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Structural Properties of Tensors and Complementarity Problems

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Abstract In this paper, one of our main purposes is to prove the boundedness of the solution set of tensor complementarity problems such that the specific bounds depend only on the structural properties of such a tensor. To achieve this purpose, firstly, we prove that this class of structured tensors is strictly semi-positive. Subsequently, the strictly lower and upper bounds of operator norms are given for two positively homogeneous operators. Finally, with the help of the above upper bounds, we show that the solution set of tensor complementarity problems has the strictly lower bound. Furthermore, the upper bounds of spectral radius are obtained, which depends only on the principal diagonal entries of tensors.

Keywords Structured tensor · Tensor complementarity problems · Spectral radius · Operator norms · Upper and lower bounds

Mathematics Subject Classification 47H15 · 47H12 · 34B10 · 47A52 · 47J10 · 47H09 · 15A48 · 47H07

1 Introduction

As a natural extension of linear complementarity problem, the tensor complementarity problem is a new topic emerged from the tensor community. Meanwhile, such

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a problem is a special type of nonlinear complementarity problems. So the tensor complementarity problem seems to have similar properties to the linear complementarity problem, and to have its particular properties other than ones of the classical nonlinear complementarity problem, and to have some nice properties that depended on itself special structure. The notion of the tensor complementarity problem was used firstly by Song and Qi [1,2]. Recently, Huang and Qi [3] formulated a multi-player non-cooperative game as a tensor complementarity problem and showed that a Nash equilibrium point of the multilinear game is equivalent to a solution of the tensor complementarity problem. By using specially structured properties of tensors, the properties of the tensor complementarity problem have been well studied in the literatures. For example, see Song and Yu [4], Song and Qi [5] for strictly semi-positive tensors, Gowda et al. [6], Luo et al. [7] for Z-tensors, Ding et al. [8] for P-tensors, Wang et al. [9] for exceptionally regular tensors, Bai et al. [10] for strong P-tensors, Che et al. [11] for some special tensors, Huang et al. [12] for Q-tensors. Song and Qi [13], Ling et al. [14,15], Chen et al. [16] studied the tensor eigenvalue complementarity problems.

In past several decades, numerous mathematical works concerned with error bound analysis for the solution of linear complementarity problem by means of the special structure of the matrices. For more details, see [17–22]. Recently, motivated by the study on error bounds for linear complementarity problems, Song and Yu [4] and Song and Qi [23] extended the error bounds results of the linear complementarity problems to the tensor complementarity problems with strictly semi-positive tensors. However, there are relatively few works in the specific upper or lower bounds of the tensor complementarity problems, which is a weak link in this topics.

In this paper, we will give the boundedness of solution set of tensor complementarity problem with B-tensors. Moreover, we will present the specific lower bounds of such a problem, which depend only upon the structural properties of B-tensors.

To achieve the above goal, we need study the structured properties of B-tensors. Nowadays, miscellaneous structured tensors have been widely studied (Qi and Luo [24]), which is one of hot research topics. For more detail, see Zhang et al. [25] and Ding et al. [26] for M-tensors, Song and Qi [1] for P-tensors and B-tensors, Li and Li [27] for double B-tensors, Song and Qi [28,29] and Mei and Song [30] for Hilbert tensors. Recently, the concept of B-tensors was first used by Song and Qi [1]. They gave many nice structured properties which are similar to ones of B-matrices. For more nice properties and applications of B-matrices, see Peña [31,32]. It is well known that each B-matrix is a P-matrix. However, the same conclusion only holds for even-order symmetric B-tensors [33]. Qi and Song [34] showed that an even-order symmetric B-tensor is positive definite. Yuan and You [33] proved that a non-symmetric B-tensor is not P-tensor in general. So there are many special properties of B-tensors for further and serious consideration.

In Sect. 3, we prove that each B-tensor is strictly semi-positive. So, the solution set of tensor complementarity problem with B-tensor is bounded. In order to presenting the specific lower bounds of such a problem, in Sect. 4, we give the strictly lower and upper bounds of the norms for two positively homogeneous operators induced by B-tensors. By means of the above upper bounds, we establish the strictly lower bound of solution set of tensor complementarity problem with B-tensors. Furthermore, we achieve our another objective with the help of upper bounds of operator norms. That is, we obtain the upper bounds of spectral radius and E-spectral radius of B-tensor, which depends only on the principal diagonal entries of tensors.

2 Preliminaries and Basic Facts

An m-order n-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m} \in \mathbb{R}$, where $i_j \in [n] = \{1, 2, \ldots, n\}$ for $j \in [m] = \{1, 2, \ldots, m\}$. The set of all m-order n-dimensional tensors is denoted by $T_{m,n}$. If the entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a symmetric tensor, denoted by $S_{m,n}$. Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ and a vector $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$. Then $\mathcal{A}x^{m-1}$ is a vector with its ith component defined by

$$\left(\mathcal{A}x^{m-1}\right)_i := \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \forall i \in [n]$$

and $\mathcal{A}x^m$ is a homogeneous polynomial of degree m, defined by

$$\mathcal{A}x^m := x^T \left(\mathcal{A}x^{m-1}\right) = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

For any $q \in \mathbb{R}^n$, the tensor complementarity problem, denoted by $TCP(\mathcal{A}, q)$, is to find $x \in \mathbb{R}^n$ such that

$$x \geq 0, q + \mathcal{A}x^{m-1} \geq 0 \text{ and } x^T(q + \mathcal{A}x^{m-1}) = 0, \tag{1}$$

or to show that no such vector exists.

