J Optim Theory Appl (2018) 177:743–769
https://doi.org/10.1007/s10957-016-1027-6

Image Space Analysis to Lagrange-Type Duality for Constrained Vector Optimization Problems with Applications

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Received: 26 March 2016 / Accepted: 13 October 2016 / Published online: 19 October 2016

Abstract The main purpose of this paper is to study the duality and penalty method for a constrained nonconvex vector optimization problem. Following along with the image space analysis, a Lagrange-type duality for a constrained nonconvex vector optimization problem is proposed by virtue of the class of vector-valued regular weak separation functions in the image space. Simultaneously, some equivalent characterizations to the zero duality gap property are established including the Lagrange multiplier, the Lagrange saddle point and the regular separation. Moreover, an exact penalization is also obtained by means of a local image regularity condition and a class of particular regular weak separation functions in the image space.

Keywords Image space analysis · Vector optimization · Lagrange-type duality · Exact penalization · Image regularity condition

Mathematics Subject Classification 49N15 · 90C30 · 90C46

1 Introduction

The present paper is concerned with a Lagrange-type duality and penalty method for a constrained nonconvex vector optimization problem by virtue of the image space analysis (for short, ISA). Just as we know, the traditional Lagrange duality is an important method for constrained scalar and vector convex optimization problems. Among several crucial aspects, the zero duality gap property between the primal problem and

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the corresponding Lagrange dual problem plays a key role, such as optimality conditions, stability and sensitivity analysis and solution methods [1–3]. However, the zero duality gap property may not hold for nonconvex optimization problems. Simultaneously, the classic Lagrange penalty method may also lose effectiveness. In order to avoid the nonzero duality gap, various kinds of generalized Lagrange-type functions, especially the augmented Lagrangian function, introduced by Hestenes [4,5] and Powell [6], and the nonlinear Lagrangian function, introduced by Rubinov et al. [7], have been proposed and applied to study scalar and vector nonconvex optimization problems. We refer to [7–10] and references therein for more details.

In the last decades, the ISA has attracted great interest in the academic and professional communities and has been used as a preliminary and auxiliary step for investigating some subjects both in the optimization theory and methodology for constrained optimization problems, such as optimality conditions [11–17], dualities [18–22], variational principles [23], existence of solutions [24], penalty methods [25,26], regularities and stabilities [27–29]. Since the path-breaking paper [13], it has shown that the ISA is a unified scheme to study any kind of problem, which can be expressed by the form of the impossibility of a parametric system. Simultaneously, the image space (for short, IS) of the constrained optimization problem provides a natural environment for the Lagrange duality and penalty method. By virtue of the ISA, Giannessi [18] showed that the classic Lagrange duality for constrained scalar optimization problems can be derived from regular separations in the IS. Subsequently, this idea was further extended in [19] to investigate relationships between Wolfe and Mond–Weir dualities, especially their equivalence in the IS under some generalized convexity assumptions. Mastroeni [20] studied some duality properties of a constrained scalar optimization problem by means of the ISA and established some equivalent characterizations for the zero duality gap property in the IS. Moreover, Mastroeni [26] analyzed a nonlinear separation scheme in the IS associated with an infinite-dimensional cone constrained scalar optimization problem and established some relationships between the penalty methods and corresponding nonlinear separations. By exploiting the ISA, Zhu and Li [21,22] proposed a unified duality scheme for a constrained scalar optimization problem in terms of the class of regular weak separation functions and obtained some necessary and sufficient conditions for the zero duality gap property, including the Lagrange saddle point, the Lagrange multiplier, the lower semicontinuity of the perturbation function and a regular separation in the IS, without any convexity assumption.

However, the study on the duality and penalty method for constrained nonconvex vector optimization problems by virtue of the ISA is very limited. In [30,31], several theoretical and methodological aspects of constrained vector optimization problems and vector variational inequalities, such as optimality conditions, dualities and scalarization, were investigated by virtue of the ISA. For more details, we refer to [32–34] and reference therein.

