

Finding Robust Global Optimal Values of Bilevel Polynomial Programs with Uncertain Linear Constraints

T. D. Chuong¹ · V. Jeyakumar¹

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Abstract This paper studies a bilevel polynomial program involving box data uncertainties in both its linear constraint set and its lower-level optimization problem. We show that the robust global optimal value of the uncertain bilevel polynomial program is the limit of a sequence of values of Lasserre-type hierarchy of semidefinite linear programming relaxations. This is done by first transforming the uncertain bilevel polynomial program into a single-level non-convex polynomial program using a dual characterization of the solution of the lower-level program and then employing the powerful Putinar's Positivstellensatz of semi-algebraic geometry. We provide a numerical example to show how the robust global optimal value of the uncertain bilevel polynomial program can be calculated by solving a semidefinite programming problem using the MATLAB toolbox YALMIP.

Keywords Bilevel programming · Robust optimization · Uncertain linear constraints · Global polynomial optimization · Semidefinite program

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1 Introduction

The bilevel optimization problems arise when two independent decision makers, ordered within a hierarchical structure, try to optimize their objectives over a jointly

✉ T. D. Chuong
chuongthaidoan@yahoo.com
V. Jeyakumar
v.jeyakumar@unsw.edu.au

¹ School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia

dependent set. They appear as hierarchical decision-making problems, such as risk management and economic planning problems, in engineering, governments, and industries [1–5]. The commonly used bilevel optimization techniques (see [3, 6, 7] and the references therein) assume precise information (i.e., accurate values for the input quantities or system parameters), despite the reality that such precise knowledge is rarely available in hierarchical decision-making problems. The data of these problems are often uncertain (i.e., they are not known exactly at the time of the decision) due to estimation errors, prediction errors, or lack of information. Consequently, the development of optimization methodologies, which are capable of generating robust optimal solutions that are immunized against data uncertainty, such as the deterministic robust optimization techniques, has become more important than ever in mathematics, commerce, and engineering [8–12].

The bilevel program is a class of NP-hard optimization problems even in the case, where all the functions are linear and are free of data uncertainty [13]. A general approach for studying bilevel optimization problems is to transform them into single-level optimization problems [2, 3, 6]. The resulting single-level optimization problems are generally non-convex constrained optimization problems. It is often difficult to find global optimal solutions of non-convex optimization problems. However, Putinar's Positivstellensatz [14] together with Lasserre-type semidefinite relaxations allows us to characterize global optimal solutions and find the global optimal value of a non-convex optimization problem involving polynomials. The reader is referred to [15–20] for related recent work on single-level convex and non-convex polynomial optimization in the literature.

In this paper, we study a bilevel polynomial program that finds *robust* global optimal values under box data *uncertainties*. The main purpose of this paper is to show how the robust global optimal value of a bilevel polynomial program in the face of uncertain linear constraints can be calculated by solving a sequence of semidefinite linear programming relaxations. More concretely, under the coercivity of the objective polynomial, we prove that the values of Lasserre-type semidefinite relaxations converge to the robust global optimal value of the uncertain bilevel polynomial program.

This is done by first transforming the uncertain bilevel polynomial program into a single-level non-convex polynomial program using a dual characterization of the solution of the lower-level program and then employing Putinar's Positivstellensatz [14]. The dual characterization is achieved by using a generalized Farkas lemma with a numerical certificate for semi-infinite linear inequality systems [21], whose dual statement can be verified by solving a semidefinite linear program. Related recent work on global bilevel polynomial optimization in the absence of data uncertainty can be found in [17, 22].

We would also like to point out that some other approaches to treating uncertain bilevel optimization problems are available in the literature. For instance, the authors in [23] used a *stochastic* approach to obtain robust solutions to hierarchical problems, where the lower-level program is described by equilibrium constraints and the uncertain data include random inputs as well as discretely distributed probability distributions. A *fuzzy* approach was employed in [24] for studying a linear bilevel optimization problem, in which the concept of the solution for both levels is understood in the sense of *fuzzy*. An *approximation/perturbation* approach was used in [25] for

treating a two-level optimization problem by approximating it by a sequence of mathematical programs. However, to the best of our knowledge, a mathematical theory and the associated methods for uncertain bilevel optimization problems that employ the *deterministic robust* optimization approach [8–10, 12] do not appear to be available in the literature.

The outline of the paper is as follows. Section 2 presents preliminaries and auxiliary results, where we can find dual characterizations for robust solutions of the lower-level program and for robust feasible points of the uncertain bilevel polynomial problem. The main result is given in Sect. 3, where we prove the convergence of robust global optimal values via semidefinite linear programming relaxations for the uncertain bilevel polynomial problem. Section 4 provides conclusions and discusses further research perspectives.

2 Preliminaries and Auxiliary Results

Let us start this section by fixing some notation and definitions. The notation \mathbb{R}^n signifies the Euclidean space, whose norm is denoted by $\| \cdot \|$ for each $n \in \mathbb{N} := \{1, 2, \dots\}$. The inner product in \mathbb{R}^n is defined by $\langle x, y \rangle := x^\top y$ for all $x, y \in \mathbb{R}^n$. As usual, cone $\Omega := \{\sum_{i=1}^k \alpha_i x_i : x_i \in \Omega, \alpha_i \in \mathbb{R}_+, i = 1, \dots, k, k \in \mathbb{N}\}$ stands for the convex conical hull of Ω , where $\mathbb{R}_+ := [0, +\infty[\subset \mathbb{R}$. The notation $\text{conv } \Omega$ denotes the convex hull of $\Omega \subset \mathbb{R}^n$. A symmetric $(n \times n)$ matrix A is said to be *positive semidefinite*, denoted by $A \geq 0$, whenever $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^n$. A real-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *coercive* on \mathbb{R}^n if $\liminf_{\|x\| \rightarrow \infty} f(x) = +\infty$. In particular, it is shown in [19, Lemma 3.1] that a convex polynomial f is coercive on \mathbb{R}^n if there exists $\bar{x} \in \mathbb{R}^n$ such that the Hessian $\nabla^2 f(\bar{x})$ is positive definite, i.e., $x^\top \nabla^2 f(\bar{x})x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Some numerically checkable sufficient conditions for the coercivity of non-convex polynomials have also been given in [15].

Denote by $\mathbb{R}[x]$ the ring of polynomials in x with real coefficients. One says that (cf. [20]) $f \in \mathbb{R}[x]$ is *sum-of-squares* if there exist polynomials $f_j \in \mathbb{R}[x], j = 1, \dots, r$, such that $f = \sum_{j=1}^r f_j^2$. The set of all sum-of-squares polynomials in x is denoted by $\Sigma^2[x]$. Given polynomials $\{g_1, \dots, g_r\} \subset \mathbb{R}[x]$, the notation $\mathbf{M}(g_1, \dots, g_r)$ stands for the set of polynomials generated by $\{g_1, \dots, g_r\}$, i.e.,

$$\mathbf{M}(g_1, \dots, g_r) := \{\sigma_0 + \sigma_1 g_1 + \dots + \sigma_r g_r : \sigma_j \in \Sigma^2[x], j = 0, 1, \dots, r\}. \tag{1}$$

If there exists $h \in \mathbf{M}(g_1, \dots, g_r)$ such that the set $\{x \in \mathbb{R}^n : h(x) \geq 0\}$ is compact, then $\mathbf{M}(g_1, \dots, g_r)$ is called to be *archimedean* (cf. [26]).

The following lemma of Putinar (cf. [14, 20]), which provides a positivity representation for a polynomial over a system of polynomial inequalities under the archimedean property, can be viewed as a polynomial analog of Farkas’s lemma.

Lemma 2.1 (Putinar’s Positivstellensatz [14]) *Let $f, g_j \in \mathbb{R}[x], j = 1, \dots, r$. Suppose that $\mathbf{M}(g_1, \dots, g_r)$ is archimedean. If $f(x) > 0$ for all $x \in \{y \in \mathbb{R}^n : g_j(y) \geq 0, j = 1, \dots, r\}$, then $f \in \mathbf{M}(g_1, \dots, g_r)$, i.e., there exist $\sigma_j \in \Sigma^2[x], j = 0, 1, \dots, r$, such that $f = \sigma_0 + \sum_{j=1}^r \sigma_j g_j$.*

Consider an *uncertain* linear inequality system,

$$x \in \mathbb{R}^n, \hat{a}_j^\top x \leq \hat{b}_j, \quad j = 1, \dots, q, \tag{2}$$

where $(\hat{a}_j, \hat{b}_j), j = 1, \dots, q$ are uncertain and they belong to the *uncertainty* sets $\widehat{U}_j, j = 1, \dots, q$. The uncertainty sets are given by

$$\widehat{U}_j := \left\{ \left(a_j^0 + \sum_{i=1}^s u_j^i a_j^i, b_j^0 + \sum_{i=1}^s u_j^i b_j^i \right) : u_j := (u_j^1, \dots, u_j^s) \in U_j \right\}, \quad j = 1, \dots, q,$$

where $a_j^i \in \mathbb{R}^n, b_j^i \in \mathbb{R}, i = 0, 1, \dots, s, j = 1, \dots, q$ are fixed and $U_j, j = 1, \dots, q$ are *spectrahedra* (see, e.g., [27]) described by

$$U_j := \left\{ u_j := (u_j^1, \dots, u_j^s) \in \mathbb{R}^s : A_j^0 + \sum_{i=1}^s u_j^i A_j^i \geq 0 \right\}, \quad j = 1, \dots, q \tag{3}$$

with $A_j^i, i = 0, 1, \dots, s, j = 1, \dots, q$ symmetric $(p_j \times p_j)$ matrices.

