^3 \right) \geq \mu \|x - y\|^2.$$

So, the function F is strongly monotone on the set X defined by (17).

Therefore, when $X \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set with $0 \notin X$, from Lemma 4.1(ii), we know that the TVI$(X, \mathscr{A}, q)$ has a unique solution on X if the function $\mathscr{A}x^{m-1} + q$ is strongly monotone on X. We do not know whether the condition of strong monotonicity can be weaken or not in this case.

Before the end of this section, we illustrate that a strictly monotone function $\mathscr{A}x^{m-1} + q$ on $X \subseteq \mathbb{R}^n$ is not necessarily strongly monotone on X when $0 \notin X$.

Example 4.4 Let $\mathscr{A} \in \mathbb{T}_{m,n}$ be defined in Example 4.1 and $X := \{(u, 1)^\top : u \in \mathbb{R}\}$. Then, for any $q \in \mathbb{R}^2$, the function $\mathscr{A}x^{m-1} + q$ is strictly monotone on X but not strongly monotone on X.

First, from Example 4.1, it is obvious that the tensor $\mathscr{A}$ is strictly positive definite on X. Therefore, the function $\mathscr{A}x^{m-1} + q$ is strictly monotone on X.

Second, we show that the function $\mathscr{A}x^{m-1} + q$ is not strongly monotone on X. Suppose that $\mathscr{A}x^{m-1} + q$ is strongly monotone on X, then there exists a scalar $\mu_0 > 0$ such that

$$(x - y)^\top \left(\mathscr{A}x^3 - \mathscr{A}y^3 \right) \geq \mu_0 \|x - y\|^2 \quad \text{for any } x, y \in X. \tag{18}$$

Now, take $x^0 = (\sqrt{\mu_0}, 1)^\top \in X$ and $y^0 = (-\frac{\sqrt{\mu_0}}{2}, 1)^\top \in X$, then

$$
\begin{aligned}
&\left(x^0 - y^0 \right)^\top \left[\mathscr{A}(x^0)^3 - \mathscr{A}(y^0)^3 \right] \\
&= \left(x_1^0 - y_1^0 \right)^2 \left[(x_1^0)^2 + x_1^0 y_1^0 + (y_1^0)^2 \right] \\
&= \left[\sqrt{\mu_0} + \frac{\sqrt{\mu_0}}{2} \right]^2 \left[(\sqrt{\mu_0})^2 - \sqrt{\mu_0} \cdot \frac{\sqrt{\mu_0}}{2} + \left(\frac{\sqrt{\mu_0}}{2} \right)^2 \right] \\
&= \left(\frac{3\sqrt{\mu_0}}{2} \right)^2 \left(\mu_0 - \frac{\mu_0}{2} + \frac{\mu_0}{4} \right) = \frac{27}{16}\mu_0^2
\end{aligned}
$$

and

$$
\mu_0 \|x^0 - y^0\|^2 = \mu_0 \left[(x_1^0 - y_1^0)^2 + (x_2^0 - y_2^0)^2 \right] = \mu_0 \left[\sqrt{\mu_0} + \frac{\sqrt{\mu_0}}{2} \right]^2 = \frac{9}{4}\mu_0^2.
$$

These yield that

$$
(x^0 - y^0)^\top \left[\mathscr{A}(x^0)^3 - \mathscr{A}(y^0)^3 \right] < \mu_0 \|x^0 - y^0\|^2,
$$

which contradicts inequality (18). So, the function $\mathscr{A}x^{m-1} + q$ is not strongly monotone on X.

5 Perspectives and Open Problems

We have just done some initial research for the tensor variational inequality. Many questions need to be answered in the future. Here, we provide two questions as follows.

Question 5.1 How to design effective algorithms to solve the TVI($X, \mathscr{A}, q$) by using the specific structure of the tensor $\mathscr{A}$?

Question 5.2 In [36], the author investigated the properties of the general polynomial complementarity problem denoted by the PCP(f) with

$$
f(x) = \mathscr{A}_m x^{m-1} + \mathscr{A}_{m-1} x^{m-2} + \cdots + \mathscr{A}_2 x + \mathscr{A}_1, \tag{19}
$$

where $\mathscr{A}_k$ is a tensor of order k and $\mathscr{A}_k x^{k-1}$ is a polynomial mapping for any $k \in [m]$. If we use the polynomial function f defined by (19) to replace the function $\mathscr{A}x^{m-1} + q$ in the TVI($X, \mathscr{A}, q$), i.e., find a vector $x^* \in X$ such that

$$
\langle y - x^*, f(x^*) \rangle \geq 0 \quad \text{for all} \quad y \in X,
$$

then we call it the polynomial variational inequality, denoted by the PVI(X, f). What are the properties of solution to the PVI(X, f)?

6 Conclusions

In this paper, we studied the tensor variational inequality which is a subclass of the general variational inequalities. For the general variational inequality, it is well known that the VI(X, F) has a unique solution if $X \subseteq \mathbb{R}^n$ is nonempty, closed and convex and the continuous function $F : X \to \mathbb{R}^n$ is strongly monotone. However, we proved that F is not strongly monotone when $F(x) = \mathscr{A}x^{m-1} + q$ with $m > 2$. Hence, the above result for the general variational inequality cannot be applied to the tensor variational inequality. In order to obtain the GUS-property of the tensor variational inequality, we defined two classes of structured tensors and discussed the relationship between them. Furthermore, we showed that the tensor variational inequality TVI$(X, \mathscr{A}, q)$ has the GUS-property when the function $F(x) := \mathscr{A}x^{m-1} + q$ is strictly monotone on X and $0 \in X$. It is possible that the method proposed in this paper can be applied to investigate the GUS-property of the polynomial variational inequality proposed in Question 5.2 of Sect. 5. In addition, for the study for the polynomial variational inequality given in Question 5.2 of Sect. 5, we believe that it is a good candidate to use the tool of degree theory.

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