An n-dimensional B-matrix $B = (b_{ij})$ is a square real matrix with its entries satisfying that for all $i \in [n]$

$$\sum_{j=1}^{n} b_{ij} > 0 \text{ and } \frac{1}{n}\sum_{j=1}^{n} b_{ij} > b_{ik}, i \neq k$$

As a natural extension of B-matrices. Song and Qi [1] gave the definitions of B-tensors and B_0-tensors.

Definition 2.1 Let $\mathcal{B} = (b_{i_1 \cdots i_m}) \in T_{m,n}$. Then $\mathcal{B}$ is said to be

(i) a B-tensor iff for all $i \in [n]$, $\sum\limits_{i_2, \ldots, i_m = 1}^{n} b_{i i_2 i_3 \cdots i_m} > 0$ and

$$\frac{1}{n^{m-1}}\left(\sum_{i_2, \ldots, i_m = 1}^{n} b_{i i_2 i_3 \cdots i_m}\right) > b_{i j_2 j_3 \cdots j_m}, \text{ for all } (j_2, j_3, \ldots, j_m) \neq (i, i, \ldots, i);$$

(ii) a B_0-tensor iff for all $i \in [n]$, $\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2i_3\cdots i_m} \geq 0$ and

$$\frac{1}{n^{m-1}}\left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2i_3\cdots i_m}\right) \geq b_{ij_2j_3\cdots j_m}, \text{ for all } (j_2, j_3, \ldots, j_m) \neq (i, i, \ldots, i).$$

For each $i \in [n]$, let

$$\eta_i(\mathcal{B}) = \max\{b_{ij_2j_3\cdots j_m}; (j_2, j_3, \cdots, j_m) \neq (i, i, \cdots, i), j_2, \ldots, j_m \in [n]\}$$

and

$$\beta_i(\mathcal{B}) = \max\{0, \eta_i(\mathcal{B})\}. \tag{2}$$

Lemma 2.1 (Song and Qi [1, Theorems 5.1, 5.2, 5.3]) *Let $\mathcal{B} = (b_{i_1\cdots i_m}) \in T_{m,n}$. If $\mathcal{B}$ is a B-tensor, then, for each $i \in [n]$,*

(i) $b_{ii\cdots i} > |b_{ij_2j_3\cdots j_m}|$ for all $(j_2, j_3, \ldots, j_m) \neq (i, i, \ldots, i),\ j_2, j_3, \ldots, j_m \in [n]$;
(ii) $\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2i_3\cdots i_m} > n^{m-1}\beta_i(\mathcal{B})$;
(iii) $b_{ii\cdots i} > \sum_{b_{ii_2\cdots i_m}<0} |b_{ii_2i_3\cdots i_m}|$.

If $\mathcal{B}$ is a B_0-tensor, then, the above three inequalities hold with "$>$" being replaced by "$\geq$".

The concepts of tensor eigenvalues were introduced by Qi [35,36] to the higher-order symmetric tensors, and the existence of the eigenvalues and some applications were studied there. Lim [37] independently introduced the concept of real tensor eigenvalues and obtained some existence results using a variational approach.

Definition 2.2 Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$.

(i) A number $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{A}$ iff there is a nonzero vector x such that
$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{3}$$
and x is called an eigenvector of $\mathcal{A}$ associated with λ. Such an eigenvalue is called H-eigenvalue if it is real and has a real eigenvector x, and such a real eigenvector x is called H-eigenvector.
(ii) A number $\mu \in \mathbb{C}$ is said to be an E-eigenvalue of $\mathcal{A}$ iff there exists a nonzero vector x such that
$$\mathcal{A}x^{m-1} = \mu x(x^T x)^{\frac{m-2}{2}}, \tag{4}$$
and x is called an E-eigenvector of $\mathcal{A}$ associated with μ. It is clear that if x is real, then, μ is also real. In this case, μ and x are called a Z-eigenvalue of $\mathcal{A}$ and a Z-eigenvector of $\mathcal{A}$, respectively.

We now give the definitions of (strictly) semi-positive tensors (Song and Qi [2,5]).

Definition 2.3 Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. $\mathcal{A}$ is said to be

(i) semi-positive iff for each $x \geq 0$ and $x \neq 0$, there exists an index $k \in [n]$ such that

$$x_k > 0 \text{ and } \left(\mathcal{A}x^{m-1}\right)_k \geq 0;$$

(ii) strictly semi-positive iff for each $x \geq 0$ and $x \neq 0$, there exists an index $k \in [n]$ such that

$$x_k > 0 \text{ and } \left(\mathcal{A}x^{m-1}\right)_k > 0.$$

For $x \in \mathbb{R}^n$, it is well known that

$$\|x\|_\infty := \max\{|x_i|; i \in [n]\} \text{ and } \|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \ (p \geq 1) \tag{5}$$

are two main norms defined on $\mathbb{R}^n$. Then, for a continuous, positively homogeneous operator $T : \mathbb{R}^n \to \mathbb{R}^n$, it is obvious that

$$\|T\|_p := \max_{\|x\|_p=1} \|T(x)\|_p \text{ and } \|T\|_\infty := \max_{\|x\|_\infty=1} \|T(x)\|_\infty \tag{6}$$

are two operator norms of T. For $\mathcal{A} \in T_{m,n}$, we may define a continuous, positively homogeneous operator $T_\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_\mathcal{A}(x) := \begin{cases} \|x\|_2^{2-m}\mathcal{A}x^{m-1}, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{7}$$

When m is even, we may define another continuous, positively homogeneous operator $F_\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ by for any $x \in \mathbb{R}^n$,

$$F_\mathcal{A}(x) := \left(\mathcal{A}x^{m-1}\right)^{\left[\frac{1}{m-1}\right]}. \tag{8}$$

The following upper bounds and properties of the operator norm were established by Song and Qi [23,38].

