J Optim Theory Appl (2018) 176:74–93
https://doi.org/10.1007/s10957-017-1199-8

Approximate Optimality and Approximate Duality for Quasi Approximate Solutions in Robust Convex Semidefinite Programs

Liguo Jiao[1] · **Jae Hyoung Lee**[1]

Received: 11 September 2016 / Accepted: 16 November 2017 / Published online: 27 November 2017

Abstract In this paper, we study quasi approximate solutions for a convex semidefinite programming problem in the face of data uncertainty. Using the robust optimization approach (worst-case approach), approximate optimality conditions and approximate duality theorems for quasi approximate solutions in robust convex semidefinite programming problems are explored under the robust characteristic cone constraint qualification. Moreover, some examples are given to illustrate the obtained results.

Keywords Robust convex semidefinite programming problems · Quasi approximate solutions · Robust characteristic cone constraint qualification · Approximate optimality conditions · Approximate duality theorems

Mathematics Subject Classification 90C22 · 90C46 · 90C31

1 Introduction

A convex semidefinite optimization problem has been recognized as a valuable modeling tool for control theory analysis and design and for many optimization problems based on human activities [1–4]. In particular, a semidefinite linear programming problem in the absence of data uncertainty also has attracted attention of a great num-

Communicated by Evgeni A. Nurminski.

✉ Jae Hyoung Lee
mc7558@naver.com

Liguo Jiao
hanchezi@163.com

1 Department of Applied Mathematics, Pukyong National University, Busan, Republic of Korea

ber of researchers in the past years. Moreover, recently the importance of solving semidefinite programs under data uncertainty has attracted a great deal of attention on identifying and obtaining (uncertainty-immunized) robust solutions of uncertain semidefinite programs [1,5–8].

As we know, there are mainly two approaches, i.e., the *robust optimization approach* and the *stochastic optimization approach*, dealing with mathematical programming problems under data uncertainty. The stochastic optimization approach to optimization problems under data uncertainty starts by assuming that the uncertain data have a probabilistic description that constraints are required to be satisfied up to prescribed level of probability [9], while the robust optimization approach examines a solution which simultaneously satisfies all possible realizations of the constraints. It is worth mentioning that the robust optimization approach to examining semidefinite linear programming problems under data uncertainty developed by Jeyakumar and Li [10], is to treat uncertain data as deterministic via uncertain sets which are *closed* and *convex*.

On the other hand, the fact that it may not be always possible to find the point of minimizer in optimization programming problems leads to the notion of approximate solutions (such as approximate solution, quasi approximate solution, regular approximate solution, and so on), which play an important role in algorithmic study of optimization problems. It is worth mentioning that among them, the notion of quasi approximate solutions first introduced by Loridan [11] is motivated by the well-known Ekeland's variational principle [12]. Many researchers have studied the approximate solutions in convex/nonconvex optimization programming problems and have established approximate necessary conditions under different suitable constraint qualifications, see [13–18] and the references therein, for example. In particular, in [13] the authors introduced a notion, i.e., the *modified approximate KKT point*, which is also motivated by Ekeland's variational principle [12]; moreover, they pointed out that the proposed KKT-proximity measure could be used as a termination condition to optimization algorithms. Besides, Lee and Lee [15] and Lee and Jiao [16] studied the robust convex programming problem for approximate solutions and quasi approximate solutions in the face of data uncertainty, respectively, and the latter paper analyzed the differences between approximate solutions and quasi approximate solutions in robust convex programs with the help of some simple examples.

However, as far as we know, until now it seems that no results focus on the quasi approximate solution for robust convex semidefinite optimization problems in spite of the fact that Lee and Jiao [16] obtained some results on quasi approximate solutions for convex optimization problems under data uncertainty. Therefore, it is worth exploring some properties of quasi approximate solutions for robust convex semidefinite optimization problems based on the special structure of semidefinite programs. This research paper focuses on studying quasi approximate solutions in convex semidefinite programming problems under data uncertainty.

The organization of this paper is as follows. Section 2 states some preliminaries. In Sect. 3, the robust version of Farkas's lemma for the convex semidefinite programming is given, and then, two approximate optimality conditions for quasi approximate solutions in robust convex semidefinite optimization problems are presented under the *robust characteristic cone constraint qualification* and the *weakened constraint* *qualification*, respectively. In Sect. 4, we propose the dual problem for the primal one, whereafter weak duality and strong duality are explored. Moreover, several examples are given throughout this article to illustrate the obtained results.

2 Preliminaries

This section gives some notations and preliminary results which will be used throughout the paper. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with standard Euclidean norm, that is, $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_i \geqq 0\}$. The inner product in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$ for all $x, y \in \mathbb{R}^n$. We say that a set A in $\mathbb{R}^n$ is convex whenever $ta_1 + (1-t)a_2 \in A$ for all $t \in [0,1]$, $a_1, a_2 \in A$. Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, +\infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$, and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. The domain and the epigraph of f are, respectively, defined by $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leqq r\}$. A proper function f is said to be convex if for all $t \in [0,1]$,

$$f((1-t)x + ty) \leqq (1-t)f(x) + tf(y)$$

for all $x, y \in \mathbb{R}^n$. For a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) := \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leqq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, \text{ if } x \in \operatorname{dom} f, \\ \emptyset, \text{ otherwise.} \end{cases}$$

In addition, for any $\epsilon \geqq 0$, the ϵ-subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) := \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leqq f(y) - f(x) + \epsilon, \forall y \in \mathbb{R}^n\}, \text{if } x \in \operatorname{dom} f, \\ \emptyset, \text{ otherwise.} \end{cases}$$

A function f is said to be lower semicontinuous if $\liminf_{y\to x} f(y) \geqq f(x)$ for all $x \in \mathbb{R}^n$. As usual, for any proper convex function f on $\mathbb{R}^n$, its conjugate function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) = \sup\left\{\langle x^*, x\rangle - f(x) : x \in \mathbb{R}^n\right\}$ for all $x^* \in \mathbb{R}^n$. For a given set $A \subset \mathbb{R}^n$, we denote the closure and the convex hull generated by A by $\operatorname{cl} A$ and $\operatorname{conv} A$, respectively. The indicator function δ_A of a subset A of $\mathbb{R}^n$ is defined by $\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$

Let S^n be the set of $n \times n$ symmetric matrices. For $X \in S^n$, X is said to be positive semidefinite (denoted by $X \succeq 0$) if $v^T X v \geqq 0$ for any $v \in \mathbb{R}^n$, and X is said to be positive definite (denoted by $X \succ 0$) if $v^T X v > 0$ for any $v (\neq 0) \in \mathbb{R}^n$. The set of $n \times n$ positive semidefinite and positive definite matrices are denoted by S^n_+ and S^n_{++}, respectively. For $X, Y \in S^n$, the inner product in S^n is defined by $\langle X, Y\rangle := \operatorname{tr}[XY]$, where $\operatorname{tr}[\cdot]$ is the trace operation.

The following proposition, which describes the relationship between the epigraph of a conjugate function and the ϵ-subdifferential and plays a key role in deriving the main results, is given in [19].

Proposition 2.1 [19] *If* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is a proper lower semicontinuous convex function and if* $a \in \operatorname{dom} f$, *then*

$$\operatorname{epi} f^* = \bigcup_{\epsilon \geqq 0} \{(v, \langle v, a\rangle + \epsilon - f(a)) : v \in \partial_\epsilon f(a)\}.$$

The following two propositions will also be used in our later analysis.

Proposition 2.2 [20] *Let* $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semicontinuous convex functions. If* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, *then*

$$\operatorname{epi}(f+g)^* = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} g^*).$$

Moreover, if one of the functions f *and* g *is continuous, then*

$$\operatorname{epi}(f+g)^* = \operatorname{epi} f^* + \operatorname{epi} g^*.$$

Proposition 2.3 [21,22] *Let* $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$ (*where* I *is an arbitrary index set*), *be proper lower semicontinuous convex functions. Suppose that there exists* $x_0 \in \mathbb{R}^n$ *such that* $\sup_{i\in I} g_i(x_0) < +\infty$. *Then,*

$$\operatorname{epi}(\sup_{i\in I} g_i)^* = \operatorname{cl}\left(\operatorname{conv} \bigcup_{i\in I} \operatorname{epi} g_i^*\right).$$

3 Approximate Optimality Theorems

Consider the standard form of a convex semidefinite programming problem [4]:

$$\text{(SDP)} \qquad \min f(x) \quad \text{s.t.} \quad A_0 + \sum_{i=1}^m x_i A_i \succeq 0,$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function, and A_i, $i = 0, 1, \ldots, m$, are $n \times n$ symmetric matrices.