Now, consider the affine mappings $a_j : \mathbb{R}^s \rightarrow \mathbb{R}^n$ and $b_j : \mathbb{R}^s \rightarrow \mathbb{R}, j = 1, \dots, q$ given, respectively, by

$$a_j(u_j) := a_j^0 + \sum_{i=1}^s u_j^i a_j^i, \quad b_j(u_j) := b_j^0 + \sum_{i=1}^s u_j^i b_j^i \quad \text{for } u_j := (u_j^1, \dots, u_j^s) \in \mathbb{R}^s \tag{4}$$

with $a_j^i \in \mathbb{R}^n, b_j^i \in \mathbb{R}, i = 0, 1, \dots, s, j = 1, \dots, q$ fixed as above. Then, the *robust* counterpart of the uncertain system (2),

$$x \in \mathbb{R}^n, \hat{a}_j^\top x \leq \hat{b}_j, \quad \forall (\hat{a}_j, \hat{b}_j) \in \widehat{U}_j, \quad j = 1, \dots, q,$$

can be expressed equivalently as

$$x \in \mathbb{R}^n, a_j(u_j)^\top x \leq b_j(u_j), \quad \forall u_j \in U_j, \quad j = 1, \dots, q. \tag{5}$$

The generalized non-homogeneous Farkas lemma, which provides a numerically tractable certificate for nonnegativity of an affine function over the uncertain linear inequality system (5), plays an important role in characterizing robust solutions of the lower-level optimization problem.

Lemma 2.2 (See [21, Theorem 2.1]) *Let $C := \text{cone}\{(a_j(u_j), b_j(u_j)) : u_j \in U_j, j = 1, \dots, q\}$ be closed, where $U_j, j = 1, \dots, p$, in (3) are assumed to be bounded, and let $(\varphi, r) \in \mathbb{R}^n \times \mathbb{R}$. Assume that the robust linear inequality system (5) has a solution, i.e.,*

$$X := \left\{ x \in \mathbb{R}^n : a_j(u_j)^\top x \leq b_j(u_j), \quad \forall u_j \in U_j, \quad j = 1, \dots, q \right\} \neq \emptyset.$$

Then, the following statements are equivalent:

- (i) $x \in \mathbb{R}^n, a_j(u_j)^\top x \leq b_j(u_j), \forall u_j \in U_j, j = 1, \dots, q \implies \wp^\top x - r \geq 0;$
- (ii) $\exists \lambda_j^0 \geq 0, \lambda_j^i \in \mathbb{R}, j = 1, \dots, q, i = 1, \dots, s,$ such that

$$\wp + \sum_{j=1}^q \left(\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i \right) = 0, \quad -r - \sum_{j=1}^q \left(\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i \right) \geq 0$$

$$\text{and } \lambda_j^0 A_j^0 + \sum_{i=1}^s \lambda_j^i A_j^i \geq 0, \quad j = 1, \dots, q.$$

Bilevel Polynomial Problems. Let $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a real polynomial. We consider an *uncertain* bilevel polynomial optimization problem with linear constraints as

$$\min_{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n} \left\{ f(x, y) : y \in Y(x, \hat{a}_1, \hat{b}_1, \dots, \hat{a}_q, \hat{b}_q), \tilde{a}_i^\top x + \tilde{b}_i^\top y \leq \tilde{c}_i, i = 1, \dots, l \right\}, \tag{P}$$

where $(\tilde{a}_i, \tilde{b}_i, \tilde{c}_i) \in \tilde{U}_i, i = 1, \dots, l$ and $(\hat{a}_j, \hat{b}_j) \in \hat{U}_j, j = 1, \dots, q$ are *uncertain* and

$$Y(x, \hat{a}_1, \hat{b}_1, \dots, \hat{a}_q, \hat{b}_q) := \operatorname{argmin}_{z \in \mathbb{R}^n} \left\{ c_0^\top x + d_0^\top z : c_j^\top x + \hat{a}_j^\top z \leq \hat{b}_j, j = 1, \dots, q \right\}$$

denotes the optimal solution set of the uncertain lower-level optimization problem

$$\min_{z \in \mathbb{R}^n} \left\{ c_0^\top x + d_0^\top z : c_j^\top x + \hat{a}_j^\top z \leq \hat{b}_j, j = 1, \dots, q \right\}. \tag{6}$$

In the above data, $c_0 \in \mathbb{R}^m, d_0 \in \mathbb{R}^n, c_j \in \mathbb{R}^m, j = 1, \dots, q,$ are fixed, the *uncertainty* sets $\tilde{U}_i, i = 1, \dots, l,$ are boxes given by

$$\tilde{U}_i := [\underline{a}_i, \bar{a}_i] \times [\underline{b}_i, \bar{b}_i] \times [\underline{c}_i, \bar{c}_i], \quad i = 1, \dots, l$$

with $\underline{a}_i := (\underline{a}_i^1, \dots, \underline{a}_i^m), \bar{a}_i := (\bar{a}_i^1, \dots, \bar{a}_i^m) \in \mathbb{R}^m, \underline{a}_i^k \leq \bar{a}_i^k, k = 1, \dots, m, \underline{b}_i := (\underline{b}_i^1, \dots, \underline{b}_i^n), \bar{b}_i := (\bar{b}_i^1, \dots, \bar{b}_i^n) \in \mathbb{R}^n, \underline{b}_i^k \leq \bar{b}_i^k, k = 1, \dots, n,$ and $\underline{c}_i, \bar{c}_i \in \mathbb{R}, \underline{c}_i \leq \bar{c}_i$ for $i = 1, \dots, l,$ while the *uncertainty* sets $\hat{U}_j, j = 1, \dots, q,$ are given by

$$\hat{U}_j := \left\{ \left(a_j^0 + \sum_{i=1}^s u_j^i a_j^i, b_j^0 + \sum_{i=1}^s u_j^i b_j^i \right) : u_j := (u_j^1, \dots, u_j^s) \in U_j \right\}, \quad j=1, \dots, q$$

with $U_j := [-\gamma_1, \gamma_1] \times \dots \times [-\gamma_s, \gamma_s]$, $\gamma_i > 0$, $i = 1, \dots, s$ and $a_j^i \in \mathbb{R}^n$, $b_j^i \in \mathbb{R}$, $i = 0, 1, \dots, s$, $j = 1, \dots, q$, fixed.

By considering the affine mappings $a_j : \mathbb{R}^s \rightarrow \mathbb{R}^n$, $b_j : \mathbb{R}^s \rightarrow \mathbb{R}$, $j = 1, \dots, q$, given as in (4), the uncertain lower-level optimization problem (6) can be formulated equivalently as

$$\min_{z \in \mathbb{R}^n} \left\{ c_0^\top x + d_0^\top z : c_j^\top x + a_j(u_j)^\top z \leq b_j(u_j), j = 1, \dots, q \right\}, \tag{LP}$$

where $u_j \in U_j$, $j = 1, \dots, q$. In the formulation of (LP), the boxes U_j , $j = 1, \dots, q$, play the role of uncertainty sets.

Now, the *robust* counterpart of the problem (P) is defined by

$$\min_{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n} \left\{ f(x, y) : y \in Y(x), \tilde{a}_i^\top x + \tilde{b}_i^\top y \leq \tilde{c}_i, \forall (\tilde{a}_i, \tilde{b}_i, \tilde{c}_i) \in \tilde{U}_i, i = 1, \dots, l \right\}, \tag{RP}$$

where $Y(x) := \operatorname{argmin}_{z \in \mathbb{R}^n} \{c_0^\top x + d_0^\top z : c_j^\top x + a_j(u_j)^\top z \leq b_j(u_j), \forall u_j \in U_j, j = 1, \dots, q\}$. Note that in the robust counterpart (RP), the uncertain constraint inequalities of both the lower-level and upper-level problems are enforced for every possible value of the data within the uncertainty sets U_j , $j = 1, \dots, q$, and \tilde{U}_i , $i = 1, \dots, l$.

