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When the Karush–Kuhn–Tucker Theorem Fails: Constraint Qualifications and Higher-Order Optimality Conditions for Degenerate Optimization Problems

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Abstract In this paper, we present higher-order analysis of necessary and sufficient optimality conditions for problems with inequality constraints. The paper addresses the case when the constraints are not assumed to be regular at a solution of the optimization problems. In the first two theorems derived in the paper, we show how Karush–Kuhn–Tucker necessary conditions reduce to a specific form containing the objective function only. Then we present optimality conditions of the Karush–Kuhn–Tucker type in Banach spaces under new regularity assumptions. After that, we analyze problems for which the Karush–Kuhn–Tucker form of optimality conditions does not hold and propose necessary and sufficient conditions for those problems. To formulate the optimality conditions, we introduce constraint qualifications for new classes of nonregular nonlinear optimization. The approach of p-regularity used in the paper can be applied to various degenerate nonlinear optimization problems due to its flexibility and generality.

Keywords Nonregular problems · Constrained optimization · Optimality conditions

Mathematics Subject Classification 49K27 · 49N60 · 90C30 · 90C46

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1 Introduction

The primary focus of the paper is on Karush–Kuhn–Tucker-type (KKT) optimality conditions for optimization problems with inequality constraints for the cases when regularity assumptions (constraint qualifications) known in the literature (see, for example, [1]) are not satisfied at a solution. One goal of the paper is to present KKT-type optimality conditions in Banach spaces under new regularity assumptions. Another goal is to analyze problems for which the KKT form of optimality conditions does not hold (see Example 7.1) and to propose necessary and sufficient conditions for those problems.

The regularity assumptions proposed in the paper expand constraint qualifications known in the literature (e.g., [1–4]) to new classes of optimization problems. While work [1] gives a careful comparison and classification of the existing constraint qualifications, all of them are stated using at most the first derivatives of the constraints. The regularity assumptions proposed in this paper expand considerations to new classes of problems by using higher-order derivatives. As a result, our approach allows us to analyze problems, where, for example, the first-order derivatives of all constraints are equal to zero, which are not covered by any regularity assumptions given in [1–4]. Also, in addition to the classical KKT-type optimality conditions analyzed in [1–4], our approach is applicable to cases when the KKT Theorem fails but generalized forms of the KKT conditions can be derived.

There is the extended literature on generalizations of the KKT Theorem to an infinite-dimensional setting (the relevant references can be found, for example, in [5, p. 159] or in [6–8]). Our approach is based on the construction of p-regularity introduced earlier in [9–12]. The main idea of the p-regularity is that it replaces the operator of the first derivative, which is not surjective, by a special operator that is onto. Various optimality conditions proposed for the degenerate case are given, for example, in [13–18]. An approach similar to ours is used in [19–22]. The main differences between optimality conditions proposed in this paper and those in [19–22] are that we consider a more general case of $p \geq 2$ and do not make some additional assumptions introduced in [19–22]. We give a more detailed comparison in Sect. 8.

The paper is organized as follows. We formulate the problem in Sect. 2. We start with an absolutely degenerate case in Sect. 3, when the Karush–Kuhn–Tucker necessary conditions reduce to a specific form containing the objective function only. In Sect. 4.1, we analyze some cases, when the KKT conditions hold for nonregular problems with a nonzero multiplier corresponding to the objective function and present new KKT-type optimality conditions in Theorems 4.1 and 4.2. After that, in Sect. 4.2, we analyze problems for which the Karush–Kuhn–Tucker form of optimality conditions does not hold. Necessary and sufficient conditions derived in Theorem 4.4 can be viewed as generalized KKT optimality conditions. As auxiliary results, we derive new geometric necessary conditions in Lemmas 3.1 and 4.1. In Sect. 5, we consider a general case of degeneracy, where we do not make assumption (10), which is one of the main assumptions in Sect. 4. A new approach presented in Sect. 5 was briefly announced in our paper [23] and is used to reduce degenerate optimization problems to new forms, so that one can use simpler ways to analyze those problems. Some directions for future work are briefly described in Sect. 6. We illustrate the optimality conditions by some examples in Sect. 7, give additional comparison with other results in Sect. 8, and conclude the paper with Sect. 9.

2 Formulation of the Problem

We consider a nonlinear optimization problem with inequality constraints

$$\min_{x\in X} f(x) \quad \text{s.t.} \quad g(x) = (g_1(x), \dots, g_m(x)) \leq 0, \tag{1}$$

where the functions f and g_i are sufficiently smooth functions from the Banach space X to $\mathbb{R}$. In the case when the Linear Independence Constraint Qualification is not satisfied at a solution $\bar{x}$ of the problem (1), we call the problem degenerate (nonregular) at $\bar{x}$. The Karush–Kuhn–Tucker (KKT) Theorem states that if $\bar{x}$ is a local solution of problem (1) and a regularity assumption holds, then there exist Lagrange multipliers $\lambda_1^*, \dots, \lambda_m^*$ such that

$$f'(\bar{x}) + \sum_{j=1}^{m} \lambda_j^* g_j'(\bar{x}) = 0, \quad g(\bar{x}) \leq 0, \quad \lambda_j^* \geq 0, \quad \lambda_j^* g_j(\bar{x}) = 0, \quad j = 1, \dots, m. \tag{2}$$

It is interesting to note that, while the first equation in (2) holds, the requirement that the Lagrange multipliers λ_j^* are nonnegative can be violated for degenerate optimization problems. For example, note that $\bar{x} = 0$ is a local minimizer for the problem:

$$\min_{x\in X} f(x) = -x_2 \quad \text{s.t.} \quad g_1(x) = -x_1^2 - x_2 \leq 0, \; g_2(x) = x_1^{12} + x_2^3 \leq 0.$$

Then, the first equation in (2) reduces to $\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and yields $\lambda_1 < 0$, contradicting $\lambda_j \geq 0$ in (2). For problems of this type, Theorem 4.4 and Theorem 5.1 give necessary optimality conditions, which can be viewed as a generalized form of the KKT conditions and guarantee that all Lagrange multipliers are of the same sign. Example 7.1 illustrates this case.

Notation: For some set C, we denote the span of C by span(C), and the set of all nonnegative combinations of vectors in C by cone C. We let $g_i^{(p)}(x)$ be the pth derivative of $g_i : X \to \mathbb{R}^n$ at the point x; the associated p-form is $g_i^{(p)}(x)[h]^p := g_i^{(p)}(x)(h, h, \dots, h)$. Notation $g_i^{(p)}(x)[h]^{p-1}$ means $\left(g_i^{(p-1)}(x)[h]^{p-1}\right)'_x$ (see [24] for additional details). The other notation will be introduced below as needed.

3 New Optimality Conditions for an Absolutely Degenerate Case and an Even *p*

Throughout this section, we assume that p is an even number and

$$g_i^{(j)}(\bar{x}) = 0, \quad j = 1, \ldots, p-1, \quad \forall i \in I(\bar{x}), \tag{3}$$

where $I(\bar{x})$ is the set of indices of active constraints at $\bar{x}$, $I(\bar{x}) := \{i = 1, \ldots, m : g_i(\bar{x}) = 0\}$. The case of an odd p was covered in our paper [25].

We need the following additional notation.

Let $S := \{x \in X : g_i(x) \leq 0,\ i = 1, \ldots, m\}$ denote the feasible set for problem (1), $G_p(\bar{x}, h) := \{d \in X : \langle g_i^{(p)}(\bar{x})[h]^{p-1}, d\rangle \leq 0,\ \forall i \in I(\bar{x})\}$ for some $h \in X$, and

$$F_0(\bar{x}) := \{d \in X : \langle f'(\bar{x}), d\rangle < 0\}. \tag{4}$$

The following theorem presents one of the main results of this section.

Theorem 3.1 *Let $\bar{x}$ be a local minimizer of problem (1), $f(x) \in C^2(X)$, and $g_i(x) \in C^{p+1}(X)$, $i = 1 \ldots m$. Assume that (3) holds with some even p and that there exist vectors $h \in X$, $\|h\| = 1$, and $\xi \in X$, $\|\xi\| = 1$, such that for all $i \in I(\bar{x})$,*

$$g_i^{(p)}(\bar{x})[h]^p = 0 \quad \text{and} \quad \langle g_i^{(p)}(\bar{x})[h]^{p-1}, \xi\rangle < 0. \tag{5}$$

Then $f'(\bar{x}) = 0$.

Note that assumption (5) can be viewed as a new generalization of the Mangasarian–Fromovitz constraint qualification.

Without loss of generality, we may assume that $I(\bar{x}) = \{1, \ldots, m\}$ throughout the paper, because the continuity of $g_i(x)$ for $i \notin I(\bar{x})$ prevents $g_i(\bar{x})$ from taking the value 0 on some neighborhood of $\bar{x}$. We need the following lemmas to prove Theorem 3.1.

Lemma 3.1 (Geometric Necessary Condition) *Let the assumptions of Theorem 3.1 hold. Then*

$$F_0(\bar{x}) \cap G_p(\bar{x}, h) = \emptyset. \tag{6}$$

Proof Assume on the contrary that there exists $d \in F_0(\bar{x}) \cap G_p(\bar{x}, h)$. Without loss of generality, let $\|d\| = 1$. Since $d \in G_p(\bar{x}, h)$,

$$\langle g_i^{(p)}(\bar{x})[h]^{p-1}, d\rangle \leq 0 \quad \forall\, i \in I(\bar{x}). \tag{7}$$

First, we will prove that $\langle f'(\bar{x}), h\rangle = 0$. Assume on the contrary that $\langle f'(\bar{x}), h\rangle \neq 0$. Let $\xi \in X$ satisfy (5) and consider $\bar{x} + th + t^{3/2}\xi$ and $\bar{x} - th - t^{3/2}\xi$ for some sufficiently small t. Then, for $i \in I(\bar{x})$, we get the following inequalities with $\bar{r}_i(t)$ and $\tilde{r}_i(t)$, $\|\bar{r}_i(t)\| = o(t^{p+1/2})$, $\|\tilde{r}_i(t)\| = o(t^{p+1/2})$,

$$g_i(\bar{x} + th + t^{3/2}\xi) = \frac{1}{p!}\left(g_i^p(\bar{x})[th]^p + t^{p+1/2}\langle g_i^p(\bar{x})[h]^{p-1}, \xi\rangle\right) + \bar{r}_i(t) \leq 0,$$

$$g_i(\bar{x} - th - t^{3/2}\xi) = \frac{1}{p!}\left(g_i^p(\bar{x})[th]^p + t^{p+1/2}\langle g_i^p(\bar{x})[-h]^{p-1}, -\xi\rangle\right) + \tilde{r}_i(t) \leq 0.$$

The inequalities imply that $\bar{x} + th + t^{3/2}\xi \in S$ and $\bar{x} - th - t^{3/2}\xi \in S$ for all sufficiently small t. Then $\langle f'(\bar{x}), h\rangle \neq 0$ yields $f(\bar{x} + th + t^{3/2}\xi) < f(\bar{x})$ or $f(\bar{x} - th - t^{3/2}\xi) < f(\bar{x})$, which contradicts the assumption that $\bar{x}$ is a local minimizer and proves $\langle f'(\bar{x}), h\rangle = 0$.

