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On the Weak Semi-continuity of Vector Functions and Minimum Problems

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Abstract Lower semi-continuity from above or upper semi-continuity from below has been used by many authors in recent papers. In this paper, we first study the weak semi-continuity for vector functions having particular form as that of Browder in ordered normed vector spaces; we obtain several new results on the lower semi-continuity from above or upper semi-continuity from below for these vector functions. Our results generalize some well-known results of Browder in scalar case. Secondly, we study the minimum or maximum problems for vector functions satisfying lower semi-continuous from above or upper semi-continuous from below conditions; several new results on the existence of minimal points or maximal points are obtained. We also use these results to study vector equilibrium problems and von Neumann's minimax principle in ordered normed vector spaces.

Keywords Lower semi-continuous from above · Upper semi-continuous from below · Minimum · von Newman's minimax principle

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1 Introduction

Vector optimization problems have been extensively studied by many authors in the past years; see Cesari and Suryanarayana [1–3], Yu [4], Wagner [5], Hartley [6], and Corley [7,8], see also [9]. Semi-continuity plays important role in variational problems, see [10,11] and the references therein. A definition of generalized semi-continuity, which is called lower semi-continuous from above, was introduced by Chen et al. [12] to study the minimum of a convex function in reflexive Banach spaces. Lower semi-continuous from above functions have been used and generalized by many authors to study variational problems, optimize problems, equilibrium problems and fixed point problems, see [13–23]. This concept has also been generalized by Khanh and Quy [24–26] to study variational problems for vector functions, see also Chen et al. [27]. Until now, the study of a function to be lower semi-continuous from above is few. In this paper, we first study the lower (respectively, upper) semi-continuity from above (respectively, below) for vector functions in ordered normed vector spaces; then, we study minimum problems and equilibrium problems for vector functions under these semi-continuity conditions. We obtain some results on semi-continuity of vector functions which generalize the corresponding results of Browder [10] in scalar case. Results on the existence of minimal (respectively, maximal) points are also obtained under lower (respectively, upper) semi-continuous from above (respectively, below) conditions. Applications to equilibrium problems and von Newman's minimax principle are also given.

The rest part of this paper is organized as follows: In Sect. 2, we recall some definitions and notations. In Sect. 3, we discuss the lower (respectively, upper) semi-continuity from above (respectively, below) of some vector functions in ordered normed spaces. In Sect. 4, we study the existence of minimal (respectively, maximal) points for lower (respectively, upper) semi-continuous from above (respectively, below) functions; applications to vector equilibrium problems and von Newman's minimax principle are also given. Section 5 contains the concluding remarks.

2 Definitions of Semi-continuous Functions and Notations

In this section, we recall some definitions that will be used in the rest of this paper. Let F be a topological vector space, $C \subset F$ is said to be a closed cone iff C is closed and convex, and $\lambda C \subseteq C$ for all $\lambda > 0$, and $C \cap (-C) = \{0\}$. The partial order $\leq$ induced by C on F is defined by $x \leq y$ iff $y - x \in C$.

Definition 2.1 (*see* [4]) Let X, F be topological vector spaces, $C \subset F$ a cone. $\leq$ is the partial order induced by C on F.

(i) A vector-valued function $\phi : X \to F$ is said to be cone convex iff $\phi(\alpha x + \beta y) \leq \alpha\phi(x) + \beta\phi(y)$, equivalently, $\alpha\phi(x) + \beta\phi(y) - \phi(\alpha x + \beta y) \in C$ for all $x, y \in D(\phi)$, $\alpha > 0$, $\beta > 0$ satisfying $\alpha + \beta = 1$, where and after $D(\phi)$ is always the domain of ϕ;

(ii) A vector-valued function $\phi : X \to F$ is said to be cone concave iff $\alpha\phi(x) + \beta\phi(y) \leq \phi(\alpha x + \beta y)$, equivalently, $\phi(\alpha x + \beta y) - \alpha\phi(x) - \beta\phi(y) \in C$ for all $x, y \in D(\phi)$, $\alpha > 0$, $\beta > 0$ satisfying $\alpha + \beta = 1$;

(iii) A vector-valued function $\phi : X \to F$ is said to be quasi cone convex iff $\{x; \phi(x) < \alpha\}$, equivalently, $\{x; \alpha - \phi(x) \in C \setminus \{0\}\}$, is convex for each $\alpha \in F$;
(iv) A vector-valued function $\phi : X \to F$ is said to be quasi cone concave iff $\{x; \alpha < \phi(x)\}$, equivalently, $\{x; \phi(x) - \alpha \in C \setminus \{0\}\}$, is convex for each $\alpha \in F$.