Lemma 2.2 (Song and Qi [38, Theorem 4.3] and [23, Lemma 3, Lemma 4]) *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. *Then,*

(i) $\|T_\mathcal{A}\|_\infty \leq \max\limits_{i \in [n]} \sum\limits_{i_2,\ldots,i_m=1}^n |a_{i i_2 \cdots i_m}|$;

(ii) $\|F_{\mathcal{A}}\|_\infty \le \max_{i\in[n]} \left(\sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\cdots i_m}|\right)^{\frac{1}{m-1}}$ *if m is even;*

(iii) $\|T_{\mathcal{A}}\|_p \le n^{\frac{m-2}{p}} \left(\sum_{i=1}^{n}\left(\sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\cdots i_m}|\right)^{p}\right)^{\frac{1}{p}}$;

(iv) $\|F_{\mathcal{A}}\|_p \le \left(\sum_{i=1}^{n}\left(\sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\cdots i_m}|\right)^{\frac{p}{m-1}}\right)^{\frac{1}{p}}$ *if m is even.*

3 The Strictly Semi-Positivity of B-Tensors

Theorem 3.1 *Let $\mathcal{B}$ be an m-order n-dimensional B-tensor. Then, $\mathcal{B}$ is strictly semi-positive.*

Proof Suppose that $\mathcal{B}$ is not strictly semi-positive. Then, there exists $x \ge 0$ and $x \neq 0$, for all $k \in [n]$ with $x_k > 0$ such that

$$\left(\mathcal{B}x^{m-1}\right)_k \le 0.$$

Choose $x_i > 0$ with $x_i \ge x_k$ for all $k \in [n]$. Clearly, $\left(\mathcal{B}x^{m-1}\right)_i \le 0$. Then, we have

$$\begin{aligned}
0 \ge \left(\mathcal{B}x^{m-1}\right)_i &= \sum_{i_2,i_3,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} x_{i_2}x_{i_3}\cdots x_{i_m}\\
&= b_{ii\cdots i}x_i^{m-1} + \sum_{b_{ii_2\cdots i_m}<0} b_{ii_2\cdots i_m} x_{i_2}x_{i_3}\cdots x_{i_m}\\
&\quad + \sum_{b_{ii_2\cdots i_m}>0} b_{ii_2\cdots i_m} x_{i_2}x_{i_3}\cdots x_{i_m}\\
&\ge b_{ii\cdots i}x_i^{m-1} + \sum_{b_{ii_2\cdots i_m}<0} b_{ii_2\cdots i_m} x_i^{m-1}\\
&\quad + \sum_{b_{ii_2\cdots i_m}>0} b_{ii_2\cdots i_m} x_{i_2}x_{i_3}\cdots x_{i_m}\\
&\ge \left(b_{ii\cdots i} + \sum_{b_{ii_2\cdots i_m}<0} b_{ii_2\cdots i_m}\right) x_i^{m-1}.
\end{aligned}$$

By Lemma 2.1 (iii), we obtain

$$0 < \left(b_{ii\cdots i} + \sum_{b_{ii_2\cdots i_m}<0} b_{ii_2\cdots i_m}\right) x_i^{m-1} \le \left(\mathcal{B}x^{m-1}\right)_i \le 0,$$

a contradiction. Consequently, $\mathcal{B}$ must be strictly semi-positive. □

Using the similar proof technique, it is easy to prove the following conclusion.

Theorem 3.2 *Let $\mathcal{B}$ be an m-order n-dimensional B_0-tensor. Then, $\mathcal{B}$ is semi-positive.*

Clearly, by Theorem 3.2 of Song and Yu [4] and Corollary 3.3 of Song and Qi [2], we easily obtain the following.

Corollary 3.3 *Let $\mathcal{B}$ be an m-order n-dimensional B-tensor. Then, for each $q \in \mathbb{R}^n$, the tensor complementarity problem $TCP(\mathcal{B}, q)$ has always a solution. Furthermore, the solution set of the $TCP(\mathcal{B}, q)$ is bounded for each $q \in \mathbb{R}^n$.*

4 Boundedness About B-Tensors

We now present the upper and lower bounds of the operator norm associated with a B-tensor.

Theorem 4.1 *Let $\mathcal{B}$ be an m-order n-dimensional B tensor. Then,*

(i) $n^{\frac{m}{2}} \max_{i\in[n]} \beta_i(\mathcal{B}) < n^{\frac{2-m}{2}} \max\limits_{i\in[n]} \sum\limits_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} \leq \|T_{\mathcal{B}}\|_\infty < n^{\frac{m}{2}} \max\limits_{i\in[n]} b_{ii\cdots i}$;

(ii) $n^{\frac{mp-2}{2p}} \left(\sum\limits_{i=1}^{n} (\beta_i(\mathcal{B}))^p\right)^{\frac{1}{p}} < n^{\frac{2p-pm-2}{2p}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}\right)^p\right)^{\frac{1}{p}} \leq \|T_{\mathcal{B}}\|_p <$ $n^{\frac{mp-2}{2p}} (\sum\limits_{i=1}^{n} b_{ii\cdots i}^p)^{\frac{1}{p}}$ *if* $p \geq 1$,

where $\beta_i(\mathcal{B})$ is defined by (2).