Motivated by the work reported in [9,12,21,22,30,31], the purpose of this paper is to study the duality and penalty method for a constrained nonconvex vector optimization problem by virtue of the ISA. To the end, we establish a Lagrange-type dual problem via the class of vector-valued regular weak separation functions in the IS. Under some appropriate assumptions, we obtain some equivalent characterizations to the zero duality gap property, including the Lagrange multiplier, the Lagrange saddle point and the regular separation. Simultaneously, we also establish an exact penalization result by means of a local image regularity condition and a class of particular regular weak separation functions in the IS.

The organization of this paper is as follows: In Sect. 2, we recall some preliminaries, particularly the concept of separation functions in the ISA. In Sect. 3, we propose a Lagrange-type dual problem via the class of vector-valued regular weak separation functions in the IS and study the zero duality gap property. In Sect. 4, we consider the applications to exact penalization methods. In Sect. 5, some conclusions are given.

2 Preliminaries and Separation Functions in the ISA

Throughout this paper, all vectors are viewed as column vectors. Let $\mathbb{R}^\ell$ be an ℓ-dimensional Euclidean space, $C = \mathbb{R}^\ell_+$, and $\operatorname{int} C$ be the interior of C. When there is no fear of confusion, we always denote by 0 the origin of different spaces. As usual, we denote by x^T the transpose of x for every $x \in \mathbb{R}^\ell$. In this paper, we use the following orderings: for all $x, y \in \mathbb{R}^\ell$,

$$\begin{aligned} x \leq_C y &\Leftrightarrow y - x \in C, \quad x \nleq_C y \Leftrightarrow y - x \notin C;\\ x \leq_{C\setminus\{0\}} y &\Leftrightarrow y - x \in C \setminus \{0\}, \quad x \nleq_{C\setminus\{0\}} y \Leftrightarrow y - x \notin C \setminus \{0\};\\ x \leq_{\operatorname{int} C} y &\Leftrightarrow y - x \in \operatorname{int} C, \quad x \nleq_{\operatorname{int} C} y \Leftrightarrow y - x \notin \operatorname{int} C. \end{aligned}$$

In addition, $y \geq_C x$ ($y \ngeq_C x$) means that $x \leq_C y$ ($x \nleq_C y$). For other above-mentioned orderings, analogous relations can be defined. Let $\bar{\mathbb{R}}^\ell := \mathbb{R}^\ell \cup \{\pm\infty\}$, where $+\infty$ ($-\infty$) is an imaginary point with each coordinate $+\infty$ ($-\infty$). This shows $-\infty \leq_{C\setminus\{0\}} z \leq_{C\setminus\{0\}} +\infty$ and $-\infty \leq_{\operatorname{int} C} z \leq_{\operatorname{int} C} +\infty$ for every $z \in \mathbb{R}^\ell$. Moreover, $-\infty \leq_C z \leq_C +\infty$ and $+\infty \nleq_{C\setminus\{0\}} z \nleq_{C\setminus\{0\}} -\infty$ hold for every $z \in \bar{\mathbb{R}}^\ell$. Without possibility of confusion, we shall not differentiate the $-\infty$ and $+\infty$ in $\bar{\mathbb{R}}^\ell$, and the $-\infty$ and $+\infty$ in the extended real number space $\bar{\mathbb{R}}$. We extend the addition and the scalar multiplication from $\mathbb{R}^\ell$ to $\bar{\mathbb{R}}^\ell$ by using the following conventions:

$$\begin{aligned} &z + (\pm\infty) = (\pm\infty) + z = \pm\infty, \quad \forall z \in \mathbb{R}^\ell,\\ &\lambda(\pm\infty) = \pm\infty, \ \forall \lambda > 0 \ \text{ and } \ \lambda(\pm\infty) = \mp\infty, \ \forall \lambda < 0,\\ &(\pm\infty)e = \pm\infty, \ \forall e \in \operatorname{int} C. \end{aligned}$$

Definition 2.1 [9,35] Let $A \subset \mathbb{R}^\ell$ be a nonempty subset. Denote the set of all infimum points of A by $\operatorname{Inf} A$. For any $z^* \in \operatorname{Inf} A$, we mean that

(i) $z^* \in \bar{\mathbb{R}}^\ell$;
(ii) $z \nleq_{C\setminus\{0\}} z^*, \forall z \in A$;
(iii) there exists a sequence $\{z^k\} \subset A$ such that $z^k \to z^*$ as $k \to +\infty$.

