Journal of Optimization Theory and Applications (2019) 181:383–410
https://doi.org/10.1007/s10957-018-01471-x

Equilibrium for a Time-Inconsistent Stochastic Linear–Quadratic Control System with Jumps and Its Application to the Mean-Variance Problem

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Received: 28 June 2018 / Accepted: 31 December 2018 / Published online: 22 January 2019

Abstract

This paper studies a kind of time-inconsistent linear–quadratic control problem in a more general framework with stochastic coefficients and random jumps. The time inconsistency comes from the dependence of the terminal cost on the current state as well as the presence of a quadratic term of the expected terminal state in the objective functional. Instead of finding a global optimal control, we look for a time-consistent locally optimal equilibrium solution within the class of open-loop controls. A general sufficient and necessary condition for equilibrium controls via a flow of forward–backward stochastic differential equations is derived. This paper further develops a new methodology to cope with the mathematical difficulties arising from the presence of stochastic coefficients and random jumps. As an application, we study a mean-variance portfolio selection problem in a jump-diffusion financial market; an explicit equilibrium investment strategy in a deterministic coefficients case is obtained and proved to be unique.

Keywords Time-inconsistent linear–quadratic control · Stochastic coefficients and random jumps · Equilibrium control · Forward–backward stochastic differential equation · Mean-variance portfolio selection

Mathematics Subject Classification 91G80 · 93E20

Communicated by Negash G. Medhin.

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1 Introduction

In [1], the time-inconsistent decision-making problem was initially formulated and discussed by viewing the whole problem as a game between incarnations of the decision maker at different time points. From then on, it has inspired hundreds of extensions and applications. Generally speaking, there are three main sources of time inconsistency: the appearance of the conditional expected state in a nonlinear way, the non-exponential discounting and the dependence of the objective functional on the initial state. Motivated by a class of mean-variance portfolio selection problems, the case with nonlinear appearance of conditional expected terminal cost has been studied by many authors including [2,3] and so on. The non-exponential discounting situation was discussed by [4,5], in which the main motivation is to try to catch people's subjective preference on the discounting. The last case depending on the initial state is motivated by a state-dependent utility function in economics (see, e.g., [6]).

One way to cope with the time-inconsistent optimal control problems is to consider them within a game theoretic framework. The basic idea is that an action made by the controller at each point in time is viewed as a game against all the actions made by future incarnations of himself. An equilibrium control is thus characterized by the property that any deviation from it will be worse off. From this point of view, Ekeland and Pirvu [4] investigate the optimal investment and consumption problem in the context of non-exponential discounting. A precise definition of equilibrium strategies is given by virtue of the so-called local spike variation. Yong [7] considers a time-inconsistent optimal control problem with the cost functional covering both the non-exponential and hyperbolic discounting situations. Under suitable conditions, he establishes sufficient conditions for equilibrium controls via a system of partial differential equations. For a general controlled Markovian system, Björk and Murgoci [8] solve the underlying time-inconsistent stochastic control problems by deriving an extended Hamilton–Jacobi–Bellman (HJB) equation system. In addition, they prove a verification theorem, showing that the equilibrium strategy is given by the optimizer in the equation system.

Motivated by some practical applications in economics, time-inconsistent linear–quadratic (LQ) control problems have become an important research topic and many efforts have been made to seek equilibrium controls; see, for example, [7,9,10] and references therein. Recently, Hu et al. [11,12] study a general time-inconsistent LQ control problem in a non-Markovian system with random parameters. They derive a necessary and sufficient condition for equilibrium controls and then present two special cases including a mean-variance portfolio selection problem, in which explicit solutions are constructed and proved to be unique. Alia et al. [13] extend the time-inconsistent LQ problem from the pure diffusion setting in [11,12] to the jump-diffusion one. A general stochastic maximum principle in a deterministic coefficients case is derived. For the time-inconsistent jump-diffusion LQ model with stochastic coefficients, Wu and Zhuang [14] derive a sufficient condition for equilibrium controls when the jump part coefficients do not contain the control variable. Based on their own assumptions on model coefficients, Alia et al. [13] and Wu and Zhuang [14] establish the existence of equilibrium controls by using the same method as that of [11,12]. It should be noted that the above-mentioned literature shares one common characteristic, that is, the first coefficient $A(\cdot)$ of the state in the controlled system is all assumed to be deterministic. However, many stochastic interest rate models in finance do not satisfy this assumption. Consequently, one motivation of this paper is to relax this deterministic assumption by proving a sharper estimate for the first-order variational equation (we refer to Lemma A.1). We deem the sharper estimate derived here as one of our major contributions and make our paper different from not only [11–14] but also other preceding works (for example, [15–17]) relying on the maximum principle approach to solve stochastic optimization problems.

In this paper, we further consider a class of time-inconsistent stochastic LQ control problems, where the state process is driven by a Brownian motion and an independent Poisson random measure. The control is allowed to enter into all the coefficients, and the objective functional contains both a state-dependent term and a quadratic term of the expected terminal cost. Also, we study these problems within the framework of stochastic coefficients. To cope with the mathematical difficulties caused by the presence of stochastic coefficients and random jumps, we introduce a new methodology to derive a sufficient condition for equilibrium controls (see Proposition 3.1), which distinguishes significantly from that in [11–14]. Then, by further developing the methodology proposed in [18], we prove that the existence of equilibrium controls is equivalent to the existence of solutions to a flow of forward–backward stochastic differential equations (FBSDEs) with constraints. Finally, we apply the established results to study a mean-variance portfolio selection problem in a jump-diffusion financial market; an explicit solution to equilibrium investment strategy in a deterministic coefficients case is obtained and proved to be unique.

Compared with [11–14], the main difficulties of this paper are to give an appropriate estimate for the first-order variational equation, which is crucial for deriving the necessary condition of equilibrium controls. Actually, our results generalize those in [11,12] by including Poisson random jumps and those in [13] by taking into consideration the stochastic coefficients. Also, it is a logical continuation of [14] to the case when the jump part coefficients contain control variables.

The rest of the paper is organized as follows: In Sect. 2, we give the formulation of the time-inconsistent LQ control problem and the definition of equilibrium control. A general sufficient and necessary condition for equilibrium controls through a system of FBSDEs with constraints is derived in Sect. 3. Section 4 is devoted to showing how to apply our theoretical results through an illustrating example. Section 5 concludes the paper and suggests some potential extensions of our work. Finally, an essential estimate for the first-order variational equation is placed in "Appendix".

2 Problem Formulation

Let $T > 0$ be a fixed finite time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space, where $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is the natural filtration generated by the Brownian motion and the Poisson random measure defined below satisfying the usual conditions. Let $\mathbb{R}_+ := [0, +\infty[$ be the time index set and $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. Suppose that $N_i(dt, de)$, $i = 1, \ldots, m$, are independent Poisson random measures on $(\mathbb{R}_+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0))$, here $\mathcal{B}(\mathbb{R}_+)$ and $\mathcal{B}(\mathbb{R}_0)$ are the Borel σ-fields generated by open subsets of $\mathbb{R}_+$ and $\mathbb{R}_0$, respectively. Assume further that the Poisson random measure $N_i(dt, de)$ has the following compensator

$$n_i(dt, de) := \nu_i(de)dt,$$

where, for each $i = 1, \dots, m$, ν_i is assumed to be a σ-finite measure on $\mathbb{R}_0$ satisfying $\nu_i(O) < \infty$ for all $O \in \mathcal{B}(\mathbb{R}_0)$, and $\int_{\mathbb{R}_0}(1 \wedge e^2)\nu_i(de) < \infty$. Moreover, denote $\nu(de) = (\nu_1(de), \dots, \nu_m(de))^T$ and the compensated Poisson random measure $\widetilde{N}(dt, de)$ by

$$\widetilde{N}(dt, de) := (N_1(dt, de) - n_1(dt, de), \dots, N_m(dt, de) - n_m(dt, de))^T.$$

On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, denote by $\mathcal{L}^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu; \mathbb{R}^m)$ the set of integrable functions $a(\cdot) : \mathbb{R}_0 \to \mathbb{R}^m$ such that $||a(\cdot)||^2_{\mathcal{L}^2} := \sum_{j=1}^m \int_{\mathbb{R}_0} |a_j(e)|^2 \nu_j(de) < \infty$ and $F_p^2(t, T; \mathbb{R}^m)$ the set of $\{\mathcal{F}_s\}_{s\in[t,T]}$-predictable processes $f(\cdot, \cdot) : \Omega \times [t, T] \times \mathbb{R}_0 \to \mathbb{R}^m$ such that $\mathbb{E}\Big[\int_t^T ||f(s, \cdot)||^2_{\mathcal{L}^2} \mathrm{d}s\Big] < \infty$. In addition, we also consider the following spaces of processes, with $\mathcal{H}$ being a finite-dimensional vector or matrix space and $k \geq 2$.

$$L^k_{\mathcal{F}_t}(\Omega; \mathcal{H}) := \Big\{\xi : \mathcal{H}\text{-valued } \mathcal{F}_t\text{-measurable random variables, s.t.} \mathbb{E}[|\xi|^k] < \infty\Big\};$$

$$L^k_{\mathcal{F}}(t, T; \mathcal{H}) := \Bigg\{f : \mathcal{H}\text{-valued } \{\mathcal{F}_s\}_{s\in[t,T]}\text{-adapted càdlàg processes, } s.t.\mathbb{E}\Big[\sup_{t\leq s\leq T} |f(s)|^k\Big] < \infty\Bigg\};$$

$$L^k_{\mathcal{F},p}(t, T; \mathcal{H}) := \Bigg\{f : \mathcal{H}\text{-valued } \{\mathcal{F}_s\}_{s\in[t,T]}\text{-predictable processes, s.t. } \mathbb{E}\Big[\int_t^T |f(s)|^k \mathrm{d}s\Big] < \infty\Bigg\}.$$

Now we introduce the model under consideration in this paper.

Let $\{W(t)\}_{t\in[0,T]} = \{(W_1(t), \dots, W_d(t))^T\}_{t\in[0,T]}$ be a d-dimensional standard Brownian motion, which is assumed to be stochastically independent of the Poisson random measure under $\mathbb{P}$. The controlled system starting from time $t \in [0, T]$ and state $x(t)$ is governed by the following linear stochastic differential equation (SDE) with jumps:

$$\begin{aligned} \mathrm{d}X(s) &= \big\{A(s)X(s) + B(s)u(s) + b(s)\big\}\mathrm{d}s \\ &\quad + \sum_{i=1}^d \big\{C_i(s)X(s) + D_i(s)u(s) + \sigma_i(s)\big\}dW_i(s) \\ &\quad + \sum_{j=1}^m \int_{\mathbb{R}_0} \big\{E_j(s, e)X(s-) + F_j(s, e)u(s-) + \eta_j(s, e)\big\}\widetilde{N}_j(ds, de), \\ X(t) &= x(t), \quad s \in [t, T]. \end{aligned} \tag{1}$$

Here $A(\cdot)$, $B(\cdot)$, $b(\cdot)$, $C_i(\cdot)$, $D_i(\cdot)$, $\sigma_i(\cdot)$, $E_j(\cdot, \cdot)$, $F_j(\cdot, \cdot)$ and $\eta_j(\cdot, \cdot)$ are uniformly bounded and $\mathbb{F}$-predictable processes on $[0, T]$ with values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times l}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times l}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times l}$ and $\mathbb{R}^n$, respectively. Moreover, to guarantee the auxiliary process $\{\Lambda(s)\}_{s\in[t,T]}$ defined in Lemma A.1 make sense, we further assume that each component of $E_j(\cdot, \cdot)$ is almost surely bigger than -1. It is worth mentioning that we do not need to require $A(\cdot)$ to be a deterministic function here, which makes our paper different from some proceeding works (for example, [11–14]) on time-inconsistent stochastic LQ control problems. In the above, $u(\cdot)$, valued in $\mathbb{R}^l$, is the control process, and $X(\cdot)$, valued in $\mathbb{R}^n$, is the state process.