The following definition of a cone lower semi-continuous function was essentially introduced by Corley [7].

Definition 2.2 Let X be a topological space, F a topological vector space, $C \subset F$ a cone. $\leq$ is the partial order induced by C on F.

(i) A vector-valued function $f : X \to F$ is said to be cone lower semi-continuous iff $f^{-1}(y - C)$ is closed for all $y \in F$;
(ii) A vector-valued function $f : X \to F$ is said to be cone upper semi-continuous iff $f^{-1}(y + C)$ is closed for all $y \in F$.

Definition 2.3 (*see* [12]) Let X be a topological space. A function $f : X \to \mathbb{R}$ is said to be sequentially lower semi-continuous from above at x_0 iff for any $x_n \to x_0$, $f(x_{n+1}) \leq f(x_n)$ implies that $f(x_0) \leq \lim_{n\to\infty} f(x_n)$. Similarly, f is said to be sequentially upper semi-continuous from below at x_0 iff for any $x_n \to x_0$, $f(x_{n+1}) \geq f(x_n)$ implies that $f(x_0) \geq \lim_{n\to\infty} f(x_n)$.

One can easily see that a strictly increasing or decreasing function on $\mathbb{R}$ is both sequentially lower semi-continuous form above and upper semi-continuous from below. It was proved in [12] that a sequentially lower semi-continuous from above convex function in a reflexive Banach space obtains it is minimum if it is also coercive. Some well-known results such as Ekland's variational principle, Caristi's fixed point theorem and von Neumann's minimax principle are also true under lower semi-continuous from above condition, see [12,14] for details. Recently, lower semi-continuous from above functions has been used and generalized by many authors to study variational problems, optimization problems, fixed point problems, equilibrium problems and Ekland's variational principle, see [13,15–23]. Khanh and Quy have generalized this concept to study variational problem for vector functions; we recall it as the following.

Definition 2.4 (*see* [24,27]) Let X be a topological space, F a topological vector space, and $C \subset F$ a cone. $\leq$ is the partial order induced by C on F.

(i) A function $f : X \to F$ is said to be sequentially cone lower semi-continuous from above at x_0 iff for any $x_n \to x_0$, $f(x_{n+1}) \leq f(x_n)$ imply that $f(x_0) \leq f(x_n)$ for $n = 1, 2, \ldots$;
(ii) f is said to be sequentially cone upper semi-continuous from below at x_0 iff $x_n \to x_0$, $f(x_{n+1}) \geq f(x_n)$ imply that $f(x_0) \geq f(x_n)$ for $n = 1, 2, \ldots$.

Clearly, cone lower semi-continuous introduced by [7] implies that cone lower semi-continuous from above, but the reverse is not true. See [12,24].

The following result follows directly from Definition 2.4.

Proposition 2.1 *Let X be a topological space, F a topological vector space, $C \subset F$ a closed cone. $\leq$ is the partial order induced by C on F, and $T : X \to F$, $f : F \to F$ are two maps. We have the following conclusions:*

(i) Suppose $f(x) > f(y)$ if and only if $x > y$. Then f is both sequentially cone lower semi-continuous from above and cone upper semi-continuous from below on F.
(ii) If T is continuous and f is sequentially cone lower semi-continuous from above (respectively, cone upper semi-continuous from below); then, $f(Tx) : X \to F$ is sequentially cone lower semi-continuous from above (respectively, cone upper semi-continuous from below).

Definition 2.5 Let X be a topological space, F a topological vector space, $C \subset F$ a closed cone. $\leq$ is the partial order induced by C on F. Let $\phi(x) : D(\phi) \subseteq X \to F$ be a function, and $epi\phi = \{(x, \alpha) \in X \times F, \phi(x) \leq \alpha\}$.