The point $z^* \in \operatorname{Inf} A$ is said to be an infimum point of A. Similarly, we can define the set of all supremum points of A by $\operatorname{Sup} A$, and $z^* \in \operatorname{Sup} A$ if and only if $-z^* \in \operatorname{Inf}(-A)$.

Remark 2.1 Clearly, $\mathrm{Sup}\,A = -\mathrm{Inf}\,(-A)$ and $\mathrm{Sup}\,(z + A) = z + \mathrm{Sup}\,A$ hold for all nonempty subset $A \subset \mathbb{R}^\ell$ and all $z \in \mathbb{R}^\ell$.

Let $F : X \rightrightarrows \mathbb{R}^\ell$ be a set-valued map. We denote the set of all infimum (supremum) points for F on X by $\inf_{x\in X} F(x)$ ($\sup_{x\in X} F(x)$), that is,

$$\inf_{x\in X} F(x) := \mathrm{Inf} \bigcup_{x\in X} F(x) \quad \Big(\sup_{x\in X} F(x) := \mathrm{Sup} \bigcup_{x\in X} F(x)\Big).$$

Therefore, for every $z^* \in \inf_{x\in X} F(x)$, it follows from Definition 2.1 that $z^* \in \bar{\mathbb{R}}^\ell$, $(z^* - F(x)) \cap (C\setminus\{0\}) = \varnothing$, $\forall x \in X$ and $\exists\, \{x^k\} \subset X$, $\exists\, z^k \in F(x^k)$ such that $z^k \to z^*$ as $k \to +\infty$. Analogously, we can give the similar explanation to $\sup_{x\in X} F(x)$. Specially, when $F = f : X \to \mathbb{R}^\ell$ is a vector-valued function, the set of all infimum (supremum) points for f on X is denoted by $\inf_{x\in X} f(x)$ ($\sup_{x\in X} f(x)$).

Next, we recall the following useful concept called nonlinear scalarization function and some of its properties.

Lemma 2.1 *[10,36] Given $e \in \mathrm{int}\, C$, the nonlinear scalarization function $\xi_e : \mathbb{R}^\ell \to \mathbb{R}$, defined by*

$$\xi_e(y) := \inf\{\alpha \in \mathbb{R} : y \in \alpha e - C\}, \quad \forall y \in \mathbb{R}^\ell,$$

is convex, strictly int C*-monotone, C-monotone, nonnegative homogeneous and globally Lipschitz. Simultaneously, for every $\alpha \in \mathbb{R}$, it follows that*

$$\{y \in \mathbb{R}^\ell : \xi_e(y) \le \alpha\} = \alpha e - C \quad \textit{and} \quad \{y \in \mathbb{R}^\ell : \xi_e(y) < \alpha\} = \alpha e - \mathrm{int}\, C.$$

Consider the following constrained vector optimization problem

$$\text{(VOP)} \qquad \min f(x), \quad \text{s.t. } x \in X,\, g_j(x) \ge 0, \quad j = 1, 2, ..., m,$$

where $X \subset \mathbb{R}^n$ is a nonempty subset, $f = (f_1, f_2, ..., f_\ell) : X \to \mathbb{R}^\ell$ and $g = (g_1, g_2, ..., g_m) : X \to \mathbb{R}^m$ are vector-valued functions with each component function $f_i : X \to \mathbb{R}$, $i \in \{1, 2, ..., \ell\}$, and $g_j : X \to \mathbb{R}$, $j \in \{1, 2, ..., m\}$, respectively. As usual, we denote by $\mathcal{R} = \{x \in X : g_j(x) \ge 0, j = 1, 2, ..., m\}$ the feasible set of (VOP). Clearly, one has $\mathcal{R} = \{x \in X : g(x) \ge_D 0\}$ where $D = \mathbb{R}^m_+$. Throughout this paper, we assume that $\mathcal{R} \ne \varnothing$.