The convex semidefinite programming problem in the face of data uncertainty in the constraints can be captured by the problem

$$\text{(USDP)} \qquad \min f(x) \quad \text{s.t.} \quad A_0 + \sum_{i=1}^m x_i A_i \succeq 0,$$

where for each $i = 0, 1, \ldots, m$, A_i is uncertain and belongs to an uncertain set $\mathcal{V}_i$ which is defined by $\mathcal{V}_i := \{A_i^0 + \sum_{j=1}^l u_i^j A_i^j : (u_i^1, \ldots, u_i^l) \in \mathcal{U}_i\}$, where for each $i = 0, 1, \ldots, m$, $\mathcal{U}_i$ is a compact convex set in $\mathbb{R}^l$, and A_i^j, $j = 0, 1, \ldots, l$, are $n \times n$

symmetric matrices. For the worst case of (USDP), the robust counterpart of (USDP) is given as follows [23,24]:

$$\text{(RSDP)} \qquad \min f(x) \quad \text{s.t.} \quad A_0 + \sum_{i=1}^{m} x_i A_i \succeq 0, \ \forall A_i \in \mathcal{V}_i, \ i \in I,$$

where the index set $I := \{0, 1, 2, \ldots, m\}$. Let F be the feasible set of (RSDP), where

$$F := \left\{ x \in \mathbb{R}^m : A_0 + \sum_{i=1}^{m} x_i A_i \succeq 0, \ \forall A_i \in \mathcal{V}_i, \ i \in I \right\}.$$

Also, we use D to denote the robust characteristic cone as follows:

$$D := \bigcup_{\substack{A_i \in \mathcal{V}_i, i \in I \\ Z \succeq 0, \delta \in \mathbb{R}_+}} \left\{ (-\text{tr}\,[ZA_1], \ldots, -\text{tr}\,[ZA_m], \text{tr}\,[ZA_0] + \delta) \right\}.$$

Indeed, D is a cone in $\mathbb{R}^{m+1}$ [10]; moreover, applying Proposition 2.3, we can easily see that

$$\text{epi}\,\delta_F^* = \text{cl conv}\, D$$

whenever $\mathcal{V}_i \subset S^n$ is compact, and $F \neq \emptyset$.

Now, we give the following lemma which is the *robust version of Farkas's lemma* for the convex semidefinite programming. It can be straightforwardly obtained from Theorem 6.5 in [25] (see also [26]).

Lemma 3.1 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and $A_i \in S^n$, $i \in I$. Assume that for each $i \in I$, $\mathcal{V}_i \subset S^n$ is compact, and the feasible set $F \neq \emptyset$. Then, the following statements are equivalent*:

(i) *the robust characteristic cone D is closed and convex;*
(ii) $F \subset \{x \in \mathbb{R}^m : f(x) \geqq 0\} \ \Leftrightarrow \ (0, 0) \in \text{epi}\, f^* + D$.

Remark 3.1 Under the same assumptions as in Lemma 3.1, we can also easily see that

$$F \subset \{x \in \mathbb{R}^m : f(x) \geqq 0\} \ \Leftrightarrow \ (0, 0) \in \text{epi}\, f^* + \text{cl conv}\, D.$$

For the convex semidefinite programming problem (SDP) in the absence of data uncertainty, the characteristic cone D is convex whenever $A_i \in S^n$, $i \in I$. However, in general, the robust characteristic cone D is not convex. Jeyakumar and Li [10] provided a sufficient condition for the convexity of the robust characteristic cone D under some suitable assumptions (see Proposition 3.1).

Proposition 3.1 [10] *For each $i \in I$, let $A_i \in \mathcal{V}_i := \{A_i^0 + \sum_{j=1}^{l} u_i^j A_i^j : (u_i^1, \ldots, u_i^l) \in \mathcal{U}_i\}$, where $\mathcal{U}_i$ is a compact convex set in $\mathbb{R}^l$, $A_0^j \in S^n$ and $A_i^j \in S_+^n$, $i = 1, \ldots, m$, $j = 1, \ldots, l$. Then, the robust characteristic cone D is a convex subset of $\mathbb{R}^{m+1}$.*

Jeyakumar and Li [10] showed the closedness of the robust characteristic cone D under the robust Slater condition, we state this result in the following for convenience.

Proposition 3.2 [20] *Let $\mathcal{V}_i \subseteq S^n$, $i \in I$, be compact and convex. Assume that the robust Slater condition holds, that is, $\{x \in \mathbb{R}^m : A_0 + \sum_{i=1}^m x_i A_i \succ 0,\ \forall A_i \in \mathcal{V}_i,\ i \in I\} \neq \emptyset$. Then, the robust characteristic cone D is closed.*

Remark 3.2 We say a *robust characteristic cone constraint qualification* holds if the robust characteristic cone D is closed and convex.

Definition 3.1 Let $\epsilon \geqq 0$ be given, then $\bar{x}$ is said to be

(i) an ϵ-solution of (RSDP) if $f(\bar{x}) \leqq f(x) + \epsilon$ for all $x \in F$;
(ii) a quasi ϵ-solution of (RSDP) if $f(\bar{x}) \leqq f(x) + \sqrt{\epsilon}\|x - \bar{x}\|$ for all $x \in F$;
(iii) a regular ϵ-solution of (RSDP) if $\bar{x}$ is an ϵ-solution as well as a quasi ϵ-solution of (RSDP).

Remark 3.3 For the differences of ϵ-solution, quasi ϵ-solution, and regular ϵ-solution, one can see [16]. If $\epsilon = 0$, then both ϵ-solution and quasi ϵ-solution $\bar{x}$ deduce to be an exact optimal solution (if exists) of (RSDP). This paper mainly focuses on studying the characterizations of quasi ϵ-solutions of (RSDP).

Now, we give the following approximate optimality theorem under the robust characteristic cone constraint qualification.

Theorem 3.1 (Approximate Optimality Theorem) *Let $\bar{x} \in F$, and let $\epsilon \geqq 0$ be given. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and let $A_i^j \in S^n$, $i \in I$, $j = 0, 1, \ldots, l$. Let $\mathcal{U}_i$ be a compact set in $\mathbb{R}^l$ for each $i \in I$. Suppose that the robust characteristic cone D is closed and convex. Then, the following statements are equivalent*:

(i) *$\bar{x}$ is a quasi ϵ-solution of* (RSDP);
(ii) *there exist $\bar{Z} \in S_+^n$, $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$, and $\bar{\delta} \in \mathbb{R}_+$ such that*

$$\begin{aligned}&(0, -\sqrt{\epsilon}\|\bar{x}\| - f(\bar{x}))\\ &\in \operatorname{epi} f^* + \left(-\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, -\operatorname{tr}[\bar{Z}\bar{A}_m], \operatorname{tr}[\bar{Z}\bar{A}_0] + \bar{\delta}\right) + \sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+;\end{aligned}$$

(iii) *there exist $\bar{Z} \in S_+^n$ and $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$ such that*

$$\begin{aligned}&0 \in \partial f(\bar{x}) - \left(\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, \operatorname{tr}[\bar{Z}\bar{A}_m]\right) + \sqrt{\epsilon}\mathbb{B} \ \text{ and}\\ &\operatorname{tr}\left[\bar{Z}\left(\bar{A}_0 + \sum_{i=1}^m \bar{x}_i \bar{A}_i\right)\right] = 0,\end{aligned}$$

where $\bar{A}_i = A_i^0 + \sum_{j=1}^l \bar{u}_i^j A_i^j \in \mathcal{V}_i$, $i \in I$, and $\mathbb{B}$ is the unit ball in $\mathbb{R}^m$.

Proof Assume that $\bar{x}$ is a quasi ϵ-solution of (RSDP), that is, $f(x) + \sqrt{\epsilon}\|x - \bar{x}\| \geqq f(\bar{x})$ for any $x \in F$. It means that $F \subseteq \{x \in \mathbb{R}^m : f(x) + \sqrt{\epsilon}\|x - \bar{x}\| - f(\bar{x}) \geqq 0\}$. Let

$\phi(x) := f(x) + \sqrt{\epsilon}\|x - \bar{x}\| - f(\bar{x})$. It follows from the assumptions for D and Lemma 3.1 that

$$(0,0) \in \operatorname{epi}\phi^* + \bigcup_{\substack{A_i \in \mathcal{V}_i, i \in I \\ Z \succeq 0, \delta \in \mathbb{R}_+}} \left\{ (-\operatorname{tr}[ZA_1], \ldots, -\operatorname{tr}[ZA_m], \operatorname{tr}[ZA_0] + \delta) \right\},$$

which implies that there exist $\bar{Z} \in S^n_+$, $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$, and $\bar{\delta} \in \mathbb{R}_+$ such that

$$(0,0) \in \operatorname{epi}\phi^* + \left(-\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, -\operatorname{tr}[\bar{Z}\bar{A}_m], \operatorname{tr}[\bar{Z}\bar{A}_0] + \bar{\delta}\right), \tag{1}$$

where $\bar{A}_i = A_i^0 + \sum_{j=1}^l \bar{u}_i^j A_i^j \in \mathcal{V}_i$, $i \in I$.