Now, consider $x(t) = \bar{x} + th + t^{3/2}d + t^{7/4}\xi$. For every $i = 1, \ldots, m$, by assumptions $i \in I(\bar{x})$ and (3), there exist $\delta_i > 0$ and r_i $:]0, \delta_i[\to \mathbb{R}$ such that $\|r_i(t)\| = o(t^{p+3/4})$ and

$$\begin{aligned} g_i(x(t)) &= g_i(\bar{x} + th + t^{3/2}d + t^{7/4}\xi) \\ &= \frac{1}{p!}\left(g_i^p(\bar{x})[th]^p + t^{p+1/2}\langle g_i^p(\bar{x})[h]^{p-1}, d\rangle + t^{p+3/4}\langle g_i^p(\bar{x})[h]^{p-1}, \xi\rangle\right) \\ &\quad + r_i(t), \ \forall t \in]0, \delta_i[. \end{aligned}$$

Hence, by (5) and (7), there exists $\varepsilon_i \in]0, \delta_i[$ such that $g_i(x(t)) \leq 0$ for all $t \in]0, \varepsilon_i[$. Taking $\varepsilon = \min_{i=1,\ldots,m} \varepsilon_i$, we get $g_i(x(t)) \leq 0$ for all $i = 1, \ldots, m$, and, therefore, $x(t)$ is feasible for problem (1) for any $t \in]0, \varepsilon[$. Then $d \in F_0(\bar{x})$ and $\langle f'(\bar{x}), h\rangle = 0$ yield $f(x(t)) < f(\bar{x})$ for all $t \in]0, \varepsilon[$, which contradicts the assumption that $\bar{x}$ is a local minimizer and proves (6). □

Lemma 3.2 *Let X be a Banach space and X^* be its dual space. Given a set of vectors $\eta_i \in X^*$, $i = 1, \ldots r$, let $z \in X$ be a vector such that $\langle z, \eta_i\rangle < 0$, $i = 1, \ldots, r$. Assume also that for some vector $\eta \in X^*$, there exist numbers $\alpha_i \geq 0$ and $\beta_i \leq 0$, $i = 1, \ldots, r$, such that $\eta = \sum_{i=1}^{r} \alpha_i \eta_i$ and $\eta = \sum_{i=1}^{r} \beta_i \eta_i$. Then $\eta = 0$.*

Proof By assumptions of the lemma, $\sum_{i=1}^{r} \alpha_i \eta_i = \sum_{i=1}^{r} \beta_i \eta_i$ and

$$0 \geq \sum_{i=1}^{r} \alpha_i \langle z, \eta_i\rangle = \sum_{i=1}^{r} \beta_i \langle z, \eta_i\rangle \geq 0,$$

since $\langle z, \eta_i\rangle < 0$, $\alpha_i \geq 0$, and $\beta_i \leq 0$, $i = 1, \ldots, r$. Therefore, $\alpha_i = \beta_i = 0$, $i = 1, \ldots, r$, and $\eta = 0$, which proves the lemma. □

We will need the following generalization of the Farkas Lemma in Banach spaces.

Lemma 3.3 (Farkas) *Consider $c, \eta_1, \ldots, \eta_r \in X^*$. Exactly one of the following holds:*

(*) *there exists $x \in X$ with $\langle \eta_i, x\rangle \leq 0 \ \forall i = 1, \ldots, r$, and $\langle c, x\rangle > 0$.*
(**) *there exists nonnegative scalars $\mu_1, \ldots, \mu_r$ such that $c = \mu_1\eta_1 + \ldots + \mu_r\eta_r$.*

Now we are ready to prove Theorem 3.1.

Proof (Theorem 3.1) Let $\eta_i = g_i^{(p)}(\bar{x})[h]^{p-1}$, $i \in I(\bar{x})$, and $c = -f'(\bar{x})$. Then, by Lemma 3.1, part (*) in Lemma 3.3 does not hold, so, by part (**), there exist scalars $\beta_i \leq 0$, $i \in I(\bar{x})$, such that

$$f'(\bar{x}) = \sum_{i \in I(\bar{x})} \beta_i g_i^{(p)}(\bar{x})[h]^{p-1}. \tag{8}$$

Note that the assumptions of the theorem also hold with the vector $-h$. Indeed, (5) can be written as $g_i^{(p)}(\bar{x})[-h]^p = 0$ and $\langle g_i^{(p)}(\bar{x})[-h]^{p-1}, -\xi \rangle < 0$. Then, similar to considerations above, by Lemma 3.3, there exist $\gamma_i \leq 0$, $i \in I(\bar{x})$, such that

$$f'(\bar{x}) = \sum_{i \in I(\bar{x})} \gamma_i g_i^{(p)}(\bar{x})[-h]^{p-1} = \sum_{i \in I(\bar{x})} (-\gamma_i) g_i^{(p)}(\bar{x})[h]^{p-1}. \tag{9}$$

Introducing $\alpha_i = -\gamma_i \geq 0$ and using Lemma 3.2 with η_i defined above, $\eta = f'(\bar{x})$, and $z = -\xi$, we get $f'(\bar{x}) = 0$, which finishes the proof of the theorem. □

Note that conditions (8) and (9) can be viewed as generalizations of the KKT-type optimality conditions. However, the constraint qualification in the form (5) implies $f'(\bar{x}) = 0$. The following theorem is a simple corollary of Theorem 3.1.

Theorem 3.2 *Let $\bar{x}$ be a local minimizer of problem* (1)*, $f(x) \in C^2(X)$, and $g_i(x) \in C^{p+1}(X)$, $i = 1 \ldots m$. Assume that* (3) *holds with some even p and that the vectors $g_i^{(p)}(\bar{x})[h]^{p-1}$, $i \in I(\bar{x})$, are linearly independent for some h satisfying $g_i^{(p)}(\bar{x})[h]^p = 0$, $i \in I(\bar{x})$. Then $f'(\bar{x}) = 0$.*

4 Optimality Conditions in the General Case of Degeneracy with $p = 2$

In this section, we analyze some cases when the KKT Theorem holds for nonregular problems. Then, we introduce generalizations of the KKT conditions for some cases when the KKT Theorem does not hold. For now, we assume that there exists a number $r \in \{1, \ldots, m-1\}$ such that

$$g_i'(\bar{x}) \neq 0, \ i = 1, \ldots, r, \quad g_i'(\bar{x}) = 0, \ i = r+1, \ldots, m. \tag{10}$$

A more general case without assumption (10) is considered in Sect. 5 of the paper.

4.1 When the KKT Theorem Holds

We start with a special case when there exists a vector $h \in X$, $h \neq 0$, such that

$$\langle g_i'(\bar{x}), h \rangle = 0, \ i = 1, \ldots, r, \quad \langle g_i''(\bar{x})h, h \rangle = 0, \ i = r+1, \ldots, m. \tag{11}$$

We use the following notation and assumptions in this section:

$$\begin{aligned} G_2(\bar{x}, h) := \{d \in X \ : \ &\langle g_i'(\bar{x}), d \rangle \leq 0, \ i = 1, \ldots, r; \\ &\langle g_i''(\bar{x})[h], d \rangle \leq 0, \ i = r+1, \ldots, m\}. \end{aligned} \tag{12}$$

Assumption 1 (*A generalized MFCQ-type 2-regularity assumption*) For a vector h satisfying (11), assume the following

Part A. There exists a vector $\xi \in X$, $\|\xi\| = 1$, such that

$$\langle g_i'(\bar{x}), \xi \rangle < 0, i = 1, \dots, r, \quad \langle g_i''(\bar{x})h, \xi \rangle < 0, \; i = r+1, \dots, m. \tag{13}$$

Part B. There exists a vector $\eta \in X$, $\|\eta\| = 1$, such that

$$\langle g_i'(\bar{x}), \eta \rangle < 0, \; i = 1, \dots, r, \quad \langle g_i''(\bar{x})h, \eta \rangle > 0, \; i = r+1, \dots, m.$$

Assumption 2 For some h, satisfying (11), assume that $C_1 \cap C_2 = \{0\}$, where $C_1 := \text{span}\{g_1'(\bar{x}), \dots, g_r'(\bar{x})\}$ and $C_2 := \text{cone}\{g_{r+1}''(\bar{x})[h], \dots, g_m''(\bar{x})[h]\}$.

The following theorem can be viewed as a generalization of the KKT Theorem.

Theorem 4.1 (Necessary optimality conditions) *Assume that $\bar{x}$ is a local minimizer of problem (1), $f(x) \in C^1(X)$, and $g_i(x) \in C^2(X)$, $i = 1, \dots, m$. Assume that (10) holds and that there exists a vector $h \in X$, $h \neq 0$, such that (11) holds. Suppose that Assumptions 1 and 2 hold for problem (1). Then there exist $\lambda_i^* \geq 0$, $i = 1, \dots, r$, such that*

$$f'(\bar{x}) + \sum_{i=1}^{r} \lambda_i^* g_i'(\bar{x}) = 0. \tag{14}$$

Remark 4.1 Note the following:

1. The multipliers λ_i^* in Theorem 4.1 do not depend on the vector h.
2. Only $g_1(\bar{x}), \dots, g_r(\bar{x})$ are used in equation (14).
3. In the case when $g_i'(\bar{x}) = 0$, $i = 1, \dots, r$, equation (14) reduces to $f'(\bar{x}) = 0$.
4. The generalized MFCQ-type 2-regularity assumption (Assumption 1) is a new constraint qualification.