Given $x \in \mathbb{R}^m$, we first focus on the *robust* counterpart of the lower-level optimization problem (LP) given by

$$\min_{z \in \mathbb{R}^n} \left\{ c_0^\top x + d_0^\top z : c_j^\top x + a_j(u_j)^\top z \leq b_j(u_j), \forall u_j \in U_j, j = 1, \dots, q \right\}. \tag{RLP}$$

As usual, a point $y \in \mathbb{R}^n$ is called to be a *robust solution* of the lower-level optimization problem (LP) (see, e.g., [8]) if it is an optimal solution of the problem (RLP), i.e., $y \in Y(x)$.

The next lemma establishes a characterization for robust solutions of the lower-level optimization problem (LP). In what follows, for the sake of simplicity, we denote by V_s the boxes U_j , $j = 1, \dots, q$, and use $\{\check{u}_k := (\check{u}_k^1, \dots, \check{u}_k^s) \in \mathbb{R}^s \mid k = 1, \dots, 2^s\}$ to indicate the set of *extreme points* of the box V_s .

Lemma 2.3 (Tractable characterization for robust solutions of (LP)) *Let $x \in \mathbb{R}^m$. Then, $y \in Y(x)$ if and only if*

$$(I) \left\{ \begin{array}{l} y \in \mathbb{R}^n, c_j^\top x + \left(a_j^0 + \sum_{i=1}^s \check{u}_k^i a_j^i \right)^\top y \\ - \left(b_j^0 + \sum_{i=1}^s \check{u}_k^i b_j^i \right) \leq 0, k = 1, \dots, 2^s, j = 1, \dots, q, \\ \exists \mu_0 > 0, \mu_j \geq 0, \mu_j^i \in \mathbb{R}, j = 1, \dots, q, i = 1, \dots, s \\ \text{such that } \sum_{j=0}^q (\mu_j)^2 + \sum_{j=1}^q \sum_{i=1}^s (\mu_j^i)^2 = 1, \\ \mu_0 d_0 + \sum_{j=1}^q \left(\mu_j a_j^0 + \sum_{i=1}^s \mu_j^i a_j^i \right) = 0, \\ -\mu_0 d_0^\top y - \sum_{j=1}^q \left(\mu_j b_j^0 - \mu_j c_j^\top x + \sum_{i=1}^s \mu_j^i b_j^i \right) \geq 0 \\ \text{and } (\mu_j \gamma_i)^2 - (\mu_j^i)^2 \geq 0, i = 1, \dots, s, j = 1, \dots, q. \end{array} \right.$$

Proof For each $j \in \{1, \dots, q\}$, put

$$A_j^0 := \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}, \quad A_j^i := \begin{pmatrix} E_i & 0 \\ 0 & -E_i \end{pmatrix}, \quad i = 1, \dots, s, \tag{7}$$

where E_0 is the $(s \times s)$ diagonal matrix with the diagonal entries, say $\gamma_i > 0, i = 1, \dots, s$, and E_i is the $(s \times s)$ diagonal matrix with one in the (i, i) th entry and zeros elsewhere. Then, it follows that

$$\begin{aligned} & \left\{ u_j := (u_j^1, \dots, u_j^s) \in \mathbb{R}^s : A_j^0 + \sum_{i=1}^s u_j^i A_j^i \geq 0 \right\} \\ &= \left\{ u_j := (u_j^1, \dots, u_j^s) \in \mathbb{R}^s : \gamma_i + u_j^i \geq 0, \gamma_i - u_j^i \geq 0, i = 1, \dots, s \right\} \\ &= \left\{ u_j := (u_j^1, \dots, u_j^s) \in \mathbb{R}^s : |u_j^i| \leq \gamma_i, i = 1, \dots, s \right\} = V_s, \quad j = 1, \dots, q, \end{aligned}$$

which shows how the box V_s can be expressed in terms of spectrahedra in (3).

Since $U_j, j = 1, \dots, q$, are boxes, the cone $C(x) := \text{cone}\{(a_j(u_j), b_j(u_j) - c_j^\top x) : u_j \in U_j, j = 1, \dots, q\}$ is closed (see, e.g., [21, Proposition 2.2(i)]). Invoking Lemma 2.2, we can verify that $y \in Y(x)$ if and only if the following conditions hold:

$$y \in \mathbb{R}^n, c_j^\top x + a_j(u_j)^\top y \leq b_j(u_j), \forall u_j \in U_j, j = 1, \dots, q \text{ and} \tag{8}$$

$\exists \lambda_j^0 \geq 0, \lambda_j^i \in \mathbb{R}, j = 1, \dots, q, i = 1, \dots, s$ such that

$$d_0 + \sum_{j=1}^q \left(\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i \right) = 0, \quad -d_0^\top y - \sum_{j=1}^q \left(\lambda_j^0 b_j^0 - \lambda_j^0 c_j^\top x + \sum_{i=1}^s \lambda_j^i b_j^i \right) \geq 0$$

$$\text{and } \lambda_j^0 A_j^0 + \sum_{i=1}^s \lambda_j^i A_j^i \geq 0, j = 1, \dots, q. \tag{9}$$

It is easy to see that (8) amounts to the following one

$$y \in \mathbb{R}^n, \max_{u_j \in U_j} \left\{ \left\langle \left(a_j^{1\top} y - b_j^1, \dots, a_j^{s\top} y - b_j^s \right), u_j \right\rangle + a_j^{0\top} y + c_j^\top x - b_j^0 \right\} \leq 0, \\ j = 1, \dots, q,$$

which in turn is equivalent to the assertion

$$y \in \mathbb{R}^n, \check{u}_k^\top \left(a_j^{1\top} y - b_j^1, \dots, a_j^{s\top} y - b_j^s \right) + a_j^{0\top} y + c_j^\top x - b_j^0 \leq 0, \\ k = 1, \dots, 2^s, j = 1, \dots, q,$$

where $\{\check{u}_k := (\check{u}_k^1, \dots, \check{u}_k^s) \in \mathbb{R}^s : k = 1, \dots, 2^s\}$ stands for the set of extreme points of the box V_s as above. Therefore, (8) becomes

$$y \in \mathbb{R}^n, c_j^\top x + \left(a_j^0 + \sum_{i=1}^s \check{u}_k^i a_j^i \right)^\top y \\ - \left(b_j^0 + \sum_{i=1}^s \check{u}_k^i b_j^i \right) \leq 0, k = 1, \dots, 2^s, j = 1, \dots, q.$$

In addition, it can be checked that, under our setting, (9) is equivalent to

$$\left(\lambda_j^0 \gamma_i \right)^2 - \left(\lambda_j^i \right)^2 \geq 0, i = 1, \dots, s, j = 1, \dots, q. \tag{10}$$

So, we come to the assertion that $y \in Y(x)$ if and only if

$$(II) \left\{ \begin{array}{l} y \in \mathbb{R}^n, c_j^\top x + \left(a_j^0 + \sum_{i=1}^s \check{u}_k^i a_j^i \right)^\top y - \left(b_j^0 + \sum_{i=1}^s \check{u}_k^i b_j^i \right) \\ \leq 0, k = 1, \dots, 2^s, j = 1, \dots, q, \\ \exists \lambda_j^0 \geq 0, \lambda_j^i \in \mathbb{R}, j = 1, \dots, q, i = 1, \dots, s \text{ such that} \\ d_0 + \sum_{j=1}^q \left(\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i \right) = 0, \\ -d_0^\top y - \sum_{j=1}^q \left(\lambda_j^0 b_j^0 - \lambda_j^0 c_j^\top x + \sum_{i=1}^s \lambda_j^i b_j^i \right) \geq 0 \\ \text{and } \left(\lambda_j^0 \gamma_i \right)^2 - \left(\lambda_j^i \right)^2 \geq 0, i = 1, \dots, s, j = 1, \dots, q. \end{array} \right.$$

To complete the proof, it remains to show that (II) and (I) are equivalent to each other.