Now, we claim that $\operatorname{epi}\phi^* = \operatorname{epi} f^* + \sqrt{\epsilon}\mathbb{B} \times [\sqrt{\epsilon}\|\bar{x}\| + f(\bar{x}), +\infty)$. By Proposition 2.2,

$$\operatorname{epi}\phi^* = \operatorname{epi}(f + \sqrt{\epsilon}\| \cdot - \bar{x}\| - f(\bar{x}))^* = \operatorname{epi} f^* + \operatorname{epi}(\sqrt{\epsilon}\| \cdot - \bar{x}\| - f(\bar{x}))^*. \tag{2}$$

Since

$$(\sqrt{\epsilon}\| \cdot - \bar{x}\| - f(\bar{x}))^*(a) = \begin{cases} \sqrt{\epsilon}\|\bar{x}\| + f(\bar{x}), & \text{if } \|a\| \leqq \sqrt{\epsilon}, \\ +\infty, & \text{otherwise,} \end{cases}$$

along with (2), we have,

$$\operatorname{epi}\phi^* = \operatorname{epi} f^* + \sqrt{\epsilon}\mathbb{B} \times [\sqrt{\epsilon}\|\bar{x}\| + f(\bar{x}), +\infty).$$

It follows from (1) that there exist $\bar{Z} \in S^n_+$, $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$, and $\bar{\delta} \in \mathbb{R}_+$ such that

$$\begin{aligned} &(0, -\sqrt{\epsilon}\|\bar{x}\| - f(\bar{x})) \\ &\in \operatorname{epi} f^* + \left(-\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, -\operatorname{tr}[\bar{Z}\bar{A}_m], \operatorname{tr}[\bar{Z}\bar{A}_0] + \bar{\delta}\right) + \sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+, \end{aligned}$$

where $\bar{A}_i = A_i^0 + \sum_{j=1}^l \bar{u}_i^j A_i^j \in \mathcal{V}_i$, $i \in I$. Thus, (i) implies (ii).

Now, we assume that (ii) holds. Then, there exist $\bar{Z} \in S^n_+$, $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$, and $\bar{\delta} \in \mathbb{R}_+$ such that

$$\begin{aligned} &(0, -\sqrt{\epsilon}\|\bar{x}\| - f(\bar{x})) \\ &\in \operatorname{epi} f^* + \left(-\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, -\operatorname{tr}[\bar{Z}\bar{A}_m], \operatorname{tr}[\bar{Z}\bar{A}_0] + \bar{\delta}\right) + \sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+, \end{aligned}$$

where $\bar{A}_i = A_i^0 + \sum_{j=1}^l \bar{u}_i^j A_i^j \in \mathcal{V}_i$, $i \in I$. So, by Proposition 2.1,

$$\begin{aligned} (0, -\sqrt{\epsilon}\|\bar{x}\| - f(\bar{x})) \in \bigcup_{\epsilon_0 \geqq 0} &\{(\xi_0, \langle \xi_0, \bar{x}\rangle + \epsilon_0 - f(\bar{x})) : \xi_0 \in \partial_{\epsilon_0} f(\bar{x})\} + \\ &\left(-\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, -\operatorname{tr}[\bar{Z}\bar{A}_m], \operatorname{tr}[\bar{Z}\bar{A}_0] + \bar{\delta}\right) + \sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+, \end{aligned}$$

which means that there exist $\bar{\xi}_0 \in \partial_{\bar{\epsilon}_0} f(\bar{x})$, $\bar{\epsilon}_0 \geqq 0$, $\bar{b} \in \mathbb{B}$, and $\bar{r} \in \mathbb{R}_+$ such that

$$0 = \bar{\xi}_0 - \left(\mathrm{tr}\,[\bar{Z}\bar{A}_1], \ldots, \mathrm{tr}\,[\bar{Z}\bar{A}_m]\right) + \sqrt{\epsilon}\bar{b} \;\text{ and}$$
$$-\sqrt{\epsilon}\|\bar{x}\| = \langle \bar{\xi}_0, \bar{x}\rangle + \bar{\epsilon}_0 + \mathrm{tr}\,[\bar{Z}\bar{A}_0] + \bar{\delta} + \bar{r}.$$

So, there exist $\bar{\xi}_0 \in \partial_{\bar{\epsilon}_0} f(\bar{x})$, $\bar{\epsilon}_0 \geqq 0$, $\bar{b} \in \mathbb{B}$, and $\bar{r} \in \mathbb{R}_+$ such that

$$\bar{\xi}_0 = \left(\mathrm{tr}\,[\bar{Z}\bar{A}_1], \ldots, \mathrm{tr}\,[\bar{Z}\bar{A}_m]\right) - \sqrt{\epsilon}\bar{b} \;\text{ and}$$
$$0 \leqq \bar{\epsilon}_0 \leqq \sqrt{\epsilon}\|\bar{x}\| - \sqrt{\epsilon}\langle \bar{b}, \bar{x}\rangle + \bar{\epsilon}_0 + \bar{\delta} + \bar{r} = -\mathrm{tr}\left[\bar{Z}\left(\bar{A}_0 + \sum_{i=1}^m \bar{x}_i\bar{A}_i\right)\right] \leqq 0,$$

i.e., $\bar{\epsilon}_0 = 0$. Hence, there exist $\bar{Z} \in S^n_+$ and $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$ such that

$$0 \in \partial f(\bar{x}) - \left(\mathrm{tr}\,[\bar{Z}\bar{A}_1], \ldots, \mathrm{tr}\,[\bar{Z}\bar{A}_m]\right) + \sqrt{\epsilon}\mathbb{B} \;\text{ and tr}\left[\bar{Z}\left(\bar{A}_0 + \sum_{i=1}^m \bar{x}_i\bar{A}_i\right)\right] = 0,$$

where $\bar{A}_i = A_i^0 + \sum_{j=1}^l \bar{u}_i^j A_i^j \in \mathcal{V}_i$, $i \in I$. Thus, (ii) implies (iii).

Finally, we will prove that (iii) implies (i). Assume that there exist $\bar{Z} \in S^n_+$ and $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$ such that

$$0 \in \partial f(\bar{x}) - \left(\mathrm{tr}\,[\bar{Z}\bar{A}_1], \ldots, \mathrm{tr}\,[\bar{Z}\bar{A}_m]\right) + \sqrt{\epsilon}\mathbb{B} \text{ and tr}\left[\bar{Z}\left(\bar{A}_0 + \sum_{i=1}^m \bar{x}_i\bar{A}_i\right)\right] = 0.$$

Then, there exist $\xi_0 \in \partial f(\bar{x})$, $\bar{Z} \in S^n_+$, $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$, and $b \in \mathbb{B}$ such that

$$\xi_0 = \left(\mathrm{tr}\,[\bar{Z}\bar{A}_1], \ldots, \mathrm{tr}\,[\bar{Z}\bar{A}_m]\right) - \sqrt{\epsilon}b \text{ and tr}\left[\bar{Z}(\bar{A}_0 + \sum_{i=1}^m \bar{x}_i\bar{A}_i)\right] = 0. \quad (3)$$

By the definition of the subdifferential of f at $\bar{x}$, we have, for any $x \in \mathbb{R}^m$,

$$f(x) - f(\bar{x}) \geqq \langle \xi_0, x - \bar{x}\rangle. \quad (4)$$

Combining (3) and (4), it follows that, for any $x \in \mathbb{R}^m$,

$$\begin{aligned} f(x) - f(\bar{x}) &\geqq \langle\left(\mathrm{tr}\,[\bar{Z}\bar{A}_1], \ldots, \mathrm{tr}\,[\bar{Z}\bar{A}_m]\right), x - \bar{x}\rangle - \sqrt{\epsilon}\langle b, x - \bar{x}\rangle \\ &\geqq \mathrm{tr}\left[\bar{Z}\left(\sum_{i=1}^m x_i\bar{A}_i\right)\right] - \mathrm{tr}\left[\bar{Z}\left(\sum_{i=1}^m \bar{x}_i\bar{A}_i\right)\right] - \sqrt{\epsilon}\|b\| \cdot \|x - \bar{x}\| \\ &= \mathrm{tr}\left[\bar{A}_0 + \bar{Z}\left(\sum_{i=1}^m x_i\bar{A}_i\right)\right] - \sqrt{\epsilon}\|b\| \cdot \|x - \bar{x}\|. \end{aligned}$$

Since for any $x \in F$, $\operatorname{tr}[\bar{A}_0 + \bar{Z}(\sum_{i=1}^m x_i \bar{A}_i)] \geqq 0$ and $\|b\| \leqq 1$, the above inequality yields

$$f(x) - f(\bar{x}) \geqq -\sqrt{\epsilon}\|x - \bar{x}\|, \quad \forall x \in F.$$

Thus, $\bar{x}$ is a quasi ϵ-solution of (RSDP). □

Corollary 3.1 *Let $\bar{x} \in F$, and let $\epsilon \geqq 0$ be given. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and let $A_i^j \in S^n$, $i \in I$, $j = 0, 1, \ldots, l$. Let $\mathcal{U}_i$ be a compact set in $\mathbb{R}^l$ for each $i \in I$. Suppose that the robust characteristic cone D is convex and the robust Slater condition holds. Then, the following statements are equivalent*:

(i) *$\bar{x}$ is a quasi ϵ-solution of* (RSDP);

(ii) *there exist $\bar{Z} \in S_+^n$, $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$, and $\bar{\delta} \in \mathbb{R}_+$ such that*

$$\begin{aligned}&(0, -\sqrt{\epsilon}\|\bar{x}\| - f(\bar{x}))\\ &\in \operatorname{epi} f^* + \left(-\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, -\operatorname{tr}[\bar{Z}\bar{A}_m], \operatorname{tr}[\bar{Z}\bar{A}_0] + \bar{\delta}\right) + \sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+;\end{aligned}$$

(iii) *there exist $\bar{Z} \in S_+^n$ and $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$, such that*

$$0 \in \partial f(\bar{x}) - \left(\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, \operatorname{tr}[\bar{Z}\bar{A}_m]\right) + \sqrt{\epsilon}\mathbb{B} \ \text{ and}$$
$$\operatorname{tr}\left[\bar{Z}(\bar{A}_0 + \sum_{i=1}^m \bar{x}_i \bar{A}_i)\right] = 0,$$

where $\bar{A}_i = A_i^0 + \sum_{j=1}^l \bar{u}_i^j A_i^j \in \mathcal{V}_i$, $i \in I$.

Proof By Proposition 3.2, the robust characteristic cone D is closed, and the proof is completed immediately with the aid of Theorem 3.1. □

Note that the result of Theorem 3.1 holds under the robust Slater condition and the convexity of D. However, the robust Slater condition is not necessary. The following example illustrates that Theorem 3.1 holds, while the robust characteristic cone D is closed and convex, whereas the robust Slater condition fails.

Example 3.1 Consider the following robust convex semidefinite program:

$$\begin{aligned}(\text{RSDP})_1 \qquad &\min |x_1| + x_2^2\\ &\text{s.t. } A_0 + x_1 A_1 + x_2 A_2 \succeq 0, \ \forall A_i \in \mathcal{V}_i, i = 0, 1, 2,\end{aligned}$$

where $\mathcal{V}_0 = \left\{\begin{pmatrix} u_0^1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} : u_0^1 \in [0, 1]\right\}$, $\mathcal{V}_1 = \left\{\begin{pmatrix} 0 & 0 & 0\\ 0 & u_1^1 & 0\\ 0 & 0 & -u_1^1\end{pmatrix} : u_1^1 \in [-1, 1]\right\}$, and $\mathcal{V}_2 = \left\{\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}\right\}$. Then, we have, for any $A_i \in \mathcal{V}_i$, $i = 0, 1, 2$,

$$A_0 + x_1 A_1 + x_2 A_2 = \begin{pmatrix} u_0^1 + x_2 & 0 & 0\\ 0 & u_1^1 x_1 + x_2 & x_2\\ 0 & x_2 & -u_1^1 x_1 + x_2\end{pmatrix}.$$

Indeed, we can easily see that $F^1 := \{(x_1, x_2) : x_1 = 0,\ x_2 \geqq 0\}$, which is the set of all robust feasible solutions of $(\text{RSDP})_1$. Let $\epsilon \geqq 0$ be given. Then, $S_{F^1} := \{(x_1, x_2) \in F^1 : x_1 = 0,\ x_2 \in [0, \frac{\sqrt{\epsilon}}{2}]\}$ is the set of all quasi ϵ-solutions of $(\text{RSDP})_1$. Taking $u_0^1 = 0 \in [0, 1]$ and $u_1^1 \in [-1, 1]$, we can easily see that $A_0 + x_1 A_1 + x_2 A_2$ is not positive definite, that is, the robust Slater condition fails. However, the robust characteristic cone

$$
\begin{aligned}
D &= \bigcup_{\substack{A_i \in \mathcal{V}_i, i=0,1,2 \\ Z \succeq 0, \delta \in \mathbb{R}_+}} \left\{ (-\mathrm{tr}\,[ZA_1], -\mathrm{tr}\,[ZA_2], \mathrm{tr}\,[ZA_0] + \delta) \right\} \\
&= \Big\{ \Big(u_1^1 (z_4 - z_6), z_1 + z_4 + 2z_5 + z_6, u_0^1 z_1 + \delta \Big) :\ u_0^1 \in [0, 1],\ u_1^1 \in [-1, 1], \\
&\qquad \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} \succeq 0,\ \delta \geqq 0 \Big\} \\
&= \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+
\end{aligned}
$$

is closed and convex.

Let $(\bar{x}_1, \bar{x}_2) := (0, \frac{\sqrt{\epsilon}}{2}) \in S_{F^1}$. Taking $\bar{u}_0^1 = 0 \in [0, 1]$, $\bar{u}_1^1 \in [-1, 1]$, $(\bar{b}_1, \bar{b}_2) = (0, -1) \in \mathbb{B}$, and $\bar{Z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \succeq 0$, we can see that $\mathrm{tr}\,[\bar{Z}(\bar{A}_0 + \bar{x}_1 \bar{A}_1 + \bar{x}_2 \bar{A}_2)] = 0$ and

$$
(0, 0) \in \partial f(\bar{x}) - \big(\mathrm{tr}\,[\bar{Z}\bar{A}_1], \mathrm{tr}\,[\bar{Z}\bar{A}_2]\big) + \sqrt{\epsilon}(0, -1) = [-1, 1] \times \{0\},
$$

where $\bar{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\bar{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{u}_1 & 0 \\ 0 & 0 & -\bar{u}_1 \end{pmatrix}$, and $\bar{A}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.
Thus, Theorem 3.1 holds.

Now, we give an approximate optimality theorem under a *weakened constraint qualification*, that is, the robust characteristic cone D is convex but not necessarily closed.

Theorem 3.2 *Let $\bar{x} \in F$, and let $\epsilon \geqq 0$ be given. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and let $A_i^j \in S^n$, $i \in I$, $j = 0, 1, \ldots, l$. Let $\mathcal{U}_i$ be a compact set in $\mathbb{R}^l$ for each $i \in I$. Suppose that the robust characteristic cone D is convex. Then, the following statements are equivalent:*

(i) *$\bar{x}$ is a quasi ϵ-solution of* (RSDP);

(ii) *there exist* $\{Z_k\} \subset S^n_+$, $\{((u_i^1)_k, \ldots, (u_i^l)_k)\} \subset \mathcal{U}_i$, $i \in I$, *and* $\{\delta_k\} \subset \mathbb{R}_+$ *such that*

$$0 \in \partial f(\bar{x}) - \lim_{k\to\infty} ((\mathrm{tr}\,[Z_k(A_1)_k], \ldots, \mathrm{tr}\,[Z_k(A_m)_k])) + \sqrt{\epsilon}\mathbb{B} \text{ and}$$
$$\lim_{k\to\infty} \Big(\mathrm{tr}\,[Z_k((A_0)_k + \sum_{i=1}^m x_i (A_i)_k)] + \delta_k\Big) = 0,$$

where $(A_i)_k = A_i^0 + \sum_{j=1}^l (u_i^j)_k A_i^j \in \mathcal{V}_i$, $i \in I$.

Proof Assume that $\bar{x}$ is a quasi ϵ-solution of (RSDP). Then, from the assumption for D and Remark 3.1, we can easily see that

$$\begin{aligned}&(0, -\sqrt{\epsilon}\|\bar{x}\| - f(\bar{x}))\\ &\in \mathrm{epi}\, f^* + \mathrm{cl}\left(\bigcup_{\substack{A_i\in\mathcal{V}_i, i\in I\\ Z\succeq 0, \delta\in\mathbb{R}_+}} \Big\{(-\mathrm{tr}\,[ZA_1], \ldots, -\mathrm{tr}\,[ZA_m], \mathrm{tr}\,[ZA_0] + \delta)\Big\}\right)\\ &+\sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+.\end{aligned}$$

By Proposition 2.1,

$$\begin{aligned}&(0, -\sqrt{\epsilon}\|\bar{x}\| - f(\bar{x}))\\ &\in \bigcup_{\epsilon_0 \geqq 0} \{(\xi_0, \langle \xi_0, \bar{x}\rangle + \epsilon_0 - f(\bar{x})) : \xi_0 \in \partial_{\epsilon_0} f(\bar{x})\} + \sqrt{\epsilon}\mathbb{B} \times \mathbb{R}_+\\ &+\mathrm{cl}\left(\bigcup_{\substack{A_i\in\mathcal{V}_i, i\in I\\ Z\succeq 0, \delta\in\mathbb{R}_+}} \Big\{(-\mathrm{tr}\,[ZA_1], \ldots, -\mathrm{tr}\,[ZA_m], \mathrm{tr}\,[ZA_0] + \delta)\Big\}\right).\end{aligned}$$