We will need the following lemma to prove Theorem 4.1. Recall that the sets $F_0(\bar{x})$ and $G_2(\bar{x}, h)$ are introduced in (4) and (12), respectively.

Lemma 4.1 (Geometric Necessary Condition) *Let the assumptions of Theorem 4.1 hold. Then*

$$F_0(\bar{x}) \cap G_2(\bar{x}, h) = \emptyset. \tag{15}$$

Proof First, we will prove that $\langle f'(\bar{x}), h \rangle = 0$ for h that satisfies Assumption 1 and $\|h\| = 1$. Assume on the contrary that $\langle f'(\bar{x}), h \rangle \neq 0$.

1. If $i \in \{1, \dots, r\}$, then $g_i(\bar{x}) = 0$ and, by Assumption 1, (11), and Taylor's expansion, there exist vectors ξ and η, $\|\xi\| = 1$, $\|\eta\| = 1$, a sufficiently small $\delta > 0$ and ω_{i_j} :$]0, \delta[\to \mathbb{R}$ such that $|\omega_{i_j}(t)| = o(t^{3/2})$, $j = 1, 2$, and $g_i(\bar{x}+th+t^{3/2}\xi) = \langle g_i'(\bar{x}), th \rangle + \langle g_i'(\bar{x}), t^{3/2}\xi \rangle + \omega_{i_1}(t) = \langle g_i'(\bar{x}), t^{3/2}\xi \rangle + \omega_{i_1}(t) < 0$, $g_i(\bar{x}-th+t^{3/2}\eta) = -\langle g_i'(\bar{x}), th \rangle + \langle g_i'(\bar{x}), t^{3/2}\eta \rangle + \omega_{i_2}(t) = \langle g_i'(\bar{x}), t^{3/2}\eta \rangle + \omega_{i_2}(t) < 0$ for all $t \in]0, \delta[$.

2. If $i \in \{r+1, \dots, m\}$, then $g_i(\bar{x}) = 0$, $g_i'(\bar{x}) = 0$, and similarly to the above, there exist functions ω_{i_j} :$]0, \delta[\to \mathbb{R}$ such that $|\omega_{i_j}(t)| = O(t^3)$, $j = 3, 4$, and $g_i(\bar{x} + th + t^{3/2}\xi) = g_i(\bar{x}) + \frac{1}{2}\langle g_i''(\bar{x})th, t^{3/2}\xi \rangle + \omega_{i_3}(t) < 0$,

$g_i(\bar{x} - th + t^{3/2}\eta) = g_i(\bar{x}) - \frac{1}{2}\langle g_i''(\bar{x})th, t^{3/2}\eta\rangle + \omega_{i_4}(t) < 0$ for all $t \in]0, \delta[$. Thus, $\bar{x} + th + t^{3/2}\xi \in S$ and $\bar{x} - th + t^{3/2}\eta \in S$ for all $t \in]0, \delta[$. Then the assumption $\langle f'(\bar{x}), h\rangle \neq 0$ above implies that either $f(\bar{x} + th + t^{3/2}\xi) < f(\bar{x})$ or $f(\bar{x} - th + t^{3/2}\eta) < f(\bar{x})$, which contradicts the minimality of $\bar{x}$ and proves $\langle f'(\bar{x}), h\rangle = 0$.

To prove (15), assume on the contrary that there exists $d \in F_0(\bar{x}) \cap G_2(\bar{x}, h)$. Then similarly to the above and by using Assumption 1, there exists a sufficiently small $\delta > 0$ such that $x(t) = \bar{x} + th + t^{3/2}d + t^{7/4}\xi \in S$ for all $t \in]0, \delta[$. This inclusion, together with $d \in F_0(\bar{x})$ and $\langle f'(x^*), h\rangle = 0$, yields $f(x(t)) < f(\bar{x})$ for all $t \in]0, \delta[$, which contradicts the minimality of $\bar{x}$ and proves (15). □

Now we are ready to prove Theorem 4.1.

Proof (Theorem 4.1) Let $\eta_1 = g_1'(\bar{x}), \ldots, \eta_r = g_r'(\bar{x})$, $\eta_{r+1} = g_{r+1}''(\bar{x})[h], \ldots, \eta_m = g_m''(\bar{x})[h]$, and $c = -f'(\bar{x})$. Then, by Lemma 4.1, part (*) in Lemma 3.3 does not hold, so by part (**), there exist scalars $\lambda_i^* \geq 0$, $i = 1, \ldots, r$, and $\gamma_i^* \geq 0$, $i = r+1, \ldots, m$, such that

$$f'(\bar{x}) = -\sum_{i=1}^{r} \lambda_i^* g_i'(\bar{x}) - \sum_{i=r+1}^{m} \gamma_i^* g_i''(\bar{x})[h]. \tag{16}$$

Note that the assumptions of Theorem 4.1 also hold with the vector $-h$. Then, using similar arguments, there exist $\bar{\lambda}_i^* \geq 0$, $i = 1, \ldots, r$, and $\bar{\gamma}_i^* \geq 0$, $i = r+1, \ldots, m$, such that

$$f'(\bar{x}) = -\sum_{i=1}^{r} \bar{\lambda}_i^* g_i'(\bar{x}) - \sum_{i=r+1}^{m} \bar{\gamma}_i^* g_i''(\bar{x})(-h). \tag{17}$$

Equations (16) and (17) imply

$$\sum_{i=1}^{r} \bar{\lambda}_i^* g_i'(\bar{x}) - \sum_{i=1}^{r} \lambda_i^* g_i'(\bar{x}) = \sum_{i=r+1}^{m} \bar{\gamma}_i^* g_i''(\bar{x})[h] + \sum_{i=r+1}^{m} \gamma_i^* g_i''(\bar{x})[h]. \tag{18}$$

Consider two cases. In the first case, we assume that

$$\sum_{i=1}^{r} \lambda_i^* g_i'(\bar{x}) = \sum_{i=1}^{r} \bar{\lambda}_i^* g_i'(\bar{x}). \tag{19}$$

Then by Lemma 3.2, $\gamma_i^* = \bar{\gamma}_i^* = 0$, $i = r+1, \ldots, m$, so (14) holds and we are done.

In the second case, suppose that (19) does not hold and recall that the sets C_1 and C_2 are defined in Assumption 2. In this case, equation (18) implies that there exists a vector $d \in C_1 \cap C_2$ such that $d = \sum_{i=1}^{r} (\bar{\lambda}_i^* - \lambda_i^*) g_i'(\bar{x}) = \sum_{i=r+1}^{m} (\bar{\gamma}_i^* + \gamma_i^*) g_i''(\bar{x})[h]$. Hence, Assumption 2 is violated and (19) holds, which finishes the proof of the theorem. □

In the following theorem, we propose necessary conditions for optimality without using Assumption 2. Instead, without loss of generality, we assume that the vectors $g_1'(\bar{x}), \ldots, g_r'(\bar{x}), g_{r+1}''(\bar{x})[h], \ldots, g_m''(\bar{x})[h]$ are the *extreme directions* for the cone

$$K(\bar{x}) := \text{cone}\{g_1'(\bar{x}), \ldots, g_r'(\bar{x}), g_{r+1}''(\bar{x})[h], \ldots g_m''(\bar{x})[h]\}. \quad (20)$$

(Recall that an extreme direction is the direction of a ray that cannot be expressed as a conic combination of any ray directions in the cone distinct from it.)

Theorem 4.2 (Necessary optimality conditions) *Assume that $\bar{x}$ is a local minimizer of problem* (1)*, $f(x) \in C^1(X)$, and $g_i(x) \in C^2(X)$, $i = 1, \ldots, m$. Assume that* (10) *holds and that there exists a vector $h \in X$, $h \neq 0$, such that relations* (11) *and Assumption 1 hold. Then there exist $\lambda_i^* \geq 0$, $i = 1, \ldots, r$, such that*

$$f'(\bar{x}) + \sum_{i=1}^{r} \lambda_i^* g_i'(\bar{x}) = 0. \quad (21)$$

Proof Let h, $h \neq 0$, satisfy (11) and (13). Note that since Assumption 1 holds, the proof of Lemma 4.1 implies $\langle f'(\bar{x}), h\rangle = 0$. Also, (16) was proved using Assumption 1 only; hence, there exist scalars $\lambda_i^* \geq 0, i = 1, \ldots, r$, and $\gamma_i^* \geq 0, i = r+1, \ldots, m$, such that

$$f'(\bar{x}) = -\sum_{i=1}^{r} \lambda_i^* g_i'(\bar{x}) - \sum_{i=r+1}^{m} \gamma_i^* g_i''(\bar{x})[h]. \quad (22)$$

Assume on the contrary that (21) does not hold. Then $-f'(\bar{x}) \notin \text{cone}\{g_1'(\bar{x}), \ldots, g_r'(\bar{x})\}$. Also, by (20) and (22), $-f'(\bar{x}) \in K(\bar{x})$ and there exists an index j such that $\gamma_j^* > 0$ in (22). Then by Part B of Assumption 1, there exists a vector $\bar{h} \neq 0$ such that

$$\langle f'(\bar{x}), \bar{h}\rangle > 0, \quad \langle g_i'(\bar{x}), \bar{h}\rangle > 0, \ i = 1, \ldots, r, \quad \langle g_j''(\bar{x})[h], \bar{h}\rangle < 0, \ j = r+1, \ldots, m. \quad (23)$$

Hence, by (11) and (23), for $i = 1, \ldots, r$, there exists a sufficiently small $0 < \varepsilon < 1$ and $\alpha > 0$ such that

$$g_i(\bar{x} - \alpha h - \alpha^{1+\varepsilon}\bar{h}) = -\langle g_i'(\bar{x}), \alpha h\rangle - \langle g_i'(\bar{x}), \alpha^{1+\varepsilon}\bar{h}\rangle + \omega_i(\alpha) < 0,$$

where $|\omega_i(\alpha)| = O(\alpha^2)$. Similarly, by (10), (11), and (23), for $i = r+1, \ldots, m$,

$$g_i(\bar{x} - \alpha h - \alpha^{1+\varepsilon}\bar{h}) = \frac{1}{2}\langle g_i''(\bar{x})\alpha^{1+\varepsilon}\bar{h}, \alpha^{1+\varepsilon}\bar{h}\rangle + \frac{1}{2}\langle g_i''(\bar{x})\alpha h, \alpha^{1+\varepsilon}\bar{h}\rangle + \xi_i(\alpha) < 0,$$

where $|\xi_i(\alpha)| = O(\alpha^3)$. Therefore, $\bar{x} - \alpha h - \alpha^{1+\varepsilon}\bar{h} \in S$,and, by using (23) and $\langle f'(\bar{x}), h\rangle = 0$, we get

$$f(\bar{x} - \alpha h - \alpha^{1+\varepsilon}\bar{h}) = f(\bar{x}) - \langle f'(\bar{x}), \alpha h\rangle - \langle f'(\bar{x}), \alpha^{1+\varepsilon}\bar{h}\rangle + \eta(\alpha) < f(\bar{x}),$$

where $|\eta(\alpha)| = O(\alpha^2)$, which contradicts the assumption that $\bar{x}$ is a local minimizer. Hence, (21) holds. □

In the following theorem, we present sufficient conditions for optimality. To simplify the consideration, we derive the sufficient conditions for problem (1) in the case when X is a finite dimensional space. However, a similar result will also be true in Banach space X under an assumption of strong p-regularity (see [26]).

