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Existence Theorems in Vector Optimization with Generalized Order

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Abstract In the present paper, we establish some results for the existence of optimal solutions in vector optimization in infinite-dimensional spaces, where the optimality notion is understood in the sense of generalized order (may not be convex and/or conical). This notion is induced by the concept of set extremality and covers many of the conventional notions of optimality in vector optimization. Some sufficient optimality conditions for optimal solutions of a class of vector optimization problems, which satisfies the free disposal hypothesis, are also examined.

Keywords Vector optimization · Generalized order optimality · Efficient point · Existence theorem

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1 Introduction

The investigation of the existence of optimal solutions is one of the most attractive topics in optimization theory. For vector optimization, a classical result states that the set of Pareto minimum points of a *nonempty* and *compact* set in a finite-dimensional space, with respect to a closed convex cone, is always nonempty; see [1] for more details. However, the compactness assumption is too strict, especially when solving problems with data in infinite-dimensional spaces. The results on the existence of Pareto minimum points in infinite-dimensional spaces to avoid the restrictions of compactness can be found in [2–6] and the references therein. Borwein [2] weakens the compactness by requiring the cone to be *Daniell*, while Jahn [3] uses the *weakly compactness*, instead of compactness. Using the so-called *cone-completeness*, Luc [4] studied the existence of Pareto efficient solutions for vector optimization problems. The results of Luc[4] cover many facts already obtained in the papers [2,3,5,6]. Another approach to obtain the existence of Pareto efficiency is to use some kind of coerciveness assumption; see [7–9] for more details. We would like to remark that all the results in the cited papers are established with convex ordering cones.

In [10, Definition 5.53], Mordukhovich has introduced the notion of *generalized order optimality* of vector optimization problems, where the ordering set is an arbitrary set containing the zero vector. This notion goes back to the early work by Kruger and Mordukhovich (see [11, p. 685]); it is directly induced by the concept of local extremal points for systems of sets and covers many of the conventional notions of optimality in vector optimization (see [12, Sect. 2]). It is important to emphasize that we do not generally assume that the ordering set is either convex or conical in response to the increasing needs for practical and theoretical applications in vector optimization and especially in economics modeling; see [13–17]. For example, by reducing the welfare economic models to a special set-valued optimization problem, Bao and Mordukhovich [13] derived new versions of the second fundamental theorem of welfare economics to nonconvex economies with *general preference relations*. We note here that the preference relations in the cited paper do not necessarily satisfy the almost transitivity property. Thus, it can be induced by an abstract set, which may not be convex or conical.

Another important application of the concept of generalized order optimality is that of deriving necessary optimality conditions for minimax problems; see [10, Theorem 5.62]. The new approach of Mordukhovich in [10, Theorem 5.62], based on the reduction to generalized order optimality, seems to be more appropriate and convenient to handle minimax problems involving maximization over a weak* compact subset of a dual space.

To the best of our knowledge, until now, there are only a few works studying the concept of generalized order optimality. In [10, Theorem 5.59], Mordukhovich established some preliminary necessary conditions to vector optimization problems with geometric constraints. In [15,18], the authors established some new subdifferential necessary conditions to set-valued optimization problems with equilibrium constraints. Huy, Mordukhovich, and Yao [19] gave some estimates of coderivatives of frontier and efficient solution maps in parametric multiobjective problems with respect to generalized order optimality. Later, Tuyen and Yen [12, Section 4] provided some sufficient conditions for a point satisfying the necessary optimality condition of Mordukhovich [10, Theorem 5.59] for being a generalized order solution of the optimization problem under consideration. Recently, Tuyen [20] investigated some criteria for the closedness and the connectedness of the set of generalized order solutions to vector optimization problems.

However, none of the above-mentioned works examines the existence of efficient solutions for vector optimization problems with respect to generalized order optimality. This paper is aimed at solving the problem. The rest of this paper is organized as follows. Section 2 investigates the notion of generalized order optimality and compares it with the traditional notions of Pareto efficiency and weakly Pareto efficiency. Some examples are given to show that three concepts can be very different. In Sect. 3, we prove the main theorems about the existence of generalized efficiency. Then, we consider vector optimization problems with generalized order optimality and provide the conditions under which optimal solutions exist. These results cover some classical existence results for efficient points with convex ordering cones. Furthermore, some sufficient optimality conditions for a generalized efficient point of a set satisfying the free disposal hypothesis are also examined in this section. Examples are given to illustrate the obtained results. The conclusions are presented in Sect. 4.

2 Generalized Order Optimality

Let Z be a real Banach space with the norm $\|\cdot\|$. Let $A \subset Z$ be a nonempty set. The topological interior, the topological closure, the topological boundary, the complement, the convex hull, and the cone hull of A are denoted, respectively, by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, A^c, $\operatorname{conv} A$, and $\operatorname{cone} A$. The zero vector of a Banach space is denoted by $\mathbf{0}$. The closed ball with center x and radius ρ is denoted by $B(x, \rho)$. Besides, $l(A)$ denotes the set $A \cap (-A)$.

Definition 2.1 Let A be a nonempty set in Z, and $\Theta \subset Z$ be a set containing the zero vector. A point $\bar{z} \in A$ is said to be a *generalized efficient point* of A with respect to Θ iff there is a sequence $\{z_k\} \subset Z$ with $\|z_k\| \to 0$ as $k \to \infty$ such that

$$A \cap (\Theta + \bar{z} - z_k) = \emptyset \quad \forall k \in \mathbb{N}. \tag{1}$$

The set of generalized efficient points of A with respect to Θ is denoted by $GE(A \mid \Theta)$.

By Definition 2.1, $\bar{z} \in GE(A \mid \Theta)$ iff $\bar{z}$ is an extremal point of the system $\{A, \bar{z}+\Theta\}$; see [21, Definition 2.1]. Loosely speaking, the extremality of sets at a common point means that they can be "pushed apart" by a small perturbation of even one of them and illustrated in Fig. 1.

The following result gives a geometric characterization of the set of generalized efficient points with respect to Θ.

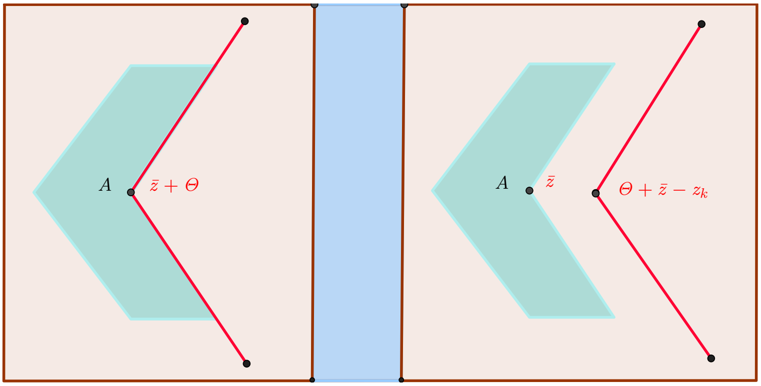

Fig. 1 Illustration of the notion of generalized efficient point

Theorem 2.1 *Let A be a nonempty set in Z, and $\Theta \subset Z$ be a set containing the zero vector. Then*

$$GE(A\,|\,\Theta) = A \cap \operatorname{bd}(A - \Theta).$$

Moreover, if A is a closed set, then so does $GE(A\,|\,\Theta)$.