By the definition of the closure, there exist $\bar{\xi}_0 \in \partial_{\bar{\epsilon}_0} f(\bar{x})$, $\bar{\epsilon}_0 \geqq 0$, $\bar{b} \in \mathbb{B}$, $\bar{r} \in \mathbb{R}_+$, $\{Z_k\} \subset S^n_+$, $\{((u_i^1)_k, \ldots, (u_i^l)_k)\} \subset \mathcal{U}_i$, $i \in I$, and $\{\delta_k\} \subset \mathbb{R}_+$ such that

$$0 = \bar{\xi}_0 + \lim_{k\to\infty} \big((-\mathrm{tr}\,[Z_k(A_1)_k], \ldots, -\mathrm{tr}\,[Z_k(A_m)_k])\big) + \sqrt{\epsilon}\bar{b}, \tag{5}$$
$$-\sqrt{\epsilon}\|\bar{x}\| = \langle \bar{\xi}_0, \bar{x}\rangle + \bar{\epsilon}_0 + \lim_{k\to\infty} (\mathrm{tr}\,[Z_k(A_0)_k] + \delta_k) + \bar{r}, \tag{6}$$

where $(A_i)_k = A_i^0 + \sum_{j=1}^{l}(u_i^j)_k A_i^j \in \mathcal{V}_i,\ i \in I$. Combining (5) and (6), we have,

$$
\begin{aligned}
0 &\geqq -\bar{\epsilon}_0 \\
&= \langle \bar{\xi}_0, \bar{x}\rangle + \lim_{k\to\infty}\big(\operatorname{tr}[Z_k((A_0)_k)] + \delta_k\big) + \sqrt{\epsilon}\|\bar{x}\| + \bar{r} \\
&= \langle \lim_{k\to\infty}\big((\operatorname{tr}[Z_k(A_1)_k], \ldots, \operatorname{tr}[Z_k(A_m)_k])\big) - \sqrt{\epsilon}\bar{b}, \bar{x}\rangle \\
&\quad + \lim_{k\to\infty}\big(\operatorname{tr}[Z_k(A_0)_k] + \delta_k\big) + \sqrt{\epsilon}\|\bar{x}\| + \bar{r} \\
&= \lim_{k\to\infty}\left(\operatorname{tr}\left[Z_k((A_0)_k + \sum_{i=1}^{m}\bar{x}_i(A_i)_k)\right] + \delta_k\right) - \sqrt{\epsilon}\langle\bar{b}, \bar{x}\rangle + \sqrt{\epsilon}\|\bar{x}\| + \bar{r} \\
&\geqq \lim_{k\to\infty}\left(\operatorname{tr}\left[Z_k((A_0)_k + \sum_{i=1}^{m}\bar{x}_i(A_i)_k)\right] + \delta_k\right) \\
&\geqq 0,
\end{aligned}
$$

i.e., $\bar{\epsilon}_0 = 0$. It means that there exist $\bar{\xi}_0 \in \partial f(\bar{x})$, $\bar{b} \in \mathbb{B}$, $\{Z_k\} \subset S_+^n$, $\{((u_i^1)_k, \ldots, (u_i^l)_k)\} \subset \mathcal{U}_i,\ i \in I$, and $\{\delta_k\} \subset \mathbb{R}_+$ such that

$$
0 \in \partial f(\bar{x}) - \lim_{k\to\infty}\big((\operatorname{tr}[Z_k(A_1)_k], \ldots, \operatorname{tr}[Z_k(A_m)_k])\big) + \sqrt{\epsilon}\mathbb{B} \quad \text{and}
$$

$$
\lim_{k\to\infty}\left(\operatorname{tr}[Z_k((A_0)_k + \sum_{i=1}^{m}\bar{x}_i(A_i)_k)] + \delta_k\right) = 0,
$$

where $(A_i)_k = A_i^0 + \sum_{j=1}^{l}(u_i^j)_k A_i^j \in \mathcal{V}_i,\ i \in I$.

Conversely, assume that (ii) holds. Then, there exist $\xi_0 \in \partial f(\bar{x})$, $\{Z_k\} \subset S_+^n$, $\{((u_i^1)_k, \ldots, (u_i^l)_k)\} \subset \mathcal{U}_i,\ i \in I$, $\{\delta_k\} \subset \mathbb{R}_+$, and $b \in \mathbb{B}$ such that

$$
\xi_0 = \lim_{k\to\infty}((\operatorname{tr}[Z_k(A_1)_k], \ldots, \operatorname{tr}[Z_k(A_m)_k])) - \sqrt{\epsilon}b, \tag{7}
$$

$$
\lim_{k\to\infty}\big(\operatorname{tr}[Z_k((A_0)_k + \sum_{i=1}^{m}x_i(A_i)_k)] + \delta_k\big) = 0. \tag{8}
$$

By the definition of the subdifferential of f at $\bar{x}$, we have, for any $x \in \mathbb{R}^m$,

$$
f(x) - f(\bar{x}) \geqq \langle \xi_0, x - \bar{x}\rangle. \tag{9}
$$

Combining (7), (8), and (9), it follows that for any $x \in \mathbb{R}^m$,

$$
\begin{aligned}
f(x) - f(\bar{x}) &\geqq \langle \lim_{k\to\infty}\big(\operatorname{tr}[Z_k(A_1)_k], \ldots, \operatorname{tr}[Z_k(A_m)_k]\big), x - \bar{x}\rangle - \sqrt{\epsilon}\langle b, x - \bar{x}\rangle \\
&\geqq \lim_{k\to\infty}\operatorname{tr}\left[Z_k(\sum_{i=1}x_i(A_i)_k)\right] - \lim_{k\to\infty}\operatorname{tr}\left[Z_k(\sum_{i=1}\bar{x}_i(A_i)_k)\right] \\
&\quad -\sqrt{\epsilon}\|b\|\cdot\|x - \bar{x}\|
\end{aligned}
$$

$$= \lim_{k\to\infty}\left(\mathrm{tr}\left[Z_k((A_0)_k + \sum_{i=1}^m x_i(A_i)_k)\right] + \delta_k\right) - \sqrt{\epsilon}\|b\|\cdot\|x-\bar{x}\|$$

$$\geqq \lim_{k\to\infty}\mathrm{tr}\,[Z_k((A_0)_k + \sum_{i=1}^m x_i(A_i)_k)] - \sqrt{\epsilon}\|x-\bar{x}\|,$$

where the last inequality follows from the fact that $b \in \mathbb{B}$ and $\{\delta_k\} \subset \mathbb{R}_+$. Since for each $i \in I$, $(A_i)_k \in \mathcal{V}_i$ and $\mathcal{V}_i$ is compact, for any $x \in F$,

$$\lim_{k\to\infty}\mathrm{tr}\left[Z_k((A_0)_k + \sum_{i=1}^m x_i(A_i)_k)\right] \geqq 0.$$

So, the above inequality yields

$$f(x) - f(\bar{x}) \geqq -\sqrt{\epsilon}\|x-\bar{x}\|, \quad \forall x \in F.$$

Thus, $\bar{x}$ is a quasi ϵ-solution of (RSDP). □

We now give the following example to illustrate Theorem 3.2.

Example 3.2 Consider the following robust convex semidefinite program:

$$(\mathrm{RSDP})_2 \qquad \min |x_1| + x_2^2 \qquad \text{s.t. } A_0 + x_1A_1 + x_2A_2 \succeq 0,\ \forall A_i \in \mathcal{V}_i, i = 0, 1, 2,$$

where $\mathcal{V}_0 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & u_0^1 \end{pmatrix} : u_0^1 \in [-1,1]\right\}$, $\mathcal{V}_1 = \left\{\begin{pmatrix} 0 & -\frac{u_1^1}{2} \\ -\frac{u_1^1}{2} & 0 \end{pmatrix} : u_1^1 \in [-1,1]\right\}$, and $\mathcal{V}_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$. Then, we have, for any $A_i \in \mathcal{V}_i$, $i = 0, 1, 2$,

$$A_0 + x_1A_1 + x_2A_2 = \begin{pmatrix} 0 & -\frac{1}{2}u_1^1x_1 \\ -\frac{1}{2}u_1^1x_1 & x_2 + u_0^1 \end{pmatrix}.$$

Indeed, we can easily see that $F^2 := \{(x_1, x_2) : x_1 = 0,\ x_2 \geqq 1\}$, which is the set of all robust feasible solutions of $(\mathrm{RSDP})_2$. Let $\epsilon \geqq 0$ be given. Then, $S_{F^2} := \{(x_1, x_2) \in F^2 : x_1 = 0,\ x_2 \in [1, \max\{1, \frac{\sqrt{\epsilon}}{2}\}]\}$ is the set of all quasi ϵ-solutions of $(\mathrm{RSDP})_2$. Moreover, the robust characteristic cone

$$D = \bigcup_{\substack{A_i\in\mathcal{V}_i, i=0,1,2\\ Z\succeq 0, \delta\in\mathbb{R}_+}} \left\{ (-\mathrm{tr}\,[ZA_1], -\mathrm{tr}\,[ZA_2], \mathrm{tr}\,[ZA_0] + \delta) \right\}$$

$$= \left\{ \left(u_1^1z_2, -z_3, u_0^1z_3 + \delta\right) : z_1, z_3 \geqq 0, z_1z_3 - z_2^2 \geqq 0, u_0^1, u_1^1 \in [-1,1], \delta \geqq 0\right\}$$

$$= \left\{ (0,0,\delta) : \delta \geqq 0\right\} \cup \left\{ (a,b,r) : a \in \mathbb{R},\ b < 0,\ r \in \mathbb{R}\right\}$$

is clearly not closed.

