Set-Valued Systems with Infinite-Dimensional Image and Applications

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Abstract In infinite-dimensional spaces, we investigate a set-valued system from the image perspective. By exploiting the quasi-interior and the quasi-relative interior, we give some new equivalent characterizations of (proper, regular) linear separation and establish some new sufficient and necessary conditions for the impossibility of the system under new assumptions, which do not require the set to have nonempty interior. We also present under mild assumptions the equivalence between (proper, regular) linear separation and saddle points of Lagrangian functions for the system. These results are applied to obtain some new saddle point sufficient and necessary optimality conditions of vector optimization problems.

1 Introduction

In recent years, generalized systems (for short, GS), as a unified framework of equilibrium problems, optimization problems, variational inequalities, and complementarity systems, have received more and more attention from the image perspective (see, e.g.,

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[1–4]). Such problems can be reduced to the impossibility of GS, which can be studied by means of separation techniques in the image space associated with GS (see, e.g., [5–10]).

Traces of the idea of studying the images of functions, involved in a constrained extremum problem, go back to the work of Carathéodory [11]. In the 1950s, Bellman, introducing his famous maximum principle [12], proposed to replace the unknowns by new ones, which run in the space of the images of the functions, that define the given problem, the image space. In the late 1970s and 1980s, Giannessi [13], Castellani, Giannessi [14] and Hestenes [15], independently of each other, brought explicitly such a study into the field of optimization.

In this paper, we shall investigate set-valued GS with infinite-dimensional image by means of the image space analysis (for short, ISA) [13]. As is well known, the "Slater" condition, as the most straightforward constraint qualification of constrained optimization problems or variational inequalities, is often not met in infinite-dimensional spaces, even in finite-dimensional ones, because it requires the existence of an interior point of a convex set, which often has empty interior (see, e.g., [16]). This is the case of constrained optimization problems or variational inequalities with infinite-dimensional image, such as optimization problems or variational inequalities connected with network equilibrium problems, the obstacle problem, the elastic–plastic torsion problem, which use positive cones of spaces of all Lebesgue measurable functions or Sobolev spaces. The use of the quasi-interior and the quasi-relative interior allows one to overcome the difficulty (see, e.g., [16–20]).

As is well known, a separation theorem plays a vital role in obtaining the necessary conditions (i.e., linear separation) for the impossibility of GS. Though some separation theorems related to the quasi-interior and the quasi-relative interior have been recently proposed and discussed by several authors (see, e.g., [16–18,20–23]), no existing separation theorems in terms of the quasi-interior and the quasi-relative interior can be used to obtain the necessary conditions for the impossibility of GS. To this aim, we first give some new equivalent characterizations of (proper, regular) linear separation for set-valued GS with infinite-dimensional image by using the quasi-interior and the quasi-relative interior, and then applied these results to establish new sufficient and necessary conditions for the impossibility of set-valued GS with infinite-dimensional image under a technical assumption (see, Theorem 4.3), which is different from the standard way. We also present under some convexity and compactness assumptions of the set-valued mapping the equivalence between (proper, regular) linear separation and saddle points of Lagrangian functions for constrained set-valued GS with infinite-dimensional image. Furthermore, we apply these results to obtain some new saddle point sufficient and necessary optimality conditions of vector optimization problems (for short, VOP), which are compared with that in [24,25]. The results presented in this paper extend and generalize corresponding results in [3,4,7,13].

The paper is organized as follows. In Sect. 2, we recall some preliminary results of the quasi-relative interior and the quasi-interior, and some concepts of set-valued mappings. We define the image of set-valued GS and the conical extension of the image, and give the equivalence between the impossibility of set-valued GS and an empty intersection of a subset of the image space and the conical extension of the image. In Sect. 3, we characterize the (proper, regular) linear separation for set-valued GS by

using the quasi-relative interior and the quasi-interior, and we present sufficient and necessary conditions for the impossibility of set-valued GS in Sect. 4. Section 5 investigates under some convexity and compactness assumptions the equivalence between (proper, regular) linear separation and saddle points of Lagrangian functions for set-valued GS. Applications to VOP are given in Sect. 6, devoting particular attention to linear separation and saddle point sufficient and necessary optimality conditions.

2 Preliminaries and ISA for Set-Valued GS

The following notations and definitions will be useful in the sequel.

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, where m is a given positive integer. Denote by $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : \text{each } x_i \geq 0\}$ and $\mathbb{R}^m_{++} := \{x \in \mathbb{R}^m : \text{each } x_i > 0\}$. Let $\mathcal{A}, \mathcal{B}$ be subsets of a Hausdorff locally convex topological linear space $\mathcal{U}$. The closure, the interior, the boundary, and the convex hull of $\mathcal{A}$ are denoted by $\text{cl } \mathcal{A}$, $\text{int } \mathcal{A}$, $\text{bd } \mathcal{A}$ and $\text{conv } \mathcal{A}$, respectively. The relative interior of $\mathcal{A}$, i.e., the interior of $\mathcal{A}$ relative to the closed affine hull of $\mathcal{A}$ (see, e.g., [26]), is denoted by $\text{ri } \mathcal{A}$. Denote by $\mathcal{A} + \mathcal{B} := \{a + b \in \mathcal{U} : a \in \mathcal{A}, b \in \mathcal{B}\}$ the Minkowski sum of $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{A} + b := \mathcal{A} + \{b\}$ and consider that $\mathcal{A} + \emptyset = \emptyset$. According to the definition, one has $\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$. Denote by $t\mathcal{A} := \{ta \in \mathcal{U} : a \in \mathcal{A}\}$, where $t \in \mathbb{R}$. The set $\mathcal{A}$ is said to be a cone, if $\lambda\mathcal{A} \subseteq \mathcal{A}$ for all $\lambda > 0$, and a convex cone if, in addition, $\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$. We say that a cone $\mathcal{A}$ is pointed, if $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$ and proper, if $\mathcal{U} \neq \mathcal{A} \neq \{0\}$. Denote by $\text{cone } \mathcal{A} := \cup_{t \in \mathbb{R}_+} t\mathcal{A}$ the cone generated by $\mathcal{A}$ and let $\text{cone}_+ \mathcal{A} := \cup_{t \in \mathbb{R}_{++}} t\mathcal{A}$. Clearly, if $\mathcal{A}$ is a cone, so is $\text{cl } \mathcal{A}$. $\mathcal{A}^* := \{a^* \in \mathcal{U}^* : \langle a^*, a \rangle \geq 0, \forall a \in \mathcal{A}\}$ is the positive polar of $\mathcal{A}$, where $\mathcal{U}^*$ is the topological dual of $\mathcal{U}$ and $\langle a^*, a \rangle$ is the value of a^* at a. Clearly, $\mathcal{A}^* = (\text{cl}\mathcal{A})^* = (\text{conv}\mathcal{A})^* = (\mathcal{A}\backslash\{0\})^*$. $N_\mathcal{A}(a)$ is the normal cone to $\mathcal{A}$ at $a \in \mathcal{A}$ and is defined by $N_\mathcal{A}(a) := \{a^* \in \mathcal{U}^* : \langle a^*, a' - a \rangle \leq 0, \forall a' \in \mathcal{A}\}$. The support function of $\mathcal{A}$ is defined by $\sigma_\mathcal{A}(a^*) := \sup_{a \in \mathcal{A}}\langle a^*, a \rangle$, where $a^* \in \mathcal{U}^*$. Clearly, $\sigma_\mathcal{A}(a^*) = \sigma_{\text{cl}\mathcal{A}}(a^*) = \sigma_{\text{conv}\mathcal{A}}(a^*)$. Let $\mathcal{U}$ and $\mathcal{V}$ be Hausdorff locally convex topological linear spaces. Define $\langle (a^*, b^*), (a, b) \rangle := \langle a^*, a \rangle + \langle b^*, b \rangle$, where $a \in \mathcal{U}, b \in \mathcal{V}, a^* \in \mathcal{U}^*$ and $b^* \in \mathcal{V}^*$. Define $(\mathcal{A}, \mathcal{B}) := \{(a, b) \in \mathcal{U} \times \mathcal{V} : a \in \mathcal{A}, b \in \mathcal{B}\}$ if $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$, where $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$.

The following definition is due to Borwein and Lewis.

Definition 2.1 [16, Definition 2.3] Let $\mathcal{A}$ be a nonempty subset of a Hausdorff locally convex topological linear space $\mathcal{V}$.

(i) We say that $a \in \mathcal{A}$ is a quasi-interior point of $\mathcal{A}$, denoted by $a \in \text{qi } \mathcal{A}$, if $\text{cl cone}(\mathcal{A} - a) = \mathcal{V}$, or equivalently, $N_\mathcal{A}(a) = \{0\}$;

(ii) We say that $a \in \mathcal{A}$ is a quasi-relative interior point of $\mathcal{A}$, denoted by $a \in \text{qri } \mathcal{A}$, if $\text{cl cone}(\mathcal{A} - a)$ is a linear subspace of $\mathcal{V}$, or equivalently, $N_\mathcal{A}(a)$ is a linear subspace of $\mathcal{V}^*$.

Recall that if $\mathcal{A} \subseteq \mathcal{V}$ is convex and $a \in \mathcal{A}$, then $\text{cl cone}(\mathcal{A} - a) = T_\mathcal{A}(a)$ (see, e.g., [27–29]), where $T_\mathcal{A}(a)$ is the contingent cone (or the Bouligand tangent cone) to $\mathcal{A}$ at a. It is easy to see that, for any $a \in \mathcal{V}$, $\text{qri }\{a\} = \{a\}$. For any convex set $\mathcal{A} \subseteq \mathcal{V}$, we have that $\text{qi } \mathcal{A} \subseteq \text{qri } \mathcal{A}$, and $\text{int } \mathcal{A} \neq \emptyset$ implies $\text{int } \mathcal{A} = \text{qri } \mathcal{A}$ [16] and $\text{int } \mathcal{A} = \text{qi } \mathcal{A}$ [19], and if $\text{qi } \mathcal{A} \neq \emptyset$, then $\text{qi } \mathcal{A} = \text{qri } \mathcal{A}$ [19,20]. Moreover, if $\mathcal{V}$ is a

finite-dimensional space, then $\text{qi } \mathcal{A} = \text{int } \mathcal{A}$ [19] and $\text{qri } \mathcal{A} = \text{ri } \mathcal{A}$ [16]. If $\mathcal{A} \subseteq \mathcal{V}$ is a closed and proper cone, then $0 \notin \text{qi } \mathcal{A}$, since $0 \in \mathcal{A}$ and $\text{cl cone } (\mathcal{A} - 0) = \mathcal{A} \neq \mathcal{V}$. If $\mathcal{A} \subseteq \mathcal{V}$ with $\mathcal{A} \neq \{0\}$ is a closed and pointed cone, then $0 \notin \text{qri } \mathcal{A}$, since $0 \in \mathcal{A}$ and $\text{cl cone } (\mathcal{A} - 0) = \mathcal{A}$ is not a linear subspace of $\mathcal{V}$.

In this paper, without other specifications, let X and V be Hausdorff locally convex topological vector spaces, Y be a parameter set, and $R \subseteq X$ be a nonempty and convex set. Let $\mathcal{H} \subseteq V$ be convex such that $0 \notin \mathcal{H}$ and $\text{cl } \mathcal{H}$ is a cone (since $\mathcal{H}$ is convex, so is $\text{cl } \mathcal{H}$), and let $F : X \times Y \rightrightarrows V$ be a set-valued mapping. We consider the following set-valued GS:

$$F(x; y) \cap \mathcal{H} \neq \emptyset, \ x \in R, \tag{1}$$

where $y \in Y$.

If $F : X \times Y \to V$ is single-valued, then system (1) collapses to the system: $F(x; y) \in \mathcal{H}, \ x \in R$, where $y \in Y$, which has been investigated in [3,4,13].

System (1) is problem (4.9.1) in [1] (see also, [30]) when $F(x; y) = F(x; z)$ for any $y, z \in Y$ and X and V are Euclidean spaces, which is related closely to the extension of the well-known Farkas lemma (see, e.g., [31,32]). The following is an example of system (1).

Example 2.1 Let X, V be Banach spaces, Y be a parameter set, and let V* be the dual space of V. Let $\mathbb{K} \subseteq$ V be closed and convex, and let $\mathbb{H} \subseteq$ V* be convex such that $0 \notin \mathbb{H}$ and $\text{cl } \mathbb{H}$ (with respect to the weak* topology) is a cone. Let $\mathbb{F} : X \times Y \to$ V* and $\mathbb{G} : X \to \mathbb{K}$. Given $y \in$ Y, consider the following parametric variational inequality (for short, VI_y): find $x \in$ X and $h^* \in \mathbb{H}$ such that

$$\langle \mathbb{F}(x, y) - h^*, z - \mathbb{G}(x) \rangle \geq 0, \ \forall z \in \mathbb{K}.$$

Then, it is easy to see that $x \in$ X and $h^* \in \mathbb{H}$ solve VI_y if and only if $-\mathbb{F}(x, y) + h^* \in N_{\mathbb{K}}(\mathbb{G}(x))$, or equivalently, $0 \in \mathbb{F}(x, y) + N_{\mathbb{K}}(\mathbb{G}(x)) - \mathbb{H}$, i.e., $(\mathbb{F}(x, y) + N_{\mathbb{K}}(\mathbb{G}(x))) \cap \mathbb{H} \neq \emptyset$. Set $X := $ X$, Y := $ Y$, V := $ V*$, R := $ X$, \mathcal{H} := \mathbb{H}$ and $F(x, y) := \mathbb{F}(x, y) + N_{\mathbb{K}}(\mathbb{G}(x))$. Then, VI_y is equivalent to system (1).

Without other specifications, we always suppose that U and Z are Hausdorff locally convex topological vector spaces, $C \subseteq U$ is convex such that $0 \notin C$ and $\text{cl } C$ is a cone (since C is convex, so is $\text{cl } C$), $D \subseteq Z$ is a closed and convex cone, $K \subseteq X$ is convex. Suppose that $G : X \times Y \rightrightarrows U$ and $H : X \rightrightarrows Z$ are set-valued mappings. In order to investigate sufficient and necessary conditions and saddle points of Lagrangian functions for system (1), we consider a special case of system (1) (i.e., system (1) with set constraints) as follows:

$$G(x; y) \cap C \neq \emptyset, \ x \in R_0 := \{x \in K : H(x) \cap D \neq \emptyset\}, \tag{2}$$

where $y \in Y$. Setting $V := U \times Z, \mathcal{H} := C \times D, R := K$ and $F(x; y) := (G(x; y), H(x))$ yields the equivalence between systems (2) and (1).

We focus on the impossibility of systems (1) and (2), which consists in finding $\bar{y} \in Y$ such that systems (1) and (2) be impossible, and in finding methods, which

prove the impossibility of such systems. As one of the most important tools, separation techniques in the image space V are useful for investigating the impossibility of systems (1) and (2).

