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Solution Existence in Bifunction-Set Optimization

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Abstract This paper establishes a bridge between set optimization problems and vector Ky Fan inequality problems. We introduce a general model, called the bifunction-set optimization problem, that provides a unifying framework for the above-mentioned problems. An existence result in our model is obtained, with the help of KKM–Fan's lemma. As applications, we derive some new or sharper existence results for set optimization problems and generalized vector Ky Fan inequalities with efficient solutions.

Keywords Bifunction-set optimization · Ky Fan inequality · KKM–Fan lemma · Solution existence

Mathematics Subject Classification 90C31 · 49J40 · 49K40

1 Introduction

In optimization with set-valued maps, the vector criterion aims at finding an efficient point of the image of an objective set-valued map, while the set criterion is based on comparisons among its values. Optimization problems with the set criterion, called set optimization problems, are first proposed by Kuroiwa [1,2]. The existence of solutions is an important research direction in the development of set optimization. In [3] (see also [2]), two existence results are proven for l-type optimal solutions

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under some l-type semicontinuity properties of the objective maps. In the first one (Theorem 4.1 of [3]), the domain of the objective map is assumed to be a compact set of a topological vector space, and in the second one (Theorem 4.2 of [3]), it is a (not necessarily compact) complete metric space. In [3], two similar existence results are also formulated for u-type optimal solutions in Theorems 4.3 and 4.4. In [4], the existence of u-type optimal solutions is established in Proposition 22 and Theorem 26 under cone-regularity assumptions, and the existence of l-type optimal solutions is obtained in Proposition 30 under cone-semicompactness assumptions. In [5], the l-domination property, that is stronger than the existence of l-type optimal solutions, is established in Theorem 5.8 and Corollary 5.6. They are improvements of Theorem 4.1 of [3] and Proposition 30 of [4]. In [6], Theorems 5.3 and 5.4 give existence of l-type optimal solutions under assumptions that are different from the l-type semicontinuity assumptions in Theorems 4.1 and 4.2 of [3], but the ordering cone is required to have a nonempty interior. The approach of [6] is based on generalizations of nonlinear scalarizing functions of Gerstewitz [7]. In [8], a similar approach is used in Theorems 5.8 and 5.9 to establish existence results in weak versions of l-type and u-type set optimization problems.

The approaches to solution existence in the above papers do not use tools, such as the KKM–Fan's lemma (see Lemma 1 of [9]), the result on the existence of maximal elements (see Theorem 5.1 of [10]), …that are very efficient in proving existence results in vector Ky Fan inequalities (see, e.g., [11]) and related problems. So, it is interesting to study solution existence in set optimization with the help of these new tools and assumptions different from the earlier ones. The Ky Fan inequality problem [12] (called also the equilibrium problem in [13] and in several other papers) provides a unifying approach to optimization, complementarity, variational inequality, saddle point and fixed point problems. The vector variational inequality, first introduced in [14], can be regarded as one of important stimulations for studying vector versions of the Ky Fan inequality. The solution existence in the theory of generalized vector Ky Fan inequality problems is intensively developed. We restrict ourselves to reviewing some results on existence of efficient solutions in generalized vector Ky Fan inequality problems that are related directly to our main result. In [15], the existence of efficient solutions in these problems is established in Theorem 3.2 under the assumption on existence of continuous strongly cone-monotonic functions. In [16], such an existence result is obtained in Theorem 3.2 for the case, where the ordering cone has a base, and the objective is a continuous set-valued map with compact convex values. If the ordering cone has a nonempty interior and the objective is single-valued, existence results for efficient solutions can be found in [17] for vector variational inequalities and in [18] for vector equilibrium problems. In [19], the existence of efficient solutions is proven in Theorems 3.1 and 3.2 for problems stronger than those of [15,16], under the assumption on existence of continuous strongly monotonic functions satisfying certain conditions.

The existence of solutions is independently developed in set optimization problems in [2–6,8] and generalized vector Ky Fan inequality problems in [15–19]. This paper establishes a bridge between these problems, showing that some traditional tools and approaches in generalized vector Ky Fan inequalities may be useful also for set optimization problems. We introduce a general model, called the bifunction-set optimization problem, that provides a unifying framework for the u-type/l-type set optimization problems and the vector Ky Fan inequality problems with efficient solutions. An existence result is obtained in our model, with the help of KKM–Fan lemma (see Lemma 1 of [9]). As applications, we derive some new or sharper existence results for the above-mentioned set optimization problems and vector Ky Fan inequalities.

2 Preliminaries

A nonempty set C of a vector space Y is called a cone, if and only if $\lambda c \in C$ for all $\lambda \geq 0$ and $c \in C$. A cone C is pointed, if and only if $C \cap (-C) = \{0\}$, where 0 denotes the origin of Y.

For a set-valued map $G : X \rightrightarrows Y$ between topological spaces X and Y, we denote by dom$G$ and im$G$ the domain and image of G:

$$\begin{aligned} \mathrm{dom}G &:= \{x \in X : G(x) \neq \emptyset\}, \\ \mathrm{im}G &:= G(X) := \cup_{x \in X} G(x), \end{aligned}$$

where Ø stands for the empty set.

We say that G is compact-valued (resp. closed-valued; convex-valued) on X, if and only if each value of G is a compact (resp. closed; convex) set in Y. We denote by $U(x_0), U'(x_0), \ldots,$ neighborhoods of $x_0 \in X$. The following definition can be found in [20].

Definition 2.1 Let $G : X \rightrightarrows Y$ be a set-valued map between a topological space X and a topological vector space Y, and let C be a convex cone of Y.

(i) G is C-usc (resp. C-lsc) at $x_0 \in \mathrm{dom}G$, if and only if we can associate with each open set W of Y with $G(x_0) \subset W$ (resp. $G(x_0) \cap W \neq \emptyset$) a neighborhood $U(x_0)$ such that $G(x) \subset W + C$ (resp. $G(x) \cap [W - C] \neq \emptyset$) for all $x \in U(x_0)$;
(ii) G is C-usc (resp. C-lsc) on X, if and only if dom$G = X$ and G is C-usc (resp. C-lsc) at each point of dom$G$.

If $C = \{0\}$, then we say that G is usc (resp. lsc) instead of saying that G is C-usc (resp. C-lsc).