Let $\epsilon \geqq 4$ be any given, and let $(\bar{x}_1, \bar{x}_2) = (0, \frac{\sqrt{\epsilon}}{2}) \in S_{F^2}$. For each $k \in \mathbb{N}$, let $(u_0^1)_k = \frac{1}{k} \in [-1, 1]$, $(u_1^1)_k = 0 \in [-1, 1]$, and let $Z_k = \begin{pmatrix} k & 1 \\ 1 & \frac{1}{k} \end{pmatrix} \in S_+^2$. Let $\delta_k = \frac{1}{k}$. Notice that $\partial f(\bar{x}_1, \bar{x}_2) = [-1, 1] \times \{\sqrt{\epsilon}\}$ and $\lim_{k\to\infty} \big(-\mathrm{tr}\,[Z_k(A_1)_k], -\mathrm{tr}\,[Z_k(A_2)_k]\big) = (0, 0)$. Let $(\bar{b}_1, \bar{b}_2) = (0, -1) \in \mathbb{B}$. Then, we have

$$
\begin{aligned}
(0, 0) &\in [-1, 1] \times \{\sqrt{\epsilon}\} + \sqrt{\epsilon}(0, -1) \\
&= [-1, 1] \times \{0\} \\
&\subset \partial f(\bar{x}_1, \bar{x}_2) - \lim_{k\to\infty} \big((-\mathrm{tr}\,[Z_k(A_1)_k], -\mathrm{tr}\,[Z_k(A_2)_k]) \big) + \sqrt{\epsilon}\mathbb{B}.
\end{aligned}
$$

Moreover, $\lim_{k\to\infty} \big(\mathrm{tr}\,[Z_k((A_0)_k + \bar{x}_1(A_1)_k + \bar{x}_2(A_2)_k)] + \delta_k\big) = \lim_{k\to\infty} \big(\frac{\sqrt{\epsilon}}{2k} + \frac{1}{k}\big) = 0$. Thus, Theorem 3.2 holds.

4 Approximate Duality Theorems

Let $\mathcal{U} := \mathcal{U}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_m = \prod_{i=0}^m \mathcal{U}_i \subset \mathbb{R}^{l\times(m+1)}$, and $u \in \mathcal{U}$ means that for each $i \in I$, $u_i := (u_i^1, \ldots, u_i^l) \in \mathcal{U}_i$.

Now, we formulate the dual problem of (RSDP) as follows:

$$
\begin{aligned}
\text{(RSDD)} \qquad & \max f(y) - \mathrm{tr}\left[Z(A_0 + \sum_{i=1}^m y_i A_i) \right] \\
& \text{s.t. } 0 \in \partial f(y) - (\mathrm{tr}\,[ZA_1], \ldots, \mathrm{tr}\,[ZA_m]) + \sqrt{\epsilon}\mathbb{B}, \\
& \quad A_i = A_i^0 + \sum_{j=1}^l u_i^j A_i^j,\ A_i^j \in S^n,\ (u_i^1, \ldots, u_i^l) \in \mathcal{U}_i, \\
& \quad i \in I,\ j = 1, \ldots, l,\ Z \in S_+^n,\ \epsilon \geqq 0.
\end{aligned}
$$

Let $F_D := \{(y, u, Z) \in \mathbb{R}^m \times \mathcal{U} \times S_+^n : 0 \in \partial f(y) - (\mathrm{tr}\,[ZA_1], \ldots, \mathrm{tr}\,[ZA_m]) + \sqrt{\epsilon}\mathbb{B},\ Z \in S_+^n,\ u \in \mathcal{U},\ \epsilon \geqq 0\}$ be the feasible set of (RSDD).

Definition 4.1 Let $\epsilon \geqq 0$ be given, then $(\bar{y}, \bar{u}, \bar{Z})$ is said to be a quasi ϵ-solution of (RSDD) if for any $(y, u, Z) \in F_D$,

$$
f(\bar{y}) - \mathrm{tr}\left[\bar{Z}(\bar{A}_0 + \sum_{i=1}^m \bar{y}_i \bar{A}_i) \right] \geqq f(y) - \mathrm{tr}\left[Z(A_0 + \sum_{i=1}^m y_i A_i) \right] - \sqrt{\epsilon}\|\bar{y} - y\|,
$$

where $\bar{A}_i = A_i^0 + \sum_{j=1}^l \bar{u}_i^j A_i^j \in \mathcal{V}_i$, $i \in I$.

Remark 4.1 The notion of a quasi ϵ-solution of (RSDP) is motivated by Ekeland's variational principle [12] as we have mentioned, and for the notion of a quasi ϵ-solution of (RSDD) which is motivated by [27] where the author introduced the notion of the quasi ϵ-saddle point. It is worth noting here that we consider the notion of a quasi

ϵ-solution over the feasible set, and it is not necessary to mention how explicitly the feasible set is defined by.

Now, we establish the approximate weak duality theorem, which holds between (RSDP) and (RSDD).

Theorem 4.1 (Approximate Weak Duality) *For any feasible solution x of* (RSDP) *and any feasible solution (y, u, Z) of* (RSDD),

$$f(x) \geqq f(y) - \operatorname{tr}\left[Z(A_0 + \sum_{i=1}^{m} y_i A_i)\right] - \sqrt{\epsilon}\|x - y\|. \tag{10}$$

Proof Let x and (y, u, Z) be feasible solutions of (RSDP) and (RSDD), respectively. Then, $\operatorname{tr}[Z(A_0 + \sum_{i=1}^{m} x_i A_i)] \geqq 0$, and there exist $\xi \in \partial f(y)$ and $b \in \mathbb{B}$ such that $\xi = (\operatorname{tr}[ZA_1], \ldots, \operatorname{tr}[ZA_m]) - \sqrt{\epsilon} b$. By the definition of the subdifferential of f, we have

$$\begin{aligned}
& f(x) - (f(y) - \operatorname{tr}\left[Z(A_0 + \sum_{i=1}^{m} y_i A_i)\right]) \\
& \geqq \langle \xi, x - y \rangle + \operatorname{tr}\left[Z(A_0 + \sum_{i=1}^{m} y_i A_i)\right] \\
& = \langle (\operatorname{tr}[ZA_1], \ldots, \operatorname{tr}[ZA_m]) - \sqrt{\epsilon} b, x - y \rangle \\
& \quad + \operatorname{tr}\left[Z(A_0 + \sum_{i=1}^{m} y_i A_i)\right] \\
& = \operatorname{tr}\left[Z(A_0 + \sum_{i=1}^{m} x_i A_i)\right] - \sqrt{\epsilon} \langle b, x - y \rangle \\
& \geqq -\sqrt{\epsilon}\|b\| \cdot \|x - y\| \\
& \geqq -\sqrt{\epsilon}\|x - y\|,
\end{aligned}$$

which implies that (10) holds. □

Now, under the *robust characteristic cone constraint qualification*, we give the approximate strong duality theorem, which holds between (RSDP) and (RSDD).

Theorem 4.2 (Approximate Strong Duality) *Suppose that the robust characteristic cone D is closed and convex. If $\bar{x}$ is a quasi ϵ-solution of* (RSDP), *then there exist $\bar{Z} \in S_+^n$, $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$ such that $(\bar{x}, \bar{u}, \bar{Z})$ is a quasi ϵ-solution of* (RSDD).