Several problems, such as optimization problems and variational inequalities, can be formulated by means of system (2).

Without other specifications, suppose always that $P \subseteq U$ is a closed, convex, and pointed cone with qri $P \neq \emptyset$, $K \subseteq X$ is convex and suppose that $S : X \rightrightarrows U$ is set-valued mapping. We consider set-valued vector optimization problem (for short, SVOP) as follows (see, e.g., [24,25,33,34]):

$$\min_{\mathcal{C}} S(x), \qquad \text{subject to} \qquad x \in R_0 := \{x \in K : H(x) \cap D \neq \emptyset\},$$

where $\mathcal{C} := P \backslash \{0\}$ or qri P, and $\min_{\mathcal{C}}$ means the minimum with respect to $\mathcal{C}$. We say that a pair (x^*, y^*) with $x^* \in R_0$ and $y^* \in S(x^*)$ is a minimizer of SVOP, if $(y^* - S(x)) \cap (P \backslash \{0\}) = \emptyset$, $\forall x \in R_0$. That is to say y^* is a minimal element of the set $S(R_0) := \cup_{x \in R_0} S(x)$ with respect to $P \backslash \{0\}$ (see, e.g., [28]). Let $x \in X$ and $y \in Y$. Set $Y := S(R_0)$, $C := P \backslash \{0\}$, $G(x; y) := y - S(x)$. Then, (x^*, y^*) with $x^* \in R_0$ and $y^* \in S(x^*)$ is a minimizer of SVOP iff system (2) is impossible with $y^* \in Y$. Similarly, we can define the weak minimizer of SVOP by using $\mathcal{C} = $ qri P.

If S and H collapse to single-valued mappings $f : X \rightarrow U$ and $g : X \rightarrow Z$, respectively, then SVOP reduces to the vector optimization problem (for short, VOP) as follows:

$$\min_{\mathcal{C}} f(x), \qquad \text{subject to} \qquad x \in R_0 := \{x \in K : g(x) \in D\},$$

where $\mathcal{C} := P \backslash \{0\}$ or qri P. We say that $x^* \in R_0$ is an efficient (res., a weakly efficient) solution of VOP, if $f(x) - f(x^*) \notin -(P \backslash \{0\})$ (res., $f(x) - f(x^*) \notin -$qri P), $\forall x \in R_0$.

In the following, we first recall some concepts of set-valued mappings.

Definition 2.2 Let $\mathcal{C} \subseteq U$ be a closed and convex cone and $K \subseteq X$ a convex set. A set-valued mapping $M : X \rightrightarrows U$ is said to be

(i) $\mathcal{C}$-map on K, if $t M(\mathbf{x}) + (1 - t) M(\mathbf{y}) \subseteq M(t\mathbf{x} + (1 - t)\mathbf{y}) + \mathcal{C}$, $\forall \mathbf{x}, \mathbf{y} \in K, t \in$]0, 1[;
(ii) $\mathcal{C}$-convexlike on K, if $M(K) + \mathcal{C}$ is convex.

Remark 2.1 If M is a single-valued mapping, then (i) and (ii) reduce to the classical definitions of $\mathcal{C}$-map and $\mathcal{C}$-convexlike mappings, respectively. We also have the following:

(a) Clearly, (i)$\Rightarrow$(ii); If M is $\mathcal{C}$-map on K, then $M + \mathcal{C}$ is convex-valued on K, i.e., $M(\mathbf{x}) + \mathcal{C}$ is convex for each $\mathbf{x} \in K$; Moreover, $-M$ is $\mathcal{C}$-map on K iff M is $-\mathcal{C}$-map on K (see, e.g., [7,35]);
(b) If H is $-D$-map on K, then the feasible set R_0 of system (2) and SVOP is convex. Let $\mathbf{x}, \mathbf{y} \in R_0, t \in$]0, 1[and set $\mathbf{z} := t\mathbf{x} + (1 - t)\mathbf{y}$. Then, $H(\mathbf{x}) \cap D \neq \emptyset$ and $H(\mathbf{y}) \cap D \neq \emptyset$, i.e., $0 \in H(\mathbf{x}) - D$ and $0 \in H(\mathbf{y}) - D$, and $\mathbf{z} \in K$, since K is convex. Since H is $-D$-map on K, it follows that

$$0 \in t(H(\mathbf{x}) - D) + (1 - t)(H(\mathbf{y}) - D) = tH(\mathbf{x}) + (1 - t)H(\mathbf{y})$$
$$-[tD + (1 - t)D] \subseteq H(t\mathbf{x} + (1 - t)\mathbf{y}) - D - D \subseteq H(\mathbf{z}) - D,$$

or equivalently, $H(\mathbf{z}) \cap D \neq \emptyset$, i.e., $\mathbf{z} \in R_0$. This proves R_0 is convex.

We now define the image of system (1).

Definition 2.3 We call $\mathcal{K}_y := F(R; y)$, $y \in Y$, the image associated with system (1) and call the space V the image space.

Let $\bar{y} \in Y$. Observe that system (1) is impossible with $\bar{y}$ iff the intersection of $\mathcal{H}$ and the image $\mathcal{K}_{\bar{y}}$ is empty, i.e.,

$$\mathcal{K}_{\bar{y}} \cap \mathcal{H} = \emptyset. \tag{3}$$

It is well known that if the image $\mathcal{K}_{\bar{y}}$ is convex, then the classic method can be used to prove (3) by showing that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ lie in two disjoint level sets of a functional; when such functional can be found linear, $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ will be said "linearly separable," or equivalently, we call a hyperplane $\mathbf{H}$ separates $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$, if $\mathcal{K}_{\bar{y}}$ is contained in one of the closed half-spaces associated with $\mathbf{H}$ and $\mathcal{H}$ lies in the opposite closed half-space. We call $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable, if a hyperplane $\mathbf{H}$ separates $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$, and $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are not both actually contained in $\mathbf{H}$ itself. However, the image $\mathcal{K}_{\bar{y}}$ is not convex in general, even if the mapping $F(\cdot; \bar{y})$ is $-(\mathrm{cl}\,\mathcal{H})$-map or $-(\mathrm{cl}\,\mathcal{H})$-convexlike on R. To overcome this difficulty, similar to [3,4], we introduce a regularization of the image $\mathcal{K}_{\bar{y}}$, namely the extension with respect to the closed and convex cone $\mathrm{cl}\,\mathcal{H}$, denoted by $\mathcal{E}_{\bar{y}}$: $\mathcal{E}_{\bar{y}} := \mathcal{K}_{\bar{y}} - \mathrm{cl}\,\mathcal{H}$. The introduction of the extension of the image $\mathcal{K}_{\bar{y}}$ allows us to obtain an equivalent formulation of (3) under mild assumptions.

It is easy to check the following proposition (see, e.g., [1,3,6–8]).

Proposition 2.1 *Let $\bar{y} \in Y$. Then the following statements are true:*

(i) *Assume that*

$$\mathcal{H} + \mathrm{cl}\,\mathcal{H} = \mathcal{H}. \tag{4}$$

Then, system (1) is impossible with $\bar{y}$, i.e., (3) holds, iff

$$\mathcal{E}_{\bar{y}} \cap \mathcal{H} = \emptyset; \tag{5}$$

(ii) *Let $\mathcal{H} := C \times D$ and assume that*

$$C + \mathrm{cl}\,C = C. \tag{6}$$

Then, system (2) is impossible with $\bar{y}$, i.e., (3) holds, or equivalently, (5) holds, iff

$$\mathcal{E}_{\bar{y}} \cap \mathcal{H}_u = \emptyset, \tag{7}$$

where $\mathcal{H}_u = C \times \{0\}$.

Assumptions (4) and (6) play a vital role in proving the equivalence between (3), (5), and (7). The following two examples show that (4) does not hold in Euclidean and Hausdorff locally convex topological vector spaces.

Example 2.2 Let $\mathcal{H} := \mathbb{R}^2_{++} \cup \{(0, 1)\}$. Then, $0 \notin \mathcal{H}$, $\mathcal{H}$ is convex and $\mathrm{cl}\,\mathcal{H} = \mathbb{R}^2_+$. It is easy to see that $\mathcal{H} + \mathrm{cl}\,\mathcal{H} \supseteq \mathcal{H}$. However, $(0, 2) = (0, 1) + (0, 1) \in \mathcal{H} + \mathrm{cl}\,\mathcal{H}$ and $(0, 2) \notin \mathcal{H}$, which yields the converse inclusion does not hold.

Example 2.3 Let

$$l^2 := \{(x_n)_{n \geq 1} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} x_n^2 < +\infty\},$$

$$l^2_+ := \{(x_n)_{n \geq 1} \in l^2 : \text{ each } x_n \in \mathbb{R}_+\},$$

$$l^2_{++} := \{(x_n)_{n \geq 1} \in l^2 : \text{ each } x_n \in \mathbb{R}_{++}\}.$$

Then, l^2 is a Hilbert space, l^2_+ is a closed and convex cone and $\mathrm{cl}\,l^2_{++} = l^2_+$. Let $\mathcal{H} := l^2_{++} \cup \{(1, 0, \ldots)\}$. Then, it is easy to check that $0 \notin \mathcal{H}$, $\mathcal{H}$ is convex, $\mathrm{cl}\,\mathcal{H} = l^2_+$ and as a consequence, $\mathcal{H} + \mathrm{cl}\,\mathcal{H} \supseteq \mathcal{H}$. But, $(1, 1, 0, \ldots) = (1, 0, 0, \ldots) + (0, 1, 0, \ldots) \in \mathcal{H} + \mathrm{cl}\,\mathcal{H}$ and $(1, 1, 0, \ldots) \notin \mathcal{H}$, which implies that the converse inclusion does not hold.

The following propositions provide some conditions such that assumptions (4) and (6) hold.

Proposition 2.2 *Assume that $A \subseteq V$ is convex with $0 \notin A$, $\mathcal{H} := \mathrm{cone}_+ A$ and $\mathcal{H} + \mathrm{cl}\,\mathcal{H} \backslash \mathcal{H} \subseteq \mathcal{H}$. Then, assumption (4) is true.*

Proof Since A is convex, so is $\mathcal{H} = \mathrm{cone}_+ A$ (see, e.g., [3]) and $0 \notin \mathcal{H}$ in view of $0 \notin A$. Then, $\mathrm{cl}\,\mathcal{H} = \mathrm{cl}\,\mathrm{cone}_+ A$ and $\mathrm{cl}\,\mathcal{H}$ is a closed and convex cone. It follows from $0 \in \mathrm{cl}\,\mathcal{H}$ that $\mathcal{H} \subseteq \mathcal{H} + \mathrm{cl}\,\mathcal{H}$. We declare that $\mathcal{H} + \mathcal{H} \subseteq \mathcal{H}$. Let $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Then, there are $a, b \in A$ and $s > 0, t > 0$ such that $\mathbf{x} = sa$ and $\mathbf{y} = tb$. Since A is convex and $\mathcal{H} = \mathrm{cone}_+ A$, it follows that

$$\mathbf{x} + \mathbf{y} = sa + tb = (s + t)(\frac{s}{s + t}a + \frac{t}{s + t}b) \in (s + t)A \subseteq \mathcal{H},$$

which yields $\mathcal{H} + \mathcal{H} \subseteq \mathcal{H}$. Now, from the assumption $\mathcal{H} + \mathrm{cl}\,\mathcal{H} \backslash \mathcal{H} \subseteq \mathcal{H}$, we have $\mathcal{H} + \mathrm{cl}\,\mathcal{H} \subseteq \mathcal{H}$. This proves assumption (4) holds.   □

Proposition 2.3 *Let $\mathcal{H} := C \times D$ and assume that $C = \mathrm{cone}_+ C$ and $C + \mathrm{cl}\,C \backslash C \subseteq C$. Then, assumption (4) is true. Especially, if any of the following statements holds:*

(i) $C := B \backslash \{0\}$;
(ii) $C := \mathrm{qri}\, B \neq \emptyset$,

where $B \subseteq U$ is a closed and convex cone, then assumptions (4) and (6) are true.

Proof Note that $0 \notin C$, $\mathrm{cl}\, C$ and D are closed and convex cones. Then, from the assumption $C = \mathrm{cone}_+\, C$ we have $\mathcal{H} = C \times D = \mathrm{cone}_+\, (C \times D) = \mathrm{cone}_+\, \mathcal{H}$. Since $C + \mathrm{cl}\, C \backslash C \subseteq C$,

$$\mathcal{H} + \mathrm{cl}\, \mathcal{H} \backslash \mathcal{H} = C \times D + \mathrm{cl}\, (C \times D) \backslash (C \times D) = C \times D + (\mathrm{cl}\, C \times D) \backslash (C \times D)$$
$$= C \times D + (\mathrm{cl}\, C \backslash C) \times D \subseteq C \times D = \mathcal{H}.$$

Now, Proposition 2.2 implies assumption (4) holds.

(i) If $C := B \backslash \{0\}$, then $\mathrm{cone}_+\, C = \mathrm{cone}_+\, (B \backslash \{0\}) = (\mathrm{cone}_+\, B) \backslash \{0\} = B \backslash \{0\} = C$ and $\mathrm{cl}\, C = B$, since B is a closed and convex cone. Therefore, $C + \mathrm{cl}\, C \backslash C = B \backslash \{0\} + \{0\} = B \backslash \{0\} = C$. As a consequence, previous conclusions yield that assumptions (4) and (6) hold.

(ii) If $C := \mathrm{qri}\, B$, then from [3,6,16,19,21,22], one has C is convex, $\mathrm{cl}\, C = \mathrm{cl}\, (\mathrm{qri}\, B) = B$, $C = \mathrm{cone}_+\, C$ and $C + \mathrm{cl}\, C \backslash C \subseteq C + \mathrm{cl}\, C = C$. Therefore, assumptions (4) and (6) follow immediately from previous conclusions. □

The extension of the image $\mathcal{K}_{\bar{y}}$ is convex under suitable convex assumptions.

Proposition 2.4 *Let $\bar{y} \in Y$. Then, the following statements are true:*

(i) *$\mathcal{E}_{\bar{y}}$ is convex iff $F(\cdot; \bar{y})$ is $-(\mathrm{cl}\, \mathcal{H})$-convexlike on R;*
(ii) *If $G(\cdot; \bar{y})$ and H are $-(\mathrm{cl}\, C)$-map and $-D$-map on K, respectively, then the regularization of the image, i.e., $\mathcal{E}_{\bar{y}} := (G(\cdot; \bar{y}), H)(K) - (\mathrm{cl}\, C \times D)$, is convex.*

Proof Statement (i) is self-evident, and it is easy to check statement (ii) holds. □

Corollary 2.1 *For SVOP, if S and H are P-map and $-D$-map on K, respectively, then the regularization of the image, i.e., $\mathcal{E}_{\bar{y}} := (\bar{y} - S, H)(K) - (P \times D)$, is convex.*

Proof From [16,19,21,22], one has $\mathrm{cl}\, C = \mathrm{cl}\, (\mathrm{qri}\, P) = P$. Moreover, if S is P-map on K, then one has $G(\cdot; \bar{y}) := \bar{y} - S$, which is $-P$-map on K. Thus the conclusion follows immediately from Proposition 2.4 (ii). □

3 Linear Separation for Systems (1) and (2)

In this section, we shall investigate the (proper, regular) linear separation of $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ by using the quasi-relative interior and the quasi-interior. Some new results of the (proper, regular) linear separation of $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are given.