Proof (i) Choose $e = (1, 1, \ldots, 1)^\top$. Then, $\|e\|_\infty = 1$ and $\|e\|_2 = n^{\frac{1}{2}}$, and hence, by Lemma 2.1 (ii), we have

$$\begin{aligned}
\|T_{\mathcal{B}}(e)\|_\infty &= \max_{i\in[n]} \left| \|e\|_2^{2-m} \sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} \right| \\
&= n^{\frac{2-m}{2}} \max_{i\in[n]} \sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} \\
&> n^{\frac{2-m}{2}} \max_{i\in[n]} n^{m-1} \beta_i(\mathcal{B}) \\
&= n^{\frac{m}{2}} \max_{i\in[n]} \beta_i(\mathcal{B}).
\end{aligned}$$

Consequently,

$$n^{\frac{m}{2}} \max_{i\in[n]} \beta_i(\mathcal{B}) < n^{\frac{2-m}{2}} \max_{i\in[n]} \sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} \leq \|T_{\mathcal{B}}\|_\infty.$$

Now we show the right inequality. It follows from Lemma 2.1 (i) together with the fact that $\|x\|_1 \le n\|x\|_\infty$ and $\|x\|_2 \le \sqrt{n}\|x\|_\infty$ that

$$\begin{aligned}
\|T_{\mathcal{B}}\|_\infty &= \max_{\|x\|_\infty=1} \|T_{\mathcal{B}}(x)\|_\infty = \max_{\|x\|_\infty=1} \max_{i\in[n]} \left| \|x\|_2^{2-m} \sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \right| \\
&\le \max_{\|x\|_\infty=1} n^{\frac{2-m}{2}} \|x\|_\infty^{2-m} \max_{i\in[n]} \left(\sum_{i_2,\ldots,i_m=1}^{n} |b_{ii_2\cdots i_m}||x_{i_2}||x_{i_3}|\cdots|x_{i_m}| \right) \\
&< n^{\frac{2-m}{2}} \max_{\|x\|_\infty=1} \|x\|_\infty^{2-m} \max_{i\in[n]} \left(b_{ii\cdots i} \sum_{i_2,\ldots,i_m=1}^{n} |x_{i_2}||x_{i_3}|\cdots|x_{i_m}| \right) \\
&= n^{\frac{2-m}{2}} \max_{\|x\|_\infty=1} \|x\|_\infty^{2-m} \max_{i\in[n]} \left(b_{ii\cdots i} \left(\sum_{k=1}^{n} |x_k| \right)^{m-1} \right) \\
&= n^{\frac{2-m}{2}} \max_{\|x\|_\infty=1} \|x\|_\infty^{2-m} \max_{i\in[n]} (b_{ii\cdots i} \|x\|_1^{m-1}) \\
&\le n^{\frac{2-m}{2}} \max_{\|x\|_\infty=1} \|x\|_\infty^{2-m} (n\|x\|_\infty)^{m-1} \max_{i\in[n]} b_{ii\cdots i} \\
&= n^{\frac{m}{2}} \max_{i\in[n]} b_{ii\cdots i}.
\end{aligned}$$

(ii) Choose $y = (n^{-\frac{1}{p}}, n^{-\frac{1}{p}}, \ldots, n^{-\frac{1}{p}})^\top$. Then, $\|y\|_p = 1$ and $\|y\|_2 = n^{\frac{p-2}{2p}}$, and hence, by Lemma 2.1 (ii), we have

$$\begin{aligned}
\|T_{\mathcal{B}}(y)\|_p^p &= \sum_{i=1}^{n} \left| \|y\|_2^{2-m} \sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} (n^{-\frac{1}{p}})^{m-1} \right|^p \\
&= n^{\frac{(p-2)(2-m)}{2}} \sum_{i=1}^{n} n^{1-m} \left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} \right)^p \\
&> n^{\frac{(p-2)(2-m)}{2}} \sum_{i=1}^{n} n^{1-m} \left(n^{m-1} \beta_i(\mathcal{B}) \right)^p \\
&= n^{\frac{mp-2}{2}} \sum_{i=1}^{n} (\beta_i(\mathcal{B}))^p .
\end{aligned}$$

Consequently,

$$n^{\frac{mp-2}{2p}} \left(\sum_{i=1}^{n} (\beta_i(\mathcal{B}))^p \right)^{\frac{1}{p}} < n^{\frac{2p-pm-2}{2p}} \left(\sum_{i=1}^{n} \left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m} \right)^p \right)^{\frac{1}{p}} \le \|T_{\mathcal{B}}\|_p .$$

Next we show the right inequality. By Lemma 2.1 (i) together with the fact that $\|x\|_q \le \|x\|_p \le n^{\frac{1}{p}-\frac{1}{q}}\|x\|_q$ for $q > p$, we obtain

$$\begin{aligned}
\|T_{\mathcal{B}}\|_p^p &= \max_{\|x\|_p=1} \|T_{\mathcal{B}}(x)\|_p^p \\
&= \max_{\|x\|_p=1} \sum_{i=1}^n \left| \|x\|_2^{2-m} \sum_{i_2,\ldots,i_m=1}^n b_{ii_2\cdots i_m} x_{i_2} x_{i_3}\cdots x_{i_m} \right|^p \\
&\le \max_{\|x\|_p=1} \|x\|_2^{(2-m)p} \sum_{i=1}^n \left(\sum_{i_2,\ldots,i_m=1}^n |b_{ii_2\cdots i_m}||x_{i_2}||x_{i_3}|\cdots|x_{i_m}| \right)^p \\
&< \max_{\|x\|_p=1} \|x\|_2^{(2-m)p} \sum_{i=1}^n \left(b_{ii\cdots i} \sum_{i_2,\ldots,i_m=1}^n |x_{i_2}||x_{i_3}|\cdots|x_{i_m}| \right)^p \\
&= \max_{\|x\|_p=1} \|x\|_2^{(2-m)p} \sum_{i=1}^n \left(b_{ii\cdots i} \left(\sum_{k=1}^n |x_k| \right)^{m-1} \right)^p \\
&= \max_{\|x\|_p=1} \|x\|_2^{(2-m)p} \sum_{i=1}^n (b_{ii\cdots i}\|x\|_1^{m-1})^p \\
&\le \max_{\|x\|_p=1} \|x\|_2^{(2-m)p} (\sqrt{n}\|x\|_2)^{(m-1)p} \sum_{i=1}^n b_{ii\cdots i}^p \\
&= \max_{\|x\|_p=1} n^{\frac{(m-1)p}{2}} \|x\|_2^p \sum_{i=1}^n b_{ii\cdots i}^p \\
&\le \max_{\|x\|_p=1} n^{\frac{(m-1)p}{2}} (n^{\frac{1}{2}-\frac{1}{p}}\|x\|_p)^p \sum_{i=1}^n b_{ii\cdots i}^p \\
&= n^{\frac{mp-2}{2}} \sum_{i=1}^n b_{ii\cdots i}^p .
\end{aligned}$$