For the state equation (1), we introduce the following set:

$$\mathcal{U}[t,T] = \Big\{ u : \Omega \times [t,T] \to \mathbb{R}^l \;\Big|\; u(\cdot) \in \bigcup_{q>4} L^q_{\mathcal{F},p}(t,T;\mathbb{R}^l) \Big\}.$$

Any $u(\cdot) \in \mathcal{U}[t,T]$ is called an admissible control. It follows from [19] that for any initial state $x(t) \in L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and a control $u(\cdot) \in L^q_{\mathcal{F},p}(t,T;\mathbb{R}^l)$, the state equation (1) admits a unique solution $X^{t,x(t),u}(\cdot) \in L^q_{\mathcal{F}}(t,T;\mathbb{R}^n)$. Unlike some other literature, which only assumes the square integrability of the control, we require that $u(\cdot) \in \bigcup_{q>4} L^q_{\mathcal{F},p}(t,T;\mathbb{R}^l)$ to guarantee one inequality in the proof of Theorem 3.1 is satisfied (see Remark 3.3).

Denote by $\mathbb{S}^n$ the set of all symmetric $n \times n$ real matrices. At any time t with the system state $X(t) = x(t)$, our objective is to minimize the following cost functional

$$\begin{aligned} J(t,x(t);u(\cdot)) := \mathbb{E}_t\Big[\int_t^T \big[\langle Q(s)X(s), X(s)\rangle + \langle R(s)u(s), u(s)\rangle\big]\mathrm{d}s + \langle GX(T), X(T)\rangle \\ + \langle H\mathbb{E}_t[X(T)], \mathbb{E}_t[X(T)]\rangle + \langle \mu_1 x(t) + \mu_2, X(T)\rangle \Big], \end{aligned} \tag{2}$$

over $u(\cdot) \in \mathcal{U}[t,T]$, where $X(\cdot) = X^{t,x(t),u}(\cdot)$ and $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$. In the above, $Q(\cdot)$ and $R(\cdot)$ are both positive semi-definite and uniformly bounded adapted processes on $[0,T]$ with values in $\mathbb{S}^n$ and $\mathbb{S}^l$, respectively; G, H, μ_1 and μ_2 are constants taking values in $\mathbb{S}^n$, $\mathbb{S}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n$, respectively. Throughout the paper, we assume that both G and $G+H$ are positive semi-definite.

The optimal control problem can be formulated as follows.

Problem 2.1 For any given initial pair $(t,x(t)) \in [0,T] \times L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, find a control $u^*(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(t,x(t);u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x(t);u(\cdot)). \tag{3}$$

As is discussed in [8], the dependence of the terminal cost on the current state $x(t)$ and the appearance of a quadratic term of the conditional expected terminal state in the cost functional (2) make Problem 2.1 time-inconsistent. So it is more appropriate to consider the notion of "equilibrium" instead of "optimality". In this paper, we use the game theoretic approach to handle the time inconsistency in the same perspective as [8]. The controller at each point t in time, referred to as "player t", is playing a game against all his/her future incarnations. Player t can only influence the dynamics of the state process $X(\cdot)$ by choosing the control $u(t)$ exactly at time t. At another time, say s, the control $u(s)$ will be chosen by player s. An equilibrium control $u^*(\cdot)$ is characterized by the property that if every player s, such that $s > t$, uses $u^*(\cdot)$, then it is optimal for player t to use $u^*(\cdot)$. Similar to [11–14], we define an equilibrium control in the following manner.

Definition 2.1 Let $u^*(\cdot) \in \bigcup_{q>4} L^q_{\mathcal{F},p}(0,T;\mathbb{R}^l)$ be a given control and $X^*(\cdot)$ be the corresponding state process. The control $u^*(\cdot)$ is called an equilibrium for Problem 2.1 if

$$\liminf_{\varepsilon \downarrow 0} \frac{J(t, X^*(t); u^{t,\varepsilon,v}(\cdot)) - J(t, X^*(t); u^*(\cdot))}{\varepsilon} \geq 0, \tag{4}$$

where for any $t \in [0, T[$, $\varepsilon > 0$ and $v \in \bigcup_{q>4} L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$,

$$u^{t,\varepsilon,v}(\cdot) = u^*(\cdot) + v\mathbf{1}_{[t,t+\varepsilon[}(\cdot). \tag{5}$$

As was remarked by [11], the definition for an equilibrium control here differs from that in some existing works such as [4,6,8], where only feedback controls are considered. The equilibrium defined here is in the class of open-loop controls, which means that the perturbation of the control in $[t, t+\varepsilon[$ will not change the control process in $[t+\varepsilon, T[$. This is not the case with feedback controls. On the other hand, our definition is also different from that given in [7,9,10], where closed-loop type equilibrium strategy is derived using the multiperson differential games approach in a Markovian setting. Moreover, the notion of equilibrium controls adopted in this paper also performs well when the controlled system is non-Markovian.

3 Optimality Conditions for Equilibrium Controls

In this section, we present a general sufficient and necessary condition for equilibrium controls. The necessary condition is derived based on a sharper estimate for the first-order variational equation as well as a stochastic Lebesgue differentiation theorem [18, Lemma 3.5], while the sufficient condition is totally different due to the introduction of random jumps. In the following proposition, we first derive a sufficient condition for an equilibrium control, which can be seen as a generalization of [11] to the jump-diffusion setting and of [14] to the case when the jump part coefficients contain control variables. However, the approach we have used distinguishes significantly from that in the above-mentioned works (see Remark 3.1).

Proposition 3.1 *Let $u^*(\cdot) \in L^q_{\mathcal{F},p}(0,T;\mathbb{R}^l)$ be a fixed control and $X^*(\cdot)$ be the corresponding state process. For each $t \in [0, T[$, let*

$$(Y(\cdot;t), (Z_i(\cdot;t))_{i=1}^d, (K_j(\cdot,\cdot;t))_{j=1}^m) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^n)$$
$$\times (L^2_{\mathcal{F},p}(t,T;\mathbb{R}^n))^d \times (F^2_p(t,T;\mathbb{R}^n))^m$$

be the unique solution to the following BSDE:

$$\begin{aligned} dY(s;t) &= -\Big\{A(s)^T Y(s;t) + \sum_{i=1}^{d} C_i(s)^T Z_i(s;t) + \sum_{j=1}^{m} \int_{\mathbb{R}_0} E_j(s,e)^T K_j(s,e;t)\nu_j(de) \\ &\quad + Q(s)X^*(s)\Big\}ds + \sum_{i=1}^{d} Z_i(s;t)dW_i(s) + \sum_{j=1}^{m} \int_{\mathbb{R}_0} K_j(s,e;t)\widetilde{N}_j(ds,de), \\ Y(T;t) &= GX^*(T) + H\mathbb{E}_t[X^*(T)] + \tfrac{1}{2}\mu_1 X^*(t) + \tfrac{1}{2}\mu_2, \qquad s \in [t,T]. \end{aligned} \tag{6}$$

Suppose $\Lambda(s;t) := R(s)u^*(s) + B(s)^T Y(s;t) + \sum_{i=1}^{d} D_i(s)^T Z_i(s;t) + \sum_{j=1}^{m} \int_{\mathbb{R}_0} F_j(s,e)^T K_j(s,e;t)\nu_j(de)$ *satisfies the following condition:*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;t)]ds = 0, \ a.s., \forall t \in [0,T[. \tag{7}$$

Then, $u^*(\cdot)$ *is an equilibrium control for Problem 2.1.*

Proof For each fixed $t \in [0,T[$ and $v(\cdot) \in \bigcup_{q>4} L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$, let $u^{t,\varepsilon,v}(\cdot)$ be defined by (5) and $X^\varepsilon(\cdot)$ be the corresponding state process stating from $X^*(t)$. Then (suppressing $(s;t)$)

$$\begin{aligned} &J(t,X^*(t);u^{t,\varepsilon,v}(\cdot)) - J(t,X^*(t);u^*(\cdot)) \\ &\quad = \mathbb{E}_t\Big[\int_t^T \big[\langle Q(X^\varepsilon + X^*), X^\varepsilon - X^*\rangle + \langle R(u^{t,\varepsilon,v} + u^*), u^{t,\varepsilon,v} - u^*\rangle\big]ds \\ &\qquad + \langle G(X^\varepsilon(T) + X^*(T)) + H\mathbb{E}_t[X^\varepsilon(T) + X^*(T)] + \mu_1 X^*(t) + \mu_2, X^\varepsilon(T) - X^*(T)\rangle\Big]. \end{aligned} \tag{8}$$

Recalling that $(Y(\cdot;t),(Z_i(\cdot;t))_{i=1}^d,(K_j(\cdot,\cdot;t))_{j=1}^m)$ is the solution to BSDE (6), by applying Itô's formula to $s \mapsto \langle Y(s;t), X^\varepsilon(s) - X^*(s)\rangle$, we have

$$\begin{aligned} &\mathbb{E}_t\big[\langle GX^*(T) + H\mathbb{E}_t[X^*(T)] + \frac{1}{2}\mu_1 X^*(t) + \frac{1}{2}\mu_2, X^\varepsilon(T) - X^*(T)\rangle\big] \\ &\quad = \mathbb{E}_t\Big[\int_t^T \Big\langle B^T Y + \sum_{i=1}^{d} D_i^T Z_i + \sum_{j=1}^{m} \int_{\mathbb{R}_0} F_j(e)^T K_j(e)\nu_j(de), u^{t,\varepsilon,v} - u^*\Big\rangle ds\Big] \\ &\qquad - \mathbb{E}_t\Big[\int_t^T \langle QX^*, X^\varepsilon - X^*\rangle ds\Big]. \end{aligned} \tag{9}$$

Consequently,

$$\begin{aligned} &J(t,X^*(t);u^{t,\varepsilon,v}(\cdot)) - J(t,X^*(t);u^*(\cdot)) \\ &\quad = \mathbb{E}_t\Big[\int_t^T \Big[\langle Q(X^\varepsilon + X^*) - 2QX^*, X^\varepsilon - X^*\rangle \\ &\qquad + \Big\langle R(u^{t,\varepsilon,v} + u^*) + 2B^T Y + \sum_{i=1}^{d} 2D_i^T Z_i + \sum_{j=1}^{m} \int_{\mathbb{R}_0} 2F_j(e)^T K_j(e)\nu_j(de), u^{t,\varepsilon,v} - u^*\Big\rangle\Big]ds \end{aligned}$$

$$
\begin{aligned}
&+ \langle G(X^\varepsilon(T)+X^*(T)) + H\mathbb{E}_t[X^\varepsilon(T)+X^*(T)] + \mu_1 X^*(t) + \mu_2, X^\varepsilon(T)-X^*(T)\rangle \\
&- \langle 2GX^*(T) + 2H\mathbb{E}_t[X^*(T)] + \mu_1 X^*(t) + \mu_2, X^\varepsilon(T)-X^*(T)\rangle \Big] \\
=\ & \mathbb{E}_t\Big[\int_t^{t+\varepsilon} \Big\langle R(v+2u^*) + 2B^T Y + \sum_{i=1}^d 2D_i^T Z_i + \sum_{j=1}^m \int_{\mathbb{R}_0} 2F_j(e)^T K_j(e)\nu_j(de), v\Big\rangle \mathrm{d}s \\
&+ \int_t^T |Q^{\frac{1}{2}}(X^\varepsilon - X^*)|^2 \mathrm{d}s + |(G+H)^{\frac{1}{2}}\mathbb{E}_t[X^\varepsilon(T)-X^*(T)]|^2 \\
&+ |G^{\frac{1}{2}}\{X^\varepsilon(T)-X^*(T)-\mathbb{E}_t[X^\varepsilon(T)-X^*(T)]\}|^2\Big] \\
\geq\ & \mathbb{E}_t\Big[\int_t^{t+\varepsilon} \Big\langle R(v+2u^*) + 2B^T Y + \sum_{i=1}^d 2D_i^T Z_i + \sum_{j=1}^m \int_{\mathbb{R}_0} 2F_j(e)^T K_j(e)\nu_j(de), v\Big\rangle \mathrm{d}s\Big] \\
=\ & \mathbb{E}_t\Big[\int_t^{t+\varepsilon} \langle Rv + 2\Lambda(s;t), v\rangle \mathrm{d}s\Big] \geq \mathbb{E}_t\Big[\int_t^{t+\varepsilon} \langle 2\Lambda(s;t), v\rangle \mathrm{d}s\Big]. \qquad (10)
\end{aligned}
$$