Proof Fix $\bar{z} \in GE(A\,|\,\Theta)$. Let $\{z_k\} \subset Z$ be a sequence satisfying condition (1). Then, $\bar{z} - z_k \in (A - \Theta)^c$ for all $k \in \mathbb{N}$. Let U be an arbitrary neighborhood of $\bar{z}$. Then, $\bar{z} - z_k \in U$ for large enough k. It follows that $U \cap (A - \Theta)^c \neq \emptyset$. Thus $\bar{z} \in A \cap \operatorname{bd}(A - \Theta)$. Now let $\bar{z}$ be in $A \cap \operatorname{bd}(A - \Theta)$. Then

$$B\left(\bar{z}, \frac{1}{k}\right) \cap (A - \Theta)^c \neq \emptyset \quad \forall k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, choose $x_k \in B\left(\bar{z}, \frac{1}{k}\right) \cap (A - \Theta)^c$ and put $z_k := \bar{z} - x_k$. Then, the sequence $\{z_k\}$ satisfies condition (1), and so $\bar{z} \in GE(A\,|\,\Theta)$. Thus

$$GE(A\,|\,\Theta) = A \cap \operatorname{bd}(A - \Theta).$$

Finally, if A is closed, by the closedness of $\operatorname{bd}(A - \Theta)$, $GE(A\,|\,\Theta)$ is closed. □

Let us illustrate Theorem 2.1.

Example 2.1 Let $Z = l^2$ denote the Hilbert space of all square summable real sequences, $A = l^1_+ = \left\{\{x_n\} \subset \mathbb{R}_+ : \sum_{n=1}^{\infty} x_n < \infty\right\} \subset l^2$, $\bar{x} = \{\frac{1}{n}\} \in l^2$ and $\Theta = \{\lambda \bar{x} \;:\; -1 \leq \lambda \leq 0\}$. Clearly, Θ is a convex set but not a cone, and

$$l^1_+ \subset A - \Theta \subset l^2_+ + l^2_+ = l^2_+, \tag{2}$$

where $l^2_+ := \big\{\{x_n\} \in l^2 \ : \ x_n \geq 0 \ \ \forall n \in \mathbb{N}\big\}$. Since (2) and $\mathrm{cl}\,(l^1_+) = l^2_+$, we have $\mathrm{cl}\,(A - \Theta) = l^2_+$. An easy computation shows that $\mathrm{int}\,(A - \Theta) = \emptyset$. It follows that $\mathrm{bd}\,(A - \Theta) = l^2_+$. Thus

$$GE(A\,|\,\Theta) = A \cap \mathrm{bd}\,(A - \Theta) = l^1_+ \cap l^2_+ = l^1_+.$$

Remark 2.1 If $\bar{z} \in GE(A\,|\,\Theta)$, by Theorem 2.1, $\bar{z} \notin \mathrm{int}\,(A - \Theta)$. In addition, as $A - \Theta$ is a convex set with nonempty interior, by the convex separation theorem, there exists $z^* \in Z^*\backslash\{\mathbf{0}\}$ such that

$$\sup_{z\in A}\langle z^*, z - \bar{z}\rangle \leq \inf_{\theta\in\Theta}\langle z^*, \theta\rangle. \tag{3}$$

We also see that, without imposing either the convexity of $A - \Theta$ or the condition $\mathrm{int}\,(A - \Theta) \neq \emptyset$, separation property (3) always implies that $\bar{z} \in GE(A\,|\,\Theta)$. Indeed, since $z^* \neq \mathbf{0}$, there is $c \in Z$ such that $\langle z^*, c\rangle > 0$. For each $k \in \mathbb{N}$, put $z_k := -\frac{c}{k}$. It suffices to show that (1) holds with this sequence $\{z_k\}$. If otherwise, there exist $\hat{z}$, $\hat{\theta}$, and $k_0 \in \mathbb{N}$ such that $\hat{z} = \hat{\theta} + \bar{z} - z_{k_0}$. By (3), one has

$$\langle z^*, \hat{z} - \bar{z}\rangle \leq \sup_{z\in A}\langle z^*, z - \bar{z}\rangle \leq \inf_{\theta\in\Theta}\langle z^*, \theta\rangle \leq \langle z^*, \hat{\theta}\rangle = \langle z^*, \hat{z} - \bar{z} + z_{k_0}\rangle.$$

Thus $\langle z^*, z_{k_0}\rangle \geq 0$. This implies that $\langle z^*, c\rangle \leq 0$, contrary to the fact that $\langle z^*, c\rangle > 0$.

We next compare the concept of generalized efficient point with the concepts of Pareto efficiency and weakly Pareto efficiency.

Definition 2.2 (*see* [22]) Let A be a nonempty set in Z, and $\Theta \subset Z$ be a set containing $\mathbf{0} \in Z$. A point $\bar{z} \in A$ is said to be a *Pareto efficient point of* A with respect to Θ iff

$$A \cap \big(\bar{z} + \Theta\backslash\{\mathbf{0}\}\big) = \emptyset.$$

When $\mathrm{int}\,\Theta \neq \emptyset$, $\bar{z} \in A$ is said to be a *weakly Pareto efficient point* of A with respect to Θ iff

$$A \cap \big(\bar{z} + \mathrm{int}\,\Theta\big) = \emptyset.$$

The set of Pareto efficient (resp., weakly Pareto efficient) points of A is denoted by $E(A\,|\,\Theta)$ (resp., $E^w(A\,|\,\Theta)$).

Remark 2.2 The concept of Pareto efficient point in Definition 2.2 is a bit different from the conventional Pareto minimality notion for nonpointed ordering cones given by

$$(\bar{z} + \Theta) \cap A \subset (\bar{z} - \Theta). \tag{4}$$

Bao and Mordukhovich [22] show that $\bar{z}$ is a Pareto efficient point of A with respect to Θ in the sense of (4) iff it is a Pareto efficient point of A in the sense of Definition 2.2 with respect to $C := (-\Theta) \cap (Z\backslash\Theta) \cup \{\mathbf{0}\}$.

We now introduce a condition to the ordering set, which will be useful in the sequel. This condition is weaker than the requirement that the ordering set is either convex or conical.