To formulate the next theorem, we introduce a *Lagrange function*, $\mathcal{L}(x, \lambda) := f(x) + \sum_{i=1}^{r} \lambda_i g_i(x)$, and a set

$$H_2(\bar{x}) := \{h \in X \ : \ \langle g_i'(\bar{x}), h\rangle \leq 0, \ i = 1, \ldots, r, \quad \langle g_i''(\bar{x})h, h\rangle \leq 0, \ i = r+1, \ldots, m\}. \tag{24}$$

Remark 4.2 Note that we do not make Assumption 1 or any regularity assumption in Theorem 4.3.

Theorem 4.3 (Sufficient optimality conditions) *Let* $X = \mathbb{R}^n$ *and* $f(x)$, $g_i(x) \in C^2(X)$*,* $i = 1, \ldots, m$*. Assume that there exist* $\lambda_i^* \geq 0$*,* $i = 1, \ldots, r$*, such that* $\mathcal{L}_x'(\bar{x}, \lambda^*) = f'(\bar{x}) + \sum_{i=1}^{r} \lambda_i^* g_i'(\bar{x}) = 0$*. Assume also that there exists* $\beta > 0$ *such that, for any* $h \in H_2(\bar{x})$*, the following holds:* $\langle \mathcal{L}_{xx}''(\bar{x}, \lambda^*)h, h\rangle \geq \beta\|h\|^2$*. Then* $\bar{x}$ *is a strict local minimizer of problem* (1)*.*

Proof Assume on the contrary that $\bar{x}$ is not a strict local minimizer of problem (1). Then there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_k \in S \cap U(\bar{x})$, $f(x_k) \leq f(\bar{x})$, and $x_k \to \bar{x}$ as $k \to \infty$. Since $x_k \in S$, it can be represented as $x_k = \bar{x} + \alpha_k h + \omega(\alpha_k)$, where $h \in H_2(\bar{x})$, $\|h\| = 1$, $\|\omega(\alpha_k)\| = o(\alpha_k)$, and $\alpha_k \to 0$ as $k \to \infty$. Indeed, x_k can be written as $x_k = \bar{x} + h_k$, where $h_k/\|h_k\| \to h$ as $k \to \infty$, so that $h_k = \alpha_k h + \omega(\alpha_k)$ and $\|h_k\| = \alpha_k$. Then the inclusion $h \in H_2(\bar{x})$ follows from $\bar{x} + h_k \in S$ and we get

$$\begin{aligned} f(x_k) &\geq f(x_k) + \sum_{i=1}^{r} \lambda_i^* g_i(x_k) = \mathcal{L}(x_k, \lambda^*) \\ &= \mathcal{L}(\bar{x}, \lambda^*) + \langle \mathcal{L}_x'(\bar{x}, \lambda^*), x_k - \bar{x}\rangle + \frac{1}{2}\langle \mathcal{L}_{xx}''(\bar{x}, \lambda^*)(x_k - \bar{x}), (x_k - \bar{x})\rangle + o(\alpha_k^2) \\ &\geq f(\bar{x}) + \frac{\beta}{2}\|\alpha_k h + \omega(\alpha_k)\|^2 + o(\alpha_k^2) > f(\bar{x}). \end{aligned}$$

Getting the contradiction finishes the proof. □

4.2 When the KKT Theorem Fails

In this section, we consider some classes of problems, for which either the KKT conditions do not hold or there is no h satisfying (11) (see Example 7.1), and present generalized KKT conditions for those problems. We assume that (10) holds and introduce additional sets: $I_1(h) := \{i \in \{1, \ldots, r\} \ : \ \langle g_i'(\bar{x}), h\rangle = 0\}$, $I_2(h) := \{i \in \{r+1, \ldots, m\} : \langle g_i''(\bar{x})h, h\rangle = 0\}$, and $H_f(\bar{x}) := \{h \in X : \langle f'(\bar{x}), h\rangle \geq 0\}$, where $h \in H_2(\bar{x})$ and $H_2(\bar{x})$ is defined in (24).

Definition 4.1 We say that a mapping $g(x)$ is 2-regular at the point $\bar{x} \in X$ along a vector $h \in H_2(\bar{x})$ if either $I_1(h) = \emptyset$ and $I_2(h) = \emptyset$, or there exists an element $\xi = \xi(h) \in X$ such that

$$\langle g_i'(\bar{x}), \xi \rangle < 0, \ \forall\, i \in I_1(h), \quad \langle g_i''(\bar{x})h, \xi \rangle < 0, \ \forall\, i \in I_2(h). \tag{25}$$

Note that Definition 4.1 introduces a new *2-regularity constraint qualification*, which can be viewed as another generalization of the MFCQ. Note that Assumption 1 given in the previous section is a special case of Definition 4.1 for $I_1(h) = \{1, \ldots, r\}$ and $I_2(h) = \{r + 1, \ldots, m\}$.

Definition 4.2 We say that a mapping $g(x)$ is 2-regular at the point $\bar{x} \in X$ if, for every $h \in H_2(\bar{x})$, either $I_1(h) = \emptyset$ and $I_2(h) = \emptyset$, or there exists $\xi = \xi(h) \in X$, such that (25) holds.

We illustrate Definition 4.2 in Example 7.1.

For $x, h \in X$, $\|h\| = 1$, and $\lambda(h) = (\lambda_i(h))_{i \in I_1(h) \cup I_2(h)}$, introduce a *2-factor-Lagrange function* as

$$L_2(x, \lambda(h), h) := f(x) + \sum_{i \in I_1(h)} \lambda_i(h) g_i(x) + \sum_{i \in I_2(h)} \lambda_i(h) g_i'(x)h. \tag{26}$$

Now we are ready to present necessary and sufficient conditions for problem (1).

Theorem 4.4 *Let $X = \mathbb{R}^n$, $f(x) \in C^1(X)$, and $g_i(x) \in C^2(X)$, $i = 1, \ldots, m$. Assume that* (10) *holds and that mapping $g(x)$ is 2-regular at the point $\bar{x}$ along $h \in H_2(\bar{x})$.*

Necessary conditions: *If $\bar{x}$ is a minimizer to* (1)*, then either*

$$\langle f'(\bar{x}), h \rangle > 0 \tag{27}$$

or there exists $\lambda^*(h) = (\lambda_i^*(h))_{i \in I_1(h) \cup I_2(h)}$ such that

$$L_2{}'_x(\bar{x}, \lambda^*(h), h) = 0, \quad \lambda^*(h) \geq 0. \tag{28}$$

Sufficient conditions: *If, in addition, $g(x)$ is 2-regular at $\bar{x}$ and, for any $h \in H_2(\bar{x})$, either* (27) *holds or there exists $\beta > 0$ such that* (28) *holds and*

$$L_2{}''_{xx}(\bar{x}, \tilde{\lambda}^*(h), h)[h]^2 \geq \beta \|h\|^2, \qquad \tilde{\lambda}_i^*(h) = \begin{cases} \lambda_i^*(h), & \text{if } i \in I_1(h) \\ \frac{\lambda_i^*(h)}{3}, & \text{if } i \in I_2(h), \end{cases} \tag{29}$$

then $\bar{x}$ is a strict local minimizer of (1).

Proof Necessary conditions. Consider an element $h \in H_2(\bar{x})$ such that $g(x)$ is 2-regular at the point $\bar{x}$ along the vector h and divide our consideration into two following cases:

Case 1: If $I_1(h) = I_2(h) = \emptyset$, then $\langle g_i'(\bar{x}), h \rangle < 0$ for all $i = 1, \ldots, r$ and $\langle g_i''(\bar{x})h, h \rangle < 0$ for all $i = r + 1, \ldots, m$. First, we will prove that $\langle f'(\bar{x}), h \rangle \geq 0$. Assume on the contrary that $\langle f'(\bar{x}), h \rangle < 0$.

Then, by Taylor expansion, $f(\bar{x} + th) < f(\bar{x})$ and $g_i(\bar{x} + th) < 0$, $i = 1, \ldots, m$, for all sufficiently small $t > 0$, which contradicts the assumption that $\bar{x}$ is a local minimizer and proves $\langle f'(\bar{x}), h\rangle \geq 0$. If $\langle f'(\bar{x}), h\rangle > 0$, then (27) holds and we are done with the proof in Case 1. Otherwise, $\langle f'(\bar{x}), h\rangle = 0$.

If $f'(\bar{x}) \neq 0$, there exists $\bar{\xi}$ such that $\langle f'(\bar{x}), \bar{\xi}\rangle < 0$. Then $g_i(\bar{x} + th + t^{3/2}\bar{\xi}) < 0$, $i = 1, \ldots, m$, and $f(\bar{x} + th + t^{3/2}\bar{\xi}) < f(\bar{x})$, which contradicts the assumption that $\bar{x}$ is a local minimizer, so $f'(\bar{x}) = 0$, and, hence, (28) holds with $\lambda_i^*(h) = 0$, $i = 1, \ldots, m$. Thus, in Case 1, either (27) or (28) holds.