Therefore, it follows from condition (7) that

$$
\liminf_{\varepsilon\downarrow 0} \frac{J(t,X^*(t);u^{t,\varepsilon,v}(\cdot)) - J(t,X^*(t);u^*(\cdot))}{\varepsilon} \geq \liminf_{\varepsilon\downarrow 0} \frac{\int_t^{t+\varepsilon}\langle 2\mathbb{E}_t[\Lambda(s;t)], v\rangle \mathrm{d}s}{\varepsilon} = 0. \qquad (11)
$$

This completes the proof. □

We now look at the necessary condition for an equilibrium control. To this end, let $u^*(\cdot)$ be a fixed control and $X^*(\cdot)$ be the corresponding state process. For each $t \in [0,T[$, define in the time interval $[t,T]$ the processes

$$
\Big(\widehat{Y}(\cdot;t), (\widehat{Z}_i(\cdot;t))_{i=1}^d, (\widehat{K}_j(\cdot,\cdot;t))_{j=1}^m\Big) \in L^2_{\mathcal{F}}(t,T;\mathbb{S}^n) \times (L^2_{\mathcal{F},p}(t,T;\mathbb{S}^n))^d \times (F^2_p(t,T;\mathbb{S}^n))^m
$$

as the unique solution to the following BSDE:

$$
\begin{aligned}
\mathrm{d}\widehat{Y}(s;t) = -\Big\{& A(s)^T\widehat{Y}(s;t) + \widehat{Y}(s;t)A(s) \\
&+ \sum_{i=1}^d \Big(C_i(s)^T\widehat{Y}(s;t)C_i(s) + C_i(s)^T\widehat{Z}_i(s;t) + \widehat{Z}_i(s;t)C_i(s)\Big) \\
&+ \sum_{j=1}^m \int_{\mathbb{R}_0} \Big(E_j(s,e)^T(\widehat{Y}(s;t) + \widehat{K}_j(s,e;t))E_j(s,e) \\
&+ E_j(s,e)^T\widehat{K}_j(s,e;t) + \widehat{K}_j(s,e;t)E_j(s,e)\Big)\nu_j(de) + Q(s)\Big\}\mathrm{d}s \\
&+ \sum_{i=1}^d \widehat{Z}_i(s;t)dW_i(s)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\int_{\mathbb{R}_0}\widehat{K}_j(s,e;t)\widetilde{N}_j(ds,de),\\
&\widehat{Y}(T;t)=G,\quad s\in[t,T].
\end{aligned}
\tag{12}
$$

We notice that neither the coefficients nor the terminal condition of (12) depend on the initial time t, so it can be seen as a BSDE in the entire time period $[0, T]$. For $s \in [0, T]$, we denote its solution as $(\widehat{Y}(s), (\widehat{Z}_i(s))_{i=1}^d, (\widehat{K}_j(s,\cdot))_{j=1}^m)$. Thus, from the uniqueness of the solution to (12), it follows that

$$
(\widehat{Y}(s;t),(\widehat{Z}_i(s;t))_{i=1}^d,(\widehat{K}_j(s,\cdot;t))_{j=1}^m)=(\widehat{Y}(s),(\widehat{Z}_i(s))_{i=1}^d,(\widehat{K}_j(s,\cdot))_{j=1}^m),
$$

for $0 \le t \le s \le T$.

In the following proposition, we shall present an estimate for the difference of the cost functional $J(t, X^*(t); u^{t,\varepsilon,v}(\cdot)) - J(t, X^*(t); u^*(\cdot))$ under local spike variation.

Proposition 3.2 *For each $t \in [0, T[$, $\varepsilon > 0$ and $v \in \bigcup_{q>4} L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$, let $u^{t,\varepsilon,v}(\cdot)$ be defined by (5). Then, we have*

$$
\begin{aligned}
&J(t,X^*(t);u^{t,\varepsilon,v}(\cdot))-J(t,X^*(t);u^*(\cdot))\\
&\quad=\mathbb{E}_t\left[\int_t^{t+\varepsilon}\Big[\langle 2\Lambda(s;t),v\rangle+\langle\Gamma(s)v,v\rangle\Big]\mathrm{d}s\right]+o(\varepsilon),
\end{aligned}
\tag{13}
$$

where $o(\varepsilon)$ represents the higher-order terms of ε and

$$
\begin{aligned}
\Gamma(s)&:=R(s)+\sum_{i=1}^{d}D_i(s)^T\widehat{Y}(s)D_i(s)\\
&\quad+\sum_{j=1}^{m}\int_{\mathbb{R}_0}F_j(s,e)^T\Big(\widehat{Y}(s)+\widehat{K}_j(s,e)\Big)F_j(s,e)\nu_j(de).
\end{aligned}
$$

Proof Denote by $X^\varepsilon(\cdot)$ the state process corresponding to $u^{t,\varepsilon,v}(\cdot)$ stating from $X^*(t)$. It follows from the standard perturbation approach (see, for example, [20,21]) that

$$
X^\varepsilon(s)=X^*(s)+Y^\varepsilon(s)+Z^\varepsilon(s),\ \ s\in[t,T],
\tag{14}
$$

where $Y^\varepsilon(\cdot)$ and $Z^\varepsilon(\cdot)$ solve the following SDEs, respectively

$$
\begin{aligned}
\mathrm{d}Y^\varepsilon(s)&=A(s)Y^\varepsilon(s)\mathrm{d}s+\sum_{i=1}^{d}\left\{C_i(s)Y^\varepsilon(s)+D_i(s)v\mathbf{1}_{[t,t+\varepsilon[}(s)\right\}dW_i(s)\\
&\quad+\sum_{j=1}^{m}\int_{\mathbb{R}_0}\left\{E_j(s,e)Y^\varepsilon(s-)+F_j(s,e)v\mathbf{1}_{[t,t+\varepsilon[}(s)\right\}\widetilde{N}_j(ds,de),\\
Y^\varepsilon(t)&=0,\qquad s\in[t,T],
\end{aligned}
\tag{15}
$$

and

$$\begin{aligned} \mathrm{d}Z^{\varepsilon}(s) &= \left\{A(s)Z^{\varepsilon}(s) + B(s)v\mathbf{1}_{[t,t+\varepsilon[}(s)\right\}\mathrm{d}s + \sum_{i=1}^{d} C_i(s)Z^{\varepsilon}(s)dW_i(s) \\ &\quad + \sum_{j=1}^{m} \int_{\mathbb{R}_0} E_j(s,e)Z^{\varepsilon}(s-)\widetilde{N}_j(ds,de), \\ Z^{\varepsilon}(t) &= 0, \qquad s \in [t,T]. \end{aligned} \tag{16}$$

Following from standard arguments by using Gronwall's inequality and the moment inequalities for jump-diffusion processes (for example, [20, Lemma 2.1]), we have

$$\mathbb{E}_t\left[\sup_{s\in[t,T]} \left|Y^{\varepsilon}(s)\right|^2\right] = O(\varepsilon) \quad \text{and} \quad \mathbb{E}_t\left[\sup_{s\in[t,T]} \left|Z^{\varepsilon}(s)\right|^2\right] = O(\varepsilon^2).$$

Moreover, it follows from the dynamics of $Y^{\varepsilon}(\cdot)$ in (15) that $\mathbb{E}_t[Y^{\varepsilon}(s)] = \int_t^s \mathbb{E}_t[A(r)Y^{\varepsilon}(r)]\mathrm{d}r$ for all $s \in [t,T]$. Setting $\Psi(s) = A(s)$ in Lemma A.1, we obtain for some positive constants C that

$$\left|\int_t^s \mathbb{E}_t[A(r)Y^{\varepsilon}(r)]\mathrm{d}r\right|^2 \le C\int_t^s \left|\mathbb{E}_t[A(r)Y^{\varepsilon}(r)]\right|^2\mathrm{d}r \le C\varepsilon\rho(\varepsilon),$$

where $\rho : \Omega\times]0,\infty[\to]0,\infty[$ satisfies $\rho(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0,\ a.s.$ Therefore,

$$\sup_{s\in[t,T]} \left|\mathbb{E}_t[Y^{\varepsilon}(s)]\right|^2 \le C\varepsilon\rho(\varepsilon).$$

By using the above estimates, we derive (suppressing $(s;t)$)

$$\begin{aligned} &J(t,X^*(t);u^{t,\varepsilon,v}(\cdot)) - J(t,X^*(t);u^*(\cdot)) \\ &= \mathbb{E}_t\bigg[\int_t^T \left[\langle Q(2X^* + Y^{\varepsilon} + Z^{\varepsilon}), Y^{\varepsilon} + Z^{\varepsilon}\rangle + \langle R(v+2u^*), v\rangle \mathbf{1}_{[t,t+\varepsilon[}\right]\mathrm{d}s \\ &\quad + \langle G(Y^{\varepsilon}(T) + Z^{\varepsilon}(T)), Y^{\varepsilon}(T) + Z^{\varepsilon}(T)\rangle + \langle H\mathbb{E}_t[Y^{\varepsilon}(T) + Z^{\varepsilon}(T)], \mathbb{E}_t[Y^{\varepsilon}(T) \\ &\quad + Z^{\varepsilon}(T)]\rangle + 2\langle GX^*(T) + H\mathbb{E}_t[X^*(T)] + \frac{1}{2}\mu_1 X^*(t) + \frac{1}{2}\mu_2, Y^{\varepsilon}(T) + Z^{\varepsilon}(T)\rangle\bigg] \\ &= \mathbb{E}_t\bigg[\int_t^T \left[\langle Q(2X^* + Y^{\varepsilon} + Z^{\varepsilon}), Y^{\varepsilon} + Z^{\varepsilon}\rangle + \langle R(v+2u^*), v\rangle \mathbf{1}_{[t,t+\varepsilon[}\right]\mathrm{d}s \\ &\quad + 2\langle Y(T;t), Y^{\varepsilon}(T) + Z^{\varepsilon}(T)\rangle + \langle \widehat{Y}(T;t)(Y^{\varepsilon}(T) + Z^{\varepsilon}(T)), Y^{\varepsilon}(T) + Z^{\varepsilon}(T)\rangle\bigg] + o(\varepsilon). \end{aligned} \tag{17}$$

Following from (9) and (14), we have

$$
\begin{aligned}
&\mathbb{E}_t[\langle Y(T;t), Y^\varepsilon(T)+Z^\varepsilon(T)\rangle] \\
&\quad = \mathbb{E}_t\Bigg[\int_t^{t+\varepsilon}\Big\langle B^T Y + \sum_{i=1}^d D_i^T Z_i + \sum_{j=1}^m \int_{\mathbb{R}_0} F_j(e)^T K_j(e)\nu_j(de), v\Big\rangle \mathrm{d}s\Bigg] \\
&\qquad - \mathbb{E}_t\Bigg[\int_t^T \langle QX^*, Y^\varepsilon + Z^\varepsilon\rangle \mathrm{d}s\Bigg].
\end{aligned}
\tag{18}
$$

Now, by applying Itô's formula to $s \mapsto \langle \widehat{Y}(s;t)(Y^\varepsilon(s)+Z^\varepsilon(s)), Y^\varepsilon(s)+Z^\varepsilon(s)\rangle$, we get

$$
\begin{aligned}
&\mathbb{E}_t[\langle \widehat{Y}(T;t)(Y^\varepsilon(T)+Z^\varepsilon(T)), Y^\varepsilon(T)+Z^\varepsilon(T)\rangle] \\
&\quad = \mathbb{E}_t\Bigg[\int_t^{t+\varepsilon}\Big\langle\Big(\sum_{i=1}^d D_i^T \widehat{Y} D_i + \sum_{j=1}^m \int_{\mathbb{R}_0} F_j(e)^T\Big(\widehat{Y}+\widehat{K}_j(e)\Big)F_j(e)\nu_j(de)\Big)v, v\Big\rangle \mathrm{d}s\Bigg] \\
&\qquad - \mathbb{E}_t\Bigg[\int_t^T \langle Q(Y^\varepsilon+Z^\varepsilon), Y^\varepsilon+Z^\varepsilon\rangle \mathrm{d}s\Bigg] + o(\varepsilon).
\end{aligned}
\tag{19}
$$

Then, substituting (18) and (19) into (17) yields (13). The proof is complete. □

Remark 3.1 Note that from the positive semi-definite assumptions on Q, R and G, the corresponding $\Gamma(\cdot)$ in [11,12] is also positive semi-definite due to the comparison principles of BSDEs. Thus, it follows from Proposition 3.2 that a sufficient and necessary condition for a control u^* being an equilibrium is $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t[\Lambda(s;t)]\mathrm{d}s = 0$, a.s. (see [12, Corollary 3.2]). However, there is a critical difference between the estimate for the cost functional here and that in the above-mentioned literature. An additional term $\widehat{K}(\cdot,\cdot)$ appears in the expression of $\Gamma(\cdot)$ because stochastic coefficients and random jumps of the controlled system are considered. So in this paper, $\Gamma(\cdot)$ is not necessarily positive semi-definite. This is why we modify the methodology of deriving the sufficient condition for equilibrium controls (see Proposition 3.1).