Definition 2.3 (*(T) condition*) Let $\Theta \subset Z$ be a set containing $\mathbf{0} \in Z$. We say that Θ satisfies the *the condition* (T) iff

$$\Theta + \Theta = \Theta,\ l(\Theta) = \{\mathbf{0}\} \text{ and } \mathbf{0} \in \operatorname{cl}(\Theta \backslash \{\mathbf{0}\}). \tag{T}$$

Remark 2.3 (i) Clearly, if Θ is a nontrivial convex pointed cone, then condition (T) is satisfied. However, this condition does not imply that the set Θ is convex or conical. For example, if Θ is given by one of the following sets:
- $\{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 \geq 0, \theta_2 \geq -\theta_1^2\}$;
- $\{(\theta_1, \theta_2) \in \mathbb{Q}^2 : \theta_1 \geq 0, \theta_2 \geq 0\}$.

(ii) Condition (T) is closely related to the asymptotic closeness property of Θ at the origin. We recall that a set $A \subset Z$ is said to be *locally asymptotically closed* (LAC) at $\bar{z} \in \operatorname{cl} A$ if there are a neighborhood V of $\bar{z}$ and a sequence $\{z_k\}$ in Z with $\|z_k\| \to 0$ as $k \to \infty$ such that

$$(\operatorname{cl} A + z_k) \cap V \subset A \backslash \{\bar{z}\} \ \ \forall k \in \mathbb{N}.$$

This property holds in many rather general settings and is also satisfied under natural assumptions in models of welfare economics; see [13, Sect. 4]. If Θ satisfies condition (T), then it induces a partial order in Z; see Sect. 3.2. It is easy to check that, if Θ is closed and satisfies condition (T), then Θ is (LAC) at $\mathbf{0}$. This allows us to study the existence of solutions to the general model of welfare economics, where the preference relation is induced by an abstract set satisfying condition (T) (may not be convex and/or conical).

The following proposition shows that, if Θ satisfies condition (T), then any Pareto efficient point of A is also a generalized efficient point.

Proposition 2.1 *Let A be a nonempty set in Z, and $\Theta \subset Z$ be a set containing $\mathbf{0} \in Z$. If Θ satisfies condition* (T), *then we have*

$$E(A \,|\, \Theta) \subset GE(A \,|\, \Theta). \tag{5}$$

Proof On the contrary, suppose that there is $\bar{z} \in E(A \,|\, \Theta)$ but $\bar{z}$ does not belong to $GE(A \,|\, \Theta)$. By Theorem 2.1, $\bar{z} \notin \operatorname{bd}(A - \Theta)$. Thus $\bar{z} \in \operatorname{int}(A - \Theta)$. This implies that there exists a neighborhood U of $\bar{z}$ such that $U \subset A - \Theta$. Since $\mathbf{0} \in \operatorname{cl}(\Theta \backslash \{\mathbf{0}\})$, there is a sequence $\{\theta_k\} \subset \Theta \backslash \{\mathbf{0}\}$ such that $\lim_{k \to \infty} \theta_k = \mathbf{0}$. From $\lim_{k\to\infty} (\bar{z} + \theta_k) = \bar{z}$ it follows that there exists k_0 large enough satisfying $\bar{z} + \theta_{k_0} \in U \subset A - \Theta$. Thus, there exist $a \in A$ and $\theta \in \Theta$ such that $\bar{z} + \theta_{k_0} = a - \theta$. Consequently, $a = \bar{z} + \theta_{k_0} + \theta$. By condition (T), we have $\theta_{k_0} + \theta \in \Theta$ and $\theta_{k_0} + \theta \neq \mathbf{0}$. Therefore

$$\bar{z} + \theta_{k_0} + \theta \in A \cap \big(\bar{z} + \Theta \backslash \{\mathbf{0}\}\big),$$

contrary to $\bar{z} \in E(A \,|\, \Theta)$. □

Remark 2.4 Clearly, when Θ is a convex cone, condition (T) implies that Θ is nontrivial, i.e., $\Theta \neq \{\mathbf{0}\}$. If Θ is trivial and A is a closed set with a nonempty interior, then inclusion (5) does not hold. Indeed, by Theorem 2.1, $GE(A \mid \Theta) = A \cap \mathrm{bd}\,(A) = \mathrm{bd}\,(A)$. Clearly, $E(A \mid \Theta) = A \not\subseteq GE(A \mid \Theta)$.

We next show that, if A satisfies the free disposal condition, then any Pareto efficient point of A with respect to a starshaped set is also a generalized efficient point. The importance of the free disposal condition in production theory and the corresponding version of the nonsatiation assumption in consumer theory is well known; see [14,16,17] for more details. Recall that a set A is said to satisfy *the free disposal condition* with respect to Θ iff

$$A - \Theta = A,$$

and the set Θ is said to be *starshaped at* $\mathbf{0}$ iff

$$\big(t \in [0, 1],\ \theta \in \Theta\big) \Longrightarrow t\theta \in \Theta.$$

If A satisfies the free disposal condition with respect to a cone Θ and $\mathbf{0} \notin A$, then A is said to be an *improvement set*; see [9, Definition 2]. Clearly, if A is an improvement set, then the set $A \cup \{\mathbf{0}\}$ still satisfies the free disposal condition. However, this set is not an improvement set.

Proposition 2.2 *Suppose that Θ is a set containing $\mathbf{0} \in Z$ such that $\Theta \backslash \{\mathbf{0}\}$ is nonempty. If Θ is starshaped at $\mathbf{0}$ and A satisfies the free disposal condition, then*

$$E(A \mid \Theta) \subset GE(A \mid \Theta). \tag{6}$$

Proof Let $\bar{z} \in E(A \mid \Theta)$. Suppose to the contrary that $\bar{z} \notin GE(A \mid \Theta)$. Thanks to Theorem 2.1, $\bar{z} \notin \mathrm{bd}\,(A - \Theta)$. This means that $\bar{z} \in \mathrm{int}\,(A - \Theta)$. Consequently, by the free disposal assumption, one has $\bar{z} \in \mathrm{int}\, A$. Take $\theta \in \Theta \backslash \{\mathbf{0}\}$. By the starshapeness of Θ at $\mathbf{0}$, we have $t\theta \in \Theta$ for all $t \in [0, 1]$. Since $\bar{z} \in \mathrm{int}\, A$, one has $\bar{z} + t_0\theta \in A$ for some $t_0 > 0$ small enough. Thus $\bar{z} + t_0\theta \in \big[A \cap (\bar{z} + \Theta \backslash \{\mathbf{0}\})\big]$, contrary to $\bar{z} \in E(A \mid \Theta)$. □

Note that inclusion (6) may be strict, even when Θ is a closed convex pointed cone with nonempty interior. For example, let $A = \mathbb{R}^2_+$ and $\Theta = \mathbb{R}^2_-$. It is easy to see that A satisfies the free disposal condition, Θ is a closed convex pointed cone with nonempty interior, and

$$E(A \mid \Theta) = \{(0, 0)\} \subsetneq G(A \mid \Theta) = \{(x_1, x_2) \,:\, x_1 \geq 0, x_2 \geq 0 \text{ and } x_1x_2 = 0\}.$$

The following result shows that the concept of the generalized efficient point extends our well known of weakly efficient one.