Case 2: Consider the case when $I_1(h) \cup I_2(h) \neq \emptyset$. First, we will prove that $\langle f'(\bar{x}), h\rangle \geq 0$. Assume on the contrary that $\langle f'(\bar{x}), h\rangle < 0$. By (25), there exists $\xi = \xi(h)$ such that $\langle g_i'(\bar{x}), \xi\rangle < 0$, $i \in I_1(h)$, and $\langle g_i''(\bar{x})h, \xi\rangle < 0$, $i \in I_2(h)$. Hence, $g_i(\bar{x} + th + t^{3/2}\xi) < 0$, $i = 1, \ldots, m$, and $f(\bar{x} + th + t^{3/2}\xi) < f(\bar{x})$ for all sufficiently small $t > 0$, which contradicts the assumption that $\bar{x}$ is a local minimizer. Hence, $\langle f'(\bar{x}), h\rangle \geq 0$. If $\langle f'(\bar{x}), h\rangle > 0$, then (27) holds, and we are done with the proof in Case 2. If not, then $\langle f'(\bar{x}), h\rangle = 0$. In this case, we will prove that $\langle f'(\bar{x}), d\rangle \geq 0$ for every d such that

$$\langle g_i'(\bar{x}), d\rangle \leq 0, \; i \in I_1(h), \quad \langle g_i''(\bar{x})h, d\rangle \leq 0, \; i \in I_2(h). \tag{30}$$

Assume on the contrary that there exists d satisfying (30) such that $\langle f'(\bar{x}), d\rangle < 0$. Then by (25), there exists $\xi = \xi(h)$ such that $g_i(\bar{x} + th + t^{3/2}\bar{d} + t^{7/4}\xi) \leq 0$, $i = 1, \ldots, m$, and $f(\bar{x}+th+t^{3/2}\bar{d}+t^{7/4}\xi) < f(\bar{x})$, which contradicts the assumption that $\bar{x}$ is a local minimizer. Hence, $\langle f'(\bar{x}), d\rangle \geq 0$ for every d satisfying (30). Then, similarly to the proof of Theorems 4.1, we get (28) by using Lemma 3.3, which finishes the proof in the second case.

Sufficient conditions. Assume on the contrary that $\bar{x}$ is not a strict local minimizer. Then there exists a sequence $\{x_k\} \to \bar{x}$ such that $f(x_k) \leq f(\bar{x})$ and $g(x_k) \leq 0$. Using the same notation for a convergent subsequence, let $\left\{\frac{x_k-\bar{x}}{\|x_k-\bar{x}\|}\right\}$ converge to some $\tilde{h}$. Then $x_k = \bar{x} + \|x_k - \bar{x}\|\tilde{h} + w(x_k) = \bar{x} + t_k\tilde{h} + w(x_k)$, where $\|w(x_k)\| = o(\|x_k - \bar{x}\|)$ and $t_k = \|x_k - \bar{x}\|$. Note that $\tilde{h} = \frac{x_k-\bar{x}-w(x_k)}{t_k}$ satisfies the following:

$$\begin{aligned}\langle g_i'(\bar{x}), \tilde{h}\rangle &= \frac{1}{t_k}\langle g_i'(\bar{x}), x_k - \bar{x} - w(x_k)\rangle \\ &= \frac{1}{t_k}\left(g_i(x_k) - g_i(\bar{x})\right) + o(t_k)/t_k, \quad i = 1, \ldots, r,\end{aligned}$$

so that, when $k \to \infty$, $\langle g_i'(\bar{x}), \tilde{h}\rangle \leq 0$, $i = 1, \ldots, r$, and similarly, $\langle g_i''(\bar{x})\tilde{h}, \tilde{h}\rangle \leq 0$, $i = r + 1, \ldots, m$. Hence, $\tilde{h} \in H_2(\bar{x})$ and, by the assumption of the theorem, either (27) holds or there exists $\beta > 0$ such that (28) and (29) hold. Consider two cases.

Case 1. If (27) holds, that is $\langle f'(\bar{x}), \tilde{h}\rangle > 0$,
then $f(x_k) = f(\bar{x}) + \left\langle f'(\bar{x}), t_k\tilde{h} + w(x_k)\right\rangle > f(\bar{x})$. However, this is a contradiction, so this case does not hold.

Case 2. If (28) holds,
then $0 = \langle f'(\bar{x}), \tilde{h}\rangle + \sum_{i\in I_1(\tilde{h})} \lambda_i^*(\tilde{h})\langle g_i'(\bar{x}), \tilde{h}\rangle + \sum_{i\in I_2(\tilde{h})} \lambda_i^*(\tilde{h})\langle g_i''(\bar{x})\tilde{h}, \tilde{h}\rangle = \langle f'(\bar{x}), \tilde{h}\rangle$,
so that $\langle f'(\bar{x}), \tilde{h}\rangle = 0$. To simplify the notation, we let $w = w(x_k)$, $h_k = t_k\tilde{h} + w$,

$I_1 = I_1(\tilde{h})$, and $I_2 = I_2(\tilde{h})$. By the consideration above, $g_i(x_k) \le 0$ and since there exists $\lambda^*(\tilde{h}) \ge 0$ such that (28) holds, we get

$$\begin{aligned} f(x_k) - f(\bar{x}) \ge\; & f(x_k) - f(\bar{x}) + \sum_{i\in I_1} \lambda_i^*(\tilde{h}) g_i(x_k) + \sum_{i\in I_2} \frac{\lambda_i^*(\tilde{h}) g_i(x_k)}{t_k} \\ =\; & \langle f'(\bar{x}), h_k\rangle + \frac{1}{2}\langle f''(\bar{x})h_k, h_k\rangle \\ & + \sum_{i\in I_1} \lambda_i^*(\tilde{h}) \left(\langle g_i'(\bar{x}), h_k\rangle + \frac{1}{2}\langle g_i''(\bar{x})h_k, h_k\rangle \right) \\ & + \sum_{i\in I_2} \lambda_i^*(\tilde{h}) \left(\frac{1}{2} g_i''(\bar{x}) \frac{[h_k]^2}{t_k} + \frac{1}{3!} g_i'''(\bar{x}) \frac{[h_k]^3}{t_k} + \frac{o(t_k^3)}{t_k} \right) + o(t_k^2). \end{aligned}$$

Then $\langle f'(\bar{x}), \tilde{h}\rangle = 0$, $\langle g_i'(\bar{x}), \tilde{h}\rangle = 0$, $i \in I_1$, and $g_i''(\bar{x})[\tilde{h}]^2 = 0$, $i \in I_2$, yield

$$\begin{aligned} f(x_k) - f(\bar{x}) \ge & \left\langle f'(\bar{x}) + \sum_{i\in I_1} \lambda_i^*(\tilde{h}) g_i'(\bar{x}) + \sum_{i\in I_2} \lambda_i^*(\tilde{h}) g_i''(\bar{x})\tilde{h}, w \right\rangle \\ & + \frac{t_k^2}{2} \left(f''(\bar{x})[\tilde{h}]^2 + \sum_{i\in I_1} \lambda_i^*(\tilde{h}) g_i''(\bar{x})[\tilde{h}]^2 + \sum_{i\in I_2} \frac{\lambda_i^*(\tilde{h})}{3} g_i'''(\bar{x})[\tilde{h}]^3 \right) \\ & + o(t_k^2). \end{aligned}$$

The assumptions of the theorem and the last inequalities imply

$$\begin{aligned} & f(x_k) - f(\bar{x}) \ge \langle L2_x'(\bar{x}, \lambda^*(\tilde{h}), \tilde{h}), w\rangle \\ & + \frac{t_k^2}{2} L2_{xx}''(\bar{x}, \tilde{\lambda}^*(\tilde{h}), \tilde{h})\, [\tilde{h}]^2 + o(t_k^2) \ge \frac{\beta}{2} t_k^2 \|\tilde{h}\|^2 + o(t_k^2) > 0, \end{aligned}$$

which contradicts the assumption $f(x_k) \le f(\bar{x})$. Hence, $\bar{x}$ is a strict local minimizer. □

Corollary 4.1 *If* (28) *holds and* $I_1(h) \cup I_2(h) = \emptyset$, *the equation* (28) *reduces to* $f'(\bar{x}) = 0$ *in the necessary conditions of Theorem* 4.4.

The proof of the corollary follows from the proof of necessary conditions in Theorem 4.4.

5 Optimality Conditions in the General Case of Degeneracy without Assumption (10)

To simplify considerations in this part of the paper, assume that $X = \mathbb{R}^n$. In this section, we consider a general case of degeneracy without assumption (10).

We need the following additional notation.

Let $H_g(\bar{x}) := \{h \in \mathbb{R}^n \ : \ \langle g_i'(\bar{x}), h\rangle \leq 0,\ i \in I(\bar{x})\}$. For some fixed element $h \in H_g(\bar{x})$, we define a set of indices: $I_1(\bar{x}, h) := \{i \in I(\bar{x}) \ : \ \langle g_i'(\bar{x}), h\rangle = 0\}$ and assume that $|I_1(\bar{x}, h)| = m_1(h) = m_1 \neq 0$. We start with construction of $l \leq m_1$ proper cones generated by vectors $g'_j(\bar{x})$ with indices j from the set $I_1(\bar{x}, h)$. The cones are determined in such a way that, for every $j \in I_1(\bar{x}, h)$, the corresponding $g'_j(\bar{x})$ is used in defining at least one cone and all cones are different. For constructing a cone with number k, $k = 1, \ldots, l$, indices $k_1, \ldots, k_{r_k} \in I_1(\bar{x}, h)$ are used so that the corresponding vectors $g'_{k_1}(\bar{x}), \ldots, g'_{k_{r_k}}(\bar{x})$ generate the *largest proper cone* and $k_j \neq k_l$ if $j \neq l$. As a result, there exists an element $\gamma_i \in \mathbb{R}^n$ such that $\langle g'_j(\bar{x}), \gamma_i\rangle < 0$, $j = k_1, \ldots k_{r_k}$, and, for every $j \in J_k(\bar{x}, h)$, where $J_k(\bar{x}, h) := I_1(\bar{x}, h)\backslash\{k_1, \ldots, k_{r_k}\}$, the following holds: $-g'_j(\bar{x}) = \alpha_{jk_1} g'_{k_1}(\bar{x}) + \ldots + \alpha_{jk_{r_k}} g'_{k_{r_k}}(\bar{x})$, where $\alpha_{jk_1} \geq 0, \ldots, \alpha_{jk_{r_k}} \geq 0$. For each $j \in J_k(\bar{x}, h)$, introduce $\tilde{g}_j(x) := g_j(x) + \alpha_{jk_1} g_{k_1}(x) + \ldots + \alpha_{jk_{r_k}} g_{k_{r_k}}(x)$, and get l sets consisting of functions $g_{k_1}(x), \ldots, g_{k_{r_k}}(x)$, $\tilde{g}_j(x)$, $j \in J_k(\bar{x}, h)$, such that

$$\tilde{g}'_j(\bar{x}) = 0, \quad j \in J_k(\bar{x}, h). \tag{31}$$

Note that conditions (31) resemble conditions (10). For every $k = 1, \ldots, l$, define a set S_k as follows:

$$S_k := \{x \in \mathbb{R}^n \ : \ g_{k_1}(x) \leq 0, \ldots, g_{k_{r_k}}(x) \leq 0,\ \tilde{g}_j(x) \leq 0,\ j \in J_k(\bar{x}, h),$$
$$g_i(x) \leq 0,\ i \in I(\bar{x})\backslash I_1(\bar{x}, h)\}. \tag{32}$$

Consider an example.