It is easy to prove the following proposition.

Proposition 3.1 *The following statements are true:*

(i) *$\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable iff there exists $\lambda^* \in \mathcal{H}^* \backslash \{0\}$ such that $\langle \lambda^*, e \rangle \leq 0$, $\forall e \in \mathcal{K}_{\bar{y}}$, or equivalently, $\sup_{x \in R} \sigma_{F(x, \bar{y})}(\lambda^*) \leq 0$;*
(ii) *$\mathcal{E}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable iff there exists $\lambda^* \in \mathcal{H}^* \backslash \{0\}$ such that $\langle \lambda^*, e \rangle \leq 0$, $\forall e \in \mathcal{E}_{\bar{y}}$.*

Similar to Propositions 4.2 and 4.3 in [3], we have Propositions 3.2 and 3.3. The proofs are omitted.

Proposition 3.2 *Assume that* (4) *holds. Then, the following statements are equivalent:*

(i) $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are linearly separable;*
(ii) $\mathcal{E}_{\bar{y}}$ *and* $\mathcal{H}$ *are linearly separable;*
(iii) $N_{\text{cone conv } \mathcal{E}_{\bar{y}}}(0) \neq \{0\}$, *i.e.,* $0 \notin \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}})$.

Proposition 3.3 *Assume that* (4) *holds. Then, the following statements are equivalent:*

(i) $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable;*
(ii) $\mathcal{E}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable;*
(iii) $N_{\text{cone conv } \mathcal{E}_{\bar{y}}}(0)$ *is not a linear subspace of* V^*, *i.e.,* $0 \notin \text{qri}(\text{cone conv } \mathcal{E}_{\bar{y}})$.

We first prove the following lemma, which is useful in characterizing the (proper) linear separation of $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$.

Lemma 3.1 *Assume that* (4) *holds, and assume that* $0 \in \text{conv } \mathcal{E}_{\bar{y}}$ *and* $\text{qi } \mathcal{H} \neq \emptyset$. *Then, the following equalities hold:* $\text{qri}(\text{cone conv } \mathcal{E}_{\bar{y}}) = \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) = \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) - \text{cl } \mathcal{H}$.

Proof Since $\text{cl } \mathcal{H}$ is a closed and convex cone, we have $\text{cl } \mathcal{H} = \text{cl } \mathcal{H} + \text{cl } \mathcal{H}$ and so from (4) we have $\mathcal{E}_{\bar{y}} = \mathcal{K}_{\bar{y}} - \text{cl } \mathcal{H} = \mathcal{K}_{\bar{y}} - (\text{cl } \mathcal{H} + \text{cl } \mathcal{H}) = \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H}$. Since $0 \in \text{conv } \mathcal{E}_{\bar{y}}$, from [29], one has

$$\text{cone conv } \mathcal{E}_{\bar{y}} = \text{cone conv}(\mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H}) = \text{cone}(\text{conv } \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H}) = \text{cone conv } \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H}. \quad (8)$$

Since $\text{qi } \mathcal{H} \neq \emptyset$, $\text{qri } \mathcal{H} = \text{qi } \mathcal{H}$ and it follows from [3,6] and (8) that

$$\text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) = \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H}) \supseteq \text{cone conv } \mathcal{E}_{\bar{y}} - \text{qi}(\text{cl } \mathcal{H})$$
$$\supseteq \text{cone conv } \mathcal{E}_{\bar{y}} - \text{qi } \mathcal{H} \neq \emptyset.$$

As a consequence, $\text{qri}(\text{cone conv } \mathcal{E}_{\bar{y}}) = \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) = \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H})$. We declare that the equality holds: $\text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H}) = \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) - \text{cl } \mathcal{H}$. In fact, again from [3] we have $\text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H}) \supseteq \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) - \text{cl } \mathcal{H}$. Since $\text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) = \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H})$ and $0 \in \text{cl } \mathcal{H}$, it follows that $\text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}} - \text{cl } \mathcal{H}) \subseteq \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) - \text{cl } \mathcal{H}$. Consequently, $\text{qri}(\text{cone conv } \mathcal{E}_{\bar{y}}) = \text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) - \text{cl } \mathcal{H}$. This completes the proof. $\quad \square$

We next give some new characterizations of the (proper) linear separation of $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$.

Proposition 3.4 *Assume that* (4) *holds. Then, the following statements are true:*

(i) *Assume that* $0 \in \text{conv } \mathcal{E}_{\bar{y}}$ *and* $\text{qi } \mathcal{H} \neq \emptyset$. *Then,* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are linearly separable iff* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable iff* $\text{qi}(\text{cone conv } \mathcal{E}_{\bar{y}}) \cap (\text{cl } \mathcal{H}) = \emptyset$;
(ii) *Assume that* $\text{int } \mathcal{H} \neq \emptyset$. *Then,* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are linearly separable iff* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable iff* $\text{conv } \mathcal{E}_{\bar{y}} \cap \text{int } \mathcal{H} = \emptyset$, *which is equivalent to* $\text{int}(\text{cone conv } \mathcal{E}_{\bar{y}}) \cap (\text{cl } \mathcal{H}) = \emptyset$ *if the condition* $0 \in \text{conv } \mathcal{E}_{\bar{y}}$ *holds;*
(iii) *Assume that* $V := \mathbb{R}^m$. *Then,* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable iff* $\text{ri}(\text{conv } \mathcal{E}_{\bar{y}}) \cap \text{ri } \mathcal{H} = \emptyset$.

Proof (i) The conclusion follows immediately from Lemma 3.1, Propositions 3.2 and 3.3.

(ii) Suppose that int $\mathcal{H} \neq \emptyset$. By the standard separation theorem (see, e.g., [27,29]), we have conv $\mathcal{E}_{\bar{y}} \cap$ int $\mathcal{H} = \emptyset$ iff conv $\mathcal{E}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable. We declare that this is equivalent to the fact that $\mathcal{E}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable. In fact, the equivalence follows immediately from the following inequalities

$$\sup_{e \in \text{conv}\, \mathcal{E}_{\bar{y}}} \langle \lambda^*, e \rangle = \sup_{e \in \mathcal{E}_{\bar{y}}} \langle \lambda^*, e \rangle \leq \inf_{h \in \mathcal{H}} \langle \lambda^*, h \rangle = \inf_{h \in \text{cl}\, \mathcal{H}} \langle \lambda^*, h \rangle = 0, \qquad (9)$$

where $\lambda^* \in \mathcal{H}^* \setminus \{0\}$. This proves that conv $\mathcal{E}_{\bar{y}} \cap$ int $\mathcal{H} = \emptyset$ iff $\mathcal{E}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable, or equivalently, $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable in view of Proposition 3.2.

Since $0 \notin \mathcal{H}$ and cl $\mathcal{H}$ is a closed and convex cone, we have the following relations hold:

$$\inf_{e \in \mathcal{K}_{\bar{y}}} \langle \lambda^*, e \rangle \leq \sup_{e \in \mathcal{K}_{\bar{y}}} \langle \lambda^*, e \rangle \leq \inf_{h \in \mathcal{H}} \langle \lambda^*, h \rangle = \inf_{h \in \text{cl}\, \mathcal{H}} \langle \lambda^*, h \rangle = 0 < \sup_{h \in \mathcal{H}} \langle \lambda^*, h \rangle$$
$$= \sup_{h \in \text{cl}\, \mathcal{H}} \langle \lambda^*, h \rangle = +\infty.$$

This yields that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable iff $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable.

From [36], one has int (cl $\mathcal{H}$) $=$ int $\mathcal{H} \neq \emptyset$. Moreover, if $0 \in$ conv $\mathcal{E}_{\bar{y}}$, then from [37] and (8) we have

$$\text{int}(\text{cone conv}\, \mathcal{E}_{\bar{y}}) = \text{int}\,(\text{cone conv}\, \mathcal{E}_{\bar{y}} - \text{cl}\, \mathcal{H}) = \text{cone conv}\, \mathcal{E}_{\bar{y}} - \text{int}\,(\text{cl}\, \mathcal{H}) \neq \emptyset.$$

Therefore qi (cone conv $\mathcal{E}_{\bar{y}}$) $=$ int (cone conv $\mathcal{E}_{\bar{y}}$), and the conclusion follows immediately from (i).

(iii) Note that $\mathcal{H}$ is convex and cl $\mathcal{H}$ is a closed and convex cone. By the standard separation theorem (see, e.g., [38]), ri (conv $\mathcal{E}_{\bar{y}}$) $\cap$ ri $\mathcal{H} = \emptyset$ iff conv $\mathcal{E}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable. We declare that this is equivalent to the fact that $\mathcal{E}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable. In fact, the equivalence follows immediately from (9) and the following relations:

$$\inf_{e \in \text{conv}\, \mathcal{E}_{\bar{y}}} \langle \lambda^*, e \rangle = \inf_{e \in \mathcal{E}_{\bar{y}}} \langle \lambda^*, e \rangle \leq 0 < \sup_{h \in \mathcal{H}} \langle \lambda^*, h \rangle = \sup_{h \in \text{cl}\, \mathcal{H}} \langle \lambda^*, h \rangle = +\infty.$$

From Proposition 3.3, $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable iff $\mathcal{E}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable. □

Remark 3.1 If int $\mathcal{H} \neq \emptyset$ and either $0 \in$ conv $\mathcal{E}_{\bar{y}}$ or cl $\mathcal{H}$ is proper, then Proposition 3.4 (ii) can be proved by using the characterizations of the (proper) linear separation of $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ presented in Propositions 3.2 and 3.3. In fact, from the proof of Proposition 3.4 (ii), one has the following relations hold: qri (cone conv $\mathcal{E}_{\bar{y}}$) $=$ qi (cone conv $\mathcal{E}_{\bar{y}}$) $=$ cone conv $\mathcal{E}_{\bar{y}}$ $-$ int (cl $\mathcal{H}$) $\neq \emptyset$. Therefore, from Propositions 3.2 and 3.3, it follows that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable iff $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable iff

$0 \notin$ cone conv $\mathcal{E}_{\bar{y}} -$ int (cl $\mathcal{H}$), i.e., cone conv $\mathcal{E}_{\bar{y}} \cap$ int (cl $\mathcal{H}$) $= \emptyset$. We declare that this is equivalent to conv $\mathcal{E}_{\bar{y}} \cap$ int (cl $\mathcal{H}$) $= \emptyset$, or equivalently, conv $\mathcal{E}_{\bar{y}} \cap$ int $\mathcal{H} = \emptyset$. It suffices to prove that conv $\mathcal{E}_{\bar{y}} \cap$ int (cl $\mathcal{H}$) $= \emptyset$ implies cone conv $\mathcal{E}_{\bar{y}} \cap$ int (cl $\mathcal{H}$) $= \emptyset$. Suppose conv $\mathcal{E}_{\bar{y}} \cap$ int (cl $\mathcal{H}$) $= \emptyset$ and suppose to the contrary that cone conv $\mathcal{E}_{\bar{y}} \cap$ int (cl $\mathcal{H}$) $\neq \emptyset$. Then, there are $e \in$ conv $\mathcal{E}_{\bar{y}}$ and $t \geq 0$ such that $te \in$ int(cl$\mathcal{H}$). Note that cl $\mathcal{H}$ is a closed and convex cone. If $0 \in$ conv $\mathcal{E}_{\bar{y}}$, then from the assumption that conv $\mathcal{E}_{\bar{y}} \cap$ int (cl $\mathcal{H}$) $= \emptyset$ we have $0 \notin$ int (cl $\mathcal{H}$). If cl $\mathcal{H}$ is proper, then $0 \notin$ int (cl $\mathcal{H}$). As a consequence, $t > 0$ and so $e \in \frac{1}{t}$int (cl $\mathcal{H}$) $=$ int $[\frac{1}{t}$(cl $\mathcal{H}$)$] =$ int (cl $\mathcal{H}$), which is a contradiction.

Remark 3.2 Note that cl cone (cone conv$\mathcal{E}_{\bar{y}} - 0$) $=$ cl cone (conv $\mathcal{E}_{\bar{y}} - 0$). If $0 \in$ conv $\mathcal{E}_{\bar{y}}$, then it easy to check that the conclusions in Propositions 3.2-3.4 and Lemma 3.1 also hold by replacing cone conv $\mathcal{E}_{\bar{y}}$ by conv $\mathcal{E}_{\bar{y}}$.

Classically, one would appeal to Slater condition to assert the existence of Lagrange multipliers. Unfortunately, in constrained optimization problems or variational inequalities, which use the positive cone $\mathcal{P} := \{I \in L^p([0, T], \mathbb{R}) : I(t) \geq 0, \text{ a. e. on } [0, T]\}$ in $L^p([0, T], \mathbb{R})$, where $p \geq 1$ and $T > 0$, the interior of the positive cone $\mathcal{P}$ is empty and the classical Slater condition fails to hold. However, it is easy to check that qri$\mathcal{P} =$ qi$\mathcal{P} = \{I \in L^p([0, T], \mathbb{R}) : I(t) > 0, \text{ a. e. on } [0, T]\}$ (see, e.g., [16–20]). A natural alternative to the classical Slater condition would be that there is a feasible function in the quasi-interior or the quasi-relative interior of the positive cone.

Recall that the classical Slater condition related to system (2) is:

$$\exists \bar{x} \in K : H(\bar{x}) \cap \text{int } D \neq \emptyset. \qquad (10)$$

Suppose that qi $D \neq \emptyset$. Consider the following Slater condition related to system (2):

$$\exists \bar{x} \in K : H(\bar{x}) \cap \text{qi } D \neq \emptyset, \qquad (11)$$

or equivalently, $0 \in H(K) -$ qi D. Note that since $R_0 \neq \emptyset$, $0 \in H(K) - D \subseteq$ conv $(H(K) - D)$. The generalized Slater condition (see, e.g., [3,39]) related to system (2) is:

$$0 \in \text{qi } (H(K) - D), \qquad (12)$$

or equivalently, cl cone $[(H(K) - D) - 0] = Z$. We consider the following generalized Slater condition (see, e.g., [3,20,23]) related to system (2):

$$0 \in \text{qi} [\text{conv } (H(K) - D)] = \text{qi} [\text{conv } H(K) - D], \qquad (13)$$

or equivalently, cl cone $[$conv $(H(K) - D) - 0] = Z$. The equality in (13) follows from [29]. Since $Z =$ cl cone $[($conv $(H(K) - D)) - 0] =$ cl cone $[$cl cone (conv $(H(K) - D)) - 0]$, this is equivalent to $0 \in$ qi $[$cl cone (conv $(H(K) - D))]$. From [3,6], we have the following relations: (10)$\Rightarrow$ (11) $\Rightarrow$ (12) $\Rightarrow$ (13). But the converse is not true in general.