To this end, suppose that (II) holds. By setting $\mu_0 := \frac{1}{\sqrt{1 + \sum_{j=1}^q (\lambda_j^0)^2 + \sum_{j=1}^q \sum_{i=1}^s (\lambda_j^i)^2}}$

and $\mu_j := \frac{\lambda_j^0}{\sqrt{1 + \sum_{j=1}^q (\lambda_j^0)^2 + \sum_{j=1}^q \sum_{i=1}^s (\lambda_j^i)^2}}, \mu_j^i := \frac{\lambda_j^i}{\sqrt{1 + \sum_{j=1}^q (\lambda_j^0)^2 + \sum_{j=1}^q \sum_{i=1}^s (\lambda_j^i)^2}},$

$j = 1, \dots, q, i = 1, \dots, s$, we come to (I). Conversely, assume that (I) holds. Letting $\lambda_j^0 := \frac{\mu_j}{\mu_0}, \lambda_i^i := \frac{\mu_i^i}{\mu_0}, j = 1, \dots, q, i = 1, \dots, s$, we arrive at (II). \square

To proceed further, we should define concepts of robust feasible/solutions for the uncertain bilevel polynomial optimization problem (P).

Definition 2.1 (i) We say that $(\bar{x}, \bar{y}) \in \mathbb{R}^m \times \mathbb{R}^n$ is a *robust feasible* point of problem (P) if it satisfies

$$\bar{y} \in Y(\bar{x}), \tilde{a}_i^\top \bar{x} + \tilde{b}_i^\top \bar{y} \leq \tilde{c}_i, \forall (\tilde{a}_i, \tilde{b}_i, \tilde{c}_i) \in \tilde{U}_i, i = 1, \dots, l,$$

or equivalently, it is a feasible point of its robust counterpart (RP).

- (ii) Let $(\bar{x}, \bar{y}) \in \mathbb{R}^m \times \mathbb{R}^n$ be a robust feasible point of problem (P). We say that (\bar{x}, \bar{y}) is a *global robust solution* of problem (P) if $f(\bar{x}, \bar{y}) \leq f(x, y)$ for every robust feasible point (x, y) of problem (P), or equivalently, it is a global solution of its robust counterpart (RP).
- (iii) We say that the problem (P) satisfies the *lower-level Slater condition* (LSC) if for each $x \in \mathbb{R}^m$ there exists $z \in \mathbb{R}^n$ such that

$$c_j^\top x + a_j(u_j)^\top z < b_j(u_j), \forall u_j \in U_j, j = 1, \dots, q. \tag{11}$$

The forthcoming lemma supplies a characterization for robust feasible points of the uncertain bilevel polynomial optimization problem (P). Hereafter, we use notation as before, and in addition, we denote by $\left\{ \left(\check{a}_i^k, \check{b}_i^k, \check{c}_i^k \right) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : k = 1, \dots, 2^{m+n+1} \right\}$ the set of extreme points of the box \tilde{U}_i for $i = 1, \dots, l$, and let $d_0 := (d_0^1, \dots, d_0^n), a_j^i := (a_j^{i1}, \dots, a_j^{in}) \in \mathbb{R}^n, i = 0, 1, \dots, s, j = 1, \dots, q$.

Lemma 2.4 (Robust feasibility for (P))

- (i) Let $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Then, (x, y) is a robust feasible point of problem (P) if and only if there exists $\mu := (\mu_0, \mu_1, \dots, \mu_q, \mu_1^1, \dots, \mu_q^1, \dots, \mu_1^s, \dots, \mu_q^s) \in \mathbb{R}^{q(s+1)+1}$ such that

$$(III) \begin{cases} g_i(x, y, \mu) \geq 0, i = 1, \dots, l2^{m+n+1} + q2^s, \\ g_i(x, y, \mu) > 0, i = l2^{m+n+1} + q2^s + 1, \\ g_i(x, y, \mu) \geq 0, i = l2^{m+n+1} + q2^s + 2, \dots, L, \\ h_j(x, y, \mu) \geq 0, -h_j(x, y, \mu) \geq 0, j = 1, \dots, n + 1, \end{cases}$$

where $L := l2^{m+n+1} + q(2^s + s + 1) + 2$ and

$$g_i(x, y, \mu) := -\left(\check{a}_k^{i-(k-1)2^{m+n+1}}\right)^\top x - \left(\check{b}_k^{i-(k-1)2^{m+n+1}}\right)^\top y + \check{c}_k^{i-(k-1)2^{m+n+1}} \text{ for } i = (k-1)2^{m+n+1} + 1, \dots, k2^{m+n+1} \text{ with } k = 1, \dots, l,$$

$$g_i(x, y, \mu) := -c_k^\top x - \left(a_k^0 + \sum_{j=1}^s \check{u}_{i-(k-1)2^s-l2^{m+n+1}}^j a_k^j\right)^\top y + b_k^0 + \sum_{j=1}^s \check{u}_{i-(k-1)2^s-l2^{m+n+1}}^j b_k^j \text{ for } i = l2^{m+n+1} + (k-1)2^s + 1, \dots, l2^{m+n+1} + k2^s \text{ with } k = 1, \dots, q,$$

$$g_i(x, y, \mu) := \mu_0 \text{ for } i = l2^{m+n+1} + q2^s + 1,$$

$$g_i(x, y, \mu) := \mu_{i-l2^{m+n+1}-q2^s-1} \text{ for } i = l2^{m+n+1} + q2^s + 2, \dots, l2^{m+n+1} + q2^s + q + 1,$$

$$g_i(x, y, \mu) := -\mu_0 d_0^\top y - \sum_{k=1}^q \left(\mu_k b_k^0 - \mu_k c_k^\top x + \sum_{j=1}^s \mu_k^j b_k^j \right) \text{ for } i = l2^{m+n+1} + q2^s + q + 2,$$

$$g_i(x, y, \mu) := \left(\mu_k \gamma_{i-(k-1)s-(l2^{m+n+1}+q2^s+q+2)} \right)^2 - \left(\mu_k^{i-(k-1)s-(l2^{m+n+1}+q2^s+q+2)} \right)^2 \text{ for } i = (l2^{m+n+1} + q2^s + q + 2) + (k-1)s + 1, \dots, (l2^{m+n+1} + q2^s + q + 2) + ks \text{ with } k = 1, \dots, q,$$

and

$$h_j(x, y, \mu) := \mu_0 d_0^j + \sum_{k=1}^q \left(\mu_k a_k^{0j} + \sum_{i=1}^s \mu_k^i a_k^{ij} \right) \text{ for } j = 1, \dots, n,$$

$$h_j(x, y, \mu) := 1 - \sum_{k=0}^q (\mu_k)^2 - \sum_{k=1}^q \sum_{i=1}^s (\mu_k^i)^2 \text{ for } j = n + 1.$$

(ii) Assume that the (LSC) in (11) is satisfied. Then, (III) is equivalent to

$$(IV) \begin{cases} g_i(x, y, \mu) \geq 0, \quad i = 1, \dots, L, \\ h_j(x, y, \mu) \geq 0, \quad -h_j(x, y, \mu) \geq 0, \quad j = 1, \dots, n + 1 \end{cases} \quad (12)$$

for each $(x, y, \mu) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{q(s+1)+1}$.

Proof (i) Let us first show that the set

$$\left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : \tilde{a}_i^\top x + \tilde{b}_i^\top y \leq \tilde{c}_i, \quad \forall (\tilde{a}_i, \tilde{b}_i, \tilde{c}_i) \in \tilde{U}_i, \quad i = 1, \dots, l \right\} \quad (13)$$

is equivalent to the following one

$$\left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : \check{a}_i^k \top x + \check{b}_i^k \top y - \check{c}_i^k \leq 0, k = 1, \dots, 2^{m+n+1}, i = 1, \dots, l \right\}, \tag{14}$$

where $\left\{ (\check{a}_i^k, \check{b}_i^k, \check{c}_i^k) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : k = 1, \dots, 2^{m+n+1} \right\}$ denotes the set of extreme points of the box \tilde{U}_i for $i = 1, \dots, l$ as denoted above. Indeed, for each $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, by letting $X := (x, y, -1)$, we obtain that

$$\begin{aligned} & \max \left\{ \check{a}_i^\top x + \check{b}_i^\top y - \check{c}_i : (\check{a}_i, \check{b}_i, \check{c}_i) \in \tilde{U}_i, i = 1, \dots, l \right\} \\ &= \max \left\{ \check{a}_i^\top x + \check{b}_i^\top y - \check{c}_i : (\check{a}_i, \check{b}_i, \check{c}_i) \in \text{conv} \left\{ (\check{a}_i^k, \check{b}_i^k, \check{c}_i^k), k = 1, \dots, 2^{m+n+1} \right\}, i = 1, \dots, l \right\} \\ &= \max \left\{ W_i^\top X : W_i \in \text{conv} \left\{ (\check{a}_i^k, \check{b}_i^k, \check{c}_i^k), k = 1, \dots, 2^{m+n+1} \right\}, i = 1, \dots, l \right\} \\ &= \max \left\{ \check{W}_i^k \top X : k = 1, \dots, 2^{m+n+1}, i = 1, \dots, l \right\} \\ &= \max \left\{ \check{a}_i^k \top x + \check{b}_i^k \top y - \check{c}_i^k : k = 1, \dots, 2^{m+n+1}, i = 1, \dots, l \right\}, \end{aligned}$$

where $W_i := (\check{a}_i, \check{b}_i, \check{c}_i)$ and $\check{W}_i^k := (\check{a}_i^k, \check{b}_i^k, \check{c}_i^k), k = 1, \dots, 2^{m+n+1}$ for $i = 1, \dots, l$. So, we conclude that (13) is equivalent to (14). Now, let (x, y) be a robust feasible point of problem (P). It means that

$$y \in Y(x), \check{a}_i^\top x + \check{b}_i^\top y \leq \check{c}_i, \forall (\check{a}_i, \check{b}_i, \check{c}_i) \in \tilde{U}_i, i = 1, \dots, l. \tag{15}$$