(i) $epi\phi$ is said to be sequentially closed from above iff $(x_n, \alpha_n) \in epi\phi$, $n = 1, 2, \ldots$, $(x_n, \alpha_n) \to (x_0, \alpha_0)$, and $\phi(x_1) \geq \phi(x_2) \geq \cdots \phi(x_n) \geq \cdots$, then $(x_0, \alpha_0) \in epi\phi$;
(ii) $epi\phi$ is said to be sequentially closed from below iff $(x_n, \alpha_n) \in epi\phi$, $n = 1, 2, \ldots$, $(x_n, \alpha_n) \to (x_0, \alpha_0)$, and $\phi(x_1) \leq \phi(x_2) \leq \cdots \phi(x_n) \leq \cdots$, then $(x_0, \alpha_0) \in epi\phi$.

Proposition 2.2 *Let X be a topological space, F a topological vector space. $C \subset F$ a closed cone. $\leq$ is the partial order induced by C on F. Let $\phi(x) : D(\phi) \subseteq X \to F$ be a function, then we have the following conclusions:*

(i) ϕ is sequentially cone lower semi-continuous from above on X iff $epi\phi$ is sequentially closed from above;
(ii) If the range of ϕ is well ordered, then $epi\phi$ is sequentially closed iff $epi\phi$ is both sequentially closed from below and above.

Proof (i) Necessity, if $(x_n, \alpha_n) \in epi\phi$, $n = 1, 2, \ldots$, $(x_n, \alpha_n) \to (x_0, \alpha_0)$, and $\phi(x_1) \geq \phi(x_2) \geq \cdots \phi(x_n) \geq \cdots$, then by the cone lower semi-continuity from above of ϕ, we have $\phi(x_0) \leq \phi(x_n)$, $n = 1, 2, \ldots$.

In view of $\phi(x_n) \leq \alpha_n$, $n = 1, 2, \ldots$, we get $\phi(x_0) \leq \alpha_0$, i.e. $(x_0, \alpha_0) \in epi\phi$, $epi\phi$ is sequentially closed from above.

Sufficiency, if $x_n \to x_0$, and $\phi(x_1) \geq \phi(x_2) \geq \cdots \geq \phi(x_n) \geq \cdots$, then for each integer k, $(x_n, \phi(x_k)) \in epi\phi$ for $n \geq k$. It is obviously that $(x_n, \phi(x_k)) \to (x_0, \phi(x_k))$ as $n \to \infty$, by sequentially closedness from above of ϕ, we have $(x_0, \phi(x_k)) \in epi\phi$, thus $\phi(x_0) \leq \phi(x_k)$, $k = 1, 2, \ldots$, i.e. ϕ is sequentially cone lower semi-continuous from above.

(ii) Necessity is obvious; we only need to prove sufficiency. Assume that $(x_n, \alpha_n) \in epi\phi$ such that $(x_n, \alpha_n) \to (x_0, \alpha_0)$. Since the range of ϕ is well ordered, by taking a subsequence, we have either $\phi(x_{n_1}) \geq \phi(x_{n_2}) \cdots$, or $\phi(x_{n_1}) \leq \phi(x_{n_2}) \leq \cdots$. In the first case, by using that $epi\phi$ is sequentially closed from above, we get $(x_0, \alpha_0) \in epi\phi$. In the second case, by using that $epi\phi$ is sequentially closed from below, we get $(x_0, \alpha_0) \in epi\phi$. Thus, $epi\phi$ is sequentially closed. □

Definition 2.6 Let F be a normed vector space, $C \subset F$ a cone. $\leq$ is the partial order induced by C on F. C is said to be a completely regular cone iff $x_1 \leq x_2 \leq \cdots \leq x_n \cdots$, and $\{x_n, n = 1, 2, \ldots\}$ is norm bounded imply that $(x_n)_{n=1}^{\infty}$ is a convergent sequence.

Note that the cone defined by $C = \{(x_1, x_2, \ldots, x_n), x_i \geq 0, i = 1, 2, \ldots, n\} \subset \mathbb{R}^n$, and $C = \{f(x) \in L^p(\Omega), f(x) \geq 0, a.e.x \in \Omega\} \subset L^p(\Omega)$, where $p > 1$, and $\Omega \subset \mathbb{R}^n$ is a bounded measurable subset, are completely regular. One may see [28] for more details on cone and partial orders.

Let E be a normed vector space; throughout this paper, we'll use $\rightharpoonup$ to denote the weak convergence and $\to$ to denote the norm convergence.

3 Weak Lower Semi-continuity

Let E, F be two normed vector spaces, $C \subset F$ a cone, and let $f : E \to F$ be a vector-valued function. We will study the lower semi-continuity from above of f under some additional conditions. The vector-valued function that we study in this section is the generalization of the scalar function studied in Browder [10].