Example 5.1 Let $X = \mathbb{R}^3$, $g_1(x) = -x_1$, $g_2(x) = -x_2$, $g_3(x) = x_2 - x_1^2 + x_2^2 + x_3^2$, and $\bar{x} = (0, 0, 0)^T$. In this example, $h = (1, 0, 1)^T$, $m = 3$, $m_1 = 2$, $l = 2$, and $I_1(0, h) = \{2, 3\}$. For $k = 1$, we have $g_{1_1}(x) = g_2(x)$, $r_1 = 1$, and $\tilde{g}_3(x) = g_3(x) + g_2(x) = -x_1^2 + x_2^2 + x_3^2$. For $k = 2$, we get $r_2 = 1$, $g_{2_1}(x) = g_3(x)$, and $\tilde{g}_2(x) = g_2(x) + g_3(x) = -x_1^2 + x_2^2 + x_3^2$. As a result, there are two following systems of inequalities, (A) and (B), that define the sets S_1 and S_2, respectively:

$$\text{(A)} \quad \begin{cases} g_1(x) = -x_1 \leq 0 \\ g_2(x) = -x_2 \leq 0 \\ \tilde{g}_3(x) = -x_1^2 + x_2^2 + x_3^2 \leq 0. \end{cases} \qquad \text{(B)} \quad \begin{cases} g_1(x) = -x_1 \leq 0 \\ g_3(x) = x_2 - x_1^2 + x_2^2 + x_3^2 \leq 0 \\ \tilde{g}_2(x) = -x_1^2 + x_2^2 + x_3^2 \leq 0. \end{cases}$$

We will need the following lemma.

Lemma 5.1 $S = \bigcap\limits_{k=1}^{l} S_k$, *where* S_k *are defined in* (32).

Proof The proof of the lemma follows from the property that, for any $k = 1, \ldots, l$, the definition of the cone with number $k + 1$ implies that at least one function $g_j(x)$, $j \in I_1(\bar{x}, h)$, is used in the definition of the set S_k and is not in S_{k+1}. Also, at least one function $g_j(x)$, $j \in I_1(\bar{x}, h)$, is used in defining S_{k+1} and is not in S_k. The process of defining the cones also implies that each index j from the set $I_1(\bar{x}, h)$ is used at least once. Then, by the definition of S_k, $\bigcap\limits_{k=1}^{l} S_k \subseteq \bigcap\limits_{k=1}^{m} A_i = S$, where

$A_i = \{x \in \mathbb{R}^n \,:\, g_i(x) \le 0\}$, $i = 1, \dots, m$. At the same time, for every $k = 1, \dots, l$, $S \subseteq S_k$, and, hence, $S \subseteq \bigcap_{k=1}^{l} S_k$. Thus $S = \bigcap_{k=1}^{l} S_k$ holds. □

Note that the functions used in the definition of the sets S_k satisfy conditions (31) and $\langle g_j'(\bar{x}), h\rangle < 0$, $j \in I(\bar{x})\backslash I_1(\bar{x}, h)$. Since Lemma 5.1 implies that problem (1) can be written as

$$\min\ f(x), \quad \text{s.t.} \quad x \in \bigcap_{k=1}^{l} S_k,$$

optimality conditions for problem (1), given below in Theorem 5.1, are formulated in terms of the functions used in the definition of the sets S_k under an assumption that guarantees $\bigcap_{k=1}^{l} S_k \neq \bar{x}$. To state the theorem, we need to introduce some additional notation and definitions.

Assume that there exists a vector $h \in H_g(\bar{x})$, satisfying the following inequalities,

$$\langle \tilde{g}_j''(\bar{x})h, h\rangle \le 0, \quad j \in J_k(\bar{x}, h), \quad k = 1, \dots, l.$$

If such h is not found, then $\bar{x}$ is an isolated feasible point for problem (1). Recall that we consider indices k_j from the set $I_1(\bar{x}, h)$ and, for every $k = 1, \dots, l$, define

$$I_0^{1\,k}(\bar{x}, h) := \{k_1, \dots, k_{r_k}\}, \quad I_0^{2\,k}(\bar{x}, h) := \{i \in J_k(\bar{x}, h) \,:\, \langle \tilde{g}_i''(\bar{x})h, h\rangle = 0\},$$
$$I_0^{1}(\bar{x}, h) := \bigcup_{k=1}^{l} I_0^{1\,k}(\bar{x}, h), \quad I_0^{2}(\bar{x}, h) := \bigcup_{k=1}^{l} I_0^{2\,k}(\bar{x}, h).$$

Definition 5.1 We say that a mapping $g(x) : \mathbb{R}^n \to \mathbb{R}^m$ is *2–regular* at the point $\bar{x} \in \mathbb{R}^n$ along a vector $h \in H_g(\bar{x})$ if there exists an element $\xi \in \mathbb{R}^n$ satisfying the following inequalities,

$$\langle g_{k_1}'(\bar{x}), \xi\rangle < 0, \dots, \langle g_{k_{r_k}}'(\bar{x}), \xi\rangle < 0,$$
$$\langle \tilde{g}_j''(\bar{x})h, \xi\rangle < 0, \quad j \in I_0^2(\bar{x}, h), \quad k = 1, \dots, l.$$

Note that Definition 5.1 introduces another *2-regularity constraint qualification*, which can be viewed as a generalization of the MFCQ.

Definition 5.2 We say that a mapping $g(x) : \mathbb{R}^n \to \mathbb{R}^m$ is tangent 2–regular at the point $\bar{x} \in \mathbb{R}^n$ along a vector $h \in H_g(\bar{x})$ if, for any $\xi \in \mathbb{R}^n$, satisfying the following inequalities,

$$\langle g_{k_1}'(\bar{x}), \xi\rangle \le 0, \dots, \langle g_{k_{r_k}}'(\bar{x}), \xi\rangle \le 0,$$
$$\langle \tilde{g}_j''(\bar{x})h, \xi\rangle \le 0, \quad j \in I_0^2(\bar{x}, h), \quad k = 1, \dots, l, \tag{33}$$

there exists a set of feasible points $x(\alpha) \in S$ in the form $x(\alpha) = \bar{x}+\alpha h+\omega(\alpha)\xi+\eta(\alpha)$, where $\alpha > 0$ is sufficiently small, $\omega(\alpha) = o(\alpha)$, $\alpha^2/\omega(\alpha) \to 0$ as $\alpha \to 0^+$, and $\|\eta(\alpha)\| = o(\omega(\alpha))$.

An example of $\omega(\alpha)$ in Definition 5.2 is $\omega(\alpha) = \alpha^{1+\varepsilon}$ with $\varepsilon \in]0, 1[$.

To formulate the next theorem, we introduce a generalized 2-factor-Lagrange function in the form:

$$L_2(x, \lambda(h), h) := f(x) + \sum_{i\in I_0^1(x,h)} \lambda_i(h) g_i(x) + \sum_{i\in I_0^2(x,h)} \tilde{\lambda}_i(h)\langle \tilde{g}_i'(x), h\rangle.$$

Theorem 5.1 *Assume that $\bar{x}$ is a local minimizer of the problem (1), $f \in C^1(\mathbb{R}^n)$, and $g \in C^2(\mathbb{R}^n)$. Assume that $g(x)$ is tangent 2-regular at the point $\bar{x}$ along a vector $h \in H_g(\bar{x})$ and $\langle f'(\bar{x}), h\rangle = 0$. Then there exist coefficients $\lambda_i^*(h) \geq 0$, $i \in I_0^1(\bar{x}, h)$, $\tilde{\lambda}_i^*(h) \geq 0$, $i \in I_0^2(\bar{x}, h)$, such that*

$$L_2{}'_x(\bar{x}, \lambda^*(h), h) = f'(\bar{x}) + \sum_{i\in I_0^1(\bar{x},h)} \lambda_i^*(h) g_i'(\bar{x}) + \sum_{i\in I_0^2(\bar{x},h)} \tilde{\lambda}_i^*(h)\tilde{g}_i''(\bar{x})[h] = 0. \tag{34}$$

The proof of Theorem 5.1 is similar to the proof of necessary conditions in Theorem 4.4 with an assumption that $g(x)$ is tangent 2-regular at the point $\bar{x}$ along a vector $h \in H_g(\bar{x})$ and an additional property:

$$(T_S(\bar{x}))^* = \left(\bigcap_{k=1}^{l} T_{S_k}(\bar{x})\right)^* = \text{cone}\left\{g_i'(\bar{x}),\, \tilde{g}_j''(\bar{x})[h],\ i \in I_0^1(\bar{x}, h),\ \ j \in I_0^2(\bar{x}, h)\right\},$$

where $T_M(\bar{x})$ is a tangent cone to the set M at $\bar{x}$, and $(T_M(\bar{x}))^*$ is its conjugate.