Proposition 2.3 *If Θ is a convex cone with nonempty interior and A is a nonempty subset in Z, then*

$$GE(A \mid \Theta) = E^w(A \mid \Theta). \tag{7}$$

Proof By [23, Lemma 2.5], one has $\text{int}\,(A - \Theta) = A - \text{int}\,\Theta$. Thus

$$GE(A\,|\,\Theta) = A \cap \text{bd}\,(A - \Theta) = A \cap (\text{int}\,(A - \Theta))^c$$
$$= A \cap (A - \text{int}\,\Theta)^c = E^w(A\,|\,\Theta).$$

The proof is complete. □

Remark 2.5 (i) The above result is an extension of Lemma 2.3 in [24] without the closedness of A. However, it is a special case of Proposition 2.11 in [12] with respect to the generalized order optimization problem (f, Θ) whenever f is an identity function.

(ii) In [17], Jofré and Jourani wrote that, when Θ has an interior, the concepts of weakly Pareto efficiency and generalized efficiency coincide. However, this fact does not hold if Θ is a nonconvex cone. The following example shows that two concepts can be very different.

Example 2.2 Let A and Θ be two sets defined as follows

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_2 = x_1, -1 \le x_1 \le 0\},\; \Theta = \{(\theta_1, \theta_2) \;:\; \theta_1\theta_2 \le 0\}.$$

We have $\text{int}\,\Theta = \{(\theta_1, \theta_2) \;:\; \theta_1\theta_2 < 0\}$ and

$$A - \Theta = \{(x_1, x_2) \;:\; -1 \le x_1, x_2 \le 0\} \cup \{(x_1, x_2) \;:\; x_1 \le -1, -1 \le x_2\}$$
$$\cup \{(x_1, x_2) \;:\; -1 \le x_1 \le 0, 0 \le x_2\}.$$

It is easily seen that $E^w(A\,|\,\Theta) = A$ and $GE(A\,|\,\Theta) = \{(-1, -1), (0, 0)\}$. Clearly, Eq. (7) does not hold true.

Remark 2.6 (i) We claim $GE(A\,|\,\Theta) \subset \text{bd}\;A$. If not, then

$$\bar{z} \in \text{int}\,A \subset \text{int}\,(A - \Theta)$$

for some $\bar{z} \in GE(A\,|\,\Theta)$. This makes a contradiction because of Theorem 2.1.

(ii) If A is a closed set satisfying the free disposal condition, then we have $\text{bd}\,A = GE(A\,|\,\Theta)$; see [17, Theorem 2]. This fact does not hold unless A is closed. For example, let $A = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 > 0, x_2 > 0\} \cup \{\mathbf{0}\}$ and $\Theta = -A$. It is easy to see that A satisfies the free disposal condition and

$$GE(A\,|\,\Theta) = \{\mathbf{0}\} \subsetneq \text{bd}\,A = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1x_2 = 0, x_1 \ge 0, x_2 \ge 0\}.$$

3 Existence of Optimal Solutions

3.1 Optimality Conditions

In the following, we shall give several sufficient conditions for a generalized efficient point of a set which satisfies the free disposal assumption. Firstly, we recall the following notions: the Clarke tangent cone, the tangent cone.

Definition 3.1 (*see* [25]) Let A be a nonempty set in Z and $\bar{z} \in A$.

(i) *The Clarke's tangent cone* to A at $\bar{z}$ is defined by

$$T_C(A, \bar{z}) := \{h \in Z \,:\, \forall z_n \xrightarrow{A} \bar{z}, \forall t_n \downarrow 0, \exists h_n \to h;\, z_n + t_n h_n \in A \;\forall n \in \mathbb{N}\}.$$

(ii) *The tangent cone (or the adjacent cone)* to A at $\bar{z}$ is defined by

$$T_0(A, \bar{z}) := \{h \in Z \,:\, \forall t_n \downarrow 0,\; \exists h_n \to h;\, \bar{z} + t_n h_n \in A \;\forall n \in \mathbb{N}\}.$$

Remark 3.1 We always have the following inclusion:

$$T_C(A, \bar{z}) \subset T_0(A, \bar{z}).$$

Whenever A is given by a finite number of C^1-smooth inequality constraints under the assumption that the gradients of active constraints are positively independent, then, as it is well known [25, pp. 150–151],

$$T_C(A, \bar{z}) = T_0(A, \bar{z}) = \{h \in Z \,:\, \langle \nabla g_i(\bar{x}), h\rangle \leq 0 \;\forall i \in I(\bar{z})\},$$

where $A := \{z \in Z \,:\, g_i(z) \leq 0 \;\; \forall i \in I\}$, $I(\bar{z}) := \{i \in I \,:\, g_i(\bar{z}) = 0\}$, and $I := \{1, \ldots, p\}$.

The following result gives some tangential sufficient conditions for a generalized efficient point to a set satisfying the free disposal condition.

Theorem 3.1 *Let Θ be a set containing $\mathbf{0} \in Z$ such that $\Theta \backslash \{\mathbf{0}\}$ is nonempty. Suppose that Θ is starshaped at $\mathbf{0}$, A satisfies the free disposal condition, and either the tangent relation*

$$T_C(A, \bar{z}) \cap T_0(\Theta, \mathbf{0}) = \{\mathbf{0}\} \tag{8}$$

holds at $\bar{z} \in A$, or $T_C(A, \bar{z})$ is not the whole space. Then, $\bar{z}$ is a generalized efficient point of A with respect to Θ.

Proof Suppose to the contrary that $\bar{z} \notin GE(A \,|\, \Theta)$. By Theorem 2.1 and the free disposal assumption, $\bar{z} \notin \text{bd}\, A$. This implies that $\bar{z} \in \text{int}\, A$. Therefore $T_C(A, \bar{z}) = Z$. By the assumption of the theorem, we have $T_C(A, \bar{z}) \cap T_0(\Theta, \mathbf{0}) = \{\mathbf{0}\}$. Thus $T_0(\Theta, \mathbf{0}) = \{\mathbf{0}\}$. Let h be an arbitrary element in $Z \backslash \{\mathbf{0}\}$. We claim that $h \notin \Theta$. Assuming the contrary and using the fact that $h \notin T_0(\Theta, \mathbf{0})$, there exists a sequence $\{t_n\} \subset]0, 1]$ such that $t_n \to 0$ and for every sequence $h_n \to h$ we have

$$t_n h_n \notin \Theta \;\text{ for some } n \in \mathbb{N}. \tag{9}$$

□

Take $h_n := h$ for all $n \in \mathbb{N}$. By the starshapeness of Θ at $\mathbf{0}$, we have $t_n h \in \Theta$ for all $n \in \mathbb{N}$, contrary to (9). Thus $\Theta = \{\mathbf{0}\}$, contrary to $\Theta \backslash \{\mathbf{0}\} \neq \emptyset$. □

In [17], Jofré and Fourani introduced some conditions ensuring Pareto optimality as follows.