Proof Let $\bar{x}$ be a quasi ϵ-solution of (RSDP). Then, by Theorem 3.1, there exist $\bar{Z} \in S_+^n$ and $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$ such that

$$0 \in \partial f(\bar{x}) - \big(\operatorname{tr}[\bar{Z}\bar{A}_1], \ldots, \operatorname{tr}[\bar{Z}\bar{A}_m]\big) + \sqrt{\epsilon}\mathbb{B} \text{ and } \operatorname{tr}\left[\bar{Z}(\bar{A}_0 + \sum_{i=1}^{m} \bar{x}_i \bar{A}_i)\right] = 0,$$

where $\bar{A}_i = A_i^0 + \sum_{j=1}^{l} \bar{u}_i^j A_i^j \in \mathcal{V}_i,\ i \in I$. Therefore, $(\bar{x}, \bar{u}, \bar{Z})$ is feasible for (RSDD). Hence, by Theorem 4.1, for any feasible solution (y, u, Z) of (RSDD),

$$f(\bar{x}) - \mathrm{tr}\left[\bar{Z}(\bar{A}_0 + \sum_{i=1}^{m} \bar{x}_i \bar{A}_i)\right] = f(\bar{x})$$
$$\geqq f(y) - \mathrm{tr}\left[Z(A_0 + \sum_{i=1}^{m} y_i A_i)\right] - \sqrt{\epsilon}\|x - y\|.$$

Thus, $(\bar{x}, \bar{u}, \bar{Z})$ is a quasi ϵ-solution of (RSDD). □

Corollary 4.1 *Suppose that the robust characteristic cone D is convex and the robust Slater condition holds. If $\bar{x}$ is a quasi ϵ-solution of* (RSDP), *then, there exist $\bar{Z} \in S_+^n$ and $(\bar{u}_i^1, \ldots, \bar{u}_i^l) \in \mathcal{U}_i$, $i \in I$, such that $(\bar{x}, \bar{u}, \bar{Z})$ is a quasi ϵ-solution of* (RSDD).

Proof By Proposition 3.2, the robust characteristic cone D is closed, and the proof is completed immediately with the aid of Theorem 4.2. □

Note that the result of Theorem 4.2 holds under the robust Slater condition and the convexity of the robust characteristic cone D. However, the robust characteristic cone D may also be closed even though the robust Slater condition does not hold. The following exam illustrates that the approximate strong duality holds, whereas the robust Slater condition fails.

Example 4.1 Consider the following robust convex semidefinite program:

$$(\mathrm{RSDP})_1 \qquad \min |x_1| + x_2^2$$
$$\text{s.t. } A_0 + x_1 A_1 + x_2 A_2 \succeq 0,\ \forall A_i \in \mathcal{V}_i,\ i = 0, 1, 2,$$

where $\mathcal{V}_0 = \left\{ \begin{pmatrix} u_0^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : u_0^1 \in [0, 1] \right\}$, $\mathcal{V}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & u_1^1 & 0 \\ 0 & 0 & -u_1^1 \end{pmatrix} : u_1^1 \in [-1, 1] \right\}$, and $\mathcal{V}_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\}$. Let $\epsilon \geqq 0$ be given. Let $f(x_1, x_2)$ and $A_0 + x_1 A_1 + x_2 A_2$ be same as in Example 3.1. Recall that the feasible set and the set of quasi ϵ-solutions for $(\mathrm{RSDP})_1$ are $F^1 := \{(x_1, x_2) : x_1 = 0,\ x_2 \geqq 0\}$ and $S_{F^1} := \{(x_1, x_2) \in F^1 : x_1 = 0,\ x_2 \in [0, \frac{\sqrt{\epsilon}}{2}]\}$, respectively. We already have shown in Example 3.1 that for any $u_0^1 \in [0, 1]$, $u_1^1 \in [-1, 1]$, $A_0 + x_1 A_1 + x_2 A_2$ is not positive definite; however, the robust characteristic cone

$$D := \bigcup_{\substack{A_i \in \mathcal{V}_i, i=0,1,2 \\ Z \succeq 0, \delta \in \mathbb{R}_+}} \left\{ (-\mathrm{tr}\,[Z A_1], -\mathrm{tr}\,[Z A_2], \mathrm{tr}\,[Z A_0] + \delta) \right\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$$

is closed and convex.

Now, we formulate the dual problem of $(\text{RSDP})_1$ as follows:

$$(\text{RSDD})_1 \qquad \max |y_1| + y_2^2 - \operatorname{tr}[Z(A_0 + y_1 A_1 + y_2 A_2)]$$
$$\text{s.t. } 0 \in \partial f(y) - (\operatorname{tr}[ZA_1], \operatorname{tr}[ZA_2]) + \sqrt{\epsilon}\mathbb{B},$$
$$A_i \in \mathcal{V}_i,\ i = 0, 1, 2,\ u_0^1 \in [0, 1],\ u_1^1 \in [-1, 1],$$
$$Z = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} \succeq 0,\ \epsilon \geqq 0.$$

Let $\mathcal{U} := [0, 1] \times [-1, 1]$. Then, the feasible solution set of $(\text{RSDD})_1$ is $F_D^1 := F_D^a \cup F_D^b \cup F_D^c$, where

$$\begin{aligned} F_D^a = \{&((0, y_2), (u_0^1, u_1^1), Z) \in \mathbb{R}^2 \times \mathcal{U} \times S_+^3 : \\ &(0, 0) \in \partial f(0, y_2) - (\operatorname{tr}[ZA_1], \operatorname{tr}[ZA_2]) + \sqrt{\epsilon}\mathbb{B},\ \epsilon \geqq 0\} \\ = \{&((0, y_2), (u_0^1, u_1^1), Z) \in \mathbb{R}^2 \times \mathcal{U} \times S_+^3 : |(-z_4 + z_6)u_1^1 + \sqrt{\epsilon} b_1| \leqq 1, \\ &y_2 = (z_1 + z_4 + 2z_5 + z_6)\tfrac{1}{2} - \tfrac{\sqrt{\epsilon}}{2} b_2,\ b_1^2 + b_2^2 \leqq 1,\ \epsilon \geqq 0\}, \end{aligned}$$

$$\begin{aligned} F_D^b = \{&((y_1, y_2), (u_0^1, u_1^1), Z) \in \mathbb{R}^2 \times \mathcal{U} \times S_+^3 : \\ &y_1 > 0,\ (0, 0) \in \partial f(y_1, y_2) - (\operatorname{tr}[ZA_1], \operatorname{tr}[ZA_2]) + \sqrt{\epsilon}\mathbb{B},\ \epsilon \geqq 0\} \\ = \{&((y_1, y_2), (u_0^1, u_1^1), Z) \in \mathbb{R}^2 \times \mathcal{U} \times S_+^3 : \sqrt{\epsilon} b_1 = -1 + (z_4 - z_6)u_1^1, \\ &y_1 > 0,\ y_2 = (z_1 + z_4 + 2z_5 + z_6)\tfrac{1}{2} - \tfrac{\sqrt{\epsilon}}{2} b_2,\ b_1^2 + b_2^2 \leqq 1,\ \epsilon \geqq 0\}, \end{aligned}$$

and

$$\begin{aligned} F_D^c = \{&((y_1, y_2), (u_0^1, u_1^1), Z) \in \mathbb{R}^2 \times \mathcal{U} \times S_+^3 : \\ &y_1 < 0,\ (0, 0) \in \partial f(y_1, y_2) - (\operatorname{tr}[ZA_1], \operatorname{tr}[ZA_2]) + \sqrt{\epsilon}\mathbb{B},\ \epsilon \geqq 0\} \\ = \{&((y_1, y_2), (u_0^1, u_1^1), Z) \in \mathbb{R}^2 \times \mathcal{U} \times S_+^3 : \sqrt{\epsilon} b_1 = (z_4 - z_6)u_1^1 + 1, \\ &y_1 < 0,\ y_2 = (z_1 + z_4 + 2z_5 + z_6)\tfrac{1}{2} - \tfrac{\sqrt{\epsilon}}{2} b_2,\ b_1^2 + b_2^2 \leqq 1,\ \epsilon \geqq 0\}. \end{aligned}$$

Then, for any $(x_1, x_2) \in F^1$ and any $(y_1, y_2, u_0^1, u_1^1, Z) \in F_D^a$,

$$\begin{aligned} &f(x_1, x_2) - [f(y_1, y_2) - \operatorname{tr}[Z(A_0 + y_1 A_1 + y_2 A_2)]] \\ &= x_2^2 - y_2^2 + z_1 u_0^1 + (z_1 + z_4 + 2z_5 + z_6) y_2 \\ &\geqq 2y_2(x_2 - y_2) + z_1 u_0^1 + (z_1 + z_4 + 2z_5 + z_6) y_2 \\ &= z_1 u_0^1 + (z_1 + z_4 + 2z_5 + z_6) x_2 - \sqrt{\epsilon} b_2 (x_2 - y_2) \\ &\geqq \operatorname{tr}[Z(A_0 + 0 \cdot A_1 + x_2 A_2)] - \sqrt{\epsilon} |x_2 - y_2| \\ &\geqq -\sqrt{\epsilon} \|(x_1, x_2) - (y_1, y_2)\|. \end{aligned}$$