Remark 3.2 It is worth mentioning that [13,14,22] also treat the time-inconsistent optimal control problems for jump diffusions. In [13], the authors assume all the model coefficients are deterministic functions, which leads to the second-order adjoint equation reducing to a backward ordinary differential equation. Although Wu and Zhuang [14] consider the case with random parameters, the jump part coefficients of their controlled system do not contain the control variable. Therefore, the form of $\Gamma(\cdot)$ in [13,14] turns out to be the same as the one corresponding to the pure diffusion case in [11,12], which differs from ours. Shen and Siu [22] study the time-inconsistent pre-committed control strategy for a jump-diffusion mean-field model by using a maximum principle approach, while in the current paper, a time-consistent equilibrium solution within the class of open-loop controls is derived.

We are now in the position to present the necessary condition for an equilibrium control.

Proposition 3.3 *If a control $u^*(\cdot) \in L^q_{\mathcal{F},p}(0,T;\mathbb{R}^l)$ is an equilibrium, then, for any $\theta > 0$,*

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\mathbb{E}_t\bigg[\int_t^{t+\varepsilon}\big[\langle 2\Lambda(s;t), v\rangle + \theta\langle \Gamma(s)v, v\rangle\big]\mathrm{d}s\bigg] \geq 0, \ a.s. \tag{20}$$

Proof For any $v \in \bigcup_{q>4} L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$ and $\theta > 0$, set $\bar{v} = \theta v \in \bigcup_{q>4} L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$. Then, it follows from Proposition 3.2 that

$$\begin{aligned} &J(t, X^*(t); u^{t,\varepsilon,\bar{v}}(\cdot)) - J(t, X^*(t); u^*(\cdot)) \\ &= \mathbb{E}_t\bigg[\int_t^{t+\varepsilon}\big[2\theta\langle \Lambda(s;t), v\rangle + \theta^2\langle \Gamma(s)v, v\rangle\big]\mathrm{d}s\bigg] + o(\varepsilon), \end{aligned}$$

which implies (20) holds. □

Although (7) and (20) already provide a sufficient and necessary condition for an equilibrium control, they are not easily applicable as they both involve a limit. In order to get rid of it, we need the following result, which not only provides a fundamental property of the solution to BSDE (6), but also represents the process $\Lambda(s;t)$ in a special form.

Proposition 3.4 *For any given state-control pair $(X^*(\cdot), u^*(\cdot)) \in L^q_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^q_{\mathcal{F},p}(0,T;\mathbb{R}^l)$, the unique solution to BSDE (6) satisfies $Z_i(s;t_1) = Z_i(s;t_2)$ and $K_j(s,\cdot;t_1) = K_j(s,\cdot;t_2)$ for a.e. $s \geq \max(t_1,t_2)$, $i = 1,\ldots,d$ and $j = 1,\ldots,m$. Furthermore, there exist some stochastic processes λ_1, λ_2 and ξ such that*

$$\Lambda(s;t) = \lambda_1(s) + \lambda_2(s)\xi(t),$$

where λ_1 satisfies $\mathbb{E}[\int_0^T |\lambda_1(s)|^2 \mathrm{d}s] < \infty$, λ_2 is uniformly bounded on $[0,T]$ and ξ is right continuous satisfying $\mathbb{E}[\sup_{t\in[0,T]} |\xi_t|^q] < +\infty$.

Proof Denote by $\Phi(\cdot)$ the unique continuous solution to the following matrix-valued ordinary differential equation:

$$d\Phi(t) = \Phi(t)A(t)^T dt, \quad \Phi(T) = I_n,$$

where I_n denotes the $n \times n$ identity matrix. Obviously, $\Phi(\cdot)$ is invertible, and both $\Phi(\cdot)$ and $\Phi(\cdot)^{-1}$ are uniformly bounded.

For $i = 1,\ldots,d$ and $j = 1,\ldots,m$, let $\widetilde{Y}(s;t) = \Phi(s)Y(s;t) - H\mathbb{E}_t[X^*(T)] - \frac{1}{2}\mu_1 X^*(t) - \frac{1}{2}\mu_2$, $\widetilde{Z}_i(s;t) = \Phi(s)Z_i(s;t)$ and $\widetilde{K}_j(s,\cdot;t) = \Phi(s)K_j(s,\cdot;t)$. It follows that on the time interval $[t,T]$, $(\widetilde{Y}(\cdot;t), (\widetilde{Z}_i(\cdot;t))_{i=1}^d, (\widetilde{K}_j(\cdot,\cdot;t))_{j=1}^m)$ satisfies

$$\begin{cases} \mathrm{d}\widetilde{Y}(s;t) = -\bigg\{\sum_{i=1}^d \Phi(s)C_i(s)^T\Phi(s)^{-1}\widetilde{Z}_i(s;t) + \sum_{j=1}^m \int_{\mathbb{R}_0} \Phi(s)E_j(s,e)^T\Phi(s)^{-1}\widetilde{K}_j(s,e;t)\nu_j(de) \\ \qquad + \Phi(s)Q(s)X^*(s)\bigg\}\mathrm{d}s + \sum_{i=1}^d \widetilde{Z}_i(s;t)dW_i(s) + \sum_{j=1}^m \int_{\mathbb{R}_0} \widetilde{K}_j(s,e;t)\widetilde{N}_j(ds,de), \\ \widetilde{Y}(T;t) = GX^*(T), \qquad s \in [t,T]. \end{cases} \tag{21}$$

It should be noted that neither the coefficients nor the terminal condition of (21) depend on t, so it can be seen as a BSDE on the entire time period $[0, T]$. For $s \in [0, T]$, denote the solution of (21) as $(\widetilde{Y}(s), (\widetilde{Z}_i(s))_{i=1}^d, (\widetilde{K}_j(s,\cdot))_{j=1}^m) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times (L^2_{\mathcal{F},p}(t, T; \mathbb{R}^n))^d \times (F_p^2(t, T; \mathbb{R}^n))^m$. Thus, from the uniqueness of the solution to BSDE (21), it follows that for $0 \le t \le s \le T$

$$(\widetilde{Y}(s;t), (\widetilde{Z}_i(s;t))_{i=1}^d, (\widetilde{K}_j(s,\cdot;t))_{j=1}^m) = (\widetilde{Y}(s), (\widetilde{Z}_i(s))_{i=1}^d, (\widetilde{K}_j(s,\cdot))_{j=1}^m).$$

As a result, $Z_i(s;t) = \Phi(s)^{-1}\widetilde{Z}_i(s) := Z_i(s)$ and $K_j(s,\cdot;t) = \Phi(s)^{-1}\widetilde{K}_j(s,\cdot) := K_j(s,\cdot)$. This proves the first claim of the proposition.

Next, from the definition of $\widetilde{Y}(s;t)$, we derive

$$\begin{aligned} Y(s;t) &= \Phi(s)^{-1}\widetilde{Y}(s) + \Phi(s)^{-1}\Big(H\mathbb{E}_t[X^*(T)] + \frac{1}{2}\mu_1 X^*(t) + \frac{1}{2}\mu_2\Big) \\ &:= Y(s) + \Phi(s)^{-1}\xi(t), \end{aligned}$$

where $Y(s) := \Phi(s)^{-1}\widetilde{Y}(s)$ defines a process $Y(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ and

$$\xi(t) := H\mathbb{E}_t[X^*(T)] + \frac{1}{2}\mu_1 X^*(t) + \frac{1}{2}\mu_2$$

defines a right continuous process satisfying $\mathbb{E}[\sup_{t\in[0,T]}|\xi(t)|^q] < +\infty$.

Therefore,

$$\begin{aligned} \Lambda(s;t) &= R(s)u^*(s) + B(s)^T Y(s;t) + \sum_{i=1}^d D_i(s)^T Z_i(s;t) \\ &\quad + \sum_{j=1}^m \int_{\mathbb{R}_0} F_j(s,e)^T K_j(s,e;t)\nu_j(de) \\ &= R(s)u^*(s) + B(s)^T\Big(Y(s) + \Phi(s)^{-1}\xi_t\Big) \\ &\quad + \sum_{i=1}^d D_i(s)^T Z_i(s) + \sum_{j=1}^m \int_{\mathbb{R}_0} F_j(s,e)^T K_j(s,e)\nu_j(de) \\ &:= \lambda_1(s) + \lambda_2(s)\xi(t), \end{aligned}$$

where $\lambda_1(s) := R(s)u^*(s) + B(s)^T Y(s) + \sum_{i=1}^d D_i(s)^T Z_i(s) + \sum_{j=1}^m \int_{\mathbb{R}_0} F_j(s,e)^T K_j(s,e)\nu_j(de)$ satisfying $\mathbb{E}[\int_0^T |\lambda_1(s)|^2 \mathrm{d}s] < \infty$ and $\lambda_2(s) := B(s)^T\Phi(s)^{-1}$ being uniformly bounded. □

We now summarize the main result of this section in the following theorem.

Theorem 3.1 *Let* $(X^*(\cdot), u^*(\cdot)) \in L^q_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^q_{\mathcal{F},p}(0, T; \mathbb{R}^l)$ *be a given state-control pair, and let* $(Y(\cdot;t), (Z_i(\cdot;t))_{i=1}^d, (K_j(\cdot,\cdot;t))_{j=1}^m) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times$

$(L^2_{\mathcal{F},p}(t,T;\mathbb{R}^n))^d \times (F^2_p(t,T;\mathbb{R}^n))^m$ *be the unique solution to BSDE* (6). *Then,* $u^*(\cdot)$ *is an equilibrium for Problem* 2.1 *if and only if*

$$\Lambda(t;t) = 0, \; a.s., \; a.e. \; t \in [0,T]. \tag{22}$$

Proof We first prove the sufficiency of the above result. It follows from the representation of $\Lambda(s;t)$ in Proposition 3.4 that

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;t)]\mathrm{d}s - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;s)]\mathrm{d}s = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\lambda_2(s)(\xi(t)-\xi(s))]\mathrm{d}s.$$

Since λ_2 is uniformly bounded and ξ is right continuous, we have for some constant C that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;t)]\mathrm{d}s - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;s)]\mathrm{d}s\right| \\
&\quad = \lim_{\varepsilon\downarrow 0}\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\lambda_2(s)(\xi(t)-\xi(s))]\mathrm{d}s\right| \\
&\quad \le C\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[|\xi(t)-\xi(s)|]\mathrm{d}s = 0.
\end{aligned}$$

Therefore, if (22) holds, then

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;t)]\mathrm{d}s = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;s)]\mathrm{d}s = 0.$$

From Proposition 3.1, $u^*(\cdot)$ is an equilibrium control.