Due to the equivalence between (13) and (14), (15) is nothing else, but the assertion that

$$y \in Y(x), \check{a}_i^k \top x + \check{b}_i^k \top y - \check{c}_i^k \leq 0, k = 1, \dots, 2^{m+n+1}, i = 1, \dots, l. \tag{16}$$

Invoking Lemma 2.3, (16) can be equivalently

$$(V) \left\{ \begin{aligned} & (x, y) \in \mathbb{R}^m \times \mathbb{R}^n, \check{a}_i^k \top x + \check{b}_i^k \top y - \check{c}_i^k \leq 0, k = 1, \dots, 2^{m+n+1}, i = 1, \dots, l, \\ & c_j^\top x + \left(a_j^0 + \sum_{i=1}^s \check{u}_k^i a_j^i \right)^\top y - \left(b_j^0 + \sum_{i=1}^s \check{u}_k^i b_j^i \right) \leq 0, k = 1, \dots, 2^s, j = 1, \dots, q, \\ & \exists \mu_0 > 0, \mu_j \geq 0, \mu_j^i \in \mathbb{R}, j = 1, \dots, q, i = 1, \dots, s \\ & \text{such that } \sum_{j=0}^q (\mu_j)^2 + \sum_{j=1}^q \sum_{i=1}^s (\mu_j^i)^2 = 1, \\ & \mu_0 d_0 + \sum_{j=1}^q \left(\mu_j a_j^0 + \sum_{i=1}^s \mu_j^i a_j^i \right) = 0, -\mu_0 d_0^\top y - \sum_{j=1}^q \left(\mu_j b_j^0 - \mu_j c_j^\top x + \sum_{i=1}^s \mu_j^i b_j^i \right) \geq 0 \\ & \text{and } (\mu_j \gamma_i)^2 - (\mu_j^i)^2 \geq 0, i = 1, \dots, s, j = 1, \dots, q, \end{aligned} \right.$$

where $\left\{ \check{u}_k := (\check{u}_k^1, \dots, \check{u}_k^s) \in \mathbb{R}^s \mid k = 1, \dots, 2^s \right\}$ stands for the set of extreme points of the box V_s .

Denote $\mu := (\mu_0, \mu_1, \dots, \mu_q, \mu_1^1, \dots, \mu_q^1, \dots, \mu_1^s, \dots, \mu_q^s) \in \mathbb{R}^{q(s+1)+1}$, $g_i(x, y, \mu), i = 1, \dots, L$ and $h_j(x, y, \mu), j = 1, \dots, n + 1$ as stated in the lemma. Then, we conclude by (V) that (x, y) is a robust feasible point of problem (P) if and only if there exists $\mu \in \mathbb{R}^{q(s+1)+1}$ such that

$$\begin{aligned} g_i(x, y, \mu) &\geq 0, \quad i = 1, \dots, l2^{m+n+1} + q2^s, \\ g_i(x, y, \mu) &> 0, \quad i = l2^{m+n+1} + q2^s + 1, \\ g_i(x, y, \mu) &\geq 0, \quad i = l2^{m+n+1} + q2^s + 2, \dots, L, \\ h_j(x, y, \mu) &\geq 0, \quad -h_j(x, y, \mu) \geq 0, \quad j = 1, \dots, n + 1. \end{aligned}$$

So, we have proved (i).

(ii) Let the (LSC) in (11) be satisfied, and consider any $(x, y, \mu) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{q(s+1)+1}$. If (x, y, μ) satisfies (III), then it obviously satisfies (IV).

Conversely, let (x, y, μ) be such that (IV) holds. It means that

$$(VI) \left\{ \begin{aligned} &(x, y) \in \mathbb{R}^m \times \mathbb{R}^n, \\ &\mu := (\mu_0, \mu_1, \dots, \mu_q, \mu_1^1, \dots, \mu_q^1, \dots, \mu_1^s, \dots, \mu_q^s) \in \mathbb{R}^{q(s+1)+1}, \\ &\check{a}_i^k \top x + \check{b}_i^k \top y - \check{c}_i^k \leq 0, \quad k = 1, \dots, 2^{m+n+1}, \quad i = 1, \dots, l, \\ &c_j \top x + \left(a_j^0 + \sum_{i=1}^s \check{u}_k^i a_j^i \right) \top y \\ &\quad - \left(b_j^0 + \sum_{i=1}^s \check{u}_k^i b_j^i \right) \leq 0, \quad k = 1, \dots, 2^s, \quad j = 1, \dots, q, \\ &\mu_0 \geq 0, \mu_j \geq 0, \quad j = 1, \dots, q, \quad \sum_{j=0}^q (\mu_j)^2 + \sum_{j=1}^q \sum_{i=1}^s (\mu_j^i)^2 = 1, \\ &\mu_0 d_0 + \sum_{j=1}^q \left(\mu_j a_j^0 + \sum_{i=1}^s \mu_j^i a_j^i \right) = 0, \\ &-\mu_0 d_0 \top y - \sum_{j=1}^q \left(\mu_j b_j^0 - \mu_j c_j \top x + \sum_{i=1}^s \mu_j^i b_j^i \right) \geq 0 \\ &\text{and } (\mu_j \gamma_i)^2 - (\mu_j^i)^2 \geq 0, \quad i = 1, \dots, s, \quad j = 1, \dots, q. \end{aligned} \right.$$

We need to show that (x, y, μ) satisfies (III). It suffices to verify that $\mu_0 \neq 0$. Assume on the contrary that $\mu_0 = 0$. We get by (VI) that

$$\sum_{j=1}^q (\mu_j)^2 + \sum_{j=1}^q \sum_{i=1}^s (\mu_j^i)^2 = 1, \tag{17}$$

$$\sum_{j=1}^q \left(\mu_j a_j^0 + \sum_{i=1}^s \mu_j^i a_j^i \right) = 0, \quad \sum_{j=1}^q \mu_j c_j \top x - \sum_{j=1}^q \mu_j b_j^0 - \sum_{j=1}^q \sum_{i=1}^s \mu_j^i b_j^i \geq 0, \tag{18}$$

$$(\mu_j \gamma_i)^2 - (\mu_j^i)^2 \geq 0, \quad i = 1, \dots, s, \quad j = 1, \dots, q. \tag{19}$$

By (19), for each $j \in \{1, \dots, q\}$, $\mu_j^i = 0$ for $i = 1, \dots, s$ whenever $\mu_j = 0$, and hence, we conclude from (17) that there exists $j \in \{1, \dots, q\}$ such that $\mu_j \neq 0$, i.e., $\sum_{j=1}^q \mu_j \neq 0$. For each $j \in \{1, \dots, q\}$, let $\bar{u}_j := (\bar{u}_j^1, \dots, \bar{u}_j^s) \in \mathbb{R}^s$ with

$$\bar{u}_j^i := \begin{cases} 0 & \text{if } \mu_j = 0 \\ \frac{\mu_j^i}{\mu_j} & \text{if } \mu_j \neq 0, \end{cases} \quad i = 1, \dots, s.$$

Then, $|\bar{u}_j^i| \leq \gamma_i$ for $i = 1, \dots, s, j = 1, \dots, q$, and so, $\bar{u}_j \in V_s = U_j$ for all $j = 1, \dots, q$. On the one hand, in view of the (LSC) in (11), we find $z \in \mathbb{R}^n$ such that

$$c_j^\top x + a_j(\bar{u}_j)^\top z < b_j(\bar{u}_j), \quad j = 1, \dots, q.$$

Since $\sum_{j=1}^q \mu_j \neq 0$, it follows that

$$\sum_{j=1}^q \mu_j c_j^\top x + \sum_{j=1}^q \mu_j a_j(\bar{u}_j)^\top z < \sum_{j=1}^q \mu_j b_j(\bar{u}_j),$$

or equivalently,

$$\sum_{j=1}^q \mu_j c_j^\top x + \left(\sum_{j=1}^q \mu_j a_j^0 + \sum_{j=1}^q \sum_{i=1}^s \mu_j^i a_j^i \right)^\top z - \sum_{j=1}^q \mu_j b_j^0 - \sum_{j=1}^q \sum_{i=1}^s \mu_j^i b_j^i < 0. \tag{20}$$