The desired conclusions follow. □

Theorem 4.2 *Let $\mathcal{B}$ be an m-order n-dimensional B-tensor. If m is even, then,*

(i) $n\,(\beta_i(\mathcal{B}))^{\frac{1}{m-1}} < \max_{i\in[n]} \left(\sum_{i_2,\ldots,i_m=1}^n b_{ii_2\cdots i_m} \right)^{\frac{1}{m-1}} \le \|F_{\mathcal{B}}\|_\infty \le \max_{i\in[n]} \left(\sum_{i_2,\ldots,i_m=1}^n |b_{ii_2\cdots i_m}| \right)^{\frac{1}{m-1}} < n \max_{i\in[n]} b_{ii\cdots i}^{\frac{1}{m-1}}$;

(ii) $n^{\frac{p-1}{p}}\left(\sum_{i=1}^{n}(\beta_i(\mathcal{B}))^{\frac{p}{m-1}}\right)^{\frac{1}{p}} < \frac{1}{\sqrt[p]{n}}\left(\sum_{i=1}^{n}\left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}\right)^{\frac{p}{m-1}}\right)^{\frac{1}{p}} \leq \|F_{\mathcal{B}}\|_p <$ $n^{\frac{p-1}{p}}\left(\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{p}{m-1}}\right)^{\frac{1}{p}}$ $(p \geq 1)$.

Proof (i) Choose $e = (1, 1, \ldots, 1)^\top$. Then, $\|e\|_\infty = 1$, and hence, by Lemma 2.1 (ii), we have

$$\begin{aligned}\|F_{\mathcal{B}}(e)\|_\infty &= \max_{i\in[n]}\left|\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}\right|^{\frac{1}{m-1}}\\ &= \max_{i\in[n]}\left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}\right)^{\frac{1}{m-1}}\\ &> n\max_{i\in[n]}(\beta_i(\mathcal{B}))^{\frac{1}{m-1}}.\end{aligned}$$

So, by the definition of the operator norm, we have

$$n\max_{i\in[n]}(\beta_i(\mathcal{B}))^{\frac{1}{m-1}} < \max_{i\in[n]}\left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}\right)^{\frac{1}{m-1}} \leq \|F_{\mathcal{B}}\|_\infty.$$

From Lemmas 2.2 (ii) and 2.1 (i), it follows that

$$\|F_{\mathcal{B}}\|_\infty \leq \max_{i\in[n]}\left(\sum_{i_2,\ldots,i_m=1}^{n}|b_{ii_2\cdots i_m}|\right)^{\frac{1}{m-1}} < \max_{i\in[n]}\left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii\cdots i}\right)^{\frac{1}{m-1}} = n\max_{i\in[n]} b_{ii\cdots i}^{\frac{1}{m-1}}.$$

(ii) Choose $y = (n^{-\frac{1}{p}}, n^{-\frac{1}{p}}, \ldots, n^{-\frac{1}{p}})^\top$. Then, $\|y\|_p = 1$, and hence, by Lemma 2.1 (ii), we have

$$\begin{aligned}\|F_{\mathcal{B}}(y)\|_p^p &= \sum_{i=1}^{n}\left|\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}(n^{-\frac{1}{p}})^{m-1}\right|^{\frac{p}{m-1}}\\ &= \sum_{i=1}^{n}\frac{1}{n}\left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}\right)^{\frac{p}{m-1}}\\ &> \frac{1}{n}\sum_{i=1}^{n}\left(n^{m-1}\beta_i(\mathcal{B})\right)^{\frac{p}{m-1}}\\ &= n^{p-1}\sum_{i=1}^{n}(\beta_i(\mathcal{B}))^{\frac{p}{m-1}}.\end{aligned}$$

Consequently,

$$n^{\frac{p-1}{p}}\left(\sum_{i=1}^{n}(\beta_i(\mathcal{B}))^{\frac{p}{m-1}}\right)^{\frac{1}{p}} < \frac{1}{\sqrt[p]{n}}\left(\sum_{i=1}^{n}\left(\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}\right)^{\frac{p}{m-1}}\right)^{\frac{1}{p}} \leq \|F_{\mathcal{B}}\|_p.$$

Next we show the right inequality. By Lemma 2.1 (i) together with the fact that $\|x\|_1 \leq n^{1-\frac{1}{p}}\|x\|_p$ for $p > 1$, we obtain

$$\begin{aligned}
\|F_{\mathcal{B}}\|_p^p &= \max_{\|x\|_p=1}\|F_{\mathcal{B}}(x)\|_p^p = \max_{\|x\|_p=1}\sum_{i=1}^{n}\left|\sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\cdots i_m}x_{i_2}x_{i_3}\cdots x_{i_m}\right|^{\frac{p}{m-1}} \\
&< \max_{\|x\|_p=1}\sum_{i=1}^{n}\left(b_{ii\cdots i}\sum_{i_2,\ldots,i_m=1}^{n}|x_{i_2}||x_{i_3}|\cdots|x_{i_m}|\right)^{\frac{p}{m-1}} \\
&= \max_{\|x\|_p=1}\sum_{i=1}^{n}\left(b_{ii\cdots i}\left(\sum_{k=1}^{n}|x_k|\right)^{m-1}\right)^{\frac{p}{m-1}} = \max_{\|x\|_p=1}\|x\|_1^p\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{p}{m-1}} \\
&\leq \max_{\|x\|_p=1}\left(n^{1-\frac{1}{p}}\|x\|_p\right)^p\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{p}{m-1}} \\
&= n^{p-1}\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{p}{m-1}}.
\end{aligned}$$

The desired conclusions follow.