Definition 2.2 Set-valued map $G : K \rightrightarrows X$ from a nonempty subset K of a vector space X to X is called a KKM-map on K, if and only if, for any finite subset $\{x_i, i = 1, 2, \ldots, n\}$ of K and any point $x \in \mathrm{conv}\{x_i, i = 1, 2, \ldots, n\}$, there exists $j \in \{1, 2, \ldots, n\}$ such that $x \in G(x_j)$, where "conv" stands for the convex hull.

Remark 2.1 If G is a KKM-map on K, then $x \in G(x)$ for all $x \in K$.

The following KKM–Fan lemma can be found in [9].

Lemma 2.1 (The KKM–Fan lemma) *Let K be a nonempty subset of a Hausdorff topological vector space X, and $G : K \rightrightarrows X$ be a KKM-map such that G has closed values and $G(x)$ is compact for at least one $x \in K$. Then, $\cap_{x \in K} G(x) \neq \emptyset$.*

Given a convex set K and two points $\eta_i \in K, i = 1, 2$, we denote by $[\eta_1, \eta_2]$ the closed segment joining η_1 and η_2, that is, $[\eta_1, \eta_2] := \{\alpha\eta_1 + (1-\alpha)\eta_2 : 0 \leq \alpha \leq 1\}$. We set $]\eta_1, \eta_2] = [\eta_1, \eta_2]\setminus\{\eta_1\}, [\eta_1, \eta_2[= [\eta_1, \eta_2]\setminus\{\eta_2\}$ and $]\eta_1, \eta_2[= [\eta_1, \eta_2]\setminus\{\eta_1, \eta_2\}$.

We will need the following definition, due to [21].

Definition 2.3 Let K be a nonempty convex set, C be a convex cone of a vector space Y, and $G : K \rightrightarrows Y$ be a set-valued map with nonempty values.

(i) G is properly C-quasiconvex on K, if and only if, for all $\eta_i \in K, i = 1, 2$, and $\eta \in [\eta_1, \eta_2]$, there exists $j \in \{1, 2\}$ such that $G(\eta) \subset G(\eta_j) - C$;

(ii) G is properly C-quasiconcave on K, if and only if, for all $\eta_i \in K, i = 1, 2$, $\eta \in [\eta_1, \eta_2]$, and $y_i \in G(\eta_i), i = 1, 2$, there exists $j \in \{1, 2\}$ such that $y_j \in G(\eta) - C$.

Definition 2.4 Let K be a nonempty convex set, C be a convex cone of a vector space Y, and $G, H : K \rightrightarrows Y$ be set-valued maps with nonempty values. The pair (G, H) is properly C-quasiconcave–quasiconvex on K, if and only if, for all $\eta_i \in K, i = 1, 2$, $\eta \in [\eta_1, \eta_2]$, and $y_i \in G(\eta_i), i = 1, 2$, there exists $j \in \{1, 2\}$ such that $y_j \in G(\eta) - C$ and $H(\eta) \subset H(\eta_j) - C$.

Remark 2.2 (i) Obviously, the pair (G, H) is properly C-quasiconcave–quasiconvex on K, if, for all $\eta_i \in K, i = 1, 2$, and $\eta \in [\eta_1, \eta_2]$, we have

$$
\begin{aligned}
\text{either} \quad & [G(\eta_1) \subset G(\eta) - C, \quad H(\eta) \subset H(\eta_1) - C]; \\
\text{or} \quad & [G(\eta_2) \subset G(\eta) - C, \quad H(\eta) \subset H(\eta_2) - C].
\end{aligned}
$$

(ii) Let H be a constant set-valued map (that is, $H(\eta) = M, \forall \eta \in K$, where M is a fixed nonempty subset of Y). Then, the pair (G, H) is properly C-quasiconcave–quasiconvex on K, if G is properly C-quasiconcave on K.

(iii) Let G be a constant set-valued map. Then, the pair (G, H) is properly C-quasiconcave–quasiconvex on K, if H is properly C-quasiconvex on K.

3 Existence Result in the Bifunction-Set Optimization Problem

Let K be a nonempty subset of a Hausdorff topological vector space X, C be a pointed closed convex cone of a topological vector space Y, and $F_i : K \times K \rightrightarrows Y, i = 1, 2$, be set-valued-maps with nonempty values. In this paper, we are interested in the following problem, called the set optimization problem with bifunction, or, shortly, the bifunction-set optimization problem:

Problem (u-BSOP): Find a point $x \in K$ such that $F_1(x, \eta) \leq^u F_2(x, \eta)$ with some $\eta \in K$ implies that $F_2(x, \eta) \leq^u F_1(x, \eta)$.

Here the binary $\leq^u$ is taken from [1–3], that is, for nonempty sets $A_1 \subset Y$ and $A_2 \subset Y$,

$$A_1 \leq^u A_2 \Leftrightarrow A_1 \subset A_2 - C.$$

To convince the reader that Problem (u-BSOP) provides a bridge between set optimization problems and vector Ky Fan inequality problems, we consider some special cases of Problem (u-BSOP). First, we assume that $F : K \times K \rightrightarrows Y$ is a set-valued map with nonempty values and consider the following u-type and l-type set optimization problems (see [1–3]):

Problem (u-SOP): Find a point $x \in K$ such that $F(\eta) \leq^u F(x)$ with some $\eta \in K$ implies that $F(x) \leq^u F(\eta)$.

Problem (l-SOP): Find a point $x \in K$ such that $F(\eta) \leq^l F(x)$ with some $\eta \in K$ implies that $F(x) \leq^l F(\eta)$, where

$$A_1 \leq^l A_2 \Leftrightarrow A_2 \subset A_1 + C.$$

Clearly, Problem (u-SOP) is a special case of Problem (u-BSOP) with $F_1(x, \eta) = F(\eta)$ and $F_2(x, \eta) = F(x)$ for all $(x, \eta) \in K \times K$. Problem (l-SOP) is a special case of Problem (u-BSOP), where $F_1(x, \eta) = -F(x)$ and $F_2(x, \eta) = -F(\eta)$ for all $(x, \eta) \in K \times K$.