On the other hand, we get by (18) that

$$\sum_{j=1}^q \mu_j c_j^\top x + \left(\sum_{j=1}^q \mu_j a_j^0 + \sum_{j=1}^q \sum_{i=1}^s \mu_j^i a_j^i \right)^\top z - \sum_{j=1}^q \mu_j b_j^0 - \sum_{j=1}^q \sum_{i=1}^s \mu_j^i b_j^i \geq 0,$$

which contradicts (20). Therefore, we conclude that $\mu_0 > 0$. It entails that (x, y, μ) satisfies (III). The proof is complete. □

3 Robust Bilevel Polynomial Problems via Semidefinite Programming Relaxations

In this section, we present semidefinite programming relaxations for the uncertain bilevel polynomial optimization problem (P) and show how the robust global optimal

value of problem (P) can be found by solving a sequence of corresponding semidefinite programming relaxation problems.

Sum-of-Squares Relaxation Problems. For each $k \in \mathbb{N}$, we consider a relaxation problem in terms of sum of squares for the uncertain bilevel polynomial optimization problem (P) given by

$$\begin{aligned} \sup_{(t, \sigma_0, \sigma_i, \xi_j, \zeta)} \left\{ t : f - \sum_{i=1}^L \sigma_i g_i - \sum_{j=1}^{n+1} \xi_j h_j - \zeta(\kappa - f) - t = \sigma_0, \right. & \quad (D_k) \\ t \in \mathbb{R}, \zeta, \sigma_0, \sigma_i \in \Sigma^2[x, y, \mu], \xi_j \in \mathbb{R}[x, y, \mu], \\ \deg(\sigma_0) \leq k, \deg(\zeta f) \leq k, \deg(\sigma_i g_i) \leq k, \deg(\xi_j h_j) \leq k, \\ \left. i = 1, \dots, L, j = 1, \dots, n + 1 \right\}, \end{aligned}$$

where $g_i, i = 1, \dots, L := l2^{m+n+1} + q(2^s + s + 1) + 2, h_j, j = 1, \dots, n + 1$, are defined as in the statement of Lemma 2.4 and $\kappa \geq f(\bar{x}, \bar{y})$ with (\bar{x}, \bar{y}) being a robust feasible point of problem (P).

It is worth mentioning here that for each fixed $k \in \mathbb{N}$, the problem (D_k) can be regarded as a *sum of squares* relaxation problem of the problem (P), and more interestingly, it can be reformulated as a semidefinite linear programming problem [20]. We denote the robust optimal value of problem (P) and the optimal value of problem (D_k), respectively, by $\text{val}(P)$ and $\text{val}(D_k)$.

In the next theorem, we show that, under some additional conditions, the uncertain bilevel polynomial optimization problem (P) has a global robust solution and the optimal values of the relaxation problem (D_k) converge to the robust global optimal value of the problem (P) when the degree bound k goes to infinity.

Theorem 3.1 (Computing robust global optimal value by SDP) *Let f be coercive on $\mathbb{R}^m \times \mathbb{R}^n$. Assume that the (LSC) in (11) is satisfied. Then, the uncertain bilevel polynomial optimization problem (P) has a global robust solution (x_0, y_0) satisfying*

$$\text{val}(D_k) \leq \text{val}(P) = f(x_0, y_0) \text{ for all } k \in \mathbb{N}. \tag{21}$$

Moreover, we have

$$\lim_{k \rightarrow \infty} \text{val}(D_k) = \text{val}(P). \tag{22}$$

Proof (Proving the existence of global robust solutions of (P)) Thanks to Lemma 2.4(i), we assert that $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ is a robust feasible point of problem (P) if and only if there exists $\mu \in \mathbb{R}^{q(s+1)+1}$ such that

$$\begin{aligned} g_i(x, y, \mu) &\geq 0, \quad i = 1, \dots, l2^{m+n+1} + q2^s, \\ g_i(x, y, \mu) &> 0, \quad i = l2^{m+n+1} + q2^s + 1, \\ g_i(x, y, \mu) &\geq 0, \quad i = l2^{m+n+1} + q2^s + 2, \dots, L, \\ h_j(x, y, \mu) &\geq 0, \quad -h_j(x, y, \mu) \geq 0, \quad j = 1, \dots, n + 1, \end{aligned} \tag{23}$$

where $g_i, i = 1, \dots, L, h_j, j = 1, \dots, n + 1$ are defined as in the statement of Lemma 2.4.

Let $k \in \mathbb{N}$, and let $(\bar{x}, \bar{y}) \in \mathbb{R}^m \times \mathbb{R}^n$ be a robust feasible point of problem (P) as in the construction of problem (D_k). Then, one has $\kappa \geq f(\bar{x}, \bar{y})$, and there exists $\bar{\mu} \in \mathbb{R}^{q(s+1)+1}$ such that (23) holds at $(\bar{x}, \bar{y}, \bar{\mu})$. Putting $\hat{h} := \kappa - f + h_{n+1}$, we assert that

$$H := \{(x, y, \mu) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{q(s+1)+1} : \hat{h}(x, y, \mu) \geq 0\} \neq \emptyset.$$

Indeed, as $(\bar{x}, \bar{y}, \bar{\mu})$ satisfies (23), it entails that $h_{n+1}(\bar{x}, \bar{y}, \bar{\mu}) = 0$. Then, $\hat{h}(\bar{x}, \bar{y}, \bar{\mu}) = \kappa - f(\bar{x}, \bar{y}) \geq 0$, and so, $(\bar{x}, \bar{y}, \bar{\mu}) \in H$.

For each $(x, y, \mu) \in H$, we have $\hat{h}(x, y, \mu) \geq 0$. It entails that

$$f(x, y) \leq 1 + \kappa, \quad \sum_{k=0}^q (\mu_k)^2 + \sum_{k=1}^q \sum_{i=1}^s (\mu_k^i)^2 \leq 1 + \kappa - \inf_{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n} f(x, y). \tag{24}$$

Since f is coercive on $\mathbb{R}^m \times \mathbb{R}^n$, it follows that $\inf_{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n} f(x, y) > -\infty$, and hence, (24) guarantees that H is a compact set. Setting

$$\begin{aligned} K := \{ & (x, y, \mu) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{q(s+1)+1} : \\ & \kappa - f(x, y) \geq 0, \quad g_i(x, y, \mu) \geq 0, \quad i = 1, \dots, L, \\ & h_j(x, y, \mu) \geq 0, \quad -h_j(x, y, \mu) \geq 0, \quad j = 1, \dots, n + 1\}, \end{aligned} \tag{25}$$

we easily verify that $(\bar{x}, \bar{y}, \bar{\mu}) \in K$, and hence, $K \neq \emptyset$. Moreover, it can be checked that $K \subset H$, and so, K is compact as well.

Now, consider the function $\tilde{f}(x, y) := f(x, y) - f(\bar{x}, \bar{y})$ for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Since \tilde{f} is a polynomial and hence continuous, we conclude that there exists $(x_0, y_0, \mu_0) \in K$ such that

$$\tilde{f}(x_0, y_0) \leq \tilde{f}(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \text{ satisfying } (x, y, \mu) \in K, \tag{26}$$

where $\mu \in \mathbb{R}^{q(s+1)+1}$.

We claim that (x_0, y_0) is a global robust solution of problem (P). Indeed, by $(x_0, y_0, \mu_0) \in K$, it follows that

$$\begin{aligned} g_i(x_0, y_0, \mu_0) & \geq 0, \quad i = 1, \dots, L, \\ h_j(x_0, y_0, \mu_0) & \geq 0, \quad -h_j(x_0, y_0, \mu_0) \geq 0, \quad j = 1, \dots, n + 1, \end{aligned} \tag{27}$$

and that

$$\kappa \geq f(x_0, y_0). \tag{28}$$

Under the fulfillment of the (LSC) in (11), invoking Lemma 2.4(ii), we assert by (27) that (23) holds at (x_0, y_0, μ_0) and so, (x_0, y_0) is a robust feasible point of problem (P).