Conversely, if $u^*(\cdot)$ is an equilibrium control, along the same lines as the above analysis, we easily get

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\langle 2\Lambda(s;t), v\rangle]\mathrm{d}s - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\langle 2\Lambda(s;s), v\rangle]\mathrm{d}s \\
&\quad = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[2\langle \lambda_2(s)(\xi(t)-\xi(s)), v\rangle]\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\langle 2\Lambda(s;t), v\rangle]\mathrm{d}s - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}_t[\langle 2\Lambda(s;s), v\rangle]\mathrm{d}s\right| \\
&\quad \le C\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \langle \mathbb{E}_t[|\xi(t)-\xi(s)|], |v|\rangle\mathrm{d}s = 0.
\end{aligned}$$

Then, from Proposition 3.3, for any $\theta > 0$,

$$\liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}_t\left[\int_t^{t+\varepsilon}\big[\langle 2\Lambda(s;s), v\rangle + \theta\langle \Gamma(s)v, v\rangle\big]\mathrm{d}s\right] \ge 0, \; a.s. \tag{23}$$

We next verify, for any $\delta > 0$ sufficiently small, that

$$\mathbb{E}\Big[\int_0^T \big|\langle 2\Lambda(s;s), v\rangle + \theta\langle \Gamma(s)v, v\rangle\big|^{1+\delta}\mathrm{d}s\Big] < +\infty. \tag{24}$$

Indeed, in view of the definitions for $\Lambda(s;s)$ and $\Gamma(s)$ in Propositions 3.1 and 3.2, we have

$$\mathbb{E}\Big[\int_0^T \big|\Lambda(s;s)\big|^2\mathrm{d}s\Big] < +\infty \quad \text{and} \quad \mathbb{E}\Big[\int_0^T \big|\Gamma(s)\big|^2\mathrm{d}s\Big] < +\infty,$$

where $|\Gamma(s)|^2$ denotes the Frobenius norm of the matrix $(\Gamma(s))_{1\le i,j\le l}$, i.e., $|\Gamma(s)|^2 = \sum_{i,j=1}^l \Gamma_{ij}(s)^2$. Then, it follows from $v \in \bigcup_{q>4} L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$ that

$$\mathbb{E}\Big[\int_0^T \big|\langle 2\Lambda(s;s), v\rangle\big|^{1+\delta}\mathrm{d}s\Big] \le C\Big(\mathbb{E}\Big[\int_0^T \big|\Lambda(s;s)\big|^2\mathrm{d}s\Big]\Big)^{\frac{1+\delta}{2}}\Big(\mathbb{E}\Big[|v|^{\frac{2(1+\delta)}{1-\delta}}\Big]\Big)^{\frac{1-\delta}{2}} < +\infty.$$

Furthermore, denote by $v = (v_1, \dots, v_l)^T$, we get

$$\begin{aligned}\big|\langle \Gamma(s)v, v\rangle\big| &= \Big|\sum_{i=1}^l \Big(\sum_{j=1}^l \Gamma_{ij}(s)v_j\Big)v_i\Big| \le \Big|\Big(\sum_{i=1}^l \Big(\sum_{j=1}^l \Gamma_{ij}(s)v_j\Big)^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{i=1}^l v_i^2\Big)^{\frac{1}{2}}\Big| \\ &\le \Big|\Big(\sum_{i=1}^l\sum_{j=1}^l \Gamma_{ij}(s)^2\Big)^{\frac{1}{2}} \cdot \sum_{i=1}^l v_i^2\Big| = |\Gamma(s)| \cdot |v|^2.\end{aligned}$$

Therefore, if $q \ge 4(1+\delta)/(1-\delta)$, we have

$$\mathbb{E}\Big[\int_0^T \big|\theta\langle \Gamma(s)v, v\rangle\big|^{1+\delta}\mathrm{d}s\Big] \le C\Big(\mathbb{E}\Big[\int_0^T \big|\Gamma(s)\big|^2\mathrm{d}s\Big]\Big)^{\frac{1+\delta}{2}}\Big(\mathbb{E}\Big[|v|^{\frac{4(1+\delta)}{1-\delta}}\Big]\Big)^{\frac{1-\delta}{2}} < +\infty.$$

Hence (24) holds, which ensures that the condition in [18, Lemma 3.5] is satisfied for the integrand of (23). Therefore

$$2\langle \Lambda(t;t), v\rangle + \theta\langle \Gamma(t)v, v\rangle \ge 0, \; a.s., \; a.e. \; t \in [0,T].$$

Sending $\theta \to 0+$, we obtain for all $v \in \bigcup_{q>4} L^q_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$ that $\langle \Lambda(t;t), v\rangle \ge 0$, $a.s.$, $a.e.$ $t \in [0,T]$, from which we obtain (22). This completes the proof. □

Remark 3.3 In the above proof of Theorem 3.1, we require that $q \ge 4(1+\delta)/(1-\delta)$, for a sufficiently small constant $\delta > 0$, to ensure the correctness of (24), which is crucial in deriving the necessary condition of an equilibrium control. This is why we consider $\bigcup_{q>4} L^q_{\mathcal{F},p}(0,T;\mathbb{R}^l)$ as the set of all admissible controls in this paper.

From the above theorem, we end up this section with the following optimality system:

$$
\begin{aligned}
\mathrm{d}X^*(s) &= \big\{A(s)X^*(s) + B(s)u^*(s) + b(s)\big\}\mathrm{d}s \\
&\quad + \sum_{i=1}^{d} \big\{C_i(s)X^*(s) + D_i(s)u^*(s) + \sigma_i(s)\big\} dW_i(s) \\
&\quad + \sum_{j=1}^{m} \int_{\mathbb{R}_0} \big\{E_j(s,e)X^*(s-) + F_j(s,e)u^*(s-) + \eta_j(s,e)\big\} \widetilde{N}_j(ds,de), \\
\mathrm{d}Y(s;t) &= -\Big\{A(s)^T Y(s;t) + \sum_{i=1}^{d} C_i(s)^T Z_i(s;t) + \sum_{j=1}^{m} \int_{\mathbb{R}_0} E_j(s,e)^T K_j(s,e;t)\nu_j(de) \\
&\quad + Q(s)X^*(s)\Big\}\mathrm{d}s + \sum_{i=1}^{d} Z_i(s;t)dW_i(s) + \sum_{j=1}^{m} \int_{\mathbb{R}_0} K_j(s,e;t)\widetilde{N}_j(ds,de), \\
X^*(0) &= x, \quad Y(T;t) = GX^*(T) + H\mathbb{E}_t[X^*(T)] + \frac{1}{2}\mu_1 X^*(t) + \frac{1}{2}\mu_2, \quad 0 \le t \le s \le T, \\
R(t)u^*(t) &+ B(t)^T Y(t;t) + \sum_{i=1}^{d} D_i(t)^T Z_i(t;t) + \sum_{j=1}^{m} \int_{\mathbb{R}_0} F_j(t,e)^T K_j(t,e;t)\nu_j(de) = 0.
\end{aligned}
\tag{25}
$$

Note that (25) is a flow of FBSDEs with constraints. The general existence of solutions for such type of FBSDEs remains an open problem. However, in the case without Poisson random measure and all the coefficients are deterministic functions, explicit solutions of the above system are carried out in [11].

4 Application to the Mean-Variance Problem

In this section, we apply Theorem 3.1 to a mean-variance portfolio selection problem with the objective functional containing both a state-dependent term and a variance term. Hence, the problem is inherently time-inconsistent. To establish the existence and uniqueness of the equilibrium investment strategy, we assume throughout this section all the parameters are deterministic functions of t. Actually, in the absence of the state-dependent term, this problem has been discussed by [23] with an additional reinsurance strategy. However, the authors in [23] only give the existence of equilibrium strategies by using a stochastic maximum principle approach.

We consider a financial market consisting of one risk-free asset and n risky assets. The risk-free asset's price process $\{S_0(t)\}_{t\in[0,T]}$ is given by:

$$
dS_0(t) = \rho(t)S_0(t)dt, \qquad S_0(0) > 0.
$$

The price processes of the other n risky assets $\{S_k(t)\}_{t\in[0,T]}$, $k = 1,\ldots,n$, are modeled by the following SDEs:

$$dS_k(t) = S_k(t-)\left\{\mu_k(t)dt + \sum_{i=1}^{d}\sigma_{ki}(t)dW_i(t) + \sum_{j=1}^{m}\int_{\mathbb{R}_0}\eta_{kj}(t,e)\widetilde{N}_j(dt,de)\right\},$$
$$S_k(0) > 0.$$

In the above, $\rho(\cdot), \mu_k(\cdot), \sigma_{ki}(\cdot)$ and $\eta_{kj}(\cdot,\cdot)$ are assumed to be deterministic and bounded functions on $[0,T]$ satisfying $\mu_k(\cdot) > \rho(\cdot) \geq 0$. Denote by $\mu(\cdot) := (\mu_1(\cdot),\ldots,\mu_n(\cdot))^T$, $\sigma(\cdot) := (\sigma_{ki}(\cdot))_{n\times d}$ and $\eta(\cdot,\cdot) := (\eta_{kj}(\cdot,\cdot))_{n\times m}$. Moreover, we assume the following non-degeneracy condition is satisfied, that is,

$$\Theta(t) := \sigma(t)\sigma(t)^T + \int_{\mathbb{R}_0}\eta(t,e)\mathrm{Diag}(\nu(de))\eta(t,e)^T \geq \delta I_n, \tag{26}$$

for all $t \in [0,T]$ and some $\delta > 0$.

In the following, we denote by $u_k(t)$, $k = 0,1,\ldots,n$, the amount of money invested in the kth asset at time t and we call $u(t) := (u_1(t),\ldots,u_n(t))^T$ a portfolio of the investment. Given any initial time $t \in [0,T[$ and a portfolio $u(\cdot)$, we easily get the wealth process $X(\cdot)$ satisfying the following equation:

$$\begin{aligned}&\mathrm{d}X(s) = \left[\rho(s)X(s) + u(s)^T B(s)\right]\mathrm{d}s + u(s)^T\sigma(s)dW(s) + u(s)^T\\ &\int_{\mathbb{R}_0}\eta(s,e)\widetilde{N}(ds,de),\ X(t) = x(t), \quad s\in[t,T],\end{aligned} \tag{27}$$

where

$$B(s) := (\mu_1(s)-\rho(s),\ldots,\mu_n(s)-\rho(s))^T.$$

At any time t with state $X(t) = x(t)$, the objective is to achieve a balance between the conditional expectation and the conditional variance of terminal wealth, i.e., to chose a portfolio $u(\cdot)$ so as to minimize

$$\begin{aligned}J(t,x(t);u(\cdot)) &:= \frac{\gamma}{2}\mathrm{Var}_t[X(T)] - (\mu_1 x(t)+\mu_2)\mathbb{E}_t[X(T)]\\ &= \frac{\gamma}{2}\mathbb{E}_t[X(T)^2] - \frac{\gamma}{2}(\mathbb{E}_t[X(T)])^2 - (\mu_1 x(t)+\mu_2)\mathbb{E}_t[X(T)],\end{aligned} \tag{28}$$

with γ, μ_1 and μ_2 are given positive constants.

Problem (27)–(28) is obviously a special cases of our LQ problem (1)–(2). Therefore, the FBSDE system (25) becomes:

$$\begin{cases}\mathrm{d}X^*(s) = \left[\rho(s)X^*(s) + u^*(s)^T B(s)\right]\mathrm{d}s + u^*(s)^T\sigma(s)dW(s) + u^*(s)^T\int_{\mathbb{R}_0}\eta(s,e)\widetilde{N}(ds,de),\\ \mathrm{d}Y(s;t) = -\rho(s)Y(s;t)\mathrm{d}s + Z(s;t)^T dW(s) + \int_{\mathbb{R}_0}K(s,e;t)^T\widetilde{N}(ds,de),\\ X^*(0) = x, \quad Y(T;t) = \frac{\gamma}{2}X^*(T) - \frac{\gamma}{2}\mathbb{E}_t[X^*(T)] - \frac{1}{2}\mu_1 X^*(t) - \frac{1}{2}\mu_2, \quad 0\leq t\leq s\leq T,\\ B(t)Y(t;t) + \sigma(t)Z(t;t) + \int_{\mathbb{R}_0}\eta(t,e)\mathrm{Diag}(\nu(de))K(t,e;t) = 0,\end{cases} \tag{29}$$

where $Z(s;t) = (Z_1(s;t),\ldots,Z_d(s;t))^T$ and $K(s,\cdot;t) = (K_1(s,\cdot;t),\ldots,K_m(s,\cdot;t))^T$.