Consider now the generalized vector Ky Fan inequality Problem (P^1) of finding $x \in K$ such that $F_1(x, \eta) \not\subset -C\backslash\{0\}$ for all $\eta \in K$. Solutions of this problem are usually called efficient solutions. As shown in Proposition 3.1 below, Problem (P^1) is exactly a special case of Problem (u-BSOP).

Proposition 3.1 *Let* $F_2 \equiv \{0\}$. *Then, Problem* (u-BSOP) *is exactly the generalized vector Ky Fan inequality Problem* (P^1).

Proof Let x be a solution of Problem (u-BSOP) with $F_2 \equiv \{0\}$. If it is not a solution of Problem (P^1), then $F_1(x, \eta) \subset -C\backslash\{0\} \subset -C = F_2(x, \eta) - C$ for some $\eta \in K$. By the definition of x, $F_2(x, \eta) = 0 \in F_1(x, \eta) - C$, that is, $y \in C$ for some $y \in F_1(x, \eta)$. Together with the pointedness of C and the inclusion $F_1(x, \eta) \subset -C$, this yields $y = 0$. Therefore, $0 \in F_1(x, \eta)$, which is impossible.

Conversely, let x be a solution of Problem (P^1) and let $F_2 \equiv \{0\}$. Then, for each $\eta \in K$ such that $F_1(x, \eta) \subset F_2(x, \eta) - C \subset -C$, we have $0 \in F_1(x, \eta)$, since $F_1(x, \eta) \not\subset -C\backslash\{0\}$. Thus, $F_2(x, \eta) = 0 \in F_1(x, \eta) \subset F_1(x, \eta) - C$, showing that x is a solution of Problem (u-BSOP) with $F_2 \equiv \{0\}$. □

Consider now another generalized vector Ky Fan inequality Problem (P^2) of finding $x \in K$ such that $F_2(x, \eta) \cap [C\backslash\{0\}] = \emptyset$ for all $\eta \in K$.

Proposition 3.2 *Let* $F_1 \equiv \{0\}$. *Then, the set of solutions of Problem* (u-BSOP) *is contained in that of Problem* (P^2). *The converse statement holds true, if* F_2 *is single-valued.*

Proof The second conclusion of Proposition 3.2 is obvious. We need to prove only the first one. Let x be a solution of Problem (u-BSOP) with $F_1 \equiv \{0\}$. If it is not a solution of Problem (P^2), then there exist $\eta \in K$ and $y \in F_2(x, \eta)$ such that $y \in C\backslash\{0\}$. Since $0 = F_1(x, \eta) \in F_2(x, \eta) - [C\backslash\{0\}] \subset F_2(x, \eta) - C$, we get $F_2(x, \eta) \subset F_1(x, \eta) - C \subset 0 - C = -C$, by the definition of x. It follows that $y \in -C$. This is impossible, since C is pointed. □

This paper aims at giving an existence result for Problem (u-BSOP), which, as we have seen above, is a unifying framework for several problems in set optimization and vector Ky Fan inequalities. Our main result is established in Theorem 3.1, with the help of the KKM–Fan lemma in [9]. We show that this general result is useful to derive some new or sharper existence results in Problems (l-SOP), (u-SOP), (P^1) and (P^2).

In the sequel, unless otherwise specified, we always assume that X is a Hausdorff topological vector space, Y is a topological vector space, $C \subset Y$ is a pointed closed convex cone with a possibly empty interior, and $F_i : K \times K \rightrightarrows Y, i = 1, 2$, are set-valued maps with nonempty values. To provide an existence result for Problem (u-BSOP), we first define a set-valued map $\mathcal{D}_{12} : K \rightrightarrows K$ as follows:

$$\mathcal{D}_{12}(x) := \left\{\eta \in K : F_1(x, \eta) \leq^u F_2(x, \eta)\right\}, x \in K.$$

To each nonempty subset $K' \subset K$, we associate a set-valued map $D_{21} : K' \rightrightarrows K'$ as follows:

$$D_{21}(\eta) := \{x \in K' : F_2(x, \eta) \leq^u F_1(x, \eta)\}, \eta \in K'.$$

We give some verifiable conditions, under which D_{21} is a KKM-map on K' (Proposition 3.3) and D_{21} is closed-valued on K' (Proposition 3.4).

Proposition 3.3 *Let K' be a convex set. Then, under each of the following conditions (i)–(iii), D_{21} is a KKM-map on K':*

(i) *For all $x \in K'$,*

$$x \notin \text{conv}\left\{\eta \in K' : F_2(x, \eta) \nleq^u F_1(x, \eta)\right\};$$

(ii) *For all $x \in K'$, the set $\{\eta \in K' : F_2(x, \eta) \nleq^u F_1(x, \eta)\}$ is convex, and*

$$F_2(x, x) \leq^u F_1(x, x); \tag{1}$$

(iii) *For all $x \in K'$, the pair $(F_2(x, \cdot), F_1(x, \cdot))$ is properly C-quasiconcave–quasiconvex on K', and condition (1) holds.*

Proof It is easily verified that condition (i) implies that D_{21} is a KKM-map on K'. It remains to show that $(iii) \Rightarrow (ii) \Rightarrow (i)$.

$(ii) \Rightarrow (i)$. Assume to the contrary that (i) is violated. Then, there exists $x \in K'$ such that $x \in \text{conv}G(x)$, where

$$G(x) := \{\eta \in K' : F_2(x, \eta) \nleq^u F_1(x, \eta)\}.$$

By (ii), $G(x)$ is convex set. Therefore, $x \in G(x)$, a contradiction to (1).