□

Theorem 4.3 *Let $\mathcal{B}$ be an m-order n-dimensional B_0-tensor. Then,*

(i) $n^{\frac{m}{2}}\max_{i\in[n]}\beta_i(\mathcal{B}) \leq \|T_{\mathcal{B}}\|_\infty \leq n^{\frac{m}{2}}\max_{i\in[n]} b_{ii\cdots i}$;

(ii) $n^{\frac{mp-2}{2p}}\left(\sum_{i=1}^{n}(\beta_i(\mathcal{B}))^p\right)^{\frac{1}{p}} \leq \|T_{\mathcal{B}}\|_p \leq n^{\frac{mp-2}{2p}}\left(\sum_{i=1}^{n} b_{ii\cdots i}^p\right)^{\frac{1}{p}}$ *if* $p \geq 1$;

(iii) $n\,(\beta_i(\mathcal{B}))^{\frac{1}{m-1}} \leq \|F_{\mathcal{B}}\|_\infty \leq n\max_{i\in[n]} b_{ii\cdots i}^{\frac{1}{m-1}}$ *if m is even;*

(iv) $n^{\frac{p-1}{p}}\left(\sum_{i=1}^{n}(\beta_i(\mathcal{B}))^{\frac{p}{m-1}}\right)^{\frac{1}{p}} \leq \|F_{\mathcal{B}}\|_p \leq n^{\frac{p-1}{p}}\left(\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{p}{m-1}}\right)^{\frac{1}{p}}$ *if m is even and* $p \geq 1$.

For a B-tensor $\mathcal{B}$, it is obvious that the upper bounds of its operator norms are simpler in form than ones of Lemma 2.2, but we cannot compare their size relationship. By constructing following two examples, we show that there exists B-tensors such that the above upper bounds of $\|T_{\mathcal{B}}\|$ and $\|F_{\mathcal{B}}\|$ are smaller than the ones of Lemma 2.2.

Example 4.1 Let $\mathcal{B} = (b_{i_1i_2i_3i_4}) \in S_{4,3}$, where $b_{1111} = b_{3333} = 6$, $b_{2222} = 5$, $b_{1333} = b_{3133} = b_{3313} = b_{3331} = 1$, $b_{2322} = b_{2232} = b_{2223} = b_{3222} = 1.5$ and all other $b_{i_1i_2i_3i_4} = 2$. Then, $\mathcal{B}$ is a B-tensor.

Proof It is obvious that $\sum\limits_{i_2,i_3,i_4=1}^{3} b_{ii_2i_3i_4} > 0$ for $i = 1, 2, 3$.

For $i = 1$, we have

$$\frac{1}{n^{m-1}}\left(\sum_{i_2,i_3,i_4=1}^{3} b_{1i_2i_3i_4}\right) = \frac{1}{3^{4-1}} \times 57 = \frac{57}{27} > 2,$$

and hence, $\frac{1}{n^{m-1}}\left(\sum\limits_{i_2,i_3,i_4=1}^{3} b_{1i_2i_3i_4}\right) > b_{1j_2j_3j_4}$ for all $(j_2, j_3, j_4) \neq (1, 1, 1)$.

For $i = 2$, we have

$$\frac{1}{n^{m-1}}\left(\sum_{i_2,i_3,i_4=1}^{3} b_{2i_2i_3i_4}\right) = \frac{1}{3^{4-1}} \times 55.5 = \frac{55.5}{27} > 2,$$

and so, $\frac{1}{n^{m-1}}\left(\sum\limits_{i_2,i_3,i_4=1}^{n} b_{2i_2i_3i_4}\right) > b_{2j_2j_3j_4}$ for all $(j_2, j_3, j_4) \neq (2, 2, 2)$.

For $i = 3$, we have

$$\frac{1}{n^{m-1}}\left(\sum_{i_2,i_3,i_4=1}^{3} b_{3i_2i_3i_4}\right) = \frac{1}{3^{4-1}} \times 54.5 = \frac{54.5}{27} > 2,$$

and so, $\frac{1}{n^{m-1}}\left(\sum\limits_{i_2,i_3,i_4=1}^{n} b_{3i_2i_3i_4}\right) > b_{3j_2j_3j_4}$ for all $(j_2, j_3, j_4) \neq (3, 3, 3)$.

Thus, $\mathcal{B}$ is a B-tensor. □

Clearly, for the upper bounds of $\|T_{\mathcal{B}}\|_\infty$, we have

$$n^{\frac{m}{2}} \max_{i\in[n]} b_{iiii} = 3^{\frac{4}{2}} \times 6 = 54 \text{ and } \max_{i\in[n]}\left(\sum_{i_2,i_3,i_4=1}^{n} |b_{ii_2i_3i_4}|\right) = 57,$$

and hence, $n^{\frac{m}{2}} \max\limits_{i\in[n]} b_{iiii} < \max\limits_{i\in[n]}\left(\sum\limits_{i_2,i_3,i_4=1}^{n} |b_{ii_2i_3i_4}|\right)$.