We now investigate the (proper, regular) linear separation of system (2).

Proposition 3.5 *Let* $V := U \times Z$, $\mathcal{H} := C \times D$, $R := K$ *and* $F(x; \bar{y}) := (G(x; \bar{y}), H(x))$. *Assume that* $\text{cone}_+ C = C$, $C + \text{cl}\, C \backslash C \subseteq C$, $\text{qi}\, C \neq \emptyset$, $\text{qi}\, D \neq \emptyset$ *and* $0 \in \text{conv}\, \mathcal{E}_{\bar{y}}$. *Consider the following statements:*

(i) $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are linearly separable;*
(ii) $\mathcal{E}_{\bar{y}}$ *and* $\mathcal{H}$ *are linearly separable;*
(iii) $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable;*
(iv) $\mathcal{E}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable;*
(v) $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *admit a regular linear separation, i.e., there exists* $(\lambda^*, \theta^*) \in C^* \times D^*$ *with* $\lambda^* \neq 0$ *such that*

$$\langle \lambda^*, u \rangle + \langle \theta^*, v \rangle \leq 0, \ \forall (u, v) \in \mathcal{K}_{\bar{y}}; \tag{14}$$

(vi) $\mathcal{E}_{\bar{y}}$ *and* $\mathcal{H}$ *admit a regular linear separation, i.e., there exists* $(\lambda^*, \theta^*) \in C^* \times D^*$ *with* $\lambda^* \neq 0$ *such that*

$$\langle \lambda^*, u \rangle + \langle \theta^*, v \rangle \leq 0, \ \forall (u, v) \in \mathcal{E}_{\bar{y}}. \tag{15}$$

Then, (i)⇔(ii)⇔(iii)⇔(iv)⇐(v)⇔(vi). *Furthermore, if the generalized Slater condition* (13) *holds, then statements* (i)-(vi) *are equivalent.*

Proof Since $\text{cone}_+ C = C$, $C + \text{cl}\, C \backslash C \subseteq C$ and D is a closed and convex cone, from Proposition 2.3 one has assumption (4) is true. Since $\text{qi}\, C \neq \emptyset$ and $\text{qi}\, D \neq \emptyset$, from [16,19,21,22] we have $\text{qi}\, \mathcal{H} = \text{qri}\, \mathcal{H} = \text{qri}\, C \times \text{qri}\, D = \text{qi}\, C \times \text{qi}\, D \neq \emptyset$. Now Propositions 3.2, 3.3 and 3.4 (i) yield that statements (i)–(iv) are equivalent. Clearly, (vi)⇒(v)⇒(iii).

We prove (v)⇒(vi). Note that $\mathcal{E}_{\bar{y}} := \mathcal{K}_{\bar{y}} - \text{cl}\, \mathcal{H} = \mathcal{K}_{\bar{y}} - \text{cl}\,(C \times D) = \mathcal{K}_{\bar{y}} - (\text{cl}\, C) \times D$. Let $(u, v) \in \mathcal{E}_{\bar{y}}$. Then, there are $(u_0, v_0) \in \mathcal{K}_{\bar{y}}$ and $(c_0, d_0) \in (\text{cl}\, C) \times D$ such that $(u, v) = (u_0 - c_0, v_0 - d_0)$. From (v), it follows that $\langle \lambda^*, u \rangle + \langle \theta^*, v \rangle = \langle \lambda^*, u_0 \rangle + \langle \theta^*, v_0 \rangle - [\langle \lambda^*, c_0 \rangle + \langle \theta^*, d_0 \rangle] \leq 0$, which proves (vi).

We show (i)-(vi) are equivalent. Suppose that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable, i.e., there exists $(\lambda^*, \theta^*) \in C^* \times D^* \backslash \{(0, 0)\}$ such that (14) holds. Suppose to the contrary that $\lambda^* = 0$. Then, $\theta^* \neq 0$ and so from (14), one has $\langle \theta^*, v \rangle \leq 0$, $\forall v \in H(K)$. Since the generalized Slater condition (13) holds and $\theta^* \in D^*$, it follows that $\langle \theta^*, v \rangle \leq 0$, $\forall v \in \text{cl cone}\,[\text{conv}\,(H(K) - D)] = Z$ and therefore, $\theta^* = 0$, a contradiction. $\square$

The following proposition provides an equivalent characterization of regular linear separation for system (2) under some mild assumptions, which extends that for constrained optimization problems in finite-dimensional spaces given by Giannessi and Mastroeni [40]. A characterization of faces was employed in the proof of Theorem 3.5 in [40]. But here, we give a direct proof.

Proposition 3.6 *Let* $V := U \times Z$, $\mathcal{H} := C \times D$, $R := K$ *and* $F(x; \bar{y}) := (G(x; \bar{y}), H(x))$. *Assume that assumption* (4) *is true. Consider the following statements:*

(i) $\text{cl cone conv}\, \mathcal{E}_{\bar{y}} \cap \mathcal{H}_u = \emptyset$;
(ii) $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *admit a regular linear separation.*

If the generalized Slater condition (13) *holds, then* (i)⇒(ii). *If* $C := \text{qi } B$, *where* $B \subseteq U$ *is a closed and convex cone with* $\text{qi } B \neq \emptyset$, *then* (ii)⇒(i). *If* $C := B\backslash\{0\}$ *and there is* $(\lambda^*, \theta^*) \in \text{qi}(C^*) \times D^*$ *such that* (14) *holds, i.e.,* (ii) *is true, then* (i) *holds.*

Proof (i)⇒(ii). From (i), we have $\text{cl cone (cone conv } \mathcal{E}_{\bar{y}} - 0) = \text{cl cone conv } \mathcal{E}_{\bar{y}} \neq V$, or equivalently, $0 \notin \text{qi (cone conv } \mathcal{E}_{\bar{y}})$. Applying Proposition 3.2 yields that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable. Since the generalized Slater condition (13) holds, from Proposition 3.5 we have $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ admit a regular linear separation.

(ii)⇒(i). Assume that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ admit a regular linear separation, from Propositions 3.2 and 3.5 there exists $(\lambda^*, \theta^*) \in C^* \times D^*$ with $\lambda^* \neq 0$ such that (15) holds and so

$$\langle \lambda^*, u \rangle + \langle \theta^*, v \rangle \leq 0, \quad \forall (u, v) \in \text{conv } \mathcal{E}_{\bar{y}}. \tag{16}$$

Suppose to the contrary that (i) is false, or equivalently, $\text{cl cone conv } \mathcal{E}_{\bar{y}} \cap \mathcal{H}_u \neq \emptyset$. Then, there is $(u_0, 0) \in \text{cl cone conv } \mathcal{E}_{\bar{y}}$ with $u_0 \in C$, which implies that there is a net $\{(u_l, v_l)\}_{l \in I} \subseteq \text{conv } \mathcal{E}_{\bar{y}}$ and $t_l \in \mathbb{R}_+ (l \in I)$ such that $\lim_{l \in I} t_l(u_l, v_l) \rightarrow (u_0, 0)$. Now, (16) implies $\langle \lambda^*, t_l u_l \rangle + \langle \theta^*, t_l v_l \rangle \leq 0$, $\forall l \in I$, and as a consequence, $\langle \lambda^*, u_0 \rangle \leq 0$. If $C = \text{qi } B$, then from [3,6] one has $\lambda^* \in C^*\backslash\{0\} = B^*\backslash\{0\}$. Since $u_0 \in C = \text{qi} B$, it follows from [3,16] that $\langle \lambda^*, u_0 \rangle > 0$, a contradiction. If $C = B\backslash\{0\}$ and there is $(\lambda^*, \theta^*) \in \text{qi}(C^*) \times D^*$ such that (ii) is true, then $u_0 \in C = B\backslash\{0\}, C^* = B^*$ and (16) holds for $(\lambda^*, \theta^*) \in \text{qi}(C^*) \times D^*$. Since B is a closed and convex cone in Hausdorff locally convex topological vector space U, from [41] one has $B = B^{**}$ and it also follows from [3,16,19,23] that $\langle \lambda^*, u_0 \rangle > 0$, a contradiction. □

Remark 3.3 In [6], the sets $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ with $C := B\backslash\{0\}$ are said to admit a strongly regular linear separation, if there exists a point $(\lambda^*, \theta^*) \in \text{qi}(C^*) \times D^*$ such that (14) holds. Moreover, we do not require $0 \in \text{conv } \mathcal{E}_{\bar{y}}$ in Proposition 3.6. In Theorem 3.5 of [40], Giannessi and Mastroeni proved the equivalence (i)⇔(ii) for a constrained optimization problem in finite-dimensional spaces in the case that $\bar{y} \in R_0$, $X := Y := \mathbb{R}^n, Z := \mathbb{R}^m, U := \mathbb{R}, C := \mathbb{R}_{++}, D := \mathbb{R}_+^m, G(x; y) := f(y) - f(x)$, $H(x) := g(x), f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow Z$. For related works, please see [7].

4 Conditions for Impossibility of Systems (1) and (2)

In this section, we present some sufficient and necessary conditions for the impossibility of systems (1) and (2). Some results related to necessary conditions for the impossibility of system (1) are new. We first consider the case where V is a finite-dimensional space. This can easily be proved by the standard separation theorem (see, e.g., [38]). But we prove it by using Proposition 3.4 (iii).

Theorem 4.1 *Let* $V := \mathbb{R}^m$. *Assume that* $\text{cone}_+ \mathcal{H} = \mathcal{H}, \mathcal{H} + \text{cl } \mathcal{H}\backslash\mathcal{H} \subseteq \mathcal{H}$ *and assume* $F(\cdot; \bar{y})$ *is* $-(\text{cl } \mathcal{H})$-convexlike on R. *If system* (1) *is impossible with* $\bar{y}$, *then* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable.*

Proof Since $\mathcal{H}$ is convex with $0 \notin \mathcal{H}$, $\text{cone}_+ \mathcal{H} = \mathcal{H}$ and $\mathcal{H} + \text{cl } \mathcal{H}\backslash\mathcal{H} \subseteq \mathcal{H}$, it follows from Proposition 2.2 that assumption (4) is true. Since $F(\cdot; \bar{y})$ is $-(\text{cl } \mathcal{H})$-convexlike on R, Proposition 2.4 (i) yields that $\mathcal{E}_{\bar{y}}$ is convex, i.e., $\text{conv } \mathcal{E}_{\bar{y}} = \mathcal{E}_{\bar{y}}$. If

system (1) is impossible with $\bar{y}$, then from Proposition 2.1 (i) we have $\mathcal{E}_{\bar{y}} \cap \mathcal{H} = \emptyset$. As a consequence, ri (conv $\mathcal{E}_{\bar{y}}) \cap$ ri $\mathcal{H} = \emptyset$ and so from Proposition 3.4 (iii), one has $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable. □

In the case that V is an infinite-dimensional space, we also present the following necessary condition for the impossibility of system (1) under the assumption that the interior of $\mathcal{H}$ is nonempty. This can easily be proved by using the standard separation theorem (see, e.g., [27,29]). But we prove it by using Proposition 3.4 (ii).

Theorem 4.2 *Assume that* $\text{cone}_+ \mathcal{H} = \mathcal{H}$, $\mathcal{H} + \text{cl} \mathcal{H} \backslash \mathcal{H} \subseteq \mathcal{H}$ *and* $F(\cdot; \bar{y})$ *is* $-(\text{cl}\,\mathcal{H})$-*convexlike on R and* int $\mathcal{H} \neq \emptyset$. *If system* (1) *is impossible with* $\bar{y}$, *then* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable.*

Proof Since $\mathcal{H}$ is convex with $0 \notin \mathcal{H}$, $\text{cone}_+ \mathcal{H} = \mathcal{H}$ and $\mathcal{H} + \text{cl}\,\mathcal{H} \backslash \mathcal{H} \subseteq \mathcal{H}$, from Proposition 2.2 we have assumption (4) holds. Since $F(\cdot; \bar{y})$ is $-(\text{cl}\,\mathcal{H})$-convexlike on R, from Proposition 2.4 (i), one has $\mathcal{E}_{\bar{y}}$ is convex, i.e., conv $\mathcal{E}_{\bar{y}} = \mathcal{E}_{\bar{y}}$. If system (1) is impossible with $\bar{y}$, then from Proposition 2.1 (i) we have $\mathcal{E}_{\bar{y}} \cap \mathcal{H} = \emptyset$ and so $\mathcal{E}_{\bar{y}} \cap$ int $\mathcal{H} = \emptyset$. Now, the conclusion follows immediately from Proposition 3.4 (ii). □

In Theorem 4.2, the interior of $\mathcal{H}$ is supposed to be nonempty. Under the emptiness of the interior of $\mathcal{H}$, we next present the necessary condition for the impossibility of system (1) by using the quasi-relative interior.

Theorem 4.3 *Assume that* $\text{cone}_+ \mathcal{H} = \mathcal{H}$, $\mathcal{H} + \text{cl}\,\mathcal{H} \backslash \mathcal{H} \subseteq \mathcal{H}$, $F(\cdot; \bar{y})$ *is* $-(\text{cl}\,\mathcal{H})$-*convexlike on R and* qi (cone conv $\mathcal{E}_{\bar{y}}) \subseteq$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$ *(res.,* qri (cone conv $\mathcal{E}_{\bar{y}}) \subseteq$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$). *If system* (1) *is impossible with* $\bar{y}$, *then* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are (res., properly) linearly separable.*

Proof Since $\mathcal{H}$ is convex with $0 \notin \mathcal{H}$, $\text{cone}_+ \mathcal{H} = \mathcal{H}$ and $\mathcal{H} + \text{cl}\,\mathcal{H} \backslash \mathcal{H} \subseteq \mathcal{H}$, it follows from Proposition 2.2 that assumption (4) is true. Since $F(\cdot; \bar{y})$ is $-(\text{cl}\,\mathcal{H})$-convexlike on R, Proposition 2.4 (i) implies that $\mathcal{E}_{\bar{y}}$ is convex, i.e., conv $\mathcal{E}_{\bar{y}} = \mathcal{E}_{\bar{y}}$. If system (1) is impossible with $\bar{y}$, then from Proposition 2.1 (i) we have $\mathcal{E}_{\bar{y}} \cap \mathcal{H} = \emptyset$. We declare that this is equivalent to cone $\mathcal{E}_{\bar{y}} \cap \mathcal{H} = \emptyset$, or equivalently, the relations hold: $0 \notin \text{cone}\,\mathcal{E}_{\bar{y}} - \mathcal{H} = \text{cone conv}\,\mathcal{E}_{\bar{y}} - \mathcal{H}$. In fact, it suffices to prove that $\mathcal{E}_{\bar{y}} \cap \mathcal{H} = \emptyset$ implies that cone $\mathcal{E}_{\bar{y}} \cap \mathcal{H} = \emptyset$. Suppose to the contrary that cone $\mathcal{E}_{\bar{y}} \cap \mathcal{H} \neq \emptyset$. Then, letting $h \in \text{cone}\,\mathcal{E}_{\bar{y}} \cap \mathcal{H}$ yields that there are $t \geq 0$ and $e \in \mathcal{E}_{\bar{y}}$ such that $h = te \in \mathcal{H}$. Since $0 \notin \mathcal{H}, t > 0$ and it follows that $e = \frac{1}{t} h \in \frac{1}{t} \mathcal{H} \subseteq \mathcal{H}$, since $\text{cone}_+ \mathcal{H} = \mathcal{H}$. This is a contradiction. Thus from the assumption qi (cone conv$\mathcal{E}_{\bar{y}}) \subseteq$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$ (res., qri (cone conv $\mathcal{E}_{\bar{y}}) \subseteq$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$), it follows from the fact $0 \notin$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$ that $0 \notin$ qi (cone conv $\mathcal{E}_{\bar{y}}$) (res., $0 \notin$ qri (cone conv $\mathcal{E}_{\bar{y}}$)). Now Proposition 3.2 (res., Proposition 3.3) yields that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are (res., properly) linearly separable. □

The assumption qi (cone conv $\mathcal{E}_{\bar{y}}) \subseteq$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$ (res., qri (cone conv $\mathcal{E}_{\bar{y}}) \subseteq$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$) in Theorem 4.3 plays a vital role in obtaining the necessary condition for the impossibility of system (1).