Theorem 3.2 *(see* [17, Theorem 3]*) Suppose that Θ is starshaped at* $\mathbf{0}$*, A satisfies the free disposal condition and either the tangent relation*

$$T_0(A, \bar{z}) \cap T_0(\Theta, \mathbf{0}) = \{\mathbf{0}\} \tag{10}$$

holds at $\bar{z} \in A$, or $T_0(A, \bar{z})$ does not contain any line. Then $\bar{z}$ is a Pareto efficient point of A with respect to Θ.

Remark 3.2 If the conditions in Theorem 3.2 hold at $\bar{z}$ and $\Theta \backslash \{\mathbf{0}\} \neq \emptyset$, then, thanks to Proposition 2.2, $\bar{z}$ is a generalized efficient point of A with respect to Θ. Since, $T_C(A, \bar{z}) \subset T_0(A, \bar{z})$, condition (8), which guarantees that $\bar{z} \in GE(A \mid \Theta)$, is weaker than condition (10). Furthermore, the condition "$T_C(A, \bar{z})$ *is not the whole space*" is also weaker than the condition "$T_0(A, \bar{z})$ *does not contain any line*."

3.2 Existence of Generalized Efficient Points

We next provide some sufficient conditions for the nonemptiness of the set of generalized efficient points. Firstly, we introduce a partial order in Z. Let Θ be a subset in Z satisfying condition (T). Then, Θ induces a partial order in Z as follows: $z_1, z_2 \in Z, z_2 \geq z_1$ iff $z_1 - z_2 \in \Theta$. We write $z_2 > z_1$ iff $z_1 - z_2 \in \Theta \backslash \{\mathbf{0}\}$. A net $\{x_\alpha\}_{\alpha \in I}$ from Z is said to be a *decreasing* with respect to Θ iff $x_\alpha > x_\beta$ for each $\alpha, \beta \in I, \beta > \alpha$. For each $x \in Z$, put $A_x := A \cap (x + \Theta)$. The set A_x is called a *section* of A at x.

Definition 3.2 (*see* [26]) A set $A \subset Z$ is said to be Θ*-complete* iff it has no covers of the form $\{(x_\alpha + \Theta)^c\}_{\alpha \in I}$ with $\{x_\alpha\}_{\alpha \in I}$ being a decreasing net in A.

The following result gives a necessary and sufficient condition for the existence of Pareto efficient points.

Theorem 3.3 *Let A be a nonempty subset in Z. Assume that Θ satisfies condition* (T)*. Then, $E(A \mid \Theta)$ is nonempty if and only if A has a nonempty Θ-complete section.*

Proof Suppose that $E(A \mid \Theta)$ is a nonempty set. Let $\bar{z} \in E(A \mid \Theta)$. Then $A \cap (\bar{z} + \Theta) = \{\bar{z}\}$. Consequently, the section $A_{\bar{z}}$ is singleton. Thus, there is no decreasing net in there.

Conversely, assume that there exists $\bar{z} \in A$ such that $A_{\bar{z}}$ is Θ-complete. We claim that $E(A_{\bar{z}} \mid \Theta)$ is nonempty. Indeed, thanks to [6, Corollary 2.1(i)], it suffices to prove that every decreasing net $\{x_\alpha\}_{\alpha \in I}$ of $A_{\bar{z}}$ is bounded from below in $A_{\bar{z}}$. Since the Θ-completeness of $A_{\bar{z}}$, one has

$$A_{\bar{z}} \not\subseteq \bigcup_{\alpha \in I} (x_\alpha + \Theta)^c.$$

Thus, there exists $a \in A_{\bar{z}}$ such that $a \in (x_\alpha + \Theta)$ for all $\alpha \in I$. This means that $x_\alpha \geq a$ for all $\alpha \in I$. Thus, $\{x_\alpha\}_{\alpha \in I}$ is bounded from below. By condition (T), it is easy to check that $E(A_{\bar{z}} \mid \Theta) \subset E(A \mid \Theta)$ and the assertion follows. □

Remark 3.3 The proof of Theorem 3.3 follows the proof scheme of [9, Theorem 15]. In this paper, the authors established the existence of Pareto efficient points of a set with respect to an improvement set of a given cone. Note that condition (T) does not imply that Θ is convex or conical. Furthermore, an ordering set satisfying condition (T) may not be an improvement of any nontrivial cone in Z, such as $\Theta := \{(\theta_1, \theta_2) \in \mathbb{Q}^2 \,:\, \theta_1 \geq 0, \theta_2 \geq 0\}$. Thus, our result is slightly different from [9, Theorem 15] and improves [26, Theorem 3.3].

The following results give sufficient conditions for the existence of generalized efficient points. These results can easily be obtained from Theorem 3.3, so their proofs are omitted.

Corollary 3.1 *Let A be a nonempty subset in Z. Assume that Θ satisfies condition* (T)*. If A admits a nonempty Θ-complete section, then $GE(A \mid \Theta)$ is nonempty.*

Corollary 3.2 *Let A be a nonempty set in Z, and $\Theta \subset Z$ be a set satisfying $\mathbf{0} \in Z$ and $\Theta \backslash \{\mathbf{0}\} \neq \emptyset$. If $\widetilde{\Theta} :=$ conv cone Θ is pointed and A admits a nonempty $\widetilde{\Theta}$-complete section, then $GE\,(A \mid \Theta) \neq \emptyset$.*

The forthcoming example shows that the sufficient condition for the nonemptiness of $GE(A \mid \Theta)$ given by Corollary 3.1 is not a necessary one.

Example 3.1 Let $A = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 \geq 0\}$ and $\Theta = \mathbb{R}^2_-$. It is clear that Θ is a pointed closed convex cone. Thus, Θ satisfies condition (T). A trivial verification shows that $E(A \mid \Theta) = \emptyset$. By Theorem 3.3, the set A has no Θ-complete section. However, we can see that $A - \Theta = A$ and

$$A \cap \mathrm{bd}\,(A - \Theta) = \mathrm{bd}\,A = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 = 0\}.$$

Thus $GE(A \mid \Theta) = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 = 0\}$.

The next example shows that the condition $l(\Theta) = \{\mathbf{0}\}$ cannot be dropped in Corollary 3.1.

Example 3.2 (see [12]) Let $A = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 x_2 = 0\}$ and

$$\Theta = \{(\theta_1, \theta_2) \in \mathbb{R}^2 \,:\, \theta_2 = -\theta_1\}.$$

We see that A is Θ-complete and Θ satisfies all condition in Corollary 3.1 except the requirement that $l(\Theta) = \{\mathbf{0}\}$. It is easily seen that $E(A \mid \Theta) = \{\mathbf{0}\}$ and $GE(A \mid \Theta) = A \cap \mathrm{bd}\,(A - \Theta) = \emptyset$.

The following result ensures the existence of generalized efficient points of a nonempty compact subset A in an infinite-dimensional space.