4.1 Existence of a Solution to (29)

In this subsection, we construct a solution to (29). To this end, let us try a process $\{Y(s;t)\}_{0\leq t\leq s\leq T}$ of the following form:

$$Y(s;t) = \Psi(s)(X^*(s) - \mathbb{E}_t[X^*(s)]) + \phi(s)X^*(t) + \varphi(s), \tag{30}$$

where $\Psi(\cdot)$, $\phi(\cdot)$ and $\varphi(\cdot)$ are deterministic, differentiable functions with terminal values

$$\Psi(T) = \gamma/2,\ \phi(T) = -\mu_1/2,\ \varphi(T) = -\mu_2/2.$$

For each fixed t, by applying Itô's formula to (30) in the time variable s, we derive

$$\begin{aligned} \mathrm{d}Y(s;t) = &\big\{\Psi'(s)X^*(s) + \Psi(s)(\rho(s)X^*(s) + u^*(s)^T B(s)) - \Psi'(s)\mathbb{E}_t[X^*(s)] \\ &- \Psi(s)\mathbb{E}_t[\rho(s)X^*(s) + u^*(s)^T B(s)] + \phi'(s)X^*(t) + \varphi'(s)\big\}\mathrm{d}s \\ &+ \Psi(s)u^*(s)^T\sigma(s)dW(s) + \Psi(s)u^*(s)^T\int_{\mathbb{R}_0}\eta(s,e)\widetilde{N}(ds,de). \end{aligned} \tag{31}$$

Comparing the coefficients with $dY(s;t)$ in (29), we get

$$Z(s;t) = \sigma(s)^T u^*(s)\Psi(s) \quad \text{and} \quad K(s,\cdot;t) = \eta(s,\cdot)^T u^*(s)\Psi(s). \tag{32}$$

Notice that both $Z(s;t)$ and $K(s,\cdot;t)$ in (32) turn out to be independent of t.

Now substituting (32) into the last equation of (29), we obtain for $s \in [0,T]$ that

$$\begin{aligned} &B(s)(\phi(s)X^*(s) + \varphi(s)) + \sigma(s)\sigma(s)^T u^*(s)\Psi(s) \\ &\quad + \int_{\mathbb{R}_0}\eta(s,e)\mathrm{Diag}(\nu(de))\eta(s,e)^T u^*(s)\Psi(s) = 0, \end{aligned}$$

from which we formally deduce that

$$u^*(s) = \alpha(s)X^*(s) + \beta(s), \tag{33}$$

where $\alpha(s) := -\Theta(s)^{-1}\Psi(s)^{-1}B(s)\phi(s)$ and $\beta(s) := -\Theta(s)^{-1}\Psi(s)^{-1}B(s)\varphi(s)$.

Next, comparing the ds term in (31) with that in (29) and substituting for $u^*(s)$ from (33), we get the following equations governing Ψ, ϕ and φ:

$$\begin{aligned} &\Psi'(s) + [2\rho(s) + \alpha(s)^T B(s)]\Psi(s) = 0, \quad \Psi(T) = \tfrac{\gamma}{2}, \\ &\phi'(s) + \rho(s)\phi(s) = 0, \quad \phi(T) = -\tfrac{1}{2}\mu_1, \\ &\varphi'(s) + \rho(s)\varphi(s) = 0, \quad \varphi(T) = -\tfrac{1}{2}\mu_2. \end{aligned} \tag{34}$$

These ordinary differential equations are explicitly solved by

$$
\begin{aligned}
\Psi(s) &= \tfrac{\gamma}{2}\exp\left\{\int_s^T [2\rho(\tau)+\alpha(\tau)^T B(\tau)]\mathrm{d}\tau\right\},\\
\phi(s) &= -\tfrac{1}{2}\mu_1\exp\left\{\int_s^T \rho(\tau)\mathrm{d}\tau\right\},\\
\varphi(s) &= -\tfrac{1}{2}\mu_2\exp\left\{\int_s^T \rho(\tau)\mathrm{d}\tau\right\}.
\end{aligned} \tag{35}
$$

We now summarize the existence result in the following theorem.

Theorem 4.1 *Let Ψ, ϕ and φ be defined by (35). Then, $u^*(\cdot)$ given by (33) is an equilibrium investment strategy for problem (27)–(28).*

Proof Define $(Y(\cdot;\cdot), Z(\cdot;\cdot), K(\cdot,\cdot;\cdot))$ by (30) and (32), respectively. It is straightforward to verify that $(u^*(\cdot), X^*(\cdot), Y(\cdot;\cdot), Z(\cdot;\cdot), K(\cdot,\cdot;\cdot))$ satisfies the system of FBSDEs (29) with constraint.

On the other hand, we can check that both $\alpha(\cdot)$ and $\beta(\cdot)$ in (33) are uniformly bounded and thus leading to $X^*(\cdot) \in L^q_{\mathcal{F}}(0,T;\mathbb{R})$ and $u^*(\cdot) \in L^q_{\mathcal{F},p}(0,T;\mathbb{R}^n)$. Then, by Theorem 3.1, $u^*(\cdot)$ is an equilibrium investment strategy. □

4.2 Uniqueness of the Equilibrium Investment Strategy

In this subsection, we prove the equilibrium investment strategy constructed above is the only equilibrium. We define

$$
\begin{aligned}
\mathcal{M}_1 &:= \left\{Y(\cdot;\cdot) : Y(\cdot;t) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}),\ \sup_{t\in[0,T]} \mathbb{E}\left[\sup_{s\in[t,T]} |Y(s;t)|^2\right] < +\infty\right\},\\
\mathcal{M}_2 &:= \left\{Z(\cdot;\cdot) : Z(\cdot;t) \in L^2_{\mathcal{F},p}(t,T;\mathbb{R}^d),\ \sup_{t\in[0,T]} \mathbb{E}\left[\int_t^T |Z(s;t)|^2 \mathrm{d}s\right] < +\infty\right\},\\
\mathcal{M}_3 &:= \left\{K(\cdot,\cdot;\cdot) : K(\cdot,\cdot;t) \in F^2_p(t,T;\mathbb{R}^m),\ \sup_{t\in[0,T]} \mathbb{E}\left[\int_t^T ||K(s,\cdot;t)||^2_{\mathcal{L}^2} \mathrm{d}s\right] < +\infty\right\}.
\end{aligned}
$$

Before introducing the main uniqueness theorem, we first prove the uniqueness of the solution to the following general BSDE, which plays a key role in the sequel.

$$
\begin{aligned}
-\mathrm{d}\bar{Y}(s;t) &= F\Big(s, \bar{Y}(s;t), \bar{Y}(s;s), \mathbb{E}_t[l_1(s)\bar{Y}(s;s)], \bar{Z}(s;t), \mathbb{E}_t[l_2(s)\bar{Z}(s;t)],\\
&\qquad \int_{\mathbb{R}_0} l_3(s,e)\mathrm{Diag}(\nu(de))\bar{K}(s,e;t), \mathbb{E}_t\Big[\int_{\mathbb{R}_0} l_4(s,e)\mathrm{Diag}(\nu(de))\bar{K}(s,e;t)\Big]\Big)\mathrm{d}s\\
&\qquad - \bar{Z}(s;t)^T dW(s) - \int_{\mathbb{R}_0} \bar{K}(s,e;t)^T \widetilde{N}(ds,de),\\
\bar{Y}(T;t) &= 0, \quad s\in[t,T],
\end{aligned} \tag{36}
$$

where l_1, l_2, l_3, l_4 are uniformly bounded stochastic processes with suitable dimensions, $F(s,\cdots)$ is deterministic and Lipschitz continuous with respect to all variables except s. It should be noted that the standard existence and uniqueness results for BSDEs driven by Brownian motions and Poisson random measures (see, for example, [20]) cannot be applied to (36) as the generator involves the conditional expectations. However, we are able to prove the uniqueness of the solution to (36) in the following lemma.

Lemma 4.1 *BSDE (36) admits at most one solution* $(\bar{Y}(\cdot;\cdot), \bar{Z}(\cdot;\cdot), \bar{K}(\cdot,\cdot;\cdot)) \in \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3$.

Proof Suppose that there exist two solutions $(\bar{Y}^{(1)}, \bar{Z}^{(1)}, \bar{K}^{(1)})$ and $(\bar{Y}^{(2)}, \bar{Z}^{(2)}, \bar{K}^{(2)})$ belonging to $\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3$. Define $\hat{Y}(s;t) := \bar{Y}^{(1)}(s;t) - \bar{Y}^{(2)}(s;t)$, $\hat{Z}(s;t) := \bar{Z}^{(1)}(s;t) - \bar{Z}^{(2)}(s;t)$, $\hat{K}(s,\cdot;t) := \bar{K}^{(1)}(s,\cdot;t) - \bar{K}^{(2)}(s,\cdot;t)$ and

$$\begin{aligned}\Delta F(s;t) := F\bigg(&s, \bar{Y}^{(1)}(s;t), \bar{Y}^{(1)}(s;s),\\ &\mathbb{E}_t[l_1(s)\bar{Y}^{(1)}(s;s)], \bar{Z}^{(1)}(s;t), \mathbb{E}_t[l_2(s)\bar{Z}^{(1)}(s;t)],\\ &\int_{\mathbb{R}_0} l_3(s,e)\mathrm{Diag}(\nu(de))\bar{K}^{(1)}(s,e;t),\\ &\mathbb{E}_t\bigg[\int_{\mathbb{R}_0} l_4(s,e)\mathrm{Diag}(\nu(de))\bar{K}^{(1)}(s,e;t)\bigg]\bigg)\\ &- F\bigg(s, \bar{Y}^{(2)}(s;t), \bar{Y}^{(2)}(s;s),\\ &\mathbb{E}_t[l_1(s)\bar{Y}^{(2)}(s;s)], \bar{Z}^{(2)}(s;t), \mathbb{E}_t[l_2(s)\bar{Z}^{(2)}(s;t)],\\ &\int_{\mathbb{R}_0} l_3(s,e)\mathrm{Diag}(\nu(de))\bar{K}^{(2)}(s,e;t),\\ &\mathbb{E}_t\bigg[\int_{\mathbb{R}_0} l_4(s,e)\mathrm{Diag}(\nu(de))\bar{K}^{(2)}(s,e;t)\bigg]\bigg).\end{aligned}$$

Then, we have for some constant C that

$$\begin{aligned}|\Delta F(s;t)| \le C\bigg(&|\hat{Y}(s;t)| + |\hat{Y}(s;s)| + |\hat{Z}(s;t)| + \sum_{j=1}^{m}\int_{\mathbb{R}_0}|\hat{K}_j(s,e;t)|\nu_j(de)\\ &+ \mathbb{E}_t[|\hat{Y}(s;s)|] + \mathbb{E}_t[|\hat{Z}(s;t)|] + \sum_{j=1}^{m}\mathbb{E}_t\bigg[\int_{\mathbb{R}_0}|\hat{K}_j(s,e;t)|\nu_j(de)\bigg]\bigg),\end{aligned}$$

and

$$-\mathrm{d}\hat{Y}(s;t) = \Delta F(s;t)\mathrm{d}s - \hat{Z}(s;t)^T dW(s) - \int_{\mathbb{R}_0}\hat{K}(s,e;t)^T \widetilde{N}(ds,de),\ \hat{Y}(T;t) = 0. \tag{37}$$

For any $t \in [0,T]$ and $s \in [t,T]$, applying Itô's formula to $s \mapsto |\hat{Y}(s;t)|^2$, we get

$$\begin{aligned}\mathbb{E}[|\hat{Y}(s;t)|^2] + \mathbb{E}\bigg[\int_s^T |\hat{Z}(r;t)|^2\mathrm{d}r\bigg] + \mathbb{E}\bigg[\int_s^T ||\hat{K}(r,\cdot;t)||^2_{\mathcal{L}^2}\mathrm{d}r\bigg]\\ \le 2\mathbb{E}\bigg[\int_s^T |\hat{Y}(r;t)\Delta F(r;t)|\mathrm{d}r\bigg]\end{aligned}$$

$$
\begin{aligned}
&\le C\mathbb{E}\bigg[\int_s^T \Big(|\hat{Y}(r;t)|^2+|\hat{Y}(r;r)|^2\Big)\mathrm{d}r\bigg] \\
&+\frac{1}{2}\mathbb{E}\bigg[\int_s^T |\hat{Z}(r;t)|^2\mathrm{d}r\bigg]+\frac{1}{2}\mathbb{E}\bigg[\int_s^T ||\hat{K}(r,\cdot;t)||^2_{\mathcal{L}^2}\mathrm{d}r\bigg],
\end{aligned}
$$

where we have used the fundamental inequality $2ab \le a^2+b^2$ for any nonnegative, suitable constants a, b. Consequently, for any $s \in [t, T]$, we have

$$
\begin{aligned}
\mathbb{E}[|\hat{Y}(s;t)|^2] &+ \frac{1}{2}\mathbb{E}\bigg[\int_s^T |\hat{Z}(r;t)|^2\mathrm{d}r\bigg] \\
&+\frac{1}{2}\mathbb{E}\bigg[\int_s^T ||\hat{K}(r,\cdot;t)||^2_{\mathcal{L}^2}\mathrm{d}r\bigg] \\
&\le C\mathbb{E}\bigg[\int_s^T \Big(|\hat{Y}(r;t)|^2+|\hat{Y}(r;r)|^2\Big)\mathrm{d}r\bigg] \\
&\le C(T-t)\bigg[\sup_{r\in[t,T]}\mathbb{E}[|\hat{Y}(r;t)|^2]+\sup_{r\in[t,T]}\mathbb{E}[|\hat{Y}(r;r)|^2]\bigg] \\
&\le 2C(T-t)\sup_{t\le r\le s\le T}\mathbb{E}[|\hat{Y}(s;r)|^2]. \qquad (38)
\end{aligned}
$$