$(iii) \Rightarrow (ii)$. It suffices to prove that, for all $x \in K'$, the set $G(x)$ is convex. Indeed, let $\eta_i \in G(x), i = 1, 2$. We need to prove that each point $\eta \in [\eta_1, \eta_2]$ belongs to $G(x)$. Indeed, since $\eta_i \in G(x), i = 1, 2$, there exist $y_i \in F_2(x, \eta_i), i = 1, 2$, with $y_i \notin F_1(x, \eta_i) - C, i = 1, 2$. Due to the proper C-quasiconcave–quasiconvexity of the pair $(F_2(x,\cdot), F_1(x,\cdot))$ on K', there exists $j \in \{1,2\}$ such that $y_j \in F_2(x,\eta) - C$ and $F_1(x,\eta) \subset F_1(x,\eta_j) - C$. If $\eta \notin G(x)$, then $F_2(x,\eta) \subset F_1(x,\eta) - C$. Making use of the above conditions, we obtain that

$$\begin{aligned} y_j &\in F_2(x,\eta) - C \\ &\subset F_1(x,\eta) - C - C \\ &\subset F_1(x,\eta_j) - C - C - C \\ &\subset F_1(x,\eta_j) - C. \end{aligned}$$

This contradicts the above assumption that $y_j \notin F_1(x,\eta_j) - C$. The convexity of the set $G(x)$ is thus proven, as desired. □

We give an example illustrating Proposition 3.3.

Example 3.1 Let $X = \mathrm{IR}$ (the real line), $Y = \mathrm{IR}^2$ (the plane), $C = \mathrm{IR}^2_+$ (the non-negative orthant) and $K' = [0,1]$. For $(x,\eta) \in K' \times K'$, define

$$\begin{aligned} F_1(x,\eta) &= [0,1] \times [0, x(x-\eta)] \subset Y, \\ F_2(x,\eta) &= [0, x\eta] \times [0, \eta - x] \subset Y. \end{aligned}$$

It is a simple matter to verify that condition (iii) [and hence, each of conditions (ii) and (i)] of Proposition 3.3 holds.

Proposition 3.4 *Let K' be closed in X and let $\eta \in K'$ be such that the restriction of $F_2(\cdot,\eta)$ to K' is $(-C)$-lsc, and the restriction of $F_1(\cdot,\eta)$ to K' is $(-C)$-usc and compact-valued. Then, $D_{21}(\eta)$ is closed in X.*

Proof It is easy to prove that $D_{21}(\eta)$ is closed in K', and hence it is closed in X (since K' is closed in X). □

The following theorem gives verifiable conditions for the existence of solutions of Problem (u-BSOP).

Theorem 3.1 *Let the following assumption be satisfied: if* $\mathrm{dom}\mathcal{D}_{12} = K$, *then there exists a convex set K' such that* $\mathrm{im}\mathcal{D}_{12} \subset K' \subset K$ *and*

(i) *For all $\eta \in K'$, either (i)$_1$ or (i)$_2$ holds, where*
   $(i)_1$ $\{x \in K' : F_2(x,\eta) \leq^u F_1(x,\eta)\}$ is closed in X;
   $(i)_2$ K' is closed in X, the restriction of $F_2(\cdot,\eta)$ to K' is $(-C)$-lsc on K', and the restriction of $F_1(\cdot,\eta)$ to K' is $(-C)$-usc and compact-valued on K';

(ii) *At least one of the conditions (i)–(iii) of Proposition 3.3 holds;*

(iii) (Coercivity condition) *There exists a nonempty compact set $A \subset K'$ such that, for each finite set $\mathcal{U}$ of points $\eta_i \in K'$, $i = 1, 2, \ldots, n$, there exists a nonempty compact convex set $B \subset K'$ such that $B \supset \mathcal{U}$ and, for all $x \in B \setminus A$, there exists $\eta \in B$ with $F_2(x,\eta) \not\leq^u F_1(x,\eta)$.*

Then, there exists a solution of Problem (u-BSOP).

Proof If dom$\mathcal{D}_{12} \neq K$, then there exists $x \in K$ with $\mathcal{D}_{12}(x) = \emptyset$, that is,

$$F_1(x, \eta) \not\leq^u F_2(x, \eta), \forall \eta \in K.$$

By definition, x is a solution of Problem (u-BSOP).

Now, let dom$\mathcal{D}_{12} = K$ and let K' be the set mentioned in Theorem 3.1. If

$$\cap_{\eta \in K'} D_{21}(\eta) \neq \emptyset, \tag{2}$$

then there exists a solution of Problem (u-BSOP). Indeed, (2) means that there exists $x \in K'$ such that

$$\forall \eta \in K', F_2(x, \eta) \leq^u F_1(x, \eta). \tag{3}$$

Now, let $\eta \in K$ and $F_1(x, \eta) \leq^u F_2(x, \eta)$. Clearly, the last condition proves that $\eta \in \mathcal{D}_{12}(x) \subset$ im$\mathcal{D}_{12}$. Combining with the assumption that im$\mathcal{D}_{12} \subset K'$, this yields $\eta \in K'$. By (3), $F_2(x, \eta) \leq^u F_1(x, \eta)$, proving that x is exactly a solution of Problem (u-BSOP).

So, it remains to verify condition (2). By Proposition 3.4, $(i)_2$ implies $(i)_1$. Hence, $D_{21}(\eta)$ is closed in X, for all $\eta \in K'$. To prove (2), it suffices to show that $A \cap [\cap_{\eta \in K'} D_{21}(\eta)] \neq \emptyset$, where A is the set mentioned in (iii). Since A is compact and each member of the family of sets $D_{21}(\eta)$, $\eta \in K'$, is closed, it is enough to show that the intersection of A and an arbitrary finite subfamily of this family is nonempty. Indeed, assume that $D_{21}(\eta_i)$, $i = 1, 2, \ldots, n$, is such a finite subfamily, where $\eta_i \in K'$, $i = 1, 2, \ldots, n$. Let $B \subset K'$ be the compact convex set mentioned in (iii). Consider the set-valued map $Q(\cdot) := B \cap D_{21}(\cdot)$, defined on B. We claim that this is a KKM-map on B. Indeed, let $x_i \in B \subset K'$, $i = 1, 2, \ldots, n$, and let $x \in \text{conv}\{x_i, i = 1, 2, \ldots, n\}$. Then, by the convexity of B, $x \in B$. By Proposition 3.3, D_{21} is a KKM-map on K'. So, there exists $j \in \{1, 2, \ldots, n\}$ with $x \in D_{21}(x_j)$. As a result, $x \in Q(x_j)$, and hence, $Q(\cdot)$ is a KKM-map on B. In particular, for each $\eta \in B$, $Q(\eta)$ is nonempty, since it is a value of a KKM-map. Furthermore, it is a compact set, since it is the intersection of the compact B and the closed set $D_{21}(\eta)$. Applying the KKM–Fan lemma (Lemma 2.1) to $Q(\cdot)$ yields $\cap_{\eta \in B} Q(\eta) \neq \emptyset$, that is, there exists $x \in B$ with $F_2(x, \eta) \leq^u F_1(x, \eta)$ for all $\eta \in B$. By (iii), $x \in A$. Furthermore, $x \in Q(\eta_i)$ for all $i = 1, 2, \ldots, n$. Thus, $A \cap [\cap_1^n Q(\eta_i)] \neq \emptyset$, and our proof is complete. □