Remark 4.1 We have the following:

(a) If qi (cone conv $\mathcal{E}_{\bar{y}}) = \emptyset$ (res., qri (cone conv $\mathcal{E}_{\bar{y}}) = \emptyset$), then qi (cone conv $\mathcal{E}_{\bar{y}}) \subseteq$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$ (res., qri (cone conv $\mathcal{E}_{\bar{y}}) \subseteq$ cone conv $\mathcal{E}_{\bar{y}} - \mathcal{H}$) holds trivially;

(b) If $0 \in \text{conv}\,\mathcal{E}_{\bar{y}}$ and $\text{qi}\,\mathcal{H} \neq \emptyset$, then from the proof of Lemma 3.1 we have the relations hold: $\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \text{qi}\,\mathcal{H} \subseteq \text{qri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = \text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = \text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) - \text{cl}\,\mathcal{H}$. Then, $\text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) \subseteq \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \mathcal{H}$ (res., $\text{qri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) \subseteq \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \mathcal{H}$) can be replaced by the following:

$$\text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) - \text{cl}\,\mathcal{H}\backslash\mathcal{H} \subseteq \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}$$
$$-\mathcal{H}(\text{res.}, \text{qri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) - \text{cl}\,\mathcal{H}\backslash\mathcal{H} \subseteq \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \mathcal{H});$$

(c) If $0 \in \text{conv}\,\mathcal{E}_{\bar{y}}$ and $\text{int}\,\mathcal{H} \neq \emptyset$, then from the proof of Proposition 3.4 (ii) we have the relations: $\text{qri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = \text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \text{int}\,\mathcal{H}$ and as a consequence, $\text{qri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = \text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) \subseteq \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \mathcal{H}$ holds trivially;

(d) Assume that $V := \mathbb{R}^m$. Then $\text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = \text{int}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}})$ and so from (c) it follows that $\text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) \subseteq \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \mathcal{H}$ if $0 \in \text{conv}\,\mathcal{E}_{\bar{y}}$ and $\text{int}\,\mathcal{H} \neq \emptyset$. Moreover, it is easy to check the following relations hold:

$$\text{qri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = \text{ri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}})$$
$$= \text{ri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \text{cl}\mathcal{H}) = \text{ri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) - \text{ri}\,(\text{cl}\,\mathcal{H})$$
$$= \text{ri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) - \text{ri}\,\mathcal{H} \subseteq \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \mathcal{H},$$

where the second equality follows from (8) and the last two equalities follow from [38].

The following example is given to illustrate Theorem 4.3.

Example 4.1 Let X be a Hausdorff locally convex topological vector space, $Y := L^p([0, T], \mathbb{R})$, $V := L^p([0, T], \mathbb{R}) \times X$, where $\infty > p \geq 1$ and $T > 0$. Let $C := \{I \in Y : I(t) \geq 0, \text{ a. e. on } [0, T]\}$ and D be a closed and convex cone of X such that it generates X, i.e., $D - D = X$ (see, e.g., [28]). Note that C is a closed and convex cone, $Y^* = L^q([0, T], \mathbb{R})$, where $\frac{1}{q} + \frac{1}{p} = 1$, and the canonical bilinear form on $Y^* \times Y$ is given by

$$\langle \mathbf{x}^*, \mathbf{x} \rangle := \int_{[0,T]} \mathbf{x}^*(t)\mathbf{x}(t)dt, \quad \forall (\mathbf{x}^*, \mathbf{x}) \in Y^* \times Y.$$

Clearly, $C^* = \{I^* \in Y^* : I^*(t) \geq 0, \text{ a. e. on } [0, T]\}$ and $\text{qi}\,C = \{I \in Y : I(t) > 0, \text{ a. e. on } [0, T]\}$ (see, e.g., [16,17,19]). Let $\mathcal{H} := \text{qi}\,C \times D$, $R := D$, $F(x; y) := (y - (\langle y^*, y \rangle + 1)C, x - D)$ for any $(x, y) \in X \times Y$, where $y^* \in Y^*$. Assume that $\text{qi}\,D \neq \emptyset$. Then, $\text{cl}\,\mathcal{H} = C \times D$ and it is easy to check (1) is impossible with $\bar{y} := 0$, $\text{cone}_+\,\mathcal{H} = \mathcal{H}$, $\mathcal{H} + \text{cl}\,\mathcal{H}\backslash\mathcal{H} \subseteq \mathcal{H}$ and $F(\cdot; \bar{y})$ is $-(\text{cl}\,\mathcal{H})$-convexlike on R. Since $\text{qri}\,D = \text{qi}\,D \neq \emptyset$ and $D \subseteq X$, from [6] we have $\emptyset \neq \text{qi}\,D \subseteq \text{qi}\,X$. Simple computation yields $\mathcal{K}_{\bar{y}} = \mathcal{E}_{\bar{y}} = -(C \times X)$ and from [16,19,21,22] one has

$$\text{qri}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = \text{qi}\,(\text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}}) = -(\text{qi}\,C \times \text{qi}\,X)$$
$$\subseteq -(C \times X) - (\text{qi}\,C \times \text{qi}\,D) = \text{cone}\,\text{conv}\,\mathcal{E}_{\bar{y}} - \mathcal{H}.$$

Letting $(\lambda^*, 0) \in \mathcal{H}^* \setminus \{(0,0)\} = (\mathrm{qi}\, C)^* \times D^*$ gives $\langle (\lambda^*, 0), (u, v) \rangle = \langle \lambda^*, u \rangle \leq 0$, $\forall (u, v) \in \mathcal{K}_{\bar{y}}$, which shows that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ admit a regular linear separation and so are properly linearly separable.

We have the following necessary condition for the impossibility of system (2).

Theorem 4.4 *Let* $V := U \times Z$, $\mathcal{H} := C \times D$, $R := K$ *and* $F(x; \bar{y}) := (G(x; \bar{y}), H(x))$. *Assume that* $0 \in \mathcal{E}_{\bar{y}}$ *and* $(G(\cdot; \bar{y}), H)$ *is* $-(\mathrm{cl}\, C \times D)$*-convexlike on* K. *Assume that* $\mathrm{cone}_+ C = C$, $C + \mathrm{cl}\, C \setminus C \subseteq C$, $\mathrm{qi}\, C \neq \emptyset$, $\mathrm{qi}\, D \neq \emptyset$ *and* $\mathrm{qi}\, (\mathrm{cone}\, \mathrm{conv}\, \mathcal{E}_{\bar{y}}) - (\mathrm{cl}\, C \setminus C \times D) \subseteq \mathrm{cone}\, \mathrm{conv}\, \mathcal{E}_{\bar{y}} - (C \times D)$. *Then, the following statements are true:*

(i) *If system* (2) *is impossible with* $\bar{y}$, *then* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are properly linearly separable;*
(ii) *If system* (2) *is impossible with* $\bar{y}$ *and moreover, the generalized Slater condition* (13) *holds, then* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *admit a regular linear separation.*

Proof From the assumptions that $\mathrm{cone}_+ C = C$, $C + \mathrm{cl}\, C \setminus C \subseteq C$ and D is a closed and convex cone, we have $\mathrm{cone}_+ \mathcal{H} = \mathrm{cone}_+ (C \times D) = \mathcal{H}$ and hence $\mathcal{H} + \mathrm{cl}\, \mathcal{H} \setminus \mathcal{H} \subseteq \mathcal{H}$ (by Proposition 2.3). Since $(G(\cdot; \bar{y}), H)$ is $-(\mathrm{cl}\, C \times D)$-convexlike on K and $\mathrm{cl}\, \mathcal{H} = \mathrm{cl}\, (C \times D) = (\mathrm{cl}\, C) \times D$, we have $\mathrm{cl}\, \mathcal{H} \setminus \mathcal{H} = (\mathrm{cl}\, C \setminus C) \times D$ and $F(\cdot; \bar{y})$ is $-\mathrm{cl}\, \mathcal{H}$-convexlike on K and consequently, $\mathcal{E}_{\bar{y}}$ is convex, i.e., $\mathrm{conv}\, \mathcal{E}_{\bar{y}} = \mathcal{E}_{\bar{y}}$. From [16,19,21,22], we obtain $\mathrm{qi}\, \mathcal{H} = \mathrm{qri}\, \mathcal{H} = \mathrm{qri}\, C \times \mathrm{qri}\, D = \mathrm{qi}\, C \times \mathrm{qi}\, D \neq \emptyset$, since $\mathrm{qi}\, C \neq \emptyset$ and $\mathrm{qi}\, D \neq \emptyset$. Now from Remark 4.1 (b), statements (i) and (ii) follow immediately from Theorem 4.3 and Proposition 3.5. $\square$

Remark 4.2 If $0 \in \mathrm{conv}\, \mathcal{E}_{\bar{y}}$, then we can replace $\mathrm{cone}\, \mathrm{conv}\, \mathcal{E}_{\bar{y}}$ by $\mathrm{conv}\, \mathcal{E}_{\bar{y}}$ in Theorems 4.3 and 4.4.

We also give the following sufficient condition for the impossibility of system (2) by using the regular linear separation of $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$.

Theorem 4.5 *Let* $V := U \times Z$, $\mathcal{H} := C \times D$, $R := K$, $F(x; \bar{y}) := (G(x; \bar{y}), H(x))$, *and let* $B \subseteq U$ *be a closed, convex and proper cone. Then, the following statements are true:*

(i) *If* $C := \mathrm{qi}\, B \neq \emptyset$ *and* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *admit a regular linear separation, then system* (2) *is impossible with* $\bar{y}$;
(ii) *If* $C := B \setminus \{0\}$ *and there is* $(\lambda^*, \theta^*) \in \mathrm{qi}\, (C^*) \times D^*$ *such that* (14) *holds, i.e.,* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *admit a regular linear separation, then system* (2) *is impossible with* $\bar{y}$.

Proof (i) Since $C = \mathrm{qi}\, B \neq \emptyset$, from [3,6,16,19,21,22] one has C is convex and the equalities hold: $C^* = (\mathrm{qi}\, B)^* = B^*$. We also have $\mathrm{cl}\, \mathrm{cone}\, (B - 0) = B \neq U$, i.e., $0 \notin C = \mathrm{qi}\, B$, since B is a closed, convex, and proper cone. Moreover, from Proposition 2.3 we know assumption (4) holds. Suppose to the contrary that system (2) is possible with $\bar{y}$. Then, from Proposition 2.1 (ii), we have $\mathcal{K}_{\bar{y}} \cap (\mathrm{qi}\, B \times D) \neq \emptyset$, i.e., there are $x \in K$, $u_0 \in \mathrm{qi}\, B \cap G(x; \bar{y})$ and $v_0 \in D \cap H(x)$. If $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ admit a regular linear separation, then there exists $(\lambda^*, \theta^*) \in B^* \times D^*$ with $\lambda^* \neq 0$ such that $\langle \lambda^*, u_0 \rangle + \langle \theta^*, v_0 \rangle \leq 0$. Since $(\lambda^*, \theta^*) \in B^* \times D^*$ with $\lambda^* \neq 0$, it follows from [3,16,19,23] that $\langle \lambda^*, u_0 \rangle + \langle \theta^*, v_0 \rangle > 0$, a contradiction.

(ii) Since $C = B \backslash \{0\}$, $C^* = B^*$. Suppose to the contrary that system (2) is possible with $\bar{y}$. Then, from Proposition 2.1 (ii), we have $\mathcal{K}_{\bar{y}} \cap ((B \backslash \{0\}) \times D) \neq \emptyset$, i.e., there are $x \in K$, $u_0 \in B \backslash \{0\} \cap G(x; \bar{y})$ and $v_0 \in D \cap H(x)$. If there is $(\lambda^*, \theta^*) \in \mathrm{qi}(C^*) \times D^*$ such that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ admit a regular linear separation, then $\langle \lambda^*, u_0 \rangle + \langle \theta^*, v_0 \rangle \leq 0$. Since B is a closed and convex cone in Hausdorff locally convex topological vector space U, from [41], one has $B = B^{**}$ and it also follows from [3,16,19,23] that $\langle \lambda^*, u_0 \rangle + \langle \theta^*, v_0 \rangle > 0$, a contradiction.   □

5 Saddle Points of Lagrangian Functions for System (2)

In this section, we shall prove the equivalence between (proper, regular) linear separation and saddle points of Lagrangian functions for system (2) under some convexity and compactness assumptions. Without other specifications, we always suppose that for $\bar{y} \in Y$, $G(\cdot; \bar{y})$ and H are compact-valued on K, i.e., $G(x; \bar{y})$ and $H(x)$ are compact for each $x \in K$.

Let $\bar{y} \in Y$. Consider the generalized Lagrangian function associated with system (2), defined by $\mathcal{L}_{\bar{y}} : K \times C^* \times D^* \to \mathbb{R}$,

$$\mathcal{L}_{\bar{y}}(x, \lambda, \theta) := -\sigma_{G(x;\bar{y})}(\lambda) - \sigma_{H(x)}(\theta), \ \forall (x, \lambda, \theta) \in K \times C^* \times D^*.$$

If $X := Y := \mathbb{R}^n$, $Z := \mathbb{R}^m$, $U := \mathbb{R}$, $C := \mathbb{R}_{++}$ and $G(x; y) := f(y) - f(x)$, where $f : X \to \mathbb{R}$, then the generalized Lagrangian function $\mathcal{L}_{\bar{y}}$ defined above reduces to the following:

$$\mathcal{L}_{\bar{y}}(x, \lambda, \theta) := \lambda(f(x) - f(\bar{y})) - \sigma_{H(x)}(\theta), \ \forall (x, \lambda, \theta) \in K \times \mathbb{R}_+ \times D^*,$$

which has been considered in [7].