Theorem 3.1 *Let E, F be two Banach spaces, $C \subset F$ a completely regular closed cone, $g(x, y) : E \times E \to F$. Suppose that the following conditions are satisfied:*

(i) g is a bounded mapping, i.e. g maps bounded subsets to bounded subsets;
(ii) For each bounded subset B, $g(x, y)$ is uniformly weak continuous in y for $x \in B$;
(iii) For each $y \in E$, $g(x, y)$ is cone convex and continuous in x.

Then, $f(x) = g(x, x)$ is a sequentially cone lower semi-continuous from above function in the weak topology of E.

Proof Let $x_n \rightharpoonup x_0$ in E, and $f(x_1) \geq f(x_2) \geq \cdots$. By assumption (i), $\{f(x_n)\}_{n=1}^{\infty}$ is a bounded sequence. Since C is a completely regular closed cone, we know that $\{f(x_n)\}_{n=1}^{\infty}$ is a convergent sequence. We denote by $y_0 = \lim_{n\to\infty} f(x_n)$, and we have $y_0 \leq f(x_n)$ for all n.

By assumption (ii), we know that $g(x_n, x_n) - g(x_n, x_0) \to 0$ as $n \to \infty$, thus $g(x_n, x_0) \to y_0$.

Since $x_n \rightharpoonup x_0$, we have $x_0 \in cl(\text{conv}\{x_n : n \geq k\})$, $k = 1, 2, 3, \ldots$. If this is not true, by using separation theorem of convex subsets, there exists $h \in E^*$ such that $h(x_0) < \inf_{y \in cl(\text{conv}\{x_n : n \geq k\})} h(y)$, and thus, we have $h(x_0) < \inf_{n \geq k} h(x_n)$, $k = 1, 2, \ldots$, which is a contradiction to $x_n \rightharpoonup x_0$. Therefore, there exist $z_k = \Sigma \alpha_n^k x_n \in \text{conv}\{x_n : n \geq k\}$, $k = 1, 2, \ldots$, such that $z_k \to x_0$ in E, where $\alpha_i^k \geq 0$ with finite many $\alpha_n^k > 0$ satisfying $\Sigma \alpha_n^k = 1$.

By cone convexity of g, we have

$$g(z_k, x_0) \leq \Sigma_n \alpha_n^k g(x_n, x_0).$$

By letting $k \to \infty$, the continuity of $g(x, x_0)$ implies that $g(x_0, x_0) \leq y_0$. So we have $f(x_0) \leq f(x_n)$ for $n = 1, 2, \ldots$. This completes the proof. □

Corollary 3.1 *Let E, F be two Banach spaces, $C \subset F$ a completely regular closed cone, $g(x, y) : E \times E \to F$, and let $M : E \to E$ be a linear compact mapping. Suppose that the following conditions are satisfied:*

(i) g is a bounded mapping, i.e. g maps bounded subsets to bounded subsets;
(ii) For each bounded subset B, $g(x, y)$ is uniformly weak continuous in y for $x \in B$;
(iii) For each $y \in E$, $g(x, y)$ is cone convex and continuous in x.

Then, $f(x) = g(x, Mx)$ is a sequentially cone lower semi-continuous from above function in the weak topology of E.

Proof Note that M is a linear compact mapping, so T maps a weak convergent sequence to a strong convergent sequence. Thus, $g(x, Mx)$ satisfies all the assumptions in Theorem 3.1, and the conclusion holds. □

Remark 3.1 If condition (i) is replaced by g maps bounded subsets to relatively compact subsets, this is true if F is the real numbers, then the completely regularity of C is not required, and we can prove that $f(x)$ is cone lower semi-continuous. In such case, we get Theorems 1 and 2 of [10]. So Theorem 3.1 and Corollary 3.1 can be viewed as general versions of Theorems 1 and 2 of [10] in vector spaces.

By using a similar proof to Theorem 3.1, we get the following

Theorem 3.2 *Let E, F be two Banach spaces, $C \subset F$ a completely regular closed cone, $g(x, y) : E \times E \to F$. Suppose that the following conditions are satisfied:*

(i) g is a bounded mapping, i.e. g maps bounded subsets to bounded subsets;
(ii) For each bounded subset B, $g(x, y)$ is uniformly weak continuous in y for $x \in B$;
(iii) For each $y \in E$, $g(x, y)$ is cone concave and continuous in x.