Therefore,

$$
\sup_{t\le r\le s\le T}\mathbb{E}[|\hat{Y}(s;r)|^2] \le 2C(T-t)\sup_{t\le r\le s\le T}\mathbb{E}[|\hat{Y}(s;r)|^2]. \qquad (39)
$$

We now take $\delta \in (0, 1/(4C))$. Then, for any $t \in [T-\delta, T]$, we have

$$
\sup_{t\le r\le s\le T}\mathbb{E}[|\hat{Y}(s;r)|^2] \le \frac{1}{2}\sup_{t\le r\le s\le T}\mathbb{E}[|\hat{Y}(s;r)|^2], \qquad (40)
$$

which implies that $\sup_{t\le r\le s\le T}\mathbb{E}[|\hat{Y}(s;r)|^2]=0$ and thus leading to $\hat{Y}(s;r)=0$, a.s. almost everywhere in $\{(s,r): t\le r\le s\le T\}$.

On the other hand, for $t \in [T-2\delta, T-\delta]$ and $s \in [T-\delta, T]$, because for any $r \in [s, T]$, $\hat{Y}(r;r)=0$, we have from the first inequality in (38) that

$$
\begin{aligned}
\mathbb{E}[|\hat{Y}(s;t)|^2] &+ \frac{1}{2}\mathbb{E}\bigg[\int_s^T |\hat{Z}(r;t)|^2\mathrm{d}r\bigg] \\
&+\frac{1}{2}\mathbb{E}\bigg[\int_s^T ||\hat{K}(r,\cdot;t)||^2_{\mathcal{L}^2}\mathrm{d}r\bigg] \le C\mathbb{E}\bigg[\int_s^T |\hat{Y}(r;t)|^2\bigg].
\end{aligned}
$$

It follows from Gronwall's inequality that $\hat{Y}(s;t)=\hat{Z}(s;t)=\hat{K}(s,\cdot;t)=0$.

For $t \in [T-2\delta, T-\delta]$ and $s \in [t, T-\delta]$, we notice that $\hat{Y}(T-\delta;t)=0$, using the same argument as in the region $t \in [T-\delta, T]$ and $s \in [t, T]$ leads to $\hat{Y}(s;t)=\hat{Z}(s;t)=\hat{K}(s,\cdot;t)=0$.

Finally, repeating the same analysis in a backward manner to $t \in [T-3\delta, T-2\delta]$ and so on until we reach time $t=0$ yields $\hat{Y}(\cdot;\cdot) = \hat{Z}(\cdot;\cdot) = \hat{K}(\cdot,\cdot;\cdot) = 0$. This completes the proof. □

We are now in the position to state the main uniqueness theorem.

Theorem 4.2 *Let Ψ, ϕ and φ be defined by (35). Then, $u^*(\cdot)$ given by (33) is the unique equilibrium investment strategy for problem (27)–(28).*

Proof Suppose that there exists another equilibrium investment strategy $u(\cdot)$ with the corresponding state process $X(\cdot)$. Then, BSDE in (29), with X^* replaced by X, admits a unique solution $(Y(\cdot;t), Z(\cdot), K(\cdot,\cdot)) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F},p}(t,T;\mathbb{R}^d) \times F^2_p(t,T;\mathbb{R}^m)$ satisfying

$$B(s)Y(s;s) + \sigma(s)Z(s) + \int_{\mathbb{R}_0} \eta(s,e)\mathrm{Diag}(\nu(de))K(s,e) = 0, \quad \text{for a.e. } s \in [0,T].$$

Here, we have used the fact that both $Z(\cdot;t)$ and $K(\cdot,\cdot;t)$ are independent of t (Proposition 3.4).

We now define

$$\begin{aligned}
&\overline{Y}(s;t) := Y(s;t) - [\Psi(s)(X(s) - \mathbb{E}_t[X(s)]) + \phi(s)X(t) + \varphi(s)],\\
&\overline{Z}(s) := Z(s) - \sigma(s)^T u(s)\Psi(s), \quad \overline{K}(s,\cdot) = K(s,\cdot) - \eta(s,\cdot)^T u(s)\Psi(s).
\end{aligned}$$

Therefore, the equilibrium condition for the state-control pair $(X(\cdot), u(\cdot))$ becomes

$$\begin{aligned}
B(s)\big(\overline{Y}(s;s) + \phi(s)X(s) + \varphi(s)\big) + \sigma(s)\big(\overline{Z}(s) + \sigma(s)^T u(s)\Psi(s)\big)&\\
+ \int_{\mathbb{R}_0} \eta(s,e)\mathrm{Diag}(\nu(de))\big(\overline{K}(s,e) + \eta(s,e)^T u(s)\Psi(s)\big) = 0&,
\end{aligned}$$

which solve for $u(s)$ in the above equation yields

$$u(s) = \alpha(s)X(s) + \beta(s) + \zeta(s), \tag{41}$$

where

$$\zeta(s) := -\Theta(s)^{-1}\Psi(s)^{-1}\Big(B(s)\overline{Y}(s;s) + \sigma(s)\overline{Z}(s) + \int_{\mathbb{R}_0} \eta(s,e)\mathrm{Diag}(\nu(de))\overline{K}(s,e)\Big).$$

Combining the dynamics for Ψ, ϕ, φ in (34) with that of $Y(\cdot,\cdot)$ in (29), we derive the following BSDE for $(\overline{Y}(\cdot;t), \overline{Z}(\cdot), \overline{K}(\cdot,\cdot))$ (with s suppressed for $\rho, B, \Theta, \sigma, \eta$):

$$\begin{aligned}
\mathrm{d}\overline{Y}(s;t) = &-\Big\{\rho\overline{Y}(s;t) - B^T\Theta^{-1}\Big(B\overline{Y}(s;s) + \sigma\overline{Z}(s) + \textstyle\int_{\mathbb{R}_0}\eta(e)\mathrm{Diag}(\nu(de))\overline{K}(s,e)\Big)\\
&+ B^T\Theta^{-1}\mathbb{E}_t\Big[B\overline{Y}(s;s) + \sigma\overline{Z}(s) + \textstyle\int_{\mathbb{R}_0}\eta(e)\mathrm{Diag}(\nu(de))\overline{K}(s,e)\Big]\Big\}\mathrm{d}s\\
&+ \overline{Z}(s)^T dW(s) + \textstyle\int_{\mathbb{R}_0}\overline{K}(s,e)^T\widetilde{N}(ds,de),\\
\overline{Y}(T;t) = &\ 0, \quad s \in [t,T].
\end{aligned} \tag{42}$$

In addition, we can easily check from the definition of $(\overline{Y}(\cdot;t), \overline{Z}(\cdot), \overline{K}(\cdot,\cdot))$ that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\sup_{s\in[t,T]} |\overline{Y}(s;t)|^2\right] < +\infty, \ \mathbb{E}\left[\int_0^T |\overline{Z}(s)|^2 \mathrm{d}s\right] < +\infty \quad \text{and}$$
$$\mathbb{E}\left[\int_0^T ||\overline{K}(s,\cdot)||^2_{\mathcal{L}^2} \mathrm{d}s\right] < +\infty,$$

which implies $(\overline{Y}(\cdot;t), \overline{Z}(\cdot), \overline{K}(\cdot,\cdot)) \in \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3$. Then, it follows from Lemma 4.1 that $\overline{Y}(s;t) = \overline{Z}(s) = \overline{K}(s,\cdot) \equiv 0$.

Finally, substituting $\overline{Y} = \overline{Z} = \overline{K} \equiv 0$ into (41), we can see that $u(\cdot)$ has exactly the same form as $u^*(\cdot)$ in (33). This proves that $u(\cdot)$ and $u^*(\cdot)$ lead to an identical control. The proof is complete. □

5 Conclusions

In this paper, we consider some time-inconsistent LQ control problem with random coefficients and jumps. A general sufficient and necessary condition for equilibrium controls via a flow of constrained FBSDEs is derived. Due to the presence of random coefficients (including $A(\cdot)$) and jumps, essential difficulties arise and we overcome these heavy difficulties by proving a sharper estimate for the first-order variational equation. We also shed light on important application in mean-variance portfolio selection problem with deterministic coefficients and present its unique explicit equilibrium investment strategy. However, due to some technical difficulties, we only consider the case where the parameters of the underlying assets are deterministic functions. One of the potential research topics is to extend the results to the market model with random parameters. We hope to address this problem in the future research.

Acknowledgements The authors would like to thank the referees for their careful reading of the paper and helpful suggestions. This work was supported by the National Natural Science Foundation of China (NSFC Grant Nos. 11571189, 11701087, 61773411) and Shandong Provincial Natural Science Foundation, China.

Appendix

In this appendix, we provide an essential estimate assisting the proof of Proposition 3.2. To ease the explosion of the results, we only consider the case for $n = 1$, and the extension to the multidimensional case is straightforward. If not specified, we will denote by C some positive constants that may differ from line to line in the following estimates.

Lemma A.1 *For each $t \in [0,T]$, let $(\Psi(s))_{s\in[t,T]}$ be a progressively measurable process, such that, for any $k \geq 1$*

$$\mathbb{E}_t\left[\sup_{s\in[t,T]} \left|\Psi(s)\right|^k\right] \leq C. \tag{43}$$

Then, there exists a function $\rho : \Omega\times]0,\infty[\to]0,\infty[$ *with* $\rho(\varepsilon)\downarrow 0$ *as* $\varepsilon\downarrow 0$, *a.s., such that*

$$\int_t^T \left|\mathbb{E}_t[\Psi(s)Y^{\varepsilon}(s)]\right|^2 \mathrm{d}s \le \varepsilon\rho(\varepsilon). \tag{44}$$

Proof Define an auxiliary process $(\Lambda(s))_{s\in[t,T]}$ with $\Lambda(t)=1$ by:

$$\begin{aligned}\Lambda(s) = \exp\Bigg\{ \int_t^s \Bigg[-A(r) + \frac{1}{2}\sum_{i=1}^d C_i^2(r) \\ + \sum_{j=1}^m \int_{\mathbb{R}_0} \Big(E_j(r,e) + \ln\frac{1}{1+E_j(r,e)}\Big)\nu_j(de)\Bigg]\mathrm{d}r \\ - \sum_{i=1}^d \int_t^s C_i(r)\mathrm{d}W_i(r) + \sum_{j=1}^m \int_t^s \int_{\mathbb{R}_0} \ln\frac{1}{1+E_j(r,e)}\widetilde{N}_j(dr,de)\Bigg\}.\end{aligned}$$

Since A, C_i, E_j are uniformly bounded, for any $k\ge 1$, there exists a positive constant C, such that

$$\mathbb{E}_t\Bigg[\sup_{s\in[t,T]}\left(\left|\Lambda(s)\right|^k + \left|\Gamma(s)\right|^k\right)\Bigg] \le C, \tag{45}$$

where $\Gamma(s)=\Lambda(s)^{-1}$. Furthermore, in view of (43), we can easily obtain

$$\mathbb{E}_t\Bigg[\sup_{s\in[t,T]}\left|\Psi(s)\Gamma(s)\right|^k\Bigg] \le C. \tag{46}$$

Following the martingale representation theorem (see, e.g., [20, Lemma 2.3]), for every $s\in[t,T]$, there exists a unique pair $(\xi(\cdot;s),\beta(\cdot,\cdot;s))\in L^2_{\mathcal{F},p}(t,s;\mathbb{R}^d)\times F_p^2(t,s;\mathbb{R}^m)$ such that

$$\begin{aligned}\Psi(s)\Gamma(s) = \mathbb{E}_t[\Psi(s)\Gamma(s)] + \sum_{i=1}^d \int_t^s \xi_i(r;s)\mathrm{d}W_i(r) \\ + \sum_{j=1}^m \int_t^s \int_{\mathbb{R}_0} \beta_j(r,e;s)\widetilde{N}_j(dr,de),\end{aligned} \tag{47}$$

where $\xi(\cdot;s)=(\xi_1(\cdot;s),\ldots,\xi_d(\cdot;s))$ and $\beta(\cdot,\cdot;s)=(\beta_1(\cdot,\cdot;s),\ldots,\beta_m(\cdot,\cdot;s))$.