Remark 3.1 The coercivity condition (iii) is automatically satisfied, if K' is compact and convex, since (iii) holds with $A = B = K'$. The coercivity condition (iii) is motivated by the corresponding one in [22]. A similar coercivity condition is also used in [23]. Among several coercivity conditions in generalized vector Ky Fan inequalities, we mention only the coercivity condition used in Theorem 1 of [24]. A modification of this condition, applied to Problem (u-BSOP), can be formulated as follows: there exist a nonempty compact set $A \subset K'$ and a compact convex set $B' \subset K'$ such that, for each $x \in K' \backslash A$, there exists $\eta \in B'$ with $F_2(x, \eta) \not\leq^u F_1(x, \eta)$. This kind of coercivity condition implies (iii), since for each finite set $\mathcal{U}$ of points $\eta_i \in K'$, $i = 1, 2, \ldots, n$, we can take B to be the convex hull of $B' \cup \{\eta_i, i = 1, 2, \ldots, n\}$. The proof of (2) is inspired by that of Theorem 1 of [24].

Remark 3.2 From the formulation and the proof of Theorem 3.1, it is clear that, if $\mathrm{dom}\mathcal{D}_{12} \neq K$, then there always exists a solution of Problem (u-BSOP). In other words, if $\mathrm{dom}\mathcal{D}_{12} \neq K$, then, for the solution existence of Problem (u-BSOP), the assumption on existence of a set K' satisfying the conditions mentioned in Theorem 3.1 becomes superfluous.

Remark 3.3 The smallest set $K' \subset K$, that can be taken as a candidate for K' in checking conditions (i)–(iii) of Theorem 3.1, is equal to $\mathrm{im}\mathcal{D}_{12}$. An advantage of using a set $K' \subset K$, larger than $\mathrm{im}\mathcal{D}_{12}$, is that sometimes finding $\mathrm{im}\mathcal{D}_{12}$ is more difficult than majorizing it by a set $K' \subset K$ satisfying (i)–(iii). For example, if conditions (i)–(iii) hold with $K' = K$, then we can obtain a solution existence, without finding $\mathrm{im}\mathcal{D}_{12}$.

Observe furthermore that, if K' is a strict subset of K, then all the requirements, imposed on the data of our problem outside the set K', are not needed for our existence result. Example 3.5 below shows that such a case may happen.

Remark 3.4 Let us give some remarks on the assumptions of Theorem 3.1. Conditions, similar to assumptions (i)–(iii) of Theorem 3.1, are familiar and often used in the theory of vector Ky Fan inequalities. The use of conditions (i)–(iii) allows us to obtain existence results for vector Ky Fan inequalities (see Corollaries 3.3, 3.4 below) under assumptions closely related to the earlier ones. On the other hand, up to now, conditions of the kind of assumptions (ii) and (iii) did not appear in studies of solution existence in set optimization problems. So, the existence results for these problems in Corollaries 3.1 and 3.2 below are new and are different from the earlier ones in [2–6,8]. The proof of Theorem 3.1 shows that traditional tools and approaches in generalized vector Ky Fan inequalities may be useful also for set optimization problems.

To apply Theorem 3.1, it is required to know the sets $\mathrm{dom}\mathcal{D}_{12}$ and $\mathrm{im}\mathcal{D}_{12}$. However, if K is convex, then we can use Theorem 3.1 without knowing these sets, since in this case, it suffices to set $K' = K$ and to verify conditions (i)–(iii) with $K' = K$. Since the convexity of K is often encountered in practice, the above remark is useful for a broad class of practical problems (as we have seen in Remarks 3.2, 3.3, we may deal with superfluous conditions when using Theorem 3.1 with $K' = K$ or $\mathrm{dom}\mathcal{D}_{12} \neq K$).

Now, let us show some special versions of Theorem 3.1 with verifiable conditions (i)–(iii). If the set K' in this theorem is compact, then, as we have seen from Remark 3.1, we can ignore condition (iii). If we use the case $(\mathrm{i})_2$ of condition (i) of Theorem 3.1, then we can check it by using the known notions of cone-semicontinuities of the original data $F_i, i = 1, 2$. Similarly, we can check condition (ii) of Theorem 3.1 with the help of the verifiable condition (iii) of Proposition 3.3, that is, to check the proper C-quasiconcavity–quasiconvexity of the pair $(F_2(x, \cdot), F_1(x, \cdot))$ for each fixed $x \in K' \subset K$. In set optimization problems, checking the just mentioned property becomes easier. Indeed, if the objective in Problem (l-SOP) is described by the set-valued map $F : K \rightrightarrows Y$, then $F_1(x, \eta) = -F(x)$ and $F_2(x, \eta) = -F(\eta)$ for all $(x, \eta) \in K \times K$. Therefore, for each fixed $x \in K' \subset K$, $F_1(x, \cdot)$ is a constant map, and checking the proper C-quasiconcavity–quasiconvexity of the pair $(F_2(x, \cdot), F_1(x, \cdot))$ leads to checking the proper C- quasiconcavity of set-valued map $-F(\cdot)$ (see Remark 2.2(ii)), which is a familiar notion of [21]. Similarly, checking the mentioned property of the pair $(F_2(x,\cdot), F_1(x,\cdot))$ in Problem (u-SOP) leads to checking the proper C-quasiconvexity of $F(\cdot)$. Checking the proper C-quasiconcavity–quasiconvexity of the pair $(F_2(x,\cdot), F_1(x,\cdot))$ in the case of generalized vector Ky Fan inequalities becomes easier, too (see condition (ii)$_3$ of Corollaries 3.3, 3.4).

In the rest of this section, we will derive from Theorem 3.1 some important consequences providing sufficient conditions for the existence of solutions in set optimization problems and vector Ky Fan inequalities. We begin with the case of Problem (l-SOP).