Then, $f(x) = g(x, x)$ is a sequentially cone upper semi-continuous from below function in the weak topology of E.

Theorem 3.3 *Let E, F be two Banach spaces, $C \subset F$ a completely regular closed cone, $g_n(x, y) : E \times E \to F$, $n = 1, 2, \ldots$. Suppose that the following conditions are satisfied:*

(i) g_n is a bounded mapping, i.e. g maps bounded subsets to bounded subsets, $n = 1, 2, \ldots$;
(ii) For each bounded subset B, $g_n(x, y)$ is uniformly weak continuous in y for $x \in B$, $n = 1, 2, \ldots$;
(iii) For each $y \in E$, $g_n(x, y)$ is cone convex and continuous in x, $n = 1, 2, \ldots$;
(iv) $g_1(x, x) \leq g_2(x, x) \leq \cdots$ is bounded for each $x \in E$, $f(x) = \lim_{n\to\infty} g_n(x, x)$, and $f(x) \leq f(y)$ implies that $g_m(x, x) \leq g_m(y, y)$ for all $m = 1, 2, \ldots$.

Then, $f(x)$ is a sequentially cone lower semi-continuous from above function in the weak topology of E.

Proof By Theorem 3.1, $g_n(x, x)$ is a sequentially cone lower semi-continuous from above function in the weak topology of E for each $n = 1, 2, \ldots$.

By condition (iv), and C is complete regular, we know that
$f(x) = \lim_{n\to\infty} g_n(x, x)$ is well defined, and $g_n(x, x) \le f(x)$ for all $x \in E$, $n = 1, 2, \ldots$.

Suppose that $x_n \rightharpoonup x_0$, and $f(x_1) \ge f(x_2) \ge \cdots$. Again by (iv), we have
$g_n(x_1, x_1) \ge g_n(x_2, x_2) \ge \cdots$, for all $n = 1, 2, \ldots$. So we have
$g_n(x_0, x_0) \le g_n(x_k, x_k)$, for all $n, k = 1, 2, \ldots$.

By letting $n \to \infty$, we get $f(x_0) \le f(x_k)$, for all $k = 1, 2, \ldots$, and the proof is complete. □

Theorem 3.4 *Let E, F be two Banach spaces, $C \subset F$ a completely regular closed cone, and let $f_n(x) : E \to F$ be a cone lower semi-continuous function, $n = 1, 2, \ldots$. Suppose that $f_1(x) \le f_2(x) \le \cdots$, and $\{f_n(x)\}_{n=1}^{\infty}$ is bounded for each $x \in E$. Then, $f(x) = \lim_{n\to\infty} f_n(x)$ is cone lower semi-continuous.*

Proof Since $f_1(x) \le f_2(x) \le \cdots$, and $\{f_n(x)\}_{n=1}^{\infty}$ is bounded for each $x \in E$, and C is completely regular, $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$. It is obviously that $f_n(x) \le f(x)$ for $n = 1, 2, \ldots$.

For each $y \in F$, we prove that $\{x \in E : f(x) \le y\}$ is closed.

Let $x_n \to x_0$ in E, $f(x_n) \le y$ for $n = 1, 2, \ldots$. Then $f_m(x_n) \le f(x_n) \le y$ for $n, m = 1, 2, \ldots$. By using the cone lower semi-continuity of f_m, we get $f_m(x_0) \le y$. Then by letting $m \to \infty$, we get $f(x_0) \le y$. This completes the proof. □

Theorem 3.5 *Let E, F be two Banach spaces, $C \subset F$ a completely regular closed cone, $g_n(x, y) : E \times E \to F$, $n = 1, 2, \ldots$. Suppose that the following conditions are satisfied*

(i) g_n is a bounded mapping, i.e. g maps bounded subsets to bounded subsets, $n = 1, 2, \ldots$;
(ii) For each bounded subset B, $g_n(x, y)$ is uniformly weak continuous in y for $x \in B$, $n = 1, 2, \ldots$;
(iii) For each $y \in E$, $g_n(x, y)$ is cone concave and continuous in x, $n = 1, 2, \ldots$;
(iv) $g_1(x, x) \ge g_2(x, x) \ge \cdots$ is bounded for each $x \in E$, $f(x) = \lim_{n\to\infty} g_n(x, x)$, and $f(x) \le f(y)$ implies that $g_m(x, x) \le g_m(y, y)$ for all $m = 1, 2, \ldots$.