It is obvious that for the upper bounds of $\|F_{\mathcal{B}}\|_p$ and $p = 1$,

$$3^{\frac{p-1}{p}}\left(\sum_{i=1}^{3} b_{iiii}^{\frac{p}{m-1}}\right)^{\frac{1}{p}} < \left(\sum_{i=1}^{3}\left(\sum_{i_2,i_3,i_4=1}^{3} b_{ii_2i_3i_4}\right)^{\frac{p}{m-1}}\right)^{\frac{1}{p}}.$$

Example 4.2 Let $\mathcal{B} = (b_{i_1 i_2 i_3 i_m}) \in S_{4,4}$, where $b_{1111} = b_{2222} = b_{3333} = b_{4444} = 3$, $b_{1444} = b_{4144} = b_{4414} = b_{4441} = 0.7$, $b_{2333} = b_{3233} = b_{3323} = b_{3332} = 0.5$ and all other $b_{i_1 i_2 i_3 i_4} = 1$. Then, $\mathcal{B}$ is a B-tensor.

Proof It is obvious that $\sum_{i_2,i_3,i_4=1}^{4} b_{i i_2 i_3 i_4} > 0$ for $i = 1, 2, 3, 4$.

For $i = 1$,

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,i_3,i_4=1}^{n} b_{1 i_2 i_3 i_4} \right) = \frac{1}{4^{4-1}} \times 65.7 = \frac{65.7}{64} > 1.$$

So we have $\frac{1}{n^{m-1}} \left(\sum_{i_2,i_3,i_4=1}^{n} b_{1 i_2 i_3 i_4} \right) > b_{1 j_2 j_3 j_4}$ for all $(j_2, j_3, j_4) \neq (1, 1, 1)$.

For $i = 2$,

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,i_3,i_4=1}^{n} b_{2 i_2 i_3 i_4} \right) = \frac{1}{4^{4-1}} \times 65.5 = \frac{65.5}{64} > 1.$$

So we have $\frac{1}{n^{m-1}} \left(\sum_{i_2,i_3,i_4=1}^{n} b_{2 i_2 i_3 i_4} \right) > b_{2 j_2 j_3 j_4}$ for all $(j_2, j_3, j_4) \neq (2, 2, 2)$.

For $i = 3$,

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,i_3,i_4=1}^{n} b_{3 i_2 i_3 i_4} \right) = \frac{1}{4^{4-1}} \times 64.5 = \frac{64.5}{64} > 1.$$

So we have $\frac{1}{n^{m-1}} \left(\sum_{i_2,i_3,i_4=1}^{n} b_{3 i_2 i_3 i_4} \right) > b_{3 j_2 j_3 j_4}$ for all $(j_2, j_3, j_4) \neq (3, 3, 3)$.

For $i = 4$

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,i_3,i_4=1}^{n} b_{4 i_2 i_3 i_4} \right) = \frac{1}{4^{4-1}} \times 65.1 = \frac{65.1}{64} > 1.$$

So we have $\frac{1}{n^{m-1}} \left(\sum_{i_2,i_3,i_4=1}^{n} b_{4 i_2 i_3 i_4} \right) > b_{4 j_2 j_3 j_4}$ for all $(j_2, j_3, j_4) \neq (4, 4, 4)$.

Therefore, $\mathcal{B}$ is a B-tensor. □

It is obvious that for the upper bounds of $\|T_{\mathcal{B}}\|_p$,

$$n^{\frac{mp-2}{2p}} \left(\sum_{i=1}^{n} b_{iiii}^{p} \right)^{\frac{1}{p}} = 4^{\frac{4p-2}{2p}} \times (3^p \times 4)^{\frac{1}{p}} = 4^{\frac{2p-1}{p}} \times 3 \times 4^{\frac{1}{p}} = 48$$

and

$$n^{\frac{m-2}{p}}\left(\sum_{i=1}^{n}\left(\sum_{i_2,i_3,i_4=1}^{n}|b_{ii_2i_3i_4}|\right)^{p}\right)^{\frac{1}{p}} = 4^{\frac{4-2}{p}} \times (65.7^p + 65.5^p + 64.5^p + 65.1^p)^{\frac{1}{p}}$$
$$> 4^{\frac{4-2}{p}} \times (64^p \times 4)^{\frac{1}{p}}$$
$$= 64 \times 4^{\frac{3}{p}}.$$

Since $4^{\frac{3}{p}} > 1$, we have $n^{\frac{mp-2}{2p}}\left(\sum_{i=1}^{n} b_{iiii}^{p}\right)^{\frac{1}{p}} < n^{\frac{m-2}{p}}\left(\sum_{i=1}^{n}\left(\sum_{i_2,i_3,i_4=1}^{n}|b_{ii_2i_3i_4}|\right)^{p}\right)^{\frac{1}{p}}$.