Corollary 3.1 *Let $K \subset X$ be a nonempty convex set and let $F : K \rightrightarrows Y$ have nonempty values. Let the following conditions be satisfied:*

(i) *Either for all $\eta \in K$, $\{x \in K : F(x) \preceq^l F(\eta)\}$ is closed in X; or K is closed in X and F is C-usc and compact-valued on K;*
(ii) *Either for all $x \in K$, $\{\eta \in K : F(x) \npreceq^l F(\eta)\}$ is convex; or F is properly $(-C)$-quasiconcave on K;*
(iii) (Coercivity condition) *There exists a nonempty compact set $A \subset K$ such that, for each finite set $\mathcal{U}$ of points $\eta_i \in K$, $i = 1, 2, \ldots, n$, there exists a nonempty compact convex set $B \subset K$ such that $B \supset \mathcal{U}$ and, for all $x \in B \setminus A$, there exists $\eta \in B$ with $F(x) \npreceq^l F(\eta)$.*

Then, there exists a solution of Problem (l-SOP).

Proof This is a direct consequence of Theorem 3.1 with $F_1(x,\eta) = -F(x)$ and $F_2(x,\eta) = -F(\eta)$ for all $(x,\eta) \in K \times K$. Observe that, due to $-F(x) \preceq^l -F(x), \forall x \in K$, we obtain $\mathrm{dom}\mathcal{D}_{12} = K$, $\mathrm{im}\mathcal{D}_{12} = K$, and (1) is automatically satisfied. □

Remark 3.5 The first existence result for Problem (l-SOP), in the case of non-compactness of K, is given in Theorem 4.2 of [3], where K is a complete metric space. For the more general case of K (that is, the case where K is not necessarily a complete metric space), an existence result is established in Proposition 30 of [4] under some cone-semicompactness assumption. This assumption is difficult to handle (see Section 6 of [5]), and, in [4,5], we do not find sufficient conditions for cone-semicompactness, except for the case of compactness of K (see Proposition 29 of [4] and Theorem 5.3 of [5], where K is compact and F is upper C-semicontinuous). If K is compact and F satisfies some C-semicontinuity assumption, existence results are formulated in Theorems 5.8–5.9 and Corollaries 5.5–5.7 of [5]. We recall that the first existence result for Problem (l-SOP), in the case of compactness of K, is given in Theorem 4.1 of [3]. Under an extra assumption that C has a nonempty interior, existence results for the general case of K can be found also in Theorems 5.3 and 5.4 of [6].

The assumptions (and the approach) we use in Corollary 3.1 are originated from those in the theory of vector Ky Fan inequalities and are quite different from the corresponding ones in the above-mentioned results of [3–6]. Example 3.2 below shows that sometimes our Corollary 3.1 is useful, while Theorem 4.2 of [3] and Theorems 5.3 and 5.4 of [6] are not. Observe, furthermore, that all the existence results, obtained, e.g., in Theorem 4.1 of [3], Theorems 5.8–5.9 and Corollaries 5.5–5.7 of [5], cannot be applied in this example, since the assumption on compactness of K in these results is violated.

Example 3.2 In Problem (l-SOP), take $X = \mathrm{IR}$, $C = \{0\} \times \mathrm{IR}_+ \subset Y = \mathrm{IR}^2$, and $K = \mathrm{IR}_+$ (the non-negative half-line). Furthermore, we set

$$F(x) = \begin{cases}]0,1[\times]x,\infty[, & \text{if} \quad x \in K\backslash\{0\}, \\ ]0,1[\times\{0\}, & \text{if} \quad x = 0. \end{cases}$$

Obviously, the first part of condition (i) of Corollary 3.1 is satisfied, since the set $\{x \in K : F(x) \leq^l F(\eta)\} = [0, \eta]$ is closed for all $\eta \in K$. Furthermore, we can see easily that F is properly $(-C)$-quasiconcave on K, that is, the second part of condition (ii) holds. To apply Corollary 3.1, it remains to verify condition (iii). We take $A = [0, 1]$, and we set $\eta' = \max\{\eta_i, i = 1, 2, \ldots, n\}$, where η_i, $i = 1, 2, \ldots, n$, are arbitrary points of K. Define

$$B = \begin{cases} A, & \text{if} \quad \eta' \leq 1, \\ [0, \eta'], & \text{if} \quad \eta' > 1. \end{cases}$$

For $x \in [B\backslash A] =]1, \eta']$, we take $\eta = 1 \in B$, and we have $F(x) \nleq^l F(\eta)$. This shows that condition (iii) of Corollary 3.2 holds, as required.

Observe that, in our example, C has an empty interior, and so, Theorems 5.3 and 5.4 of [6] are inapplicable. Furthermore, $F(\cdot) + C$ is not closed-valued on K, and hence, Theorem 4.2 of [3] is not useful. Finally, as we said above, the non-compactness of K in our example excludes the possibility of applying the existence results in Theorem 4.1 of [3], Theorems 5.8–5.9 and Corollaries 5.5–5.7 of [5].

Corollary 3.2 *Let $K \subset X$ be a nonempty convex set and let $F : K \rightrightarrows Y$ have nonempty values. Let the following conditions be satisfied:*

(i) *Either for all $\eta \in K$, $\{x \in K : F(x) \leq^u F(\eta)\}$ is closed in X; or K is closed in X and F is $(-C)$-lsc and compact-valued on K;*
(ii) *Either for all $x \in K$, $\{\eta \in K : F(x) \nleq^u F(\eta)\}$ is convex; or F is properly C-quasiconvex on K;*
(iii) (Coercivity condition) *There exists a nonempty compact set $A \subset K$ such that, for each finite set $\mathcal{U}$ of points $\eta_i \in K$, $i = 1, 2, \ldots, n$, there exists a nonempty compact convex set $B \subset K$ such that $B \supset \mathcal{U}$ and, for all $x \in B\backslash A$, there exists $\eta \in B$ with $F(x) \nleq^u F(\eta)$.*

Then, there exists a solution of Problem (u-SOP).