Then, $f(x)$ is a sequentially cone upper semi-continuous from below function in the weak topology of E.

4 Minimum and Equilibrium Problems

In this section, let E, F be two normed vector spaces, and $C \subset F$ a cone, and let $\phi(x) : D \subseteq E \to F$ be a vector function. We will study the minimum (respectively, maximum) problem and find $x_0 \in D$ such that $\phi(x_0) = \min_{x\in D} \phi(x)$, (respectively, $\phi(x_0) = \max_{x\in D} \phi(x)$). Several existence results are obtained under lower (respectively, upper) semi-continuity from above (respectively, below) condition. Applications to equilibrium problems and von Neumann's minimax principle are also given.

Theorem 4.1 *Let E be a Banach space, F a normed vector space, $C \subset F$ a closed cone, $D \subset E$ a weakly compact subset, and $\phi : D \to F$ a sequentially cone lower semi-continuous from above function in the weak topology of E. Suppose that $\phi(D)$ is separable. Then there exists $y_0 \in D$ such that*

$$\phi(y_0) = \min\{\phi(x) : x \in D\}.$$

Proof For each totally ordered subset $T \subset \phi(D)$, we prove that T has a lower bound. Since $\phi(D)$ is separable, there exist $x_n \in D$, $n = 1, 2, \ldots$ such that $cl\{\phi(x_n) : n = 1, 2, \ldots\} = T$. We first prove that there exists $x_0 \in D$ such that $\phi(x_0) \leq \phi(x_n)$ for $n = 1, 2, \ldots$. Without loss of generality, we may assume that $\phi(x_1) \geq \phi(x_2) \geq \cdots$, since T is totally ordered.

By using the assumption that D is weakly compact, there exist x_{n_k}, $k = 1, 2, \ldots$, such that $x_{n_k} \rightharpoonup x_0$ as $k \to \infty$. The cone lower semi-continuity from above of ϕ in the weak topology implies that $\phi(x_0) \leq \phi(x_{n_k})$. Since $\phi(x_{n_k}) \leq \phi(x_n)$ for all $k \geq n$, so we have

$$\phi(x_0) \leq \phi(x_n) \tag{1}$$

for all $n = 1, 2, \ldots$.

We next prove that $\phi(x_0) \leq \phi(y)$ for all $\phi(y) \in T$. For each $\phi(y) \in T$, there exist $\phi(x_{n_m})$, $m = 1, 2, \ldots$, such that $\phi(x_{n_m}) \to \phi(y)$ as $m \to \infty$. By (1), we get $\phi(x_0) \leq \phi(y)$, so $\phi(x_0)$ is the lower bound for T. By Zorn's Lemma, $\phi(D)$ has a minimal element. This completes the proof. □

By using a similar argument to Theorem 4.1, we get the following

Theorem 4.2 *Let E be a Banach space, F a normed vector space, $C \subset F$ a closed cone, $D \subset E$ a weakly compact subset, and $\phi : D \to F$ a sequentially cone upper semi-continuous from below function in the weak topology of E. Suppose that $\phi(D)$ is separable. Then there exists $y_0 \in D$ such that*

$$\phi(y_0) = \max\{\phi(x) : x \in D\}.$$

Theorem 4.3 *Let E be a reflexive Banach space, F a normed vector space, $C \subset F$ a cone, $cl(B(0, r))$ the closed ball centred at zero with radius $r > 0$, $D \subset E$ a closed and convex subset, and $\phi : D \to F$ a cone convex and cone lower semi-continuous mapping. Suppose that $\phi(D \cap cl(B(0, r)))$ is separable. Then there exists $y_0 \in D \cap cl(B(0, r))$ such that*

$$\phi(y_0) = \min\{\phi(x) : x \in D \cap cl(B(0, r))\}.$$

Proof For each totally ordered subset $T \subset \phi(D \cap cl(B(0, r))$, we prove that T has a lower bound. Since $\phi(D \cap cl(B(0, r)))$ is separable, there exist $x_n \in D \cap cl(B(0, r))$, $n = 1, 2, \ldots$ such that $cl\{\phi(x_n) : n = 1, 2, \ldots\} = T$. We first prove that there exists $x_0 \in D \cap cl(B(0, r))$ such that $\phi(x_0) \leq \phi(x_n)$ for $n = 1, 2, \ldots$. Without loss of generality, we may assume that $\phi(x_1) \geq \phi(x_2) \geq \cdots$, since T is totally ordered.