Corollary 3.3 *Let A be a nonempty compact set in Z, and $\Theta \subset Z$ be a set satisfying* $\mathbf{0} \in Z$ *and* $\Theta \backslash \{\mathbf{0}\} \neq \emptyset$. *If* $\widehat{\Theta} :=$ cl conv cone Θ *is pointed, then* $GE\,(A \mid \Theta) \neq \emptyset$.

Proof By the compactness of A and [26, Lemma 3.5], A is $\widehat{\Theta}$-complete. Clearly, the cone $\widehat{\Theta}$ satisfies condition (T). Thanks to Corollary 3.1, the set $GE\left(A \mid \widehat{\Theta}\right)$ is nonempty. It is easily seen that $GE\left(A \mid \widehat{\Theta}\right) \subset GE\,(A \mid \Theta)$. Thus, $GE\,(A \mid \Theta)$ is nonempty. □

Remark 3.4 In [26, Corollary 3.11], Luc shows that "*If Z is of finite-dimension, then $E(A \mid \Theta)$ is nonempty whatever a nonempty compact set A and a convex cone Θ be*." However, this fact does not hold true for an infinite-dimension space. Corollary 3.3 shows that a compact set in an infinite-dimension space admits a generalized efficient point provided that the closed convex cone hull of the ordering set Θ is pointed.

Example 3.3 (see [26, Example 3.13] and [5]) Let Z be the vector space of all sequences $x = \{x_n\}$ of real numbers such that $x_n = 0$ for all but a finite number of choices for n. It is a normed space, if we provide it with the norm

$$\|x\| = \max\{|x_n| \,:\, n = 1, 2, \ldots\}.$$

Let Θ be the cone composed of zero and of sequences, whose last nonzero term is negative. Then, Θ is a convex pointed cone. It is called a *ubiquitous cone* because the linear space spanned by Θ is the whole Z. Let e_n stay for the vector with the unique nonzero component being -1 at the nth place. Consider the set

$$A = \{x_0\} \cup \left\{ \bigcup_{n=1}^{\infty} \sum_{i=1}^{n} x_i \,:\, n = 1, 2, \ldots \right\},$$

where

$$x_0 = e_1,\; x_n = \sum_{i=1}^{n-1} \frac{e_i}{2^{n-1}} - \frac{e_n}{2^{n-1}}, n \geq 1.$$

Then, A is compact because $\lim_{n\to\infty} \sum_{i=1}^{n} x_i = x_0$. Furthermore,

$$x_0 > \sum_{i=1}^{n} x_i > \sum_{i=1}^{n+1} x_i,$$

which shows that $E(A \mid \Theta) = \emptyset$. However, $GE(A \mid \Theta)$ is a nonempty subset in Z. Indeed, we have $\widehat{\Theta} :=$ cl conv cone $\Theta = \{z = \{z_n\} \in Z \,:\, z_n \leq 0\}$. Clearly, $\widehat{\Theta}$ is a closed pointed cone. Thanks to Corollary 3.3, $GE(A \mid \Theta)$ is nonempty.

When the sets are polyhedral, Luc [26, Theorem 3.18] introduced a necessary and sufficient condition for the existence of nonempty complete sections in terms of recession cones. We recall that the *recession cone* of a nonempty set A is defined by

$$\mathrm{Rec}\,(A) := \{v \in Z \ : \ a + tv \in A \ \ \forall a \in A, \forall t \geq 0\}.$$

The set A is said to be a *polyhedral* iff it is the sum of a polyhedron and a polyhedral cone; see [26] for more details. Clearly, every polyhedral set is closed. The following result ensures the existence for generalized efficient points of a polyhedral set in an infinite-dimensional space. Its proof follows directly from [26, Theorem 3.18], Theorem 3.3, and Proposition 2.1, so omitted.

Proposition 3.1 *Let Z be a real Banach space, $A \subsetneq Z$ be a polyhedral set, and $\Theta \subset Z$ be a pointed convex cone. If*

$$\mathrm{Rec}\,(A) \cap \Theta = \{\mathbf{0}\},$$

then $GE\,(A\,|\,\Theta) \neq \emptyset$.

By using the coercive condition, we present some results on the existence of Pareto (generalized) efficient points of unbounded sets in a reflexive Banach space with respect to an arbitrary ordering set. The first result is an extension of Theorem 26 in [9] without the improvement property of $\Theta\backslash\{\mathbf{0}\}$, but the second one is a new result.

Theorem 3.4 *Let Z be a reflexive Banach space, $\Theta \subset Z$ be an arbitrary set containing $\mathbf{0} \in Z$, and $A \subset Z$ be an unbounded nonempty set. Moreover, suppose that the following conditions hold:*

(i) *there exists $z^* \in Z^*$ satisfying $\langle z^*, \theta\rangle < 0$ for all $\theta \in \Theta\backslash\{\mathbf{0}\}$;*
(ii) $\lim\limits_{\substack{a\in A \\ \|a\|\to\infty}} \langle z^*, a\rangle = +\infty$;
(iii) *either A or $A - \Theta$ is weakly closed. Then, $E(A\,|\,\Theta)$ is nonempty.*

Proof It follows from assumption (ii) that

$$\lambda_{z^*}(A) := \inf_{a\in A} \langle z^*, a\rangle > -\infty.$$

For each $k \in \mathbb{N}$, let $a_k \in A$ such that

$$\langle z^*, a_k\rangle < \lambda_{z^*}(A) + \frac{1}{k}.$$

Thanks to assumption (ii), the sequence $\{a_k\}$ is bounded. By the reflexivity of Z and the weakly closedness of $A - \Theta$ (or A), without any loss of generality, we may assume that a_k weakly converges to $\bar{z} - \bar{\theta}$, where $\bar{z} \in A$ and $\bar{\theta} \in \Theta$. For each $\theta \in \Theta\backslash\{\mathbf{0}\}$, by assumption (i), we have

$$\langle z^*, \bar{z} + \theta\rangle < \langle z^*, \bar{z}\rangle = \langle z^*, (\bar{z} - \bar{\theta}) + \bar{\theta}\rangle \leq \langle z^*, \bar{z} - \bar{\theta}\rangle = \lim_{k\to\infty} \langle z^*, a_k\rangle = \lambda_{z^*}(A).$$

Thus $\bar{z} + \theta \notin A$. This implies that

$$A \cap (\bar{z} + \Theta \backslash \{\mathbf{0}\}) = \emptyset,$$

or, equivalently, $\bar{z} \in E(A \,|\, \Theta)$. □

The following example shows that Theorem 3.4 improves the corresponding result in [9, Theorem 26].

Example 3.4 Let A and Θ be two subsets of $\mathbb{R}^2$ defined as follows

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 0, -1 \leq x_2 \leq 0\}, \Theta = \{(\theta_1, \theta_2) \,:\, \theta_1\theta_2 = 0\}.$$

Let $z^* = (1, 1)$. Then A, Θ, and z^* satisfy all assumptions of Theorem 3.4. Thus $E(A \,|\, \Theta) \neq \emptyset$. It is easy to check that $E := \Theta \backslash \{\mathbf{0}\}$ cannot be an improvement set with respect to any nontrivial cone. Therefore, [9, Theorem 26] cannot be applied for this example.