Proof This is immediate from Theorem 3.1 with $F_1(x, \eta) = F(\eta)$ and $F_2(x, \eta) = F(x)$ for all $(x, \eta) \in K \times K$. Observe that, due to $F(x) \leq^u F(x)$, $\forall x \in K$, we obtain $\mathrm{dom}\mathcal{D}_{12} = K$, $\mathrm{im}\mathcal{D}_{12} = K$, and (1) is automatically satisfied. □

Remark 3.6 Our discussion is similar to that in Remark 3.5. The first existence result for Problem (u-SOP), in the case of non-compactness of K, is given in Theorem

4.4 of [3], where K is a complete metric space. For normed spaces, it is established under some cone-regularity assumptions in Proposition 22 of [4] (see also Theorem 2.6 of [4]). However, these cone-regularity assumptions are difficult to handle, and in [4], we do not find sufficient conditions for cone-regularity, except for the case of compactness of K (see Proposition 21 of [4], where K is compact and F is a lower C-semicontinuous set-valued map such that $F(\cdot) - C$ is closed-valued). If K is compact, existence results are given in Theorem 4.3 of [3] and Corollary 24 of [4].

Our Corollary 3.2 is useful as a new choice in verifying existence of solutions in Problem (u-SOP), with assumptions originated in the theory of vector Ky Fan inequalities and different from the known ones in set optimization. Example 3.3 below shows that sometimes our Corollary 3.2 is applicable, while Theorem 4.4 of [3] is not. Furthermore, since the set K in this example is non-compact, all the known results for Problem (u-SOP), given in Theorem 4.3 of [3] and Corollary 24 of [4], cannot be used here.

Example 3.3 In Problem (u-SOP), take $X = \mathrm{IR}$, $C = \{0\} \times \mathrm{IR}_+ \subset Y = \mathrm{IR}^2$, and $K = \mathrm{IR}_+$. Furthermore, we set

$$F(x) = \begin{cases}]0,1[\times]0,x[, & \text{if} \quad x \in K\backslash\{0\}, \\ ]0,1[\times\{0\}, & \text{if} \quad x = 0. \end{cases}$$

Obviously, the first part of condition (i) of Corollary 3.2 is satisfied, since the set $\{x \in K : F(\eta) \leq^l F(x)\} = [0, \eta]$ is closed for all $\eta \in K$. Furthermore, we can see easily that F is properly $(-C)$-quasiconvex on K, that is, the second part of condition (ii) holds. To apply Corollary 3.2, it remains to verify condition (iii). Let A and B be defined as in Example 3.2. For $x \in [B\backslash A] =]1, \eta']$, we set $\eta = 0 \in B$, and we obtain $F(x) \not\leq^u F(\eta)$, as desired.

In our example, $F(\cdot) - C$ is not closed-valued on K, and so, Theorem 4.4 of [3] is not useful. Furthermore, as we said above, the non-compactness of K excludes the possibility of applying Theorem 4.3 of [3] and Corollary 24 of [4] to our example.

Remark 3.7 The second part of both assumptions (i) and (ii) of Corollary 3.2 is verifiable; see our discussion in Remark 3.4.

The following example illustrates Remark 3.7.

Example 3.4 Consider Problem (u-SOP), where $X = \mathrm{IR}$, $Y = \mathrm{IR}^2$, $C = \mathrm{IR}^2_+$ and $F(x) = \{x^2\} \times [0, x] \subset Y$ for all $x \in K = \mathrm{IR}_+$. Clearly, K is closed and convex, and F is lsc and compact-valued on K. Furthermore, F is properly C-quasiconvex on K. So, the second part of both assumptions (i) and (ii) of Corollary 3.2 is satisfied. To apply this corollary, it remains to verify condition (iii). Let $A = [0, 1]$, and, for all $\eta_i \in K, i = 1, 2, \ldots, n$, let B be defined as in Example 3.3. For $x \in [B\backslash A] =]1, \eta']$, we take $\eta = 1 \in B$, and we have $F(x) \not\leq^u F(\eta)$, proving that condition (iii) of Corollary 3.2 holds, as required.

For $F_2 \equiv 0$, we write $\mathcal{D}_1$ instead of $\mathcal{D}_{12}$. More precisely,

$$\mathcal{D}_1(x) := \{\eta \in K : F_1(x, \eta) \subset -C\}, x \in K.$$

Corollary 3.3 *Let the following assumption be satisfied: if* $\mathrm{dom}\mathcal{D}_1 = K$*, then there exists a convex set* K' *such that* $\mathrm{im}\mathcal{D}_1 \subset K' \subset K$ *and*

(i) *For all* $\eta \in K'$*, either* $(i)_1$ *or* $(i)_2$ *holds, where*
   $(i)_1$ $\{x \in K' : F_1(x,\eta) \cap C \neq \emptyset\}$ *is closed in* X*;*
   $(i)_2$ K' *is closed in* X*, and the restriction of* $F_1(\cdot,\eta)$ *to* K' *is* $(-C)$*-usc and compact-valued on* K'*;*

(ii) *At least one of the conditions* $(ii)_1$*–*$(ii)_3$ *holds, where*
   $(ii)_1$ *For all* $x \in K'$, $x \notin \mathrm{conv}\{\eta \in K' : F_1(x,\eta) \cap C = \emptyset\}$*;*
   $(ii)_2$ *For all* $x \in K'$, $F_1(x,x) \cap C \neq \emptyset$*, and the set* $\{\eta \in K' : F_1(x,\eta) \cap C = \emptyset\}$ *is convex;*
   $(ii)_3$ *For all* $x \in K'$, $F_1(x,x) \cap C \neq \emptyset$*, and the set-valued map* $F_1(x,\cdot)$ *is properly* C*-quasiconvex on* K'*;*

(iii) (Coercivity condition) *There exists a nonempty compact set* $A \subset K'$ *such that, for each finite set* $\mathcal{U}$ *of points* $\eta_i \in K'$, $i = 1, 2, \ldots, n$*, there exists a nonempty compact convex set* $B \subset K'$ *such that* $B \supset \mathcal{U}$ *and, for all* $x \in B \setminus A$*, there exists* $\eta \in B$ *with* $F_1(x,\eta) \cap C = \emptyset$*.*

Then, there exists a solution of Problem (P^1).

This is immediate from Proposition 3.1 and Theorem 3.1.