By using the assumption that E is reflexive, there exist $y_k = x_{n_k}$, $k = 1, 2, \ldots$, such that $y_k \rightharpoonup x_0$ as $k \to \infty$. We have $x_0 \in cl(\text{conv}\{y_m : m \geq k\})$ for $k = 1, 2, \ldots$, there exist $z_k = \Sigma \alpha_i^k y_i \in \text{conv}\{y_m : m \geq k\}$, $k = 1, 2, \ldots$, such that $z_k \to x_0$ in E, where $\alpha_i^k \geq 0$ with only finite many $\alpha_i^k > 0$ satisfying $\Sigma \alpha_i^k = 1$. By using cone convexity of ϕ, we get

$$\phi(z_k) \leq \Sigma \alpha_i^k \phi(y_i) \leq \phi(y_k) = \phi(x_{n_k}) \leq \phi(x_n), \text{ for all } k \geq n.$$

By using the cone lower semi-continuity of ϕ, we get

$$\phi(x_0) \leq \phi(x_n), n = 1, 2, \ldots. \tag{2}$$

We next prove that $\phi(x_0) \leq \phi(y)$ for all $\phi(y) \in T$. For each $\phi(y) \in T$, there exist $\phi(x_{n_m})$, $m = 1, 2, \ldots$, such that $\phi(x_{n_m}) \to \phi(y)$ as $m \to \infty$. By (2), we get $\phi(x_0) \leq \phi(y)$, so $\phi(x_0)$ is the lower bound for T. By Zorn's Lemma, $\phi(D)$ has a minimal element. This completes the proof. □

By using a similar argument to Theorem 4.3, we get the following

Theorem 4.4 *Let E be a reflexive Banach space, F a normed vector space, $C \subset F$ a closed cone, $cl(B(0, r))$ the closed ball centred at zero with radius $r > 0$, $D \subset E$ a closed and convex subset, and $\phi : D \to F$ a cone concave and cone upper semi-continuous mapping. Suppose that $\phi(D \cap cl(B(0, r)))$ is separable. Then there exists $y_0 \in D \cap cl(B(0, r)))$ such that*

$$\phi(y_0) = \max\{\phi(x) : x \in D \cap cl(B(0, r))\}.$$

In the following, let E, F be two normed vector spaces, $C \subset F$ a closed cone. $\leq$ is the order induced by C on F. Assume that $D \subset E$, and $f(x, y) : D \times D \to F$, we consider the vector equilibrium problem and find $x_0 \in D$ such that

$f(x_0, y) \not< 0$, for all $y \in D$. (VEP)

Theorem 4.5 *Assume that D is a weakly compact subset, and $f(x, y) \leq f(x, z) + f(z, y)$ for all $x, y, z \in D$. If there exists $z_0 \in D$ such that $f(x, z_0)) : D \to F$ is sequentially cone upper semi-continuous from below in the weak topology of E, and $f(D, z_0)$ is separable, then there exists $x_0 \in D$ such that (VEP) holds.*

Proof Since $f(x, z_0) : D \to F$ is sequentially cone upper semi-continuous from below in the weak topology of E, and $f(D, z_0)$ is separable, by Theorem 3.2, there exists $x_0 \in D$ such that $f(x_0, z_0) = max\{f(x, z_0) : x \in D\}$.

We also have $f(x_0, y) \geq f(x_0, z_0) - f(y, z_0)$, for all $y \in D$. Therefore, $f(x_0, y) \not< 0$ for all $y \in D$. □

Remark 4.1 When F is the real numbers, Theorem 4.5 is proved in [20].