Definition 5.1 We say that the point $(\bar{x}, \lambda^*, \theta^*) \in K \times C^* \times D^*$ is a saddle point for $\mathcal{L}_{\bar{y}}$ on $K \times C^* \times D^*$, if

$$\mathcal{L}_{\bar{y}}(\bar{x}, \lambda, \theta) \leq \mathcal{L}_{\bar{y}}(\bar{x}, \lambda^*, \theta^*) \leq \mathcal{L}_{\bar{y}}(x, \lambda^*, \theta^*), \quad \forall (x, \lambda, \theta) \in K \times C^* \times D^*.$$

We have the following proposition:

Proposition 5.1 *The function* $(\lambda, \theta) \mapsto \mathcal{L}_{\bar{y}}(x, \lambda, \theta)$ *is concave on* $C^* \times D^*$. *If* $G(\cdot; \bar{y})$ *and* H *are* $-(\mathrm{cl}\, C)$-*map and* $-D$-*map on* K, *respectively, then the function* $x \mapsto \mathcal{L}_{\bar{y}}(x, \lambda, \theta)$ *is convex on* K.

Proof Clearly, the function $(\lambda, \theta) \mapsto \mathcal{L}_{\bar{y}}(x, \lambda, \theta)$ is concave on $C^* \times D^*$. Suppose that $G(\cdot; \bar{y})$ and H are $-(\mathrm{cl}\, C)$-map and $-D$-map on K, respectively. It suffices to prove that $x \mapsto \sigma_{G(x;\bar{y})}(\lambda)$ and $x \mapsto \sigma_{H(x)}(\theta)$ are concave on K. Let $\lambda \in C^*$, $\mathbf{x}, \mathbf{y} \in K$, $t \in]0, 1[$ and set $\mathbf{z} := t\mathbf{x} + (1 - t)\mathbf{y}$. Then, $\mathbf{z} \in K$, since K is convex. Since $G(\cdot; \bar{y})$ is $-(\mathrm{cl}\, C)$-map on K, $tG(\mathbf{x}; \bar{y}) + (1 - t)G(\mathbf{y}; \bar{y}) \subseteq G(\mathbf{z}; \bar{y}) - \mathrm{cl}\, C$. From the compactness of $G(\mathbf{x}; \bar{y})$ and $G(\mathbf{y}; \bar{y})$, there are $g_{\mathbf{x}} \in G(\mathbf{x}; \bar{y})$ and $g_{\mathbf{y}} \in G(\mathbf{y}; \bar{y})$ such

that $\sigma_{G(\mathbf{x};\bar{y})}(\lambda) = \langle \lambda, g_{\mathbf{x}} \rangle$ and $\sigma_{G(\mathbf{y};\bar{y})}(\lambda) = \langle \lambda, g_{\mathbf{y}} \rangle$. It follows from $tG(\mathbf{x};\bar{y}) + (1 - t)G(\mathbf{y};\bar{y}) \subseteq G(\mathbf{z};\bar{y}) - \operatorname{cl} C$ that $tg_{\mathbf{x}} + (1 - t)g_{\mathbf{y}} \in G(\mathbf{z};\bar{y}) - \operatorname{cl} C$ and consequently,

$$
\begin{aligned}
t\sigma_{G(\mathbf{x};\bar{y})}(\lambda) + (1 - t)\sigma_{G(\mathbf{y};\bar{y})}(\lambda) &= \langle \lambda, tg_{\mathbf{x}} + (1 - t)g_{\mathbf{y}} \rangle \\
&\leq \max_{g \in G(\mathbf{z};\bar{y})} \langle \lambda, g \rangle + \max_{c \in -\operatorname{cl} C} \langle \lambda, c \rangle = \sigma_{G(\mathbf{z};y)}(\lambda),
\end{aligned}
$$

since $\operatorname{cl} C$ is a cone and $\lambda \in C^*$. This proves that $x \mapsto \sigma_{G(x;\bar{y})}(\lambda)$ is concave on K. Similarly, we can prove that $x \mapsto \sigma_{H(x)}(\theta)$ is concave on K. $\square$

We next show that the linear separation of $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ for system (2) is equivalent to the existence of saddle points of the generalized Lagrangian function $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$ under suitable assumptions.

Theorem 5.1 *Let* $V := U \times Z$, $\mathcal{H} := C \times D$, $R := K$ *and* $F(x;\bar{y}) := (G(x;\bar{y}), H(x))$. *Assume that* H *is* $-D$*-map on* K *and* $G(\bar{x};\bar{y}) = \{0\}$ *for* $\bar{x} \in K$. *Then,* $\mathcal{K}_{\bar{y}}$ *and* $\mathcal{H}$ *are linearly separable and* $\bar{x} \in R_0$, *if and only if there exists* $(\lambda^*, \theta^*) \in C^* \times D^* \backslash \{(0, 0)\}$ *such that* $(\bar{x}, \lambda^*, \theta^*)$ *is a saddle point for* $\mathcal{L}_{\bar{y}}$ *defined on* $K \times C^* \times D^*$.

Proof Necessity. Suppose that $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable and $\bar{x} \in R_0$. Then, setting $\mathcal{H} := C \times D$ and $F(x;\bar{y}) := (G(x;\bar{y}), H(x))$ in Proposition 3.1 (i) yields that there is $(\lambda^*, \theta^*) \in C^* \times D^* \backslash \{(0, 0)\}$ such that

$$
\sigma_{G(x;\bar{y})}(\lambda^*) + \sigma_{H(x)}(\theta^*) \leq 0, \ \forall x \in K. \tag{17}
$$

Set $x := \bar{x}$ in (17). Since $G(\bar{x};\bar{y}) = \{0\}$, we have $\sigma_{G(\bar{x};\bar{y})}(\lambda) = 0$ for each $\lambda \in C^*$, and so $\sigma_{H(\bar{x})}(\theta^*) \leq 0$. Since $\bar{x} \in R_0$, i.e., $\bar{x} \in K$ and $H(\bar{x}) \cap D \neq \emptyset$, one has $\sigma_{H(\bar{x})}(\theta) \geq 0$ for each $\theta \in D^*$. Thus, it follows that $\sigma_{H(\bar{x})}(\theta^*) = 0$. Again from (17), we obtain

$$
\begin{aligned}
\mathcal{L}_{\bar{y}}(\bar{x}, \lambda^*, \theta^*) &= -\sigma_{G(\bar{x};\bar{y})}(\lambda^*) - \sigma_{H(\bar{x})}(\theta^*) = 0 \leq -\sigma_{G(x;\bar{y})}(\lambda^*) - \sigma_{H(x)}(\theta^*) \\
&= \mathcal{L}_{\bar{y}}(x, \lambda^*, \theta^*), \ \forall x \in K.
\end{aligned}
$$

On the other hand, since $\sigma_{H(\bar{x})}(\theta) \geq 0$ for each $\theta \in D^*$, we have

$$
\mathcal{L}_{\bar{y}}(\bar{x}, \lambda, \theta) = -\sigma_{G(\bar{x};\bar{y})}(\lambda) - \sigma_{H(\bar{x})}(\theta) \leq 0 = \mathcal{L}_{\bar{y}}(\bar{x}, \lambda^*, \theta^*), \ \forall(\lambda, \theta) \in C^* \times D^*.
$$

This proves $(\bar{x}, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$.

Sufficiency. Suppose that there is $(\lambda^*, \theta^*) \in C^* \times D^* \backslash \{(0, 0)\}$ such that $(\bar{x}, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$. Since $G(\bar{x};\bar{y}) = \{0\}$, we have $\sigma_{G(\bar{x};\bar{y})}(\lambda) = 0$ for each $\lambda \in C^*$, and hence

$$
\begin{aligned}
-\sigma_{H(\bar{x})}(\theta) &\leq -\sigma_{H(\bar{x})}(\theta^*) \leq -\sigma_{G(x;\bar{y})}(\lambda^*) \\
&\quad -\sigma_{H(x)}(\theta^*), \ \forall(x, \lambda, \theta) \in K \times C^* \times D^*. \tag{18}
\end{aligned}
$$

Setting $\theta := 0 \in D^*$ in the first inequality in (18) yields $\sigma_{H(\bar{x})}(\theta^*) \leq 0$.

We next prove that $\bar{x} \in R_0$, i.e., $H(\bar{x}) \cap D \neq \emptyset$. The proof is similar to that in [7]. For completeness, we include it here. Suppose to the contrary that $H(\bar{x}) \cap D = \emptyset$. We first show $(H(\bar{x}) - D) \cap D = \emptyset$. In fact, if not, i.e., $(H(\bar{x}) - D) \cap D \neq \emptyset$, then there exist $\bar{h} \in H(\bar{x})$ and $\bar{d} \in D$ such that $\bar{h} - \bar{d} \in D$ and so $\bar{h} \in D + \bar{d} \subseteq D$, since D is a convex cone. Thus it follows that $\bar{h} \in H(\bar{x}) \cap D$, a contradiction with the assumption $H(\bar{x}) \cap D = \emptyset$. Since H is $-D$-map on K, from Remark 2.1, one has $H(\bar{x}) - D$ is convex. Since $H(\bar{x})$ is compact and D is a closed and convex cone, $H(\bar{x}) - D$ is closed and as a consequence,

$$(H(\bar{x}) - D) - D = H(\bar{x}) - D = \mathrm{cl}\,(H(\bar{x}) - D) = \mathrm{cl}\,[(H(\bar{x}) - D) - D]. \quad (19)$$

Since $(H(\bar{x}) - D) \cap D = \emptyset$, from (19) we have $0 \notin (H(\bar{x}) - D) - D = \mathrm{cl}\,[(H(\bar{x}) - D) - D]$. By the standard separation theorem [27,29], there exists $z^* \in Z^* \backslash \{0\}$ such that

$$\sup_{e \in H(\bar{x}) - D} \langle z^*, e \rangle < \inf_{d \in D} \langle z^*, d \rangle. \quad (20)$$

We declare that $z^* \in D^*$. If not, i.e., $z^* \notin D^*$, then there is $d_0 \in D$ such that $\langle z^*, d_0 \rangle < 0$. Since D is a cone, $td_0 \in D$ for each $t > 0$. Consequently, as $t \to +\infty$, $\langle z^*, td_0 \rangle = t\langle z^*, d_0 \rangle \to -\infty$, a contradiction with (20). Since $z^* \in D^*$ and D is a cone, $\inf_{d \in D} \langle z^*, d \rangle = \min_{d \in D} \langle z^*, d \rangle = 0$ and therefore it follows from (20) that

$$\sup_{e \in H(\bar{x}) - D} \langle z^*, e \rangle < 0. \quad (21)$$

Since

$$\sup_{e \in H(\bar{x}) - D} \langle z^*, e \rangle = \sup_{h \in H(\bar{x})} \langle z^*, h \rangle + \sup_{d \in -D} \langle z^*, d \rangle$$
$$= \sup_{h \in H(\bar{x})} \langle z^*, h \rangle - \inf_{d \in D} \langle z^*, d \rangle = \sup_{h \in H(\bar{x})} \langle z^*, h \rangle,$$

from (21), one has $\sigma_{H(\bar{x})}(z^*) = \sup_{h \in H(\bar{x})} \langle z^*, h \rangle < 0$. Since D^* is a cone, $tz^* \in D^*$ for each $t > 0$. Since $\sigma_{H(\bar{x})}(\cdot)$ is positively homogeneous, as $t \to +\infty$, $-\sigma_{H(\bar{x})}(tz^*) = -t\sigma_{H(\bar{x})}(z^*) \to +\infty$, a contradiction with the first inequality in (18). This shows $\bar{x} \in R_0$, i.e., $H(\bar{x}) \cap D \neq \emptyset$.

Since $\bar{x} \in R_0$, i.e., $\bar{x} \in K$ and $H(\bar{x}) \cap D \neq \emptyset$, $\sigma_{H(\bar{x})}(\theta^*) \geq 0$ and so $\sigma_{H(\bar{x})}(\theta^*) = 0$, since $\sigma_{H(\bar{x})}(\theta^*) \leq 0$. Thus, from the second inequality in (18), we have $\sigma_{G(x;\bar{y})}(\lambda^*) + \sigma_{H(x)}(\theta^*) \leq 0$, $\forall x \in K$, which implies that the sets $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are linearly separable. □

Remark 5.1 In Theorem 5.1, we do not assume that $G(\cdot; \bar{y})$ is $-(\mathrm{cl}\, C)$-map on K and moreover the assumption $G(\bar{x}; \bar{y}) = \{0\}$ for $\bar{x} \in K$ plays a crucial role in proving the equivalence between the linear separation of $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ and the existence of saddle points for $\mathcal{L}$ defined on $K \times C^* \times D^*$. If $X := Y := \mathbb{R}^n$, $Z := \mathbb{R}^m$, $U := \mathbb{R}$, $C := \mathbb{R}_{++}$ and $G(x; y) := f(y) - f(x)$, where $f : X \to \mathbb{R}$, then Theorem 5.1 collapses to Theorem 5.1 in [7].