Remark 3.8 Existence results in some problems more general than Problem (P^1) can be found, e.g., in Theorem 3.2 of [15] and Theorem 3.2 of [16]. The main difference between Theorem 3.2 of [15] and our Corollary 3.3 is that Theorem 3.2 of [15] assumes the existence of some continuous strongly C-monotonic function, while our Corollary 3.3 does not. Theorem 3.2 in [16], applied to Problem (P^1), requires that C has a base, and F_1 is convex-valued and continuous in both variables $(x,\eta) \in K \times K$. All these requirements are not needed for our Corollary 3.3.

The following example shows that Corollary 3.3 may be applicable for K' smaller than K, while it is inapplicable for $K' = K$.

Example 3.5 In Problem (P^1), let $X = \mathrm{IR}$, $Y = \mathrm{IR}^2$, $C = \mathrm{IR}^2_+$ and K be the union of K_1 and K_2, where $K_1 = [0,2]$, $K_2 =]2,3[\cup]3,4[$. Furthermore, for $(x,\eta) \in K \times K$, we set $F_1(x,\eta) = [-1,0] \times [-1, g(x,\eta)] \subset Y$, where

$$g(x,\eta) = \begin{cases} x - \eta, & \text{if } x \in K_1, \eta \in K_1, \\ \eta - x, & \text{if } x \in K_1, \eta \in K_2, \\ \eta^2 - x, & \text{if } x \in K_2, \eta \in K. \end{cases}$$

It is easy to verify that, in our example, $\mathrm{dom}\mathcal{D}_1 = K$ and $\mathrm{im}\mathcal{D}_1 \subset [0,2] \subset K$. Taking $K' = [0,2] = K_1$, we see that K' is a compact convex set satisfying all conditions of Corollary 3.3, and hence Problem (P^1) has a solution. A direct computation shows that $x = 2$ is a solution of this problem. Observe that, if we take $K' = K$, then some assumptions of Corollary 3.3 are violated: firstly, $K' = K$ is nonconvex; and secondly, condition (i) does not hold for $\eta \in K_2$ (since $\{x \in K' = K : F_1(x,\eta) \cap C \neq \emptyset\} = K$, and K is not closed).

For $F_1 \equiv 0$, we write $\mathcal{D}_2$ instead of $\mathcal{D}_{12}$. More precisely,

$$\mathcal{D}_2(x) := \{\eta \in K : F_2(x, \eta) \cap C \neq \emptyset\}, x \in K.$$

Corollary 3.4 *Let the following assumption be satisfied: if* $\mathrm{dom}\mathcal{D}_2 = K$*, then there exists a convex set* K' *such that* $\mathrm{dom}\mathcal{D}_2 \subset K' \subset K$ *and*

(i) *For all* $\eta \in K'$*, either* $(i)_1$ *or* $(i)_2$ *holds, where*
  $(i)_1$ $\{x \in K' : F_2(x, \eta) \subset -C\}$ *is closed in* X*;*
  $(i)_2$ K' *is closed in* X*, and the restriction of* $F_2(\cdot, \eta)$ *to* K' *is* $(-C)$*-lsc on* K'*;*

(ii) *At least one of the conditions* $(ii)_1$*–*$(ii)_3$ *holds, where*
  $(ii)_1$ *For all* $x \in K'$, $x \notin \mathrm{conv}\{\eta \in K' : F_2(x, \eta) \not\subset -C\}$*;*
  $(ii)_2$ *For all* $x \in K'$, $F_2(x, x) \subset -C$*, and the set* $\{\eta \in K' : F_2(x, \eta) \not\subset -C\}$ *is convex;*
  $(ii)_3$ *For all* $x \in K'$, $F_2(x, x) \subset -C$*, and the set-valued map* $F_2(x, \cdot)$ *is properly* C*-quasiconcave on* K'*.*

(iii) (Coercivity condition) *There exists a nonempty compact set* $A \subset K'$ *such that, for each finite set* $\mathcal{U}$ *of points* $\eta_i \in K'$, $i = 1, 2, \ldots, n$*, there exists a nonempty compact convex set* $B \subset K'$ *such that* $B \supset \mathcal{U}$ *and, for all* $x \in B \backslash A$*, there exists* $\eta \in B$ *with* $F_2(x, \eta) \not\subset -C$*.*

Then, there exists a solution of Problem (P^2)*.*

This is immediate from Proposition 3.2 and Theorem 3.1.

Remark 3.9 In Corollary 3.4, F_2 is set-valued, and C may have a possibly empty interior. If F_2 is single-valued, and C has a nonempty interior, existence results for Problem (P^2) can be found in [17] for vector variational inequalities and in [18] for vector equilibrium problems. Existence results in some problems more general than Problem (P^2) can be found in Theorems 3.1 and 3.2 of [19], under the assumption on existence of a continuous strongly monotonic function satisfying certain conditions. Such an assumption is not used in Corollary 3.4.

4 Conclusions

The existence of solutions is independently developed in set optimization problems and generalized vector Ky Fan inequality problems. In this paper, we introduce a general model, that establishes a bridge between these problems, and discuss the existence of solutions of this model. Our main existence result (Theorem 3.1) is original and is the first one indicating that some traditional tools and approaches in generalized vector Ky Fan inequalities may be useful also for set optimization problems. More precisely, Theorem 3.1, applied to both Problems (l-SOP) and (u-SOP) in set optimization, yields some new results that, in some circumstances, can be useful, while some earlier ones cannot (see our detailed discussion in Remarks 3.5, 3.6 and Examples 3.2, 3.3). Moreover, with the help of Theorem 3.1, we can establish existence results in new classes of vector Ky Fan inequalities (see Corollaries 3.3, 3.4) and discover superfluous assumptions in some earlier results.

Our approach is quite different from the corresponding ones to the solution existence in set optimization. The main tool of this paper is the KKM–Fan lemma ([9]) that plays a crucial role in the theory of vector Ky Fan inequalities and, up to now, that is not used in set optimization. Since there are other powerful tools in the theory of Ky Fan inequalities, it is expected that their use in our model can lead to other significant results on existence of solutions in both set optimization and vector Ky Fan inequality theory. More general models and stability studies will be discussed in subsequent papers.

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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