Theorem 4.6 (von Neumann's minimax principle) *Let E_1, E_2 be two reflexive Banach spaces, and let $X \subset E_1$, $Y \subset E_2$ be two nonempty convex bounded subsets. Assume that F is a separable normed vector space, and $C \subset F$ is a closed cone. Suppose that $f : X \times Y \to F$ is a function satisfying the following conditions:*

(i) $y \to f(x, y)$ *is sequentially cone lower semi-continuous from above in the weak topology of* E_2 *and quasi cone convex for each fixed* $x \in X$*;*
(ii) $x \to f(x, y)$ *is sequentially cone upper semi-continuous from below in the weak topology of* E_1 *and quasi cone concave for each fixed* $y \in Y$*;*
(iii) For each $r \in F$*, there exist* $x_i \in X$*,* $i = 1, 2, \dots, n$*, such that* $A_i = \{y : f(x_i, y) > r\}$ *is open and* $Y = \cup_{i=1}^n A_i$*;*
(iv) For each $r \in F$*, there exist* $y_j \in Y$*,* $j = 1, 2, \dots, m$*, such that* $B_j = \{x : f(x, y_j) < r\}$ *is open and* $X = \cup_{j=1}^m B_j$*.*

Then, $\max_{x \in X} \min_{y \in Y} f(x, y) \not< \min_{y \in Y} \max_{x \in X} f(x, y)$.

Proof Since F is separable and X, Y are weakly compact, by assumptions (i), (ii), and Theorems 3.1 and 3.2, we know that $\max_{x \in X} \min_{y \in Y} f(x, y)$ and $\min_{y \in Y} \max_{x \in X} f(x, y)$ both exist. Now we show that

$$\max_{x \in X} \min_{y \in Y} f(x, y) \not< \min_{y \in Y} \max_{x \in X} f(x, y).$$

If this is not true, then there would be a element $r \in F$ such that

$$\max_{x \in X} \min_{y \in Y} f(x, y) < r < \min_{y \in Y} \max_{x \in X} f(x, y)$$

Define two maps $A, B : X \to 2^Y$ by $Ax = \{y : f(x, y) > r\}$ and $Bx = \{y : f(x, y) < r\}$ for $x \in X$. It is obvious that

$$Y = \cup_{i=1}^n Ax_i, \quad X = \cup_{j=1}^m B^{-1} y_j.$$

Since $f(x, y)$ is quasi cone convex in y and quasi cone concave in x, we know that $A^{-1}y$ is convex for $y \in Y$ and Bx is convex for each $x \in X$. Thus by Theorem 4.1 in [14], there exist $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in Ax_0 \cap Bx_0 \neq \phi$. Therefore, we have $f(x_0, y_0) < r < f(x_0, y_0)$, which is a contradiction. This completes the proof. □

Similarly, by using Theorems 3.3, 3.4 and 4.1 in [14], we get the following

Theorem 4.7 *Let* E_1, E_2 *be two reflexive Banach spaces, and and let* $X \subset E_1$*,* $Y \subset E_2$ *be two nonempty convex bounded subsets. Assume that* F *is a separable normed vector space, and* $C \subset F$ *is a closed cone. Suppose that* $f : X \times Y \to F$ *is a function satisfying the following conditions:*

(i) $y \to f(x, y)$ *is cone lower semi-continuous and cone convex for each fixed* $x \in X$*;*
(ii) $x \to f(x, y)$ *is cone upper semi-continuous from below and cone concave for each fixed* $y \in Y$*;*
(iii) For each $r \in F$*, there exist* $x_i \in X$*,* $i = 1, 2, \dots, n$*, such that* $A_i = \{y : f(x_i, y) > r\}$ *is open and* $Y = \cup_{i=1}^n A_i$*;*
(iv) For each $r \in F$*, there exist* $y_j \in Y$*,* $j = 1, 2, \dots, m$*, such that* $B_j = \{x : f(x, y_j) < r\}$ *is open and* $X = \cup_{j=1}^m B_j$*.*
Then, $\max_{x \in X} \min_{y \in Y} f(x, y) \not< \min_{y \in Y} \max_{x \in X} f(x, y)$.

Remark 4.2 If $f(X, Y)$ is a totally ordered subset of F, especially F is the real numbers, then we have $\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y)$ in Theorems 4.6 and 4.7.

5 Conclusions

Lower semi-continuous from above functions are very useful in recent research papers; we obtain new results on the lower semi-continuity from above of vector functions with form $g(x, x)$ in ordered normed vector spaces. Our results can be viewed as general versions of the corresponding results in [10]. By using lower semi-continuous from above and upper semi-continuous from below conditions, we prove some new results on the existence of minimal points and maximal points of vector-valued functions in ordered normed vector spaces. These results are used to study equilibrium problems and von Newman's minimax principle.

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