Moreover, for any $(x_1, x_2) \in F^1$ and any $(y_1, y_2, u_0^1, u_1^1, Z) \in F_D^b$,

$$\begin{aligned} & f(x_1, x_2) - [f(y_1, y_2) - \mathrm{tr}\,[Z(A_0 + y_1 A_1 + y_2 A_2)]] \\ & \quad = x_2^2 - y_1 - y_2^2 + z_1 u_0^1 + (z_4 - z_6) u_1^1 y_1 + (z_1 + z_4 + 2z_5 + z_6) y_2 \\ & \quad \geqq 2y_2(x_2 - y_2) + z_1 u_0^1 + (-1 + (z_4 - z_6) u_1^1) y_1 + (z_1 + z_4 + 2z_5 + z_6) y_2 \\ & \quad = z_1 u_0^1 + (z_1 + z_4 + 2z_5 + z_6) x_2 + \sqrt{\epsilon} b_1 y_1 - \sqrt{\epsilon} b_2 (x_2 - y_2) \\ & \quad \geqq \mathrm{tr}\,[Z(A_0 + 0 \cdot A_1 + x_2 A_2)] - \sqrt{\epsilon} \sqrt{b_1^2 + b_2^2} \sqrt{y_1^2 + (x_2 - y_2)^2} \\ & \quad \geqq -\sqrt{\epsilon} \|(x_1, x_2) - (y_1, y_2)\|. \end{aligned}$$

Similarly, we can show that, for any $(x_1, x_2) \in F^1$ and any $((y_1, y_2), (u_0^1, u_1^1), Z) \in F_D^c$,

$$f(x_1, x_2) \geqq f(y_1, y_2) - \mathrm{tr}\,[Z(A_0 + y_1 A_1 + y_2 A_2)] - \sqrt{\epsilon} \|(x_1, x_2) - (y_1, y_2)\|.$$

The foregoing calculations imply that, for any feasible solution (x_1, x_2) of $(\mathrm{RSDP})^1$ and any feasible solution $((y_1, y_2), (u_0^1, u_1^1), Z)$ of $(\mathrm{RSDD})^1$,

$$f(x_1, x_2) \geqq f(y_1, y_2) - \mathrm{tr}\,[Z(A_0 + y_1 A_1 + y_2 A_2)] - \sqrt{\epsilon} \|(x_1, x_2) - (y_1, y_2)\|,$$

that is, Theorem 4.1 (approximate weak duality) holds.

Let $(\bar{x}_1, \bar{x}_2) = (0, \frac{\sqrt{\epsilon}}{2}) \in S_{F^1}$. Let $\bar{u}_0^1 = 0$, $\bar{u}_1^1 \in [-1, 1]$, and let $\bar{Z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$. Then, we can easily check that $((\bar{x}_1, \bar{x}_2), (\bar{u}_0^1, \bar{u}_1^1), \bar{Z}) \in F_D^1$. Moreover, for any $((y_1, y_2), (u_0^1, u_1^1), Z) \in F_D^1$,

$$\begin{aligned} & f(\bar{x}_1, \bar{x}_2) - \mathrm{tr}\,[Z(\bar{A}_0 + \bar{x}_1 \bar{A}_1 + \bar{x}_2 \bar{A}_2)] \\ & \quad -[f(y_1, y_2) - \mathrm{tr}\,[Z(A_0 + y_1 A_1 + y_2 A_2)]] \\ & \quad \geqq -\sqrt{\epsilon} \|(\bar{x}_1, \bar{x}_2) - (y_1, y_2)\| - \mathrm{tr}\,[Z(\bar{A}_0 + \bar{x}_1 \bar{A}_1 + \bar{x}_2 \bar{A}_2)] \text{ (by Theorem 4.1)} \\ & \quad = -\sqrt{\epsilon} \|(\bar{x}_1, \bar{x}_2) - (y_1, y_2)\|, \end{aligned}$$

where $\bar{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\bar{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{u}_1^1 & 0 \\ 0 & 0 & -\bar{u}_1^1 \end{pmatrix}$, and $\bar{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Thus, Theorem 4.2 (approximate strong duality) holds.

5 Conclusions

In this research paper, we studied quasi approximate solutions for a convex semidefinite programming problem in the face of data uncertainty. By using the robust optimization approach (worst-case approach), approximate optimality conditions and approximate duality theorems for quasi approximate solutions in robust convex semidefinite programming problems were examined under the robust characteristic cone constraint qualification (Theorem 3.2 was given under a weakened constraint qualification). Throughout this article, some skillful and sightworthy examples were given to illustrate the obtained results.

Acknowledgements The second author was supported by the National Research Foundation of Korea (NRF) funded by the Korea government (MSIP) (NRF-2016R1A2B1006430). The authors would like to express their sincere thanks to both the handling editor and the anonymous referees, who have given several valuable and helpful suggestions and comments to improve the initial manuscript of the paper.

References

1. Goldfarb, D., Iyengar, G.: Robust convex quadratically constrained programs. Math. Program. **97**, 495–515 (2003)
2. Helmberg, C.: Semidefinite programming. Eur. J. Oper. Res. **137**, 461–482 (2002)
3. Vandenberghe, L., Boyd, S.: Semidefinite programming. SIAM Rev. **38**, 49–95 (1996)
4. Wolkowicz, H., Saigal, R., Vandenberghe, L. (eds.): Handbook of Semidefinite Programming: Theory, Algorithms, and Applications. Kluwer Academic, Boston (2000)
5. Bertsimas, D., Pachamanova, D., Sim, M.: Robust linear optimization aunder general norms. Oper. Res. Lett. **32**, 510–516 (2004)
6. Ghaoui, L.E., Oustry, F., Lebret, H.: Robust solutions to uncertain semi-infinite programs. SIAM J. Optim. **9**, 33–52 (1998)
7. Jeyakumar, V.: A note on strong duality in convex semidefinite optimization: necessary and sufficient conditions. Optim. Lett. **2**, 15–25 (2008)
8. Scherer, C.W.: Relaxations for robust linear matrix inequality problems with verifications for exactness. SIAM J. Matrix Anal. Appl. **27**, 365–395 (2005)
9. Houda, M.: Comparison of approximations in stochastic and robust optimization programs. In: Hušková, M., Janžura, M. (eds.) Prague Stochastics 2006, pp. 418–425. Prague, Matfyzpress (2006)
10. Jeyakumar, V., Li, G.Y.: Strong duality in robust semi-definite linear programming under data uncertainty. Optimization **63**, 713–733 (2014)
11. Loridan, P.: Necessary conditions for ϵ-optimality. Math. Program. Stud. **19**, 140–152 (1982)
12. Ekeland, I.: On the variational principle. J. Math. Anal. Appl. **47**, 324–353 (1974)
13. Dutta, J., Deb, K., Tulshyan, R., Arora, R.: Approximate KKT points and a proximity measure for termination. J. Glob. Optim. **56**, 1463–1499 (2013)
14. Gao, Y., Hou, S.H., Yang, X.M.: Existence and optimality conditions for approximate solutions to vector optimization problems. J. Optim. Theory Appl. **152**, 97–120 (2009)
15. Lee, J.H., Lee, G.M.: On ϵ-solutions for convex optimization problems with uncertainty data. Positivity **16**, 509–526 (2012)
16. Lee, J.H., Jiao, L.G.: On quasi ϵ-solution for robust convex optimization problems. Optim. Lett. **11**, 1609–1622 (2017)
17. Son, T.Q., Kim, D.S.: ϵ-Mixed type duality for nonconvex multiobjective programs with an infinite number of constraints. J. Global Optim. **57**, 447–465 (2013)
18. Son, T.Q., Strodiot, J.J., Nguyen, V.H.: ϵ-Optimality and ϵ-Lagrangian duality for a nonconvex programming problem with an infinite number of constraints. J. Optim. Theory Appl. **141**, 389–409 (2009)
19. Jeyakumar, V.: Asymptotic dual conditions characterizing optimality for convex programs. J. Optim. Theory Appl. **93**, 153–165 (1997)
20. Jeyakumar, V., Li, G.Y.: Strong duality in robust convex programming: complete characterizations. SIAM J. Optim. **20**, 3384–3407 (2010)
21. Jeyakumar, V., Lee, G.M., Dinh, N.: New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs. SIAM J. Optim. **14**, 534–547 (2003)
22. Li, C., Ng, K.F., Pong, T.K.: Constraint qualifications for convex inequality systems with applications in constrained optimization. SIAM. J. Optim. **19**, 163–187 (2008)

23. Ben-Tal, A., Ghaoui, L.E., Nemirovski, A.: Robustness. In: Wolkowicz, H., Saigal, R., Vandenberghe, L. (eds.) Handbook of Semidefinite Programming, pp. 139–162. Kluwer Academic, Boston (2000)
24. Ben-Tal, A., Ghaoui, L.E., Nemirovski, A.: Robust Optimization. Princeton Series in Applied Mathematics. Princeton University Press, Priceton (2009)
25. Fang, D.H., Li, C., Ng, K.F.: Constraint qualifications for extended Farkas's lemmas and Lagrangian dualities in convex infinite programming. SIAM J. Optim. **20**, 1311–1332 (2009)
26. Dinh, N., Goberna, M.A., López, M.A., Mo, T.H.: From the Farkas lemma to the Hahn–Banach theorem. SIAM J. Optim. **24**, 678–701 (2014)
27. Dutta, J.: Necessary optimality conditions and saddle points for approximate optimization in Banach space. Top **13**, 127–143 (2005)