Following from (46), the Burkholder–Davis–Gundy inequality, and Doob's maximal inequality, we get for $k>1$

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{i=1}^{d}\int_t^s |\xi_i(r;s)|^2\mathrm{d}r+\sum_{j=1}^{m}\int_t^s\int_{\mathbb{R}_0}|\beta_j(r,e;s)|^2\nu_j(de)\mathrm{d}r\Big)^{\frac{k}{2}}\Big]\\
&\le C\mathbb{E}\Big[\sup_{u\in[t,s]}\Big|\sum_{i=1}^{d}\int_t^u \xi_i(r;s)\mathrm{d}W_i(r)+\sum_{j=1}^{m}\int_t^u\int_{\mathbb{R}_0}\beta_j(r,e;s)\widetilde{N}_j(dr,de)\Big|^k\Big]\\
&\le C\Big(\frac{k}{k-1}\Big)^k\mathbb{E}\Big[\Big|\sum_{i=1}^{d}\int_t^s \xi_i(r;s)\mathrm{d}W_i(r)+\sum_{j=1}^{m}\int_t^s\int_{\mathbb{R}_0}\beta_j(r,e;s)\widetilde{N}_j(dr,de)\Big|^k\Big]\\
&\le C\mathbb{E}\Big[\big|\Psi(s)\Gamma(s)-\mathbb{E}_t\big[\Psi(s)\Gamma(s)\big]\big|^k\Big]\le C\Big(\mathbb{E}\Big[\big|\Psi(s)\Gamma(s)\big|^k\Big]+\mathbb{E}\Big[\big|\mathbb{E}_t\big[\Psi(s)\Gamma(s)\big]\big|^k\Big]\Big)\\
&\le C\mathbb{E}\Big[\big|\Psi(s)\Gamma(s)\big|^k\Big]\le C\mathbb{E}\Big[\sup_{s\in[t,T]}\big|\Psi(s)\Gamma(s)\big|^k\Big]\le C. \qquad (48)
\end{aligned}$$

For $k=1$, by using Hölder's inequality, we can easily get

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{d}\int_t^s |\xi_i(r;s)|^2\mathrm{d}r+\sum_{j=1}^{m}\int_t^s\int_{\mathbb{R}_0}|\beta_j(r,e;s)|^2\nu_j(de)\mathrm{d}r\Big)^{\frac{1}{2}}\Big]\le C. \qquad (49)$$

Combining (48) and (49), we have for any $k\ge 1$

$$\sup_{s\in[t,T]}\mathbb{E}\Big[\Big(\sum_{i=1}^{d}\int_t^s |\xi_i(r;s)|^2\mathrm{d}r+\sum_{j=1}^{m}\int_t^s\int_{\mathbb{R}_0}|\beta_j(r,e;s)|^2\nu_j(de)\mathrm{d}r\Big)^{\frac{k}{2}}\Big]\le C. \qquad (50)$$

Now, using Itô's formula to $s\mapsto \Lambda(s)Y^{\varepsilon}(s)$ yields

$$\begin{aligned}
Y^{\varepsilon}(s)=\Gamma(s)\int_t^s\Lambda(r)\Big\{\Big[B(r)-\sum_{i=1}^{d}C_i(r)D_i(r)\\
-\sum_{j=1}^{m}\int_{\mathbb{R}_0}\frac{E_j(r,e)F_j(r,e)}{1+E_j(r,e)}\nu_j(de)\Big]v\mathbf{1}_{[t,t+\varepsilon[}(r)\Big\}\mathrm{d}r\\
+\sum_{i=1}^{d}\Gamma(s)\int_t^s\Lambda(r)D_i(r)v\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}W_i(r)\\
+\sum_{j=1}^{m}\Gamma(s)\int_t^s\int_{\mathbb{R}_0}\Lambda(r-)\frac{F_j(r,e)}{1+E_j(r,e)}v\mathbf{1}_{[t,t+\varepsilon[}(r)\widetilde{N}_j(dr,de).
\end{aligned}$$

Consider

$$\Psi(s)Y^{\varepsilon}(s):=L_1(s)+L_2(s)+L_3(s),\quad s\in[t,T],$$

with

$$L_1(s) = \Psi(s)\Gamma(s)\int_t^s \Lambda(r)\Big\{\Big[B(r) - \sum_{i=1}^d C_i(r)D_i(r) - \sum_{j=1}^m \int_{\mathbb{R}_0} \frac{E_j(r,e)F_j(r,e)}{1+E_j(r,e)}\nu_j(de)\Big]v\mathbf{1}_{[t,t+\varepsilon[}(r)\Big\}\mathrm{d}r,$$

$$L_2(s) = \sum_{i=1}^d \Psi(s)\Gamma(s)\int_t^s \Lambda(r)D_i(r)v\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}W_i(r),$$

$$L_3(s) = \sum_{j=1}^m \Psi(s)\Gamma(s)\int_t^s \int_{\mathbb{R}_0} \Lambda(r-)\frac{F_j(r,e)}{1+E_j(r,e)}v\mathbf{1}_{[t,t+\varepsilon[}(r)\widetilde{N}_j(dr,de).$$

Actually, by virtue of (45) and (46), the following estimate for $\mathbb{E}_t[L_1(s)]$ holds:

$$\int_t^T \big|\mathbb{E}_t[L_1(s)]\big|^2\mathrm{d}s \le C|v|^2\varepsilon^2, \quad s\in[t,T]. \tag{51}$$

Following the expression of $\Psi(s)\Gamma(s)$ in (47), we have

$$\begin{aligned}\mathbb{E}_t[L_2(s)] &= \sum_{i=1}^d \mathbb{E}_t\Big[\Big(\Psi(s)\Gamma(s)\Big)\cdot\int_t^s \Lambda(r)D_i(r)v\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}W_i(r)\Big]\\ &= \sum_{i=1}^d \mathbb{E}_t\Big[\int_t^s \xi_i(r;s)\Lambda(r)D_i(r)v\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\Big]\\ &\le C\varepsilon^{\frac{1}{2}}|v|\sum_{i=1}^d \mathbb{E}_t\Big[\sup_{r\in[t,T]}\Lambda(r)\cdot\Big(\int_t^s \big|\xi_i(r;s)\big|^2\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\Big)^{\frac{1}{2}}\Big]\\ &\le C\varepsilon^{\frac{1}{2}}|v|\sum_{i=1}^d \Big(\mathbb{E}_t\Big[\int_t^s \big|\xi_i(r;s)\big|^2\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\Big]\Big)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\int_t^T \big|\mathbb{E}_t[L_2(s)]\big|^2\mathrm{d}s \le C\varepsilon|v|^2\sum_{i=1}^d \mathbb{E}_t\Big[\int_t^T\int_t^s \big|\xi_i(r;s)\big|^2\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\mathrm{d}s\Big].$$

Setting $\rho_1(\varepsilon) := C|v|^2\sum_{i=1}^d \mathbb{E}_t\Big[\int_t^T\int_t^s \big|\xi_i(r;s)\big|^2\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\mathrm{d}s\Big]$ and then from (50), we have

$$\mathbb{E}\Big[\int_t^T\int_t^s \big|\xi_i(r;s)\big|^2\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\mathrm{d}s\Big] \le C\sup_{s\in[t,T]}\mathbb{E}\Big[\int_t^s \big|\xi_i(r;s)\big|^2\mathrm{d}r\Big] < \infty.$$

Thus, following the dominated convergence theorem for conditional expectations and observing the fact that $\mathbf{1}_{[t,t+\varepsilon[} \to 0$, we can easily obtain that $\rho_1(\varepsilon) \to 0$ as $\varepsilon \downarrow 0,\ a.s.$ Hence,

$$\int_t^T \left|\mathbb{E}_t[L_2(s)]\right|^2 \mathrm{d}s \leq \varepsilon\rho_1(\varepsilon). \tag{52}$$

Similarly, we get

$$\begin{aligned}
\mathbb{E}_t[L_3(s)] &= \sum_{j=1}^m \mathbb{E}_t\Big[\int_t^s \int_{\mathbb{R}_0} \beta_j(r,e;s)\Lambda(r)\frac{F_j(r,e)}{1+E_j(r,e)} v\mathbf{1}_{[t,t+\varepsilon[}(r)\nu_j(de)\mathrm{d}r\Big] \\
&\leq C\varepsilon^{\frac{1}{2}}|v| \sum_{j=1}^m \mathbb{E}_t\Big[\sup_{r\in[t,T]} \Lambda(r)\cdot\Big(\int_t^s \int_{\mathbb{R}_0} \left|\beta_j(r,e;s)\right|^2 \nu_j(de)\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\Big)^{\frac{1}{2}}\Big] \\
&\leq C\varepsilon^{\frac{1}{2}}|v| \sum_{j=1}^m \Big(\mathbb{E}_t\Big[\int_t^s \int_{\mathbb{R}_0} \left|\beta_j(r,e;s)\right|^2 \nu_j(de)\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\Big]\Big)^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$\int_t^T \left|\mathbb{E}_t[L_3(s)]\right|^2 \mathrm{d}s \leq C\varepsilon|v|^2 \sum_{j=1}^m \mathbb{E}_t\Big[\int_t^T \int_t^s \int_{\mathbb{R}_0} \left|\beta_j(r,e;s)\right|^2 \nu_j(de)\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\mathrm{d}s\Big].$$

Setting $\rho_2(\varepsilon) := C\varepsilon|v|^2 \sum_{j=1}^m \mathbb{E}_t\Big[\int_t^T \int_t^s \int_{\mathbb{R}_0} \left|\beta_j(r,e;s)\right|^2 \nu_j(de)\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\mathrm{d}s\Big]$ and then from (50) again, we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_t^T \int_t^s \int_{\mathbb{R}_0} \left|\beta_j(r,e;s)\right|^2 \nu_j(de)\mathbf{1}_{[t,t+\varepsilon[}(r)\mathrm{d}r\mathrm{d}s\Big] \\
&\quad\leq C \sup_{s\in[t,T]} \mathbb{E}\Big[\int_t^s \int_{\mathbb{R}_0} \left|\beta_j(r,e;s)\right|^2 \nu_j(de)\mathrm{d}r\Big] < \infty.
\end{aligned}$$

Thus, using the dominated convergence theorem once again and observing the fact that $\mathbf{1}_{[t,t+\varepsilon[} \to 0$, it holds that $\rho_2(\varepsilon) \to 0$ as $\varepsilon \downarrow 0,\ a.s.$ Hence,

$$\int_t^T \left|\mathbb{E}_t[L_3(s)]\right|^2 \mathrm{d}s \leq \varepsilon\rho_2(\varepsilon). \tag{53}$$

In view of (51)–(53), we have

$$\int_t^T \left|\mathbb{E}_t[\Psi(s)Y^\varepsilon(s)]\right|^2 \mathrm{d}s \leq C\big(\varepsilon^2|v|^2 + \varepsilon\rho_1(\varepsilon) + \varepsilon\rho_2(\varepsilon)\big). \tag{54}$$

Now setting $\rho(\varepsilon) := C(\varepsilon|v|^2 + \rho_1(\varepsilon) + \rho_2(\varepsilon))$ in (54), we obtain estimate (44). $\square$

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