Similarly, we have the following:

Theorem 5.2 *Let $V := U \times Z$, $\mathcal{H} := C \times D$, $R := K$ and $F(x; \bar{y}) := (G(x; \bar{y}), H(x))$. Assume that H is $-D$-map on K and $G(\bar{x}; \bar{y}) = \{0\}$ for $\bar{x} \in K$. Then, the following statements are true:*

(i) *The sets $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ are properly linearly separable and $\bar{x} \in R_0$, if and only if there exists a point $(\lambda^*, \theta^*) \in C^* \times D^* \backslash \{(0, 0)\}$ such that $(\bar{x}, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$ with $0 < \mathcal{L}_{\bar{y}}(x^0, \lambda^*, \theta^*)$ for some $x^0 \in K$;*

(ii) *The sets $\mathcal{K}_{\bar{y}}$ and $\mathcal{H}$ admit a regular linear separation and $\bar{x} \in R_0$, if and only if there exists a point $(\lambda^*, \theta^*) \in C^* \times D^*$ with $\lambda^* \neq 0$ such that $(\bar{x}, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$.*

Theorem 5.3 *Let $V := U \times Z$, $\mathcal{H} := C \times D$, $R := K$ and $F(x; \bar{y}) := (G(x; \bar{y}), H(x))$. Assume that $G(\bar{x}; \bar{y}) = \{0\}$ for $\bar{x} \in R_0$ and assume that $G(\cdot; \bar{y})$ and H are $-(\mathrm{cl}\, C)$-map and $-D$-map on K, respectively. Assume that $\mathrm{cone}_+ C = C$, $C + \mathrm{cl}\, C \backslash C \subseteq C$, $\mathrm{qi}\, C \neq \emptyset$, $\mathrm{qi}\, D \neq \emptyset$ and assume that $\mathrm{qi}\,(\mathrm{cone}\,\mathrm{conv}\,\mathcal{E}_{\bar{y}}) - (\mathrm{cl}\, C \backslash C \times D) \subseteq \mathrm{cone}\,\mathrm{conv}\,\mathcal{E}_{\bar{y}} - (C \times D)$. Then, the following statements are true:*

(i) *If system (2) is impossible with $\bar{y}$, then there is $(\lambda^*, \theta^*) \in C^* \times D^* \backslash \{(0, 0)\}$ such that $(\bar{x}, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$ with $0 < \mathcal{L}_{\bar{y}}(x^0, \lambda^*, \theta^*)$ for some $x^0 \in K$;*

(ii) *If system (2) is impossible with $\bar{y}$ and moreover, the generalized Slater condition (13) holds, then there is $(\lambda^*, \theta^*) \in C^* \times D^*$ with $\lambda^* \neq 0$ such that $(\bar{x}, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$.*

Proof The conclusion follows immediately from Theorems 4.4 and 5.2. □

Similarly, from Theorems 4.5 and 5.2, we have the following:

Theorem 5.4 *Let $V := U \times Z$, $\mathcal{H} := C \times D$, $R := K$ and $F(x; \bar{y}) := (G(x; \bar{y}), H(x))$, and let $B \subseteq U$ be a closed, convex and proper cone. Assume that H is $-D$-map on K and $G(\bar{x}; \bar{y}) = \{0\}$ for $\bar{x} \in R_0$. Then, the following statements are true:*

(i) *If $C := \mathrm{qi}\, B \neq \emptyset$ and there exists $(\lambda^*, \theta^*) \in C^* \times D^*$ with $\lambda^* \neq 0$ such that the point $(\bar{x}, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$, then system (2) is impossible with $\bar{y}$;*

(ii) *If $C := B \backslash \{0\}$ and there exists $(\lambda^*, \theta^*) \in \mathrm{qi}\,(C^*) \times D^*$ such that the point $(\bar{x}, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{\bar{y}}$ defined on $K \times C^* \times D^*$, then system (2) is impossible with $\bar{y}$.*

An anonymous referee asked an interesting and deep question: Is there any connection between the results of this section and the Mountain Pass Theorem (for short, MPT)? As is well known, the MPT plays an extremely important role in investigating a very large number of problems in many areas of nonlinear analysis, which is a very useful argument for finding critical points of an objective function. The Palais–Smale (for short, PS) condition is crucial for the MPT, which in general requires the objective function to be continuously differentiable or locally Lipschitz (see, e.g., [42]).

By comparison, both the MPT and Lagrangian functions can be used to characterize the stationary point of a optimization problem, although the former works with the minmax form under the smoothness of the objective function and the continuity of another function passing through two given points and the latter does with the minmax or maxmin form under convexity (see, e.g., [43]). Specifically, consider the following constrained extremum problem (for short, P):

$$\min f(x), \quad \text{s.t.} \quad x \in X := \{x \in \mathbb{R}^m : g(x) \geq 0\},$$

where $f, g : \mathbb{R}^m \to \mathbb{R}$. Then, P can be associated with the image space, where we have an image problem (for short, IP),

$$\max u, \quad \text{s.t.} \quad (u, v) \in \mathcal{K}_{\bar{x}}, v \geq 0,$$

where $\mathcal{K}_{\bar{x}} := \{(u, v) \in \mathbb{R}^2 : u = f(\bar{x}) - f(x), v = g(x), x \in \mathbb{R}^m\}$ and $\bar{x} \in X$.

It is easy to prove the following Proposition (see, e.g., [1,44]).

Proposition 5.2 P *is equivalent to IP, that is to say,* $(\bar{u}, \bar{v})$ *solves IP, if and only if* x' *solves P, where* $(\bar{u}, \bar{v}) = (f(\bar{x}) - f(x'), g(x'))$.

Set $\Sigma_0 := \{(u, v) \in \mathcal{K}_{\bar{x}} : v \geq 0\}$. Suppose that Σ_0 is connected (this is true if f and g are continuous and the level set $\text{lev}_{\geq 0} g := \{x \in \mathbb{R}^m : g(x) \geq 0\}$ of g is bounded) and int $\Sigma_0 \neq \emptyset$. Let $\triangle := \{\Phi : \Sigma_0 \to \mathbb{R} \cup \{+\infty\} :$ the values of Φ become infinite on bd $\Sigma_0\}$. Let $\Phi \in \triangle$ and set $\phi(x) := \Phi(f(\bar{x}) - f(x), g(x))$.

The following shows that the MPT can be applied to IP and guarantees precisely the same thing (critical point) as the saddle point (of von Neumann memory).

Proposition 5.3 *Suppose* $\Phi \in \triangle$ *is continuously differentiable in* int Σ_0 *and possesses two distinct strict relative minima* $(u^1, v^1), (u^2, v^2) \in \Sigma_0$. *Then,* Φ *possesses a critical point* $(\bar{u}, \bar{v}) \in \Sigma_0$ *(i.e.,* $\nabla \Phi(\bar{u}, \bar{v}) = 0$*) such that* $\Phi(\bar{u}, \bar{v}) > \max\{\Phi(u^1, v^1), \Phi(u^2, v^2)\}$, *characterized by*

$$\Phi(\bar{u}, \bar{v}) = \inf_{\Sigma \in \Gamma} \sup_{(u,v) \in \Sigma} \Phi(u, v),$$

where $\Gamma := \{\Sigma \subseteq \Sigma_0 : \Sigma$ *is compact and connected and* $(u^1, v^1), (u^2, v^2) \in \Sigma\}$ *and* ∇ *denotes the gradient. Moreover, if both* f *and* g *are differentiable, then there is* $x' \in \mathbb{R}^m$ *such that* $\nabla \phi(x') = 0$, *where* $(\bar{u}, \bar{v}) = (f(\bar{x}) - f(x'), g(x'))$.

Proof The first assertion follows immediately from a finite-dimensional version of the MPT by Courant [45] (see also, [42, Theorem 5.2]). Suppose that f and g are differentiable. Since $(\bar{u}, \bar{v})$ is a critical point of Φ, we have $0 = \nabla \Phi(\bar{u}, \bar{v}) = (\frac{\partial \Phi(u,v)}{\partial u}|_{(\bar{u},\bar{v})}, \frac{\partial \Phi(u,v)}{\partial v}|_{(\bar{u},\bar{v})})$, where $\frac{\partial \Phi(u,v)}{\partial u}|_{(\bar{u},\bar{v})}$ and $\frac{\partial \Phi(u,v)}{\partial v}|_{(\bar{u},\bar{v})}$ are the partial derivatives of Φ with respect to u and v at $(\bar{u}, \bar{v})$, respectively. Then, there exists $x' \in \mathbb{R}^m$ such that $(\bar{u}, \bar{v}) = (f(\bar{x}) - f(x'), g(x'))$ and it follows that

$$
\begin{aligned}
\nabla\phi(x') &= \left(\frac{\partial\phi(x)}{\partial x_1}|_{x'}, \ldots, \frac{\partial\phi(x)}{\partial x_m}|_{x'} \right) \\
&= \left(\frac{\partial\Phi(u,v)}{\partial u}|_{(\bar{u},\bar{v})} \frac{\partial u}{\partial x_1}|_{x'} + \frac{\partial\Phi(u,v)}{\partial v}|_{(\bar{u},\bar{v})} \frac{\partial v}{\partial x_1}|_{x'}, \ldots, \right. \\
&\quad \left. \frac{\partial\Phi(u,v)}{\partial u}|_{(\bar{u},\bar{v})} \frac{\partial u}{\partial x_m}|_{x'} + \frac{\partial\Phi(u,v)}{\partial v}|_{(\bar{u},\bar{v})} \frac{\partial v}{\partial x_m}|_{x'} \right) \\
&= \left(\frac{\partial\Phi(u,v)}{\partial u}|_{(\bar{u},\bar{v})} \frac{\partial(f(\bar{x})-f(x))}{\partial x_1}|_{x'} + \frac{\partial\Phi(u,v)}{\partial v}|_{(\bar{u},\bar{v})} \frac{\partial g(x)}{\partial x_1}|_{x'}, \ldots, \right. \\
&\quad \left. \frac{\partial\Phi(u,v)}{\partial u}|_{(\bar{u},\bar{v})} \frac{\partial(f(\bar{x})-f(x))}{\partial x_m}|_{x'} + \frac{\partial\Phi(u,v)}{\partial v}|_{(\bar{u},\bar{v})} \frac{\partial g(x)}{\partial x_m}|_{x'} \right) \\
&= 0.
\end{aligned}
$$

This completes the proof.   □

Proposition 5.3 also shows that IP can be used to verify the assumptions of the MPT, while P does not. As to the duality in Lagrangian sense, the dual space $\{(\lambda, \theta) \in \mathbb{R}^2 : \lambda \geq 0, \theta \geq 0\}$ is that of the linear functional, whose zero levels $\{(u, v) \in \mathbb{R}^2 : \lambda u + \theta v = 0\}$ are the manifolds used to separate $\mathcal{K}_{\bar{x}}$ and $\mathcal{H} := \mathbb{R}_{++} \times \mathbb{R}_+$ in the image space, while the critical point $(\bar{u}, \bar{v})$ of such manifolds characterizes the slope of the separating line.

6 Some Applications

In this section, we shall apply the obtained results to investigate the weak minimizer of SVOP and the efficient and weakly efficient solution of VOP.

6.1 Applications to SVOP

For SVOP, set

$$
\begin{aligned}
Y &:= S(R_0), \quad C := \operatorname{qri} P, \quad \mathcal{H} := C \times D, \\
G(x; y) &:= y - S(x), \quad F(x; y) := (G(x; y), H(x)),
\end{aligned}
$$

where $x \in X$ and $y \in Y$. In Theorems 6.1 (iii), 6.3 and 6.4, we always suppose that S and H are compact-valued on K. Let $y^* \in U$ and define $\mathcal{L}_{y^*}^\circ : K \times P^* \times D^* \to \mathbb{R}$ by

$$
\mathcal{L}_{y^*}^\circ(x, \lambda, \theta) := -\langle\lambda, y^*\rangle - \sigma_{-S(x)}(\lambda) - \sigma_{H(x)}(\theta), \quad \forall(x, \lambda, \theta) \in K \times P^* \times D^*.
$$

Theorem 6.1 *Let* $\operatorname{qi} P \neq \emptyset$, $\operatorname{qi} D \neq \emptyset$, $x^* \in R_0$ *and* $y^* \in S(x^*)$. *Then, the following statements are true:*

(i) (x^*, y^*) *is a weak minimizer of* SVOP, *if and only if* $(y^*, 0) \notin \operatorname{qi} P \times D - (-S, H)(K)$;

(ii) *The sets $\mathcal{K}_{y^*}$ and* qi $P \times D$ *are (properly) linearly separable, if and only if the relation holds* $(y^*, 0) \notin$ qi $(P \times D - \text{conv}\,(-S, H)(K))$;

(iii) *If H is $-D$-map on K and $S(x^*) = \{y^*\}$, then $\mathcal{K}_{y^*}$ and* qi $P \times D$ *are (properly) linearly separable, if and only if there exists $(\lambda^*, \theta^*) \in P^* \times D^* \backslash \{(0, 0)\}$ such that $(x^*, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}_{y^*}^{\circ}$ defined on $K \times P^* \times D^*$ with $0 < \mathcal{L}_{y^*}^{\circ}(x^0, \lambda^*, \theta^*)$ for some $x^0 \in K$;*

(iv) *If the generalized Slater condition* (13) *holds, then $\mathcal{K}_{y^*}$ and* qi $P \times D$ *admit a regular linear separation, if and only if* $(0, 0) \notin$ cl cone $[(y^*, 0) + \text{conv}\,(-S, H)(K) - P \times D] - $ qi $P \times \{0\}$.

Proof (i) Clearly, $\mathcal{K}_{y^*} = (y^* - S, H)(K) = (y^*, 0) + (-S, H)(K)$ and so the conclusion follows immediately from Proposition 2.1.

(ii) From [16,19,21,22], one has $\mathcal{E}_{y^*} = \mathcal{K}_{y^*} - \text{cl}\,\mathcal{H} = (y^*, 0) + (-S, H)(K) - (P \times D)$ and as a consequence, conv $\mathcal{E}_{y^*} = (y^*, 0) + \text{conv}\,(-S, H)(K) - (P \times D)$ follows immediately from [29]. From [3,6,16,19,21,22] we obtain

$$\begin{aligned}
\text{qi}\,(\text{conv}\,\mathcal{E}_{y^*}) &= (y^*, 0) + \text{qi}\,(\text{conv}\,(-S, H)(K) - (P \times D)) \supseteq (y^*, 0) \\
&\quad + \text{conv}\,(-S, H)(K) - \text{qi}\,(P \times D) \\
&= (y^*, 0) + \text{conv}\,(-S, H)(K) - (\text{qi}\,P \times \text{qi}\,D) \neq \emptyset.
\end{aligned}$$

Therefore, qri $(\text{conv}\,\mathcal{E}_{y^*}) = $ qi $(\text{conv}\,\mathcal{E}_{y^*}) \neq \emptyset$. Since $x^* \in R_0$ and $y^* \in S(x^*)$, we have $0 \in \mathcal{E}_{y^*}$. Now, the fact that cl cone (cone conv $\mathcal{E}_{y^*} - 0$) = cl cone (conv $\mathcal{E}_{y^*} - 0$) yields $0 \notin$ qi (cone conv $\mathcal{E}_{y^*}$) if and only if $0 \notin$ qi (conv $\mathcal{E}_{y^*}$). Then, from [16,19,21,22], the conclusion follows immediately from Propositions 3.2 and 3.3.

(iii) It follows from Theorem 5.1.

(iv) From (ii), we have conv $\mathcal{E}_{y^*} = (y^*, 0) + \text{conv}\,(-S, H)(K) - (P \times D)$ and therefore, the equality holds: cl cone conv $\mathcal{E}_{y^*} = $ cl cone $[(y^*, 0) + \text{conv}\,(-S, H)(K) - (P \times D)]$. Thus, the conclusion follows immediately from Proposition 3.6.   □

Theorem 6.2 *Let (x^*, y^*) with $x^* \in R_0$ and $y^* \in S(x^*)$. Assume that* qi $P \neq \emptyset$ *and* qi $D \neq \emptyset$. *Then, the following statements are true:*

(i) *Assume that* qi (cone conv $\mathcal{E}_{y^*}$) $- (P \backslash \text{qi}\,P \times D) \subseteq$ cone conv $\mathcal{E}_{y^*} - (\text{qi}\,P \times D)$ *and $(-S, H)$ is $-(P \times D)$-convexlike on K. If (x^*, y^*) is a weak minimizer of* SVOP, *then $\mathcal{K}_{y^*}$ and* qi $P \times D$ *are properly linearly separable;*

(ii) *If $\mathcal{K}_{y^*}$ and* qi $P \times D$ *are linearly separable and the generalized Slater condition* (13) *holds, or $\mathcal{K}_{y^*}$ and* qi $P \times D$ *admit a regular linear separation, then (x^*, y^*) is a weak minimizer of* SVOP.