Remark 5.1 Note that $I_0^1(\bar{x}, h) \subseteq I_1(\bar{x}, h)$ and $I_0^2(\bar{x}, h) \subseteq I_1(\bar{x}, h)$, and recall the definition of functions $\tilde{g}_j(x) = g_j(x)+\alpha_{jk_1} g_{k_1}(x)+\ldots+\alpha_{jk_{r_k}} g_{k_{r_k}}(x)$, $j \in J_k(\bar{x}, h)$, where $\alpha_{jk_1} \geq 0, \ldots, \alpha_{jk_{r_k}} \geq 0$. Then the statement of Theorem 5.1 can be written in the form: $f'(\bar{x}) + \sum_{i\in I_1(\bar{x},h)} \lambda_i^* g_i'(\bar{x}) + \sum_{i\in I_1(\bar{x},h)} \gamma_i^* g_i''(\bar{x})[h] = 0,$ where $\lambda_i^* \geq 0$ and $\gamma_i^* \geq 0$.

Example 5.1. (*continued*). Let $f = x_1 + x_2 - x_3$. In this example, we have $\bar{x} = 0$, $h = (1, 0, 1)^T$, $\langle f'(\bar{x}), h\rangle = 0$, and $\tilde{g}_2''(\bar{x})[h] = \tilde{g}_3''(\bar{x})[h] = (-2, 0, 2)^T$. Also, using the introduced notation, we get the following sets: $I_0^{1\,1}(\bar{x}, h) = \{2\}$, $I_0^{1\,2}(\bar{x}, h) = \{3\}$, $I_0^{2\,1}(\bar{x}, h) = \{3\}$, $I_0^{2\,2}(\bar{x}, h) = \{2\}$, $I_0^1(\bar{x}, h) = \{2, 3\}$, and $I_0^2(\bar{x}, h) = \{3, 2\}$. Note that mapping $g(x)$ is tangent 2-regular at the point $\bar{x}$ along the vector h; hence, all conditions of Theorem 5.1 are satisfied. Then, there exist multipliers $\lambda_2^*(h) = 1$ and $\lambda_3^*(h) = 0$ for $i \in I_0^1(\bar{x}, h)$, and there exist multipliers $\tilde{\lambda}_3^*(h) = 0$ and $\tilde{\lambda}_2^*(h) = \frac{1}{2}$ for $i \in I_0^2(\bar{x}, h)$, such that condition (34) holds in the following form:

$$f'(0) + \lambda_2^*(h)g_2'(0) + \lambda_3^*(h)g_3'(0) + \tilde{\lambda}_3^*(h)\tilde{g}_3''(0)[h] + \tilde{\lambda}_2(h)\tilde{g}_2''(0)[h] = 0.$$

6 Future Work for $p > 2$

Similar to the approach described in Sect. 4.2, the constraints of Problem (1) can be reduced to equivalent ones that satisfy the following relations (without notation change):

$$\begin{array}{ll} g_i'(\bar{x}) \neq 0,\ i = 1, \dots, r_1, & g_i'(\bar{x}) = 0,\ i = r_1 + 1, \dots, m, \\ g_i''(\bar{x}) \neq 0,\ i = r_1 + 1, \dots, r_2, & g_i''(\bar{x}) = 0,\ i = r_2 + 1, \dots, m, \\ \dots & \dots \\ g_i^{(p-1)}(\bar{x}) \neq 0,\ i = r_{p-2} + 1, \dots, r_{p-1}, & g_i^{(p-1)}(\bar{x}) = 0,\ i = r_{p-1} + 1, \dots, m \\ g_i^{(p)}(\bar{x}) \neq 0,\ i = r_{p-1} + 1, \dots, m. & \end{array}$$

Introduce the sets:
$I_1(\bar{x}) := \{1, \dots, r_1\}$, $I_2(\bar{x}) := \{r_1 + 1, \dots, r_2\}, \dots,$ and $I_p(\bar{x}) := \{r_{p-1} + 1, \dots, m\}$.

Definition 6.1 Assume that there exists h such that $\langle g_i'(\bar{x}), h\rangle = 0$ for all $i \in I_1(\bar{x})$, $g_i''(\bar{x})[h]^2 = 0$ for all $i \in I_2(\bar{x})$, …, and $g_i^{(p)}(\bar{x})[h]^p = 0$ for all $i \in I_p(\bar{x})$. We say that mapping $g(x) : \mathbb{R}^n \to \mathbb{R}^m$ is p–regular at the point $\bar{x} \in \mathbb{R}^n$ along the vector h, if there exists $\xi \in \mathbb{R}^n$, which satisfies the following inequalities: $\langle g_i'(\bar{x}), \xi\rangle < 0$ for all $i \in I_1(\bar{x})$, $\langle g_i''(\bar{x})h, \xi\rangle < 0$ for all $i \in I_2(\bar{x})$, …, and $\langle g_i^{(p)}(\bar{x})[h]^{p-1}, \xi\rangle < 0$ for all $i \in I_p(\bar{x})$.

Theorem 6.1 *Assume that $\bar{x}$ is a local minimizer of problem (1), $f \in C^1(\mathbb{R}^n)$, and $g \in C^{p+1}(\mathbb{R}^n)$. Assume that mapping $g(x)$ is p–regular at $\bar{x}$ along a vector $h \in H_g(\bar{x})$ and $\langle f'(\bar{x}), h\rangle = 0$. Then there exists $\lambda^*(h) = \big(\lambda_i^*(h)\big)_{i \in I_1(\bar{x}) \bigcup I_2(\bar{x}) \bigcup \cdots \bigcup I_p(\bar{x})}$ such that $\lambda^*(h) \geq 0$ and*

$$f'(\bar{x}) + \sum_{i \in I_1(\bar{x})} \lambda_i^*(h) g_i'(\bar{x}) + \sum_{i \in I_2(\bar{x})} \lambda_i^*(h) g_i''(\bar{x}) h + \cdots + \sum_{i \in I_p(\bar{x})} \lambda_i^*(h) g_i^{(p)}(\bar{x})[h]^{p-1} = 0.$$

The proof of Theorem 6.1 is similar to the proof of necessary conditions in Theorem 4.4 with an additional property:

$$(T_S(\bar{x}))^* = \mathrm{cone}\left\{ g_i'(\bar{x}), i \in I_1(\bar{x}), g_i''(\bar{x})h, i \in I_2(\bar{x}), \dots, g_i^{(p)}(\bar{x})[h]^{p-1}, i \in I_p(\bar{x}) \right\}.$$

A more general version of optimality conditions given in Theorem 6.1 can be derived under an assumption that $g(x) : \mathbb{R}^n \to \mathbb{R}^m$ is tangent p–regular ($p > 2$) at $\bar{x} \in \mathbb{R}^n$ along a vector $h \in H_g(\bar{x})$, which is a generalization of Definition 5.2. Optimality conditions given in Theorem 4.2 can also be expanded to the case $p > 2$ under a generalized version of Assumption 1:

There exist vectors $\xi, \eta \in X$, $\|\xi\| = \|\eta\| = 1$, such that $\langle g_i'(\bar{x}), \xi\rangle < 0$, $\langle g_i'(\bar{x}), \eta\rangle < 0$, $i \in I_1(\bar{x})$; $\langle g_i''(\bar{x})h, \xi\rangle < 0$, $\langle g_i''(\bar{x})h, \eta\rangle > 0$, $i \in I_2(\bar{x})$; ..., $\langle g_i^{(p)}(\bar{x})[h]^{p-1}, \xi\rangle < 0$, and $\langle g_i^{(p)}(\bar{x})h^{p-1}, \eta\rangle < 0$, if p is odd, or $\langle g_i^{(p)}(\bar{x})h^{p-1}, \eta\rangle > 0$, if p is even, $i \in I_p(\bar{x})$.

7 Examples

Example 7.1. This example illustrates Theorem 4.4. Consider the following problem:

$$\min_{x\in\mathbb{R}^3} \; 3x_1 - 3x_3 + x_1^2 + x_2^2 \quad \text{s.t.} \quad g_1(x) = -x_1 - x_2 + 2x_3 + |x_1|^{5/2} \le 0,$$
$$g_2(x) = -2x_1^2 + x_2^2 + x_3^2 + |x_2|^{7/2} \le 0, \quad g_3(x) = x_1^2 - 2x_2^2 + x_3^2 + |x_3|^{7/2} \le 0. \tag{35}$$

Necessary conditions. Note that $\bar{x} = 0$ is a minimizer in Problem (35). To verify assumptions of Theorem 4.4, notice that $r = 1$ in (10) and vectors $h = (h_1, h_2, h_3)$ in the set $H_2(0)$ satisfy the following inequalities: $\langle g_1'(0), h\rangle = -h_1 - h_2 + 2h_3 \le 0$, $\langle g_2''(0)h, h\rangle = -4h_1^2 + 2h_2^2 + 2h_3^2 \le 0$, and $\langle g_3''(0)h, h\rangle = 2h_1^2 - 4h_2^2 + 2h_3^2 \le 0$. Consider an element $h = (a, a, a)$, $a \neq 0$, which is in the set $H_2(0)$ and satisfies $\langle f'(0), h\rangle = 0$, $\langle g_1'(0), h\rangle = 0$, and $\langle g_2''(0)h, h\rangle = \langle g_3''(0)h, h\rangle = 0$. Otherwise, necessary optimality conditions hold in the form (27). The mapping $g(x) = (g_1(x), g_2(x), g_3(x))$ is 2-regular at $\bar{x} = 0$ along the vector h because there exists a vector ξ, for example, $\xi = (2, 2, 1)$, such that $\langle g_1'(\bar{x}), \xi\rangle = -\xi_1 - \xi_2 + 2\xi_3 < 0$, $\langle g_2''(\bar{x})h, \xi\rangle = -4a\xi_1 + 2a\xi_2 + 2a\xi_3 < 0$, and $\langle g_3''(\bar{x})h, \xi\rangle = 2a\xi_1 - 4a\xi_2 + 2a\xi_3 < 0$, where $a > 0$. Therefore, all assumptions of Theorem 4.4 hold. Hence, for $h = (a, a, a) \in H_f(0)$, there exist multipliers $\lambda_i^*(h) \ge 0$, $i \in I_1(h) \cup I_2(h) = \{1, 2, 3\}$, such that (28) holds. Indeed, taking $\lambda_1^*(h) = 1$, $\lambda_2^*(h) = 1/2a$, $\lambda_3^*(h) = 0$, we get $f'(0) + (1)g_1'(0) + (1/2a)g_2''(0)[h] = 0$.