Theorem 3.5 *Let Z be a reflexive Banach space, $\Theta \subset Z$ be an arbitrary set containing $\mathbf{0} \in Z$, and $A \subset Z$ be an unbounded nonempty set. Suppose that the conditions*
(i′) *there exists $z^* \in Z^* \backslash \{\mathbf{0}\}$ satisfying $\langle z^*, \theta \rangle \leq 0$ for all $\theta \in \Theta$,*
(ii)*, and* (iii) *of Theorem* 3.4 *are satisfied. Then, $GE(A \,|\, \Theta)$ is nonempty.*

Proof Let $\{a_k\} \subset A$ be a sequence as in the proof of Theorem 3.4. Since $z^* \neq \mathbf{0}$, there exists $c \in Z$ such that $\langle z^*, c \rangle > 0$. For each $k \in \mathbb{N}$, put $z_k := \frac{c}{k}$. Then, for each $k \in \mathbb{N}$ and $\theta \in \Theta$, we have

$$\langle z^*, \bar{z} + \theta - z_k \rangle < \langle z^*, \bar{z} \rangle = \langle z^*, (\bar{z} - \bar{\theta}) + \bar{\theta} \rangle \leq \langle z^*, \bar{z} - \bar{\theta} \rangle = \lim_{n \to \infty} \langle z^*, a_n \rangle = \lambda_{z^*}(A).$$

This implies that $\bar{z} + \theta - z_k \notin A$ for all $k \in \mathbb{N}$ and $\theta \in \Theta$, or, equivalently,

$$A \cap (\bar{z} + \Theta - z_k) = \emptyset \;\; \forall k \in \mathbb{N}.$$

Thus $\bar{z} \in GE(A \,|\, \Theta)$. □

Remark 3.5 (i) If $A \subset Z$ is bounded, the same proof shows that $E(A \,|\, \Theta)$ (resp., $GE(A \,|\, \Theta)$) is nonempty under the only assumptions (i) and (iii) (resp., (i′) and (iii)).
(ii) Clearly, assumption (i) of Theorem 3.4 implies assumption (i′) of Theorem 3.5. Thus, if A and Θ satisfy assumptions (i)–(iii) of Theorem 3.4, then $GE(A \,|\, \Theta) \neq \emptyset$. The following example shows that assumption (i) of Theorem 3.4 may be too strict for the existence of generalized efficient points.

Example 3.5 Let A and Θ be two subsets of $\mathbb{R}^2$ defined as follows

$$A = (]-\infty, 0] \times \{0\}) \cup (\{0\} \times [0, 1]), \Theta = \{(\theta_1, \theta_2) \in \mathbb{R}^2 \,:\, \theta_1 \geq 0, \theta_2 \geq -\theta_1^2\}.$$

Put $z^* := (-1, 0)$. It is easy to check that A, Θ, and z^* satisfy all assumptions of Theorem 3.5, and we therefore get $GE(A \,|\, \Theta) \neq \emptyset$. Moreover, an easy computation shows that $E(A \,|\, \Theta) = \emptyset$, and there is no $z^* \in \mathbb{R}^2$ satisfying assumption (i) of Theorem 3.4.

3.3 Applications to Set-Valued Optimization Problems

Let $F\colon X \rightrightarrows Z$ be a set-valued mapping between two Banach spaces. Denote by

$$\operatorname{dom} F := \{x \in X \,:\, F(x) \neq \emptyset\},\ \operatorname{gph} F := \{(x, y) \in X \times Z \,:\, y \in F(x)\}$$

the *domain* and the *graph* of F.

Definition 3.3 (*see* [26, Definition 7.1]) Let Ω be a subset of dom F, and $\Theta \subset Z$ be a set containing the zero vector. We say that F is *upper Θ-continuous at $\bar{x} \in \Omega$* iff for each neighborhood V of $F(\bar{x})$ in Z, there is a neighborhood U of $\bar{x}$ in X such that

$$F(x) \subset V - \Theta \quad \forall x \in U \cap \operatorname{dom} F.$$

We say that F is *upper Θ-continuous on Ω* iff it is upper Θ-continuous at every point of Ω. When $\Theta = \{\mathbf{0}\}$ we say simply *upper continuous* instead of upper $\{\mathbf{0}\}$-continuous.

Definition 3.4 (*see* [15, Definition 1.1]) Let $F : X \rightrightarrows Z$ be a set-valued mapping between Banach spaces and let $\Theta \subset Z$ be a set containing the origin, and a set constraint $\Omega \subset X$. We say that a pair $(\bar{x}, \bar{z}) \in \operatorname{gph} F \cap (\Omega \times Z)$ is a *generalized efficient (or extremal) solution* to F with respect to the ordering set Θ over Ω iff there is a sequence $\{z_k\} \subset Z$ with $\|z_k\| \to 0$ as $k \to \infty$ such that

$$F(\Omega) \cap (\Theta + \bar{z} - z_k) = \emptyset \quad \forall k \in \mathbb{N}.$$

The set of generalized efficient solutions to F with respect to Θ over Ω is denoted by $GE(F, \Omega, \Theta)$.

When $F = f : X \to Z$ is single-valued mapping, we omit $\bar{z}$ in the notation of generalized order efficient solutions and $\bar{x}$ is called a (f, Θ)-*optimal point* over Ω; see [10, Definition 5.53].

Proposition 3.2 *Assume that $\Omega \subset Z$ is a compact set and F is upper continuous on Ω. Let $\Theta \subset Z$ be a set containing the origin and satisfying $\Theta \backslash \{\mathbf{0}\} \neq \emptyset$. If $\widehat{\Theta} := \operatorname{cl}\operatorname{conv}\operatorname{cone}\Theta$ is pointed, then $GE(F, \Omega, \Theta)$ is nonempty.*

Proof By the compactness of Ω and the upper continuity of F, thanks to [27, Theorem 6.3], we have $F(\Omega)$ is a compact set. Thanks to Corollary 3.3, we have $GE(F(\Omega)\,|\,\Theta) \neq \emptyset$. Consequently, $GE(F, \Omega, \Theta) \neq \emptyset$. □

Proposition 3.3 *(cf.* [26, Theorem 5.3]*) Let Θ be a set satisfying condition* (T). *Assume that Ω is nonempty compact and F is upper Θ-continuous on Ω with $F(x) - \Theta$ being closed Θ-complete for every $x \in \Omega$. Then, $GE(F, \Omega, \Theta)$ is nonempty.*