Proof Since P is a closed, convex, and pointed cone with qri $P \neq \emptyset$, from [3,6,16,19,21,22] we have cl $C = $ cl (qi P) $= P$, cone$_+ C = C$, qri $C = $ qri (qri P) $= $ qri P and $C + \text{cl}\,C \backslash C \subseteq C$. Clearly, $0 \in \mathcal{E}_{y^*}$ and since $(-S, H)$ is $-(P \times D)$-convexlike on K, so is $(G(\cdot; y^*); H)$. Therefore, conclusions (i) and (ii) follow immediately from Theorems 4.4 and 4.5 and Proposition 3.5.   □

We also have the following saddle point sufficient and necessary optimality conditions for SVOP.

Theorem 6.3 *Assume that* qi $P \neq \emptyset$, qi $D \neq \emptyset$, $x^* \in R_0$ *and* $S(x^*) = \{y^*\}$ *and assume that the condition holds:* qi (cone conv $\mathcal{E}_{y^*}$) $- (P \backslash qi\, P \times D) \subseteq$ cone conv $\mathcal{E}_{y^*} -$ (qi $P \times D$). *Assume that* S *and* H *are* P-*map and* $-D$-*map on* K, *respectively. If* (x^*, y^*) *is a weak minimizer of* SVOP, *then* $\mathcal{K}_{y^*}$ *and* qi $P \times D$ *are properly linearly separable, or equivalently, there exists* $(\lambda^*, \theta^*) \in P^* \times D^* \backslash \{(0, 0)\}$ *such that* $(x^*, \lambda^*, \theta^*)$ *is a saddle point for* $\mathcal{L}_{y^*}^{\circ}$ *defined on* $K \times P^* \times D^*$ *with* $0 < \mathcal{L}_{y^*}^{\circ}(x^0, \lambda^*, \theta^*)$ *for some* $x^0 \in K$.

Proof Since P is a closed, convex, and pointed cone with qri $P \neq \emptyset$, from [3,6,16,19, 21,22] we have cl $C = $ cl (qi P) $= P$, cone$_+$ $C = C$, qri $C = $ qri (qri P) = qri P and $C + $ cl $C \backslash C \subseteq C$. Since S is P-map on K, $G(\cdot; y^*)$ is $-P$-map on K. The conclusion follows immediately from Theorem 5.3.   □

Similarly, from Theorem 5.4 we have:

Theorem 6.4 *Assume that* qi $P \neq \emptyset$ *and* qi $D \neq \emptyset$. *Assume that* H *is* $-D$-*map on* K, $x^* \in R_0$ *and* $S(x^*) = \{y^*\}$. *If* $\mathcal{K}_{y^*}$ *and* qi $P \times D$ *admit a regular linear separation, or equivalently, there exists* $(\lambda^*, \theta^*) \in P^* \times D^*$ *with* $\lambda^* \neq 0$ *such that the point* $(x^*, \lambda^*, \theta^*)$ *is a saddle point for* $\mathcal{L}_{y^*}^{\circ}$ *on* $K \times P^* \times D^*$, *then* (x^*, y^*) *is a weak minimizer of* SVOP.

We give the following remark to compare our results with Proposition 23 of [25] (see also, Theorem 3.3 of [24]).

Remark 6.1 It is easy to check the following statements for (a)-(f) and (b') of Proposition 23 in [25] hold:

(i) (a)$\Leftrightarrow$ $x^* \in R_0$ and $y^* \in S(x^*)$;

(ii) (b)$\Leftrightarrow$ $\mathcal{K}_{y^*}$ and qi $P \times D$ are linearly separable. Since $(-S, H)$ is $-(P \times D)$-convexlike on K, so is $(G(\cdot; y^*); H)$. Clearly, $0 \in \mathcal{E}_{y^*}$. From Proposition 2.4 (i), we have $\mathcal{E}_{y^*}$ is convex and so cl cone $[\mathcal{E}_{y^*} - 0] = $ cl cone $[(\text{cone conv } \mathcal{E}_{y^*}) - 0]$. From [3,6], (b)$\Leftrightarrow$ $0 \notin$ qi $\mathcal{E}_{y^*}$ and it follows that (b)$\Leftrightarrow$ $0 \notin$ qi (cone conv $\mathcal{E}_{y^*}$). Now Proposition 3.2 leads to the assertion.

(iii) (c)$\Leftrightarrow$ the generalized Slater condition (12);

(iv) (d)$\Leftrightarrow$ $\mathcal{K}_{y^*}$ and qi $P \times D$ admit a regular linear separation;

(v) (e)$\Leftarrow$ $\sigma_{H(x^*)}(\theta^*) = 0$ (see the proof of the necessity of Theorem 5.1);

(vi) (f)$\Leftrightarrow$ (x^*, y^*) is a weak minimizer of SVOP;

(vii) (b')$\Leftrightarrow$ $\mathcal{K}_{y^*}$ and qi $P \times D$ are properly linearly separable. The assertion follows from similar arguments in (ii) and Proposition 3.3.

In Theorem 3.3 of [24] and Proposition 23 of [25], the first conclusion is: (a)–(c) imply (d). This follows immediately from Proposition 3.5, where the generalized Slater condition (13) is weaker than (c) (i.e., the generalized Slater condition (12)). The second conclusion in Proposition 23 of [25] is: both (a) and (d) imply (b), (e), (f) and (b'). From Proposition 3.5, it is obvious that both (a) and (d) imply (b) and (b'), which yields $\sigma_{H(x^*)}(\theta^*) = 0$ from the proof of the necessity of Theorem 5.1 and as a consequence, (e) holds. Moreover, it follows from Theorem 6.2 (ii) that (f) is true.

Theorem 3.3 of [24] and Proposition 23 of [25] gave no necessary optimality conditions for SVOP, while Theorem 6.2 (i) does under a new assumption which do not

require the set $\mathcal{H}$ to have nonempty interior. Moreover, Theorems 6.3 and 6.4 provide saddle point sufficient and necessary optimality conditions for SVOP under certain assumptions.

6.2 Applications to VOP

For VOP, $\mathcal{K}_{f(x^*)} := (f(x^*) - f(\cdot), g(\cdot))(K)$, $\mathcal{E}_{f(x^*)} := \mathcal{K}_{f(x^*)} - (P \times D)$ and the generalized Lagrangian function $\mathcal{L}^\circ : K \times P^* \times D^* \rightarrow \mathbb{R}$ is defined by:

$$\mathcal{L}^\circ(x, \lambda, \theta) := \langle \lambda, f(x) - f(x^*) \rangle - \langle \theta, g(x) \rangle, \ \forall (x, \lambda, \theta) \in K \times P^* \times D^*,$$

where $x^* \in R_0$. From Theorems 4.4, 4.5, 5.1-5.4, we have the following:

Theorem 6.5 *Let* qi $P \neq \emptyset$, qi $D \neq \emptyset$ *and* $x^* \in R_0$. *Then, the following statements are true:*

  (i) *Assume that* g *is* $-D$-map on K. *Then,* $\mathcal{K}_{f(x^*)}$ *and* qi $P \times D$ *are (properly) linearly separable, if and only if there exists* $(\lambda^*, \theta^*) \in P^* \times D^* \backslash \{(0,0)\}$ *such that* $(x^*, \lambda^*, \theta^*)$ *is a saddle point for* $\mathcal{L}^\circ$ *defined on* $K \times P^* \times D^*$ *with* $0 < \mathcal{L}^\circ(x^0, \lambda^*, \theta^*)$ *for some* $x^0 \in K$;
  (ii) *Assume that* g *is* $-D$-map on K. *Then,* $\mathcal{K}_{f(x^*)}$ *and* qi $P \times D$ *admit a regular linear separation, if and only if there exists* $(\lambda^*, \theta^*) \in P^* \times D^*$ *with* $\lambda^* \neq 0$ *such that* $(x^*, \lambda^*, \theta^*)$ *is a saddle point for* $\mathcal{L}^\circ$ *defined on* $K \times P^* \times D^*$;
  (iii) *Assume that* qi (cone conv $\mathcal{E}_{f(x^*)}) - (P \backslash$ qi $P \times D) \subseteq$ cone conv $\mathcal{E}_{f(x^*)} - ($qi $P \times D)$ *and* $(-f, g)$ *is* $-(P \times D)$-convexlike on K. *If* x^* *is a weakly efficient solution of* VOP, *then* $\mathcal{K}_{f(x^*)}$ *and* qi $P \times D$ *are properly linearly separable;*
  (iv) *If* $\mathcal{K}_{f(x^*)}$ *and* qi $P \times D$ *are linearly separable and the generalized Slater condition* (13) *with* $H := g$ *holds, or* $\mathcal{K}_{f(x^*)}$ *and* qi $P \times D$ *admit a regular linear separation, then* x^* *is a weakly efficient solution of* VOP.
  (v) *Assume that* qi (cone conv $\mathcal{E}_{f(x^*)}) - (\{0\} \times D) \subseteq$ cone conv $\mathcal{E}_{f(x^*)} - (P \backslash \{0\} \times D)$ *and* $(-f, g)$ *is* $-(P \times D)$-convexlike on K. *If* x^* *is an efficient solution of* VOP, *then* $\mathcal{K}_{f(x^*)}$ *and* $P \backslash \{0\} \times D$ *are properly linearly separable;*
  (vi) *If there is* $(\lambda^*, \theta^*) \in$ qi $(P^*) \times D^*$ *such that* (14) *with* $\mathcal{K}_{\bar{y}} := \mathcal{K}_{f(x^*)}$ *holds, then* x^* *is an efficient solution of* VOP.

Theorem 6.6 *Assume that* ri $K \neq \emptyset$, f *and* g *are* P-map and $-D$-map on K, *respectively. Then, there exists* $(\lambda^*, \theta^*) \in P^* \times D^* \backslash \{(0,0)\}$ *such that* $(x^*, \lambda^*, \theta^*)$ *is a saddle point for* $\mathcal{L}^\circ$ *defined on* $K \times P^* \times D^*$, *if and only if it is a solution of the following system,*

$$\begin{aligned} &0 \in \partial(\lambda \circ f)(x) + \partial(-\theta \circ g)(x) + N_K(x), \\ &0 = \langle \theta, g(x) \rangle, \\ &x \in R_0, \ (\lambda, \theta) \in P^* \times D^*, \ \text{with } (\lambda, \theta) \neq (0,0), \end{aligned}$$

where $(\lambda \circ f)(x) := \langle \lambda, f(x) \rangle$, $(\theta \circ g)(x) := \langle \theta, g(x) \rangle$ *and* ∂ *denotes the subdifferential.*

Proof Similar to Theorem 5.3 in [7], the conclusion can be proved by using the well-known Moreau–Rockafellar theorem (see, e.g, Theorem 4.2.3 in [46]). □

The following conclusion follows from Theorem 6.6.

Corollary 6.1 *Assume that K is open, that is, $K = \text{int}\,K$, f and g are P-map and $-D$-map on K, respectively. Assume that for any $\lambda \in P^*$ and $\theta \in D^*$, $\lambda \circ f$ and $-\theta \circ g$ are Gâteaux derivable and locally bounded at some point of X. Then, there exists $(\lambda^*, \theta^*) \in P^* \times D^* \backslash \{(0,0)\}$ such that $(x^*, \lambda^*, \theta^*)$ is a saddle point for $\mathcal{L}^\circ$ defined on $K \times P^* \times D^*$, if and only if it is a solution of the following system,*

$$0 = \nabla(\lambda \circ f)(x) + \nabla(-\theta \circ g)(x),$$
$$0 = \langle \theta, g(x) \rangle,$$
$$x \in R_0, \ (\lambda, \theta) \in P^* \times D^*, \ \text{with} \ (\lambda, \theta) \neq (0, 0),$$

where ∇ denotes the Gâteaux derivative.

Proof Since f and g are P-map and $-D$-map on K, respectively, we have for any $\lambda \in P^*$ and $\theta \in D^*$, $\lambda \circ f : X \to \mathbb{R}$ and $-\theta \circ g : X \to \mathbb{R}$ are convex. Note that $\emptyset \neq K = \text{int}\,K \subseteq \text{int}\,X$. Since for any $\lambda \in P^*$ and $\theta \in D^*$, $\lambda \circ f$ and $-\theta \circ g$ are locally bounded at some point of X, from Theorem 4.1 in [47], we have $\lambda \circ f$ and $\theta \circ g$ are continuous on $\text{int}\,X$. Now, from Corollary 4.1.1 in [46], we have $\partial(\lambda \circ f)(x) = \{\nabla(\lambda \circ f)(x)\}$ and $\partial(-\theta \circ g)(x) = \{\nabla(-\theta \circ g)(x)\}$, where $x \in K$. Therefore, the conclusion follows immediately from Theorem 6.6. □

7 Conclusions

We have investigated a set-valued system with infinite-dimensional image by exploiting the quasi-relative interior and the quasi-interior, and we have obtained some new necessary and/or sufficient conditions for the impossibility of this set-valued system. Furthermore, these new results have been applied to the investigation of vector optimization problems.

As pointed by an anonymous referee, the main statements in this paper assumed the nonemptiness of the quasi-interior of the involved sets, which is definitively very strong. Essentially speaking, these results are based on Proposition 3.4 (i) or Lemma 3.1 or an important feature of the quasi-interior (see, e.g., Theorem 2.2 of [19], Theorem 3.10 of [16] and Proposition 2.6 of [23]), however, which does not hold when the quasi-relative interior of $\mathcal{H}$ is nonempty. In Theorem 4.3, technical assumptions related to the quasi-interior and the quasi-relative interior were given to obtain the necessary conditions for the impossibility of systems (1) and (2). Of course, it is not easy to check these assumptions because of the computational complexity of the quasi-interior and the quasi-relative interior (compared with that of the interior). As a consequence, it is interesting to obtain similar results under the nonemptiness of the quasi-relative interior of the involved sets and give the necessary conditions (i.e., linear separation) for impossibility of systems (1) and (2) under other suitable assumptions.

We investigated the impossibility of system (2) with $C := B \backslash \{0\}$, where $B \subseteq U$ is a closed, convex, and proper cone (see, Theorem 4.5 (ii) and Theorem 5.4 (ii)),

and proved existence of efficient solutions of VOP (see, Theorem 6.5 (v)) under the assumption of strongly regular linear separation (see, Remark 3.3). It is also interesting to obtain these results under other assumptions such as strict convexity. We leave these questions for future research.

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