Sufficient conditions. For $\bar{x} = 0$, consider $h = (a, a, a) \in H_f(0)$, where a is a fixed real number, $a \neq 0$, and, using (26), define a 2-factor-Lagrange function with $\tilde{\lambda}_1^*(h) = 1$, $\tilde{\lambda}_2^*(h) = 1/6a$, and $\tilde{\lambda}_3^*(h) = 0$ as $L_2(0, \tilde{\lambda}^*(h), h) = 3x_1 - 3x_3 + x_1^2 + x_2^2$ $+ (1)(-x_1 - x_2 + 2x_3 + |x_1|^{5/2}) + \frac{1}{6a}(-4x_1 + 2x_2 + 2x_3 + 7/2|x_2|^{5/2})a$. Note that, by the above, (28) holds and there exists $\beta > 0$ such that

$L_2{}''_{xx}(0, \tilde{\lambda}^*(h), h)[h]^2 = 4a^2 \ge \beta\|h\|^2$, so (29) is satisfied. Then, by Theorem 4.4, $\bar{x} = 0$ is a strict local minimizer in Problem (35).

Example 7.2. Consider the following problem that illustrates Theorem 5.1:

$$\min_{x\in\mathbb{R}^3} \; x_1 + x_2 - x_3 + x_1^2 + x_2^2 + x_3^2 \quad \text{s.t.} \quad g_1(x) = -x_1 - |x_2|^{5/2} \le 0, \quad g_2(x) = -x_2 - |x_3|^{5/2} \le 0,$$
$$g_3(x) = x_2 - x_1^2 + x_2^2 + x_3^2 - |x_1|^{5/2} \le 0, \quad g_4(x) = x_1 - x_3 \le 0.$$

Results presented in [16] cannot be applied here since there is no $p > 1$ such that the constraints are $2p$–times continuously differentiable. Optimality conditions given in [19,22] are also not applicable to this problem since their assumptions are not satisfied at $\bar{x} = (0, 0, 0)^T$. Consider $h = (1, 0, 1)^T \in H_g(0)$. Then $I_1(0, h) = \{2, 3, 4\}$, so $m_1 = |I_1(0, h)| = 3$. For $k = 1$, we get $r_1 = 2$, $g_{1_1}(x) = g_2(x)$, $g_{1_2}(x) = g_4(x)$, and $\tilde{g}_3(x) = g_3(x) + g_2(x) = -x_1^2 + x_2^2 + x_3^2 - |x_1|^{5/2} - |x_3|^{5/2}$. For $k = 2$, we get $r_2 = 2$,

$g_{2_1}(x) = g_3(x)$, $g_{2_2}(x) = g_4(x)$, and $\tilde{g}_2(x) = -x_1^2 + x_2^2 + x_3^2 - |x_1|^{5/2} - |x_3|^{5/2}$. Note that $\langle f'(\bar{x}), h\rangle = 0$, $\tilde{g}_3''(\bar{x})[h] = \tilde{g}_2''(\bar{x})[h] = (-2, 0, 2)^T$, and $I_0^2(0, h) = \{2, 3\}$.

To verify Definition 5.2, consider $\xi = (\xi_1, \xi_2, \xi_3)$. The inequalities (33) in this example, $\langle g_{1_1}'(0), \xi\rangle \leq 0$, $\langle g_{1_2}'(0), \xi\rangle \leq 0$, $\langle g_{2_1}'(0), \xi\rangle \leq 0$, $\langle \tilde{g}_3''(0)h, \xi\rangle \leq 0$, and $\langle \tilde{g}_2''(0)h, \xi\rangle \leq 0$, reduce to $\xi_2 = 0$, $\xi_1 = \xi_3$. Now let $\xi = (\xi_1, 0, \xi_1)$, $\omega(\alpha) = \alpha^{3/2}$, and $\eta(\alpha) = 0$ to define $x(\alpha) = \bar{x} + \alpha h + \omega(\alpha)\xi + \eta(\alpha) = (\alpha + \alpha^{3/2}\xi_1, 0, \alpha + \alpha^{3/2}\xi_1)$ and get

$$
\begin{aligned}
&g_1(x(\alpha)) = -\alpha - \alpha^{3/2}\xi_1 \leq 0, \quad g_2(x(\alpha)) = -|\alpha + \alpha^{3/2}\xi_1|^{5/2} \leq 0, \\
&g_3(x(\alpha)) = -|\alpha + \alpha^{3/2}\xi_1|^{5/2} \leq 0, \quad g_4(x(\alpha)) = 0,
\end{aligned}
$$

for all sufficiently small $\alpha > 0$. Therefore, $g(x)$ is tangent 2-regular at a point $\bar{x}$ along the vector h. Since all conditions of Theorem 5.1 are satisfied, there exist $\lambda_i^*(h) \geq 0$, $i \in I_0^1(0, h) = \{2, 3, 4\}$, and $\tilde{\lambda}_i^*(h) \geq 0$, $i \in I_0^2(0, h) = \{3, 2\}$, such that (34) holds:

$$
\begin{aligned}
&f'(0) + \lambda_2^*(h)g_2'(0) + \lambda_3^*(h)g_3'(0) + \lambda_4^*(h)g_4'(0) + \tilde{\lambda}_3^*(h)\tilde{g}_3''(0)[h] \\
&\quad + \tilde{\lambda}_2^*(h)\tilde{g}_2''(0)[h] = 0.
\end{aligned}
$$

Indeed, the last equation is satisfied, for example, with $\lambda_2^*(h) = 1$, $\lambda_3^*(h) = 0$, $\lambda_4^*(h) = 0$, $\tilde{\lambda}_3^*(h) = 1/2$, and $\tilde{\lambda}_2^*(h) = 0$.

8 Comparison with Other Results

We start this section comparing our results with Fritz John-type conditions proposed in [19]. The main difference is in the coefficient λ_0 of the objective function that is not guaranteed to be nonzero in [19], while optimality conditions presented in this paper have $\lambda_0 = 1$. If $\lambda_0 = 0$, then optimality conditions do not provide qualitative information about the optimization problem. Moreover, the authors in [19] make an additional assumption that a vector h used in the statements of their results belongs to the set:
$H_2(\bar{x}) := \{h \in X : \langle g_i'(\bar{x}), h\rangle \leq 0 \ \forall i \in I(\bar{x})$ and $\exists x$ such that $\langle g_i'(\bar{x}), x\rangle + g_i''(\bar{x})[h, h] \leq 0 \ \forall i \in I(\bar{x}, h)\}$, where $I(\bar{x}, h) := \{i \in I(\bar{x}) : \langle g_i'(\bar{x}), h\rangle = 0\}$. However, there is no similar requirement in the classical case. For the case when $\lambda_0 \neq 0$ in [19], there is an additional requirement that there exist ξ and $\hat{\xi}$ such that $\langle g_i'(\bar{x}), \xi\rangle + g_i''(\bar{x})[h, \hat{\xi}] < 0$. This restricts the class of problems, where the optimality conditions from [19] can be applied and give $\lambda_0 \neq 0$. Example 7.2 illustrates the case, when the above assumptions do not hold, but optimality conditions presented in the paper are satisfied. Another difference between optimality conditions given in [19] and in Theorem 4.1 is in the last term in the generalized Lagrange function, $\mathcal{L}(x, \lambda) := \lambda_0 f(x) + \sum_{j=1}^{r} \lambda_j g_j(x) + \sum_{j=r+1}^{m} \lambda_j g_j'(x)h$. Namely, Theorem 4.1 yields $\lambda_j^* = 0$, $j = r+1, \ldots, m$; and hence, the optimality conditions derived in Theorem 4.1 reduce to the classical form of the KKT conditions. Note that Theorem 4.1

also implies that either $\lambda_0 = 0$ or $\lambda_j = 0$, $j = r+1, \dots, m$, in optimality conditions presented in [19] in terms of the function $\mathcal{L}(x, \lambda)$,

Results in [22] are KKT-type optimality conditions. However, the main assumption in [22] is stronger than regularity conditions proposed in this paper. There is also an additional requirement on a vector h from some special set in [22], $\langle f'(\bar{x}), h\rangle = 0$, where $\bar{x}$ is a local minimizer of problem (1). In our paper, we prove that $\langle f'(\bar{x}), h\rangle = 0$ holds in some relevant results without making it an assumption. Moreover, a regularity assumption in [22] has the form: $\exists\, h, \bar{h}$ such that $\langle g_i'(\bar{x}), h\rangle \leq 0 \ \forall i \in I(\bar{x})$ and $g_i''(\bar{x})[h, \bar{h}] < 0 \ \forall i \in I(\bar{x}, h)$. It restricts the class of problems where optimality conditions given in [22] hold and $\lambda_0 \neq 0$ is guaranteed. Example 7.2 illustrates a case when this assumption does not hold, but optimality conditions presented in the paper are satisfied. All results derived in [19–22] are given for the case of $p = 2$ only, while we consider a more general case of $p \geq 2$. The results presented in [16] are different from ones given in this paper, since [16] requires that the constraints are $2p$–times continuously differentiable.

9 Conclusions

We showed that the Karush–Kuhn–Tucker necessary conditions can reduce to a specific form containing the objective function only in an absolutely degenerate case when (3) holds with an even p. Then we analyzed classes of nonregular problems, when the KKT conditions hold with a nonzero multiplier corresponding to the objective function. After that, we turned our attention to the degenerate optimization problems for which the KKT Theorem fails. We presented necessary and sufficient conditions that can be viewed as generalized KKT-type optimality conditions. As auxiliary results, we derived new geometric necessary conditions. A new approach presented in Sect. 5 can be used to reduce degenerate optimization problems to new forms to simplify analysis of nonregular optimization problems. The proposed optimality conditions were illustrated by some examples and compared to existing optimality conditions.

Most of the constraint qualifications (CQs) proposed in the paper can be viewed as generalizations of either MFCQ or LICQ. The main difference between CQs known in the literature (see, for example, [1–4] and references therein) and ones presented in the paper is that our regularity assumptions allowed us to derive not only classical KKT-type optimality conditions but also generalized forms of the KKT conditions.

Some directions for future research are described in Sect. 6. It would be interesting to extend the results presented in the paper to the case of $p > 2$. It remains open to determine a generalization for the approach described in Sect. 5 to new classes of optimization problems and to the case of $p > 2$.

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