Let $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ be a robust feasible point of problem (P). Then, there is $\mu \in \mathbb{R}^{q(s+1)+1}$ such that (23) holds. If in addition $\kappa - f(x, y) \geq 0$, then $(x, y, \mu) \in K$, and so, we get by (26) that $f(x_0, y_0) \leq f(x, y)$. Otherwise, $\kappa - f(x, y) < 0$, then $f(x, y) > \kappa \geq f(x_0, y_0)$, where the last inequality holds by virtue of (28). Consequently, our claim holds.

Furthermore, it holds that

$$\text{val(P)} = f(x_0, y_0). \tag{29}$$

[Verifying (21)] If the feasible set of problem (D_k) is empty, then $\text{val(D}_k) = -\infty$, and in this case, (21) holds trivially. Now, let $(t, \sigma_0, \sigma_i, \xi_j, \zeta), i = 1, \dots, L, j = 1, \dots, n + 1$ be a feasible point of problem (D_k). It means that there exist $t \in \mathbb{R}, \zeta, \sigma_0, \sigma_i \in \Sigma^2[x, y, \mu], \xi_j \in \mathbb{R}[x, y, \mu], \text{deg}(\sigma_0) \leq k, \text{deg}(\zeta f) \leq k, \text{deg}(\sigma_i g_i) \leq k, \text{deg}(\xi_j h_j) \leq k, i = 1, \dots, L, j = 1, \dots, n + 1$ such that $f - \sum_{i=1}^L \sigma_i g_i - \sum_{j=1}^{n+1} \xi_j h_j - \zeta(\kappa - f) - t = \sigma_0$ or equivalently,

$$(1 + \zeta)f = \sigma_0 + \sum_{i=1}^L \sigma_i g_i + \sum_{j=1}^{n+1} \xi_j h_j + t + \zeta f(x_0, y_0) + \zeta(\kappa - f(x_0, y_0)), \tag{30}$$

where (x_0, y_0) is the global robust solution of problem (P) as shown above. Recall here that (28) and (23) hold at (x_0, y_0, μ_0) . Due to the nonnegativity of sum-of-squares polynomials, estimating (30) at (x_0, y_0, μ_0) , we obtain that $(1 + \zeta(x_0, y_0, \mu_0))f(x_0, y_0) \geq t + \zeta(x_0, y_0, \mu_0)f(x_0, y_0)$, or equivalently, $f(x_0, y_0) \geq t$. It guarantees that $\text{val(D}_k) \leq f(x_0, y_0)$, which together with (29) proves that (21) is valid.

[Verifying (22)] Let us consider a set of polynomials $\mathbf{M}(g_1, \dots, g_L, h_1, \dots, h_{n+1}, -h_1, \dots, -h_{n+1}, \kappa - f)$ as defined in (1). It is obvious by definition that

$$\hat{h} := (\kappa - f) + h_{n+1} \in \mathbf{M}(g_1, \dots, g_L, h_1, \dots, h_{n+1}, -h_1, \dots, -h_{n+1}, \kappa - f).$$

As shown above, the set $H := \{(x, y, \mu) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{q(s+1)+1} : \hat{h}(x, y, \mu) \geq 0\}$ is compact. Hence, $\mathbf{M}(g_1, \dots, g_L, h_1, \dots, h_{n+1}, -h_1, \dots, -h_{n+1}, \kappa - f)$ is archimedean.

Let $\epsilon > 0$. We pay attention to the set K given in (25). For any $(x, y, \mu) \in K$, it follows that

$$\begin{aligned} g_i(x, y, \mu) &\geq 0, \quad i = 1, \dots, L, \\ h_j(x, y, \mu) &\geq 0, \quad -h_j(x, y, \mu) \geq 0, \quad j = 1, \dots, n + 1, \end{aligned}$$

which is nothing else but (23) under the fulfillment of the (LSC) in (11), and so, (x, y) is a robust feasible point of problem (P) by virtue of Lemma 2.4. This fact gives us that $f(x, y) \geq f(x_0, y_0)$ inasmuch as (x_0, y_0) is a global robust solution of problem (P)

as shown above. Consequently, it guarantees that

$$\hat{f}(x, y, \mu) := f(x, y) - f(x_0, y_0) + \epsilon > 0 \text{ for all } (x, y, \mu) \in K.$$

[Applying Putinar’s Positivstellensatz] Now, applying Lemma 2.1, we find sum-of-squares polynomials $\sigma_i, i = 0, 1, \dots, L, \xi_j^1, \xi_j^2, j = 1, \dots, n + 1, \zeta \in \Sigma^2[x, y, \mu]$ such that

$$\hat{f} = \sigma_0 + \sum_{i=1}^L \sigma_i g_i + \sum_{j=1}^{n+1} \xi_j^1 h_j + \sum_{j=1}^{n+1} \xi_j^2 (-h_j) + \zeta(\kappa - f). \tag{31}$$

Let $\xi_j \in \mathbb{R}[x, y, \mu], j = 1, \dots, n + 1$ be real polynomials defined by $\xi_j := \xi_j^1 - \xi_j^2, j = 1, \dots, n + 1$. Then, we deduce from (31) that $f - \sum_{i=1}^L \sigma_i g_i - \sum_{j=1}^{n+1} \xi_j h_j - \zeta(\kappa - f) - f(x_0, y_0) + \epsilon = \sigma_0$, or equivalently,

$$f - \sum_{i=1}^L \sigma_i g_i - \sum_{j=1}^{n+1} \xi_j h_j - \zeta(\kappa - f) - t_0 = \sigma_0,$$

with $t_0 := f(x_0, y_0) - \epsilon \in \mathbb{R}$. So, there exists $k_\epsilon \in \mathbb{N}$ such that $\deg(\sigma_0) \leq k_\epsilon, \deg(\zeta f) \leq k_\epsilon, \deg(\sigma_i g_i) \leq k_\epsilon, \deg(\xi_j h_j) \leq k_\epsilon, i = 1, \dots, L, j = 1, \dots, n + 1$, and that

$$\text{val}(\mathbf{D}_{k_\epsilon}) \geq f(x_0, y_0) - \epsilon.$$

Since $\epsilon > 0$ was arbitrarily taken, we conclude that $\liminf_{k \rightarrow \infty} \text{val}(\mathbf{D}_k) \geq f(x_0, y_0)$. This together with (21) establishes (22), which ends the proof of the theorem. \square

Remark 3.1 It is worth noticing that if f is a convex polynomial and there exists $(\bar{x}, \bar{y}) \in \mathbb{R}^m \times \mathbb{R}^n$ such that its Hessian $\nabla^2 f(\bar{x}, \bar{y})$ is positive definite, then f is coercive on $\mathbb{R}^m \times \mathbb{R}^n$ (see, e.g., [19, Lemma 3.1]) and hence, the above theorem can be obviously applied for this convex setting under the fulfillment of the (LSC) in (11).

The next example illustrates that if the (LSC) in (11) is violated, then the conclusion of Theorem 3.1 may go awry. Below, we consider the case of $l := 1$ and $q := 1$ for the purpose of simplicity.

Example 3.1 (The importance of the Slater condition) Consider an uncertain bilevel polynomial optimization problem of the form:

$$\min_{(x,y) \in \mathbb{R}^2} \{f(x, y) := x^2 + y^2 + 2y - 2 : y \in Y(x, u)\}, \tag{EP1}$$

where $u \in U := [-1, 1] \subset \mathbb{R}$ and $Y(x, u) := \text{argmin}_{z \in \mathbb{R}} \{x - z : (1 + u)z \leq 0\}$. The problem (EP1) can be expressed in terms of problem (P), where $\tilde{U} := [\underline{a}, \bar{a}] \times$

$[\underline{b}, \bar{b}] \times [\underline{c}, \bar{c}]$ with $\underline{a} := \bar{a} := \underline{b} := \bar{b} := \underline{c} := \bar{c} := 0 \in \mathbb{R}$, and $c_0 := 1 \in \mathbb{R}, d_0 := -1 \in \mathbb{R}, c := 0 \in \mathbb{R}$, the affine mappings $a : \mathbb{R} \rightarrow \mathbb{R}$ and $b : \mathbb{R} \rightarrow \mathbb{R}$ are given by $a(u) := a^0 + ua^1$ and $b(u) := b^0 + ub^1$ with $a^0 := a^1 := 1 \in \mathbb{R}, b^0 := b^1 := 0 \in \mathbb{R}$ for $u \in \mathbb{R}$.