Proof Thanks to Theorem 3.1, it is sufficient to verify that $F(\Omega)$ is Θ-complete. On the contrary, suppose that $F(\Omega)$ is not Θ-complete. Then there exists a decreasing net $\{a_\alpha\}_{\alpha \in I}$ from $F(\Omega)$ such that $\{(a_\alpha + \Theta)^c\}_{\alpha \in I}$ is a cover of $F(\Omega)$. For each $\alpha \in I$, let $x_\alpha \in \Omega$ such that $a_\alpha \in F(x_\alpha)$. By the compactness of Ω, without loss of generality we can assume that $\lim x_\alpha = x \in \Omega$. For every neighborhood V of $F(x)$, by the upper Θ-continuity of F on Ω, there exists $\alpha_0 \in I$ such that

$$a_\alpha \in V - \Theta \ \text{ for all } \ \alpha \geq \alpha_0. \tag{11}$$

Furthermore, since $\{a_\alpha\}_{\alpha\in I}$ is decreasing, one has $a_\alpha \in a_{\alpha_0} - \Theta$ for all $\alpha \leq \alpha_0$. This and (11) imply that $a_\alpha \in V - \Theta$ for all $\alpha \in I$. Consequently,

$$a_\alpha \in \mathrm{cl}\,(F(x) - \Theta) = F(x) - \Theta \ \text{ for all } \ \alpha \in I.$$

Thus, $\{a_\alpha\}_{\alpha\in I}$ is a decreasing net from $F(x) - \Theta$. Note that $\{(a_\alpha + \Theta)^c\}_{\alpha\in I}$ covers $F(\Omega)$, so does $F(x) - \Theta$. We arrive at the contradiction that $F(x) - \Theta$ is not Θ-complete. □

Remark 3.6 From the proof of Proposition 3.3 we also see that the set of Pareto efficient solutions to F with respect to Θ over Ω is nonempty. Thus, our result generalizes [26, Theorem 5.3] to the ordering set, which may not be convex or conical.

Example 3.6 Let $X = Z = \mathbb{R}^2$, $\Theta = \{(\theta_1, \theta_2) \in \mathbb{R}^2 \mid \theta_1 \geq 0, \theta_2 \geq -\theta_1^2\}$, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \frac{1}{2}x_1, -1 \leq x_1 \leq 0\}$, and $F(x_1, x_2) = \{(x_1, x_2)\}$ for all $(x_1, x_2) \in \Omega$. We see that Θ is neither convex nor conical. It is a simple matter to verify that all the conditions of Proposition 3.3 are satisfied. Thus, $GE(F, \Omega, \Theta)$ is nonempty. Moreover, an easy computation shows that

$$GE(F, \Omega, \Theta) = E(F, \Omega, \Theta) = \{((0, 0), (0, 0))\},$$

where $E(F, \Omega, \Theta) := \{(x, z) \in \mathrm{gph}\, F : z \in E(F(\Omega) \mid \Theta)\}$ is the set of all Pareto efficient solutions to F with respect to Θ over Ω.

Remark 3.7 If condition (T) is not satisfied, then the conclusion of Proposition 3.3 may not hold. This means that condition (T) is an essential assumption of that proposition.

Example 3.7 Let X, Z, Ω, F be as in Example 3.6, and

$$\Theta = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \leq z_1\} \cup \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \leq -z_1\}.$$

Clearly, the cone Θ is neither convex, nor pointed. Thus, we see that all the assumptions of Proposition 3.3 are satisfied except condition (T). We claim that $GE(F, \Omega, \Theta) = \emptyset$, or, equivalently, $GE(A \mid \Theta) = \emptyset$, where $A := F(\Omega)$. Indeed, we have

$$A - \Theta = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \geq z_1\} \cup \left\{(z_1, z_2) \in \mathbb{R}^2 : z_2 \geq -z_1 - \frac{3}{2}\right\},$$

and $A \subset \mathrm{int}(A - \Theta)$. Thus $A \cap \mathrm{bd}\,(A - \Theta) = \emptyset$, or, equivalently, $GE(A \mid \Theta) = \emptyset$.

We finish this section by presenting a new sufficient condition for the existence of generalized order optimal solutions to vector optimization problems with noncompact feasible sets.

Theorem 3.6 *Let X, Z be two Banach spaces, $f : X \to Z$ be a single-valued mapping, and $\Theta \subset Z$ be an arbitrary set containing $\mathbf{0}$. Assume that there exists $z^* \in Z^*\backslash\{\mathbf{0}\}$ and $\lambda \in \mathbb{R}$ such that*

(i) *$\langle z^*, \theta\rangle \leq 0$ for all $\theta \in \Theta$;*
(ii) *$D(\lambda) := \{x \in X \ : \ \langle z^*, f(x)\rangle \leq \lambda\}$ is a nonempty compact set;*
(iii) *the map $x \in X \mapsto \langle z^*, f(x)\rangle$ is lower semicontinuous. Then, $GE(f, X, \Theta)$ is nonempty.*

Proof By assumptions (ii), (iii) and Weierstrass theorem, there exists $\bar{x} \in D(\lambda)$ such that

$$\langle z^*, f(\bar{x})\rangle \leq \langle z^*, f(x)\rangle \ \ \forall x \in D(\lambda). \tag{12}$$

We claim that $\bar{x} \in GE(f, X, \Theta)$. Since $z^* \neq \mathbf{0}$, there exists $c \in Z$ satisfying $\langle z^*, c\rangle < 0$. For each $k \in \mathbb{N}$, put $z_k := -\frac{c}{k}$. Clearly, $\|z_k\| \to 0$ as $k \to \infty$. For each $\theta \in \Theta$ and $k \in \mathbb{N}$, by assumption (i), we have

$$\langle z^*, f(\bar{x}) + \theta - z_k\rangle \leq \langle z^*, f(\bar{x}) - z_k\rangle < \langle z^*, f(\bar{x})\rangle.$$

By (12), $f(\bar{x}) + \theta - z_k \notin f(D(\lambda))$. For each $x \in X\backslash D(\lambda)$, one has

$$\langle z^*, f(x)\rangle > \lambda \geq \langle z^*, f(\bar{x})\rangle > \langle z^*, f(\bar{x}) + \theta - z_k\rangle.$$

This implies that $f(x) \neq f(\bar{x}) + \theta - z_k$. Thus

$$f(X) \cap (f(\bar{x}) + \Theta - z_k) = \emptyset \ \ \forall k \in \mathbb{N},$$

or, equivalently, $f(\bar{x}) \in GE(f(X) \,|\, \Theta)$. Consequently, $\bar{x} \in GE(f, X, \Theta)$. □

Remark 3.8 If the assumptions (ii), (iii) of Theorem 3.6 hold, and there exists $z^* \in Z^*$ such that $\langle z^*, \theta\rangle < 0$ for all $\theta \in \Theta\backslash\{\mathbf{0}\}$, the same proof shows that $E(f(X) \,|\, \Theta)$ is nonempty.

4 Conclusions

In this paper, we establish some results for the existence of generalized order optimal solutions to vector optimization problems in infinite-dimensional spaces. The obtained results are compared with the well-known results of vector optimization with convex ordering cones. Some sufficient optimality conditions for a generalized efficient point to a set satisfying the free disposal hypothesis are also examined.

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