Theorem 4.4 *Let $\mathcal{B}$ be a B-tensor. Then,*

(i) $|\lambda| < \left(\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{1}{m-1}}\right)^{m-1}$ *for all eigenvalues λ of $\mathcal{B}$ if m is even;*

(ii) $|\mu| < n^{\frac{m-1}{2}}\left(\sum_{i=1}^{n} b_{ii\cdots i}^{2}\right)^{\frac{1}{2}}$ *for all E-eigenvalues μ of $\mathcal{B}$.*

Proof (i) From the definition of the operator $F_{\mathcal{B}}$, it follows that λ is an eigenvalue of $\mathcal{B}$ if and only if $\lambda^{\frac{1}{m-1}}$ is an eigenvalue of $F_{\mathcal{B}}$. By Theorem 4.2 of Song and Qi [38], we have

$$|\lambda|^{\frac{1}{m-1}} \leq \|F_{\mathcal{B}}\|_1, \text{ i.e., } |\lambda| \leq \|F_{\mathcal{B}}\|_1^{m-1}.$$

By Theorem 4.2 (ii), we have

$$\|F_{\mathcal{B}}\|_1 < \sum_{i=1}^{n} b_{ii\cdots i}^{\frac{1}{m-1}}.$$

So we obtain

$$|\lambda| \leq \|F_{\mathcal{B}}\|_1^{m-1} < \left(\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{1}{m-1}}\right)^{m-1}.$$

(ii) From the definition of the operator $T_{\mathcal{B}}$, it follows that μ is an E-eigenvalue of $\mathcal{B}$ if and only if μ is an eigenvalue of $T_{\mathcal{B}}$. By Theorem 4.2 of Song and Qi [38], we have

$$|\mu| \leq \|T_{\mathcal{B}}\|_\infty \text{ and } |\mu| \leq \|T_{\mathcal{B}}\|_2.$$

By Theorem 4.1, we obtain

$$\|T_{\mathcal{B}}\|_\infty < n^{\frac{m}{2}} \max_{i\in[n]} b_{ii\cdots i} \text{ and } \|T_{\mathcal{B}}\|_2 < n^{\frac{m-1}{2}}\left(\sum_{i=1}^{n} b_{ii\cdots i}^{2}\right)^{\frac{1}{2}}.$$

Then, we have

$$|\mu| < n^{\frac{m}{2}} \max_{i\in[n]} b_{ii\cdots i} \text{ and } |\mu| < n^{\frac{m-1}{2}} \left(\sum_{i=1}^{n} b_{ii\cdots i}^2\right)^{\frac{1}{2}}.$$

So, the desired conclusion follows. □

Similarly, we easily show the following.

Theorem 4.5 *Let $\mathcal{B}$ be a B_0-tensor. Then,*

(i) $|\lambda| \le \left(\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{1}{m-1}}\right)^{m-1}$ *for all eigenvalues λ of $\mathcal{B}$ if m is even;*

(ii) $|\mu| \le n^{\frac{m-1}{2}} \left(\sum_{i=1}^{n} b_{ii\cdots i}^2\right)^{\frac{1}{2}}$ *for all E-eigenvalues μ of $\mathcal{B}$.*

For the tensor complementarity problem with a B-tensor $\mathcal{B}$, denoted by $TCP(\mathcal{B}, q)$, we show the lower bound of the solution set of $TCP(\mathcal{B}, q)$.

Theorem 4.6 *Let $\mathcal{B}$ be a B-tensor. Assume that x is a nonzero solution of $TCP(\mathcal{B}, q)$ and $y_+ = (\max\{y_1, 0\}, \max\{y_2, 0\}, \ldots, \max\{y_n, 0\})^\top$. Then,*

(i) $\dfrac{\|(-q)_+\|_\infty}{n^{m-1} \max_{i\in[n]} b_{ii\cdots i}} < \|x\|_\infty^{m-1}$;

(ii) $\dfrac{\|(-q)_+\|_2}{n^{\frac{m-1}{2}} \left(\sum_{i=1}^{n} b_{ii\cdots i}^2\right)^{\frac{1}{2}}} < \|x\|_2^{m-1}$;

(iii) $\dfrac{\|(-q)_+\|_m}{n^{\frac{(m-1)^2}{m}} \left(\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}}} < \|x\|_m^{m-1}$ *if m is even.*

Proof Theorem 3.1 implies that each B-tensor is strictly semi-positive. From Theorem 5 of Song and Qi [23], it follows that

$$\frac{\|(-q)_+\|_\infty}{n^{\frac{m-2}{2}} \|T_\mathcal{A}\|_\infty} \le \|x\|_\infty^{m-1} \text{ and } \frac{\|(-q)_+\|_2}{\|T_\mathcal{A}\|_2} \le \|x\|_2^{m-1}. \tag{9}$$

By Theorem 4.1, we have

$$\|T_\mathcal{A}\|_\infty < n^{\frac{m}{2}} \max_{i\in[n]} b_{ii\cdots i} \text{ and } \|T_\mathcal{A}\|_2 < n^{\frac{m-1}{2}} \left(\sum_{i=1}^{n} b_{ii\cdots i}^2\right)^{\frac{1}{2}}. \tag{10}$$

Then, combining the inequalities (9) and (10), the conclusions (i) and (ii) are proved.

Similarly, we may show (iii). In fact, if m is even, Theorem 5 of Song and Qi [23] implies

$$\frac{\|(-q)_+\|_m}{\|F_{\mathcal{A}}\|_m^{m-1}} \leq \|x\|_m^{m-1}. \tag{11}$$

By Theorem 4.2, we have

$$\|F_{\mathcal{A}}\|_m < n^{\frac{m-1}{m}} \left(\sum_{i=1}^{n} b_{ii\cdots i}^{\frac{m}{m-1}} \right)^{\frac{1}{m}}. \tag{12}$$

So, the desired conclusion follows by combining the inequalities (11) and (12). □

5 Conclusions

In this paper, we show that each B-tensor is strictly semi-positive. We obtain the strictly lower bounds of the solution set of tensor complementarity problem with B-tensors. We establish the upper bounds of spectral radius and E-spectral radius of B-tensors. The following topics are worth of further and serious consideration.

– For two positively homogeneous operators induced by B-tensors, whether or not their operator norms may be found exactly.
– Do there exist similar bounds of the operator norm associated with some other structured tensors such as Z-tensors, H-tensors, M-tensors and so on?
– Are the upper bounds and lower bounds best in this paper?
– How to design an effective algorithm to compute the upper or lower bounds?

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