In this setting, it is easy to see that $Y(x) = \{0\}$ for any $x \in \mathbb{R}$, and therefore, we see that $(x_0, y_0) := (0, 0)$ is a global robust solution of problem (EP1). Obviously, f is coercive on \mathbb{R}^2 . Let $\mu := (\mu_0, \mu_1, \mu_1^1) \in \mathbb{R}^3$, and denote the functions $g_i(x, y, \mu)$ and $h_j(x, y, \mu)$ as in the statement of Lemma 2.4. For the sake of clear representation, we first remove the null functions and then relabel them as

$$\begin{aligned} g_1(x, y, \mu) &= -2y, \quad g_2(x, y, \mu) = \mu_0, \quad g_3(x, y, \mu) = \mu_1, \\ g_4(x, y, \mu) &= \mu_0y, \quad g_5(x, y, \mu) = (\mu_1)^2 - (\mu_1^1)^2, \\ h_1(x, y, \mu) &= -\mu_0 + \mu_1 + \mu_1^1, \quad h_2(x, y, \mu) = 1 - (\mu_0)^2 - (\mu_1)^2 - (\mu_1^1)^2. \end{aligned}$$

Let $\kappa \geq -2 = f(x_0, y_0)$. In this setting, the problem (D_k), $k \in \mathbb{N}$, becomes

$$\begin{aligned} \sup_{(t, \sigma_0, \sigma_i, \xi_j, \zeta)} \left\{ t : f - \sum_{i=1}^5 \sigma_i g_i - \sum_{j=1}^2 \xi_j h_j - \zeta(\kappa - f) - t = \sigma_0, \right. \\ t \in \mathbb{R}, \zeta, \sigma_0, \sigma_i \in \Sigma^2[x, y, \mu], \xi_j \in \mathbb{R}[x, y, \mu], \deg(\sigma_0) \leq k, \\ \left. \deg(\zeta f) \leq k, \deg(\sigma_i g_i) \leq k, \deg(\xi_j h_j) \leq k, i = 1, \dots, 5, j = 1, 2 \right\}. \end{aligned} \tag{32}$$

We claim that, for each $k \in \mathbb{N}$, the representation of sum-of-squares polynomials in (32) fails to hold for any $t \in]-3, +\infty[$. Indeed, assume on the contrary that there exist $k \in \mathbb{N}, t \in]-3, +\infty[$ and sum-of-squares polynomials $\zeta, \sigma_0, \sigma_i \in \Sigma^2[x, y, \mu], i = 1, \dots, 5$, as well as real polynomials $\xi_j \in \mathbb{R}[x, y, \mu], j = 1, 2$ such that

$$f - \sum_{i=1}^5 \sigma_i g_i - \sum_{j=1}^2 \xi_j h_j - \zeta(\kappa - f) - t = \sigma_0, \tag{33}$$

where $\deg(\sigma_0) \leq k, \deg(\zeta f) \leq k, \deg(\sigma_i g_i) \leq k, \deg(\xi_j h_j) \leq k, i = 1, \dots, 5, j = 1, 2$. Setting $\tilde{x} := 0, \tilde{y} := -1$ and $\tilde{\mu} := (0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$, and then substituting $(\tilde{x}, \tilde{y}, \tilde{\mu})$ into (33) we obtain that

$$-3 - 2\sigma_1(\tilde{x}, \tilde{y}, \tilde{\mu}) - \frac{1}{\sqrt{2}}\sigma_3(\tilde{x}, \tilde{y}, \tilde{\mu}) - (\kappa + 3)\zeta(\tilde{x}, \tilde{y}, \tilde{\mu}) - t = \sigma_0(\tilde{x}, \tilde{y}, \tilde{\mu}).$$

Hence, it entails that $\sigma_0(\tilde{x}, \tilde{y}, \tilde{\mu}) \leq -3 - t < 0$, which is absurd.

Consequently, the conclusion (22) of Theorem 3.1 fails due to the fact that $\text{val}(\text{D}_k) \leq -3$ for all $k \in \mathbb{N}$ and $\text{val}(\text{EP1}) = f(x_0, y_0) = -2$. The reason is that the (LSC) in (11) is violated.

Finally in this section, we provide a simple example which shows how our schemes can be applied to find the robust global optimal value of an uncertain bilevel polynomial optimization problem.

Example 3.2 (Uncertain bilevel polynomial problem) Consider an uncertain bilevel polynomial optimization problem of the form:

$$\min_{(x,y) \in \mathbb{R}^2} \left\{ f(x, y) := x^4 - 4xy + y^4 - 2 : y \in Y(x, u) \right\}, \tag{EP2}$$

where $u \in U := [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$ and $Y(x, u) := \operatorname{argmin}_{z \in \mathbb{R}} \{x - z : (1 + u)z \leq 0\}$. The problem (EP2) can be expressed in terms of problem (P), where $\tilde{U} := [\underline{a}, \bar{a}] \times [b, \bar{b}] \times [c, \bar{c}]$ with $\underline{a} := \bar{a} := \underline{b} := \bar{b} := \underline{c} := \bar{c} := 0 \in \mathbb{R}$, and $c_0 := 1 \in \mathbb{R}, d_0 := -1 \in \mathbb{R}, c := 0 \in \mathbb{R}$, the affine mappings $a : \mathbb{R} \rightarrow \mathbb{R}$ and $b : \mathbb{R} \rightarrow \mathbb{R}$ are given by $a(u) := a^0 + ua^1$ and $b(u) := b^0 + ub^1$ with $a^0 := a^1 := 1 \in \mathbb{R}, b^0 := b^1 := 0 \in \mathbb{R}$ for $u \in \mathbb{R}$. A direct calculation shows that $(x_0, y_0) := (0, 0)$ is a global robust solution of problem (EP2) with the robust global optimal value -2 .

Now, we employ the relaxation schemes formulated in Theorem 3.1 to verify this robust global optimal value. In this setting, it is easy to see that f is coercive on \mathbb{R}^2 and the (LSC) in (11) is fulfilled. Let $\mu := (\mu_0, \mu_1, \mu_1^1) \in \mathbb{R}^3$, and denote the functions $g_i(x, y, \mu)$ and $h_j(x, y, \mu)$ as in the statement of Lemma 2.4. For the sake of clear representation, we first remove the null functions and then relabel them as

$$\begin{aligned} g_1(x, y, \mu) &= -\frac{1}{2}y, \quad g_2(x, y, \mu) = -\frac{3}{2}y, \quad g_3(x, y, \mu) = \mu_0, \\ g_4(x, y, \mu) &= \mu_1, \quad g_5(x, y, \mu) = \mu_0y, \quad g_6(x, y, \mu) = \frac{1}{4}(\mu_1)^2 - (\mu_1^1)^2, \\ h_1(x, y, \mu) &= -\mu_0 + \mu_1 + \mu_1^1, \quad h_2(x, y, \mu) = 1 - (\mu_0)^2 - (\mu_1)^2 - (\mu_1^1)^2. \end{aligned}$$

Let $(\bar{x}, \bar{y}) := (1, 0)$ be a feasible point, and take $\kappa := -1 \geq -1 = f(\bar{x}, \bar{y})$. In this setting, the problem (D_k) becomes

$$\begin{aligned} \sup_{(t, \sigma_0, \sigma_i, \xi_j, \zeta)} \left\{ t : f - \sum_{i=1}^6 \sigma_i g_i - \sum_{j=1}^2 \xi_j h_j - \zeta(-1 - f) - t = \sigma_0, \right. \\ \left. t \in \mathbb{R}, \zeta, \sigma_0, \sigma_i \in \Sigma^2[x, y, \mu], \xi_j \in \mathbb{R}[x, y, \mu], \right. \\ \left. \deg(\sigma_0) \leq k, \deg(\zeta f) \leq k, \deg(\sigma_i g_i) \leq k, \deg(\xi_j h_j) \leq k, i = 1, \dots, 6, j = 1, 2 \right\}. \end{aligned}$$

Using the MATLAB toolbox YALMIP (see, e.g., [28]), we convert the above optimization problem into an equivalent semidefinite program and solve it with $k := 4$. The solver returns the true robust global optimal value as -2.000 .

4 Conclusions

We have shown that the robust global optimal value of the uncertain bilevel polynomial program with box data uncertainties is the limit of a sequence of values of Lasserre-type hierarchy of semidefinite linear programming relaxations. It has been achieved by transforming the uncertain bilevel polynomial program into a single-level non-convex polynomial program using a dual characterization of the solution of the lower-level program and Putinar's Positivstellensatz of semi-algebraic geometry. We have provided a numerical example to show how the robust global optimal value of the uncertain bilevel polynomial program can be calculated via a semidefinite programming problem using the MATLAB toolbox YALMIP.

It would be of interest to study how a global robust solution of the uncertain bilevel polynomial program can be found from its semidefinite linear programming relaxations, or to extend the obtained results to the case of spectrahedral uncertainty, where explicit sum-of-squares/SDP relaxations in terms of the original data are often not available.

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