Recall that a point $\hat{x} \in \mathcal{R}$ is said to be a strong (weak) vector minimum point (in short, *vmp*) to (VOP) iff $f(x) \not\le_{C\setminus\{0\}} f(\hat{x})$ ($f(x) \not\le_{\mathrm{int}\, C} f(\hat{x})$) for all $x \in \mathcal{R}$. The point $\hat{x} \in \mathcal{R}$ is said to be a local strong (local weak) *vmp* to (VOP) iff there exists $\delta > 0$ such that $f(x) \not\le_{C\setminus\{0\}} f(\hat{x})$ ($f(x) \not\le_{\mathrm{int}\, C} f(\hat{x})$) for all $x \in \mathcal{R} \cap \mathcal{B}(\hat{x}, \delta)$, where $\mathcal{B}(\hat{x}, \delta)$ denotes the open ball with center at $\hat{x}$ and radius δ. Clearly, every strong *vmp* (local strong *vmp*) must be a weak *vmp* (local weak *vmp*). Moreover, let $\hat{x} \in \mathcal{R}$. Then, $\hat{x}$ is a strong *vmp* to (VOP) if and only if $f(\hat{x}) \in \inf_{x\in\mathcal{R}} f(x)$.

Take arbitrary $\bar{x} \in X$. We consider the vector-valued map $A : X \to \mathbb{R}^\ell \times \mathbb{R}^m$ with

$$A(x) := (f(\bar{x}) - f(x), g(x)), \quad \forall x \in X,$$

and the sets

$$\mathcal{K} := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u = f(\bar{x}) - f(x), v = g(x), x \in X\},$$
$$\mathcal{H} := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u \geq_{C\setminus\{0\}} 0, v \geq_D 0\} = (C \setminus \{0\}) \times D.$$

Note that $\mathcal{K}$ is the image of the map A, that is, $\mathcal{K} = A(X)$. Moreover, the sets $\mathcal{K}$ and $\mathcal{H}$ are subsets of $\mathbb{R}^\ell \times \mathbb{R}^m$. We recall from [12,21,30,31] that $\mathbb{R}^\ell \times \mathbb{R}^m$ is said to be the *image space* associated with (VOP) and $\mathcal{K}$ is said to be the *image* of (VOP). Note that the image $\mathcal{K}$ does not generally enjoy some convexity property. To overcome this difficulty, a regularization of the image $\mathcal{K}$, called *conic extension* with respect to the cone $\operatorname{cl}\mathcal{H} = C \times D$, denoted by

$$\mathcal{E} := \mathcal{K} - \operatorname{cl}\mathcal{H} = \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u \leq_C f(\bar{x}) - f(x), v \leq_D g(x), x \in X\}$$

was introduced in [12,30,31]. As a result, the convexity of $\mathcal{E}$ can be verified under some appropriate conditions; for example, A is $-(\operatorname{cl}\mathcal{H})$-convexlike. We refer to [12, 19,30,31] and references therein for more details. Take specially $\bar{x} \in \mathcal{R}$. Then, it is easy to verify that $\bar{x}$ is a strong *vmp* to (VOP) if and only if $\mathcal{K} \cap \mathcal{H} = \emptyset$. Since $\mathcal{H} + \operatorname{cl}\mathcal{H} = \mathcal{H}$, it follows that $\mathcal{K} \cap \mathcal{H} = \emptyset$ if and only if $\mathcal{E} \cap \mathcal{H} = \emptyset$, or equivalently, $\mathcal{E} \cap \mathcal{H}_u = \emptyset$, where

$$\mathcal{H}_u := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u \geq_{C\setminus\{0\}} 0, v = 0\} = (C \setminus \{0\}) \times \{0\}.$$

Remark 2.2 It is worth noting that the choice of $\bar{x}$ is arbitrary throughout this paper, unless otherwise specified. Based on this fact, we use the notations A and $\mathcal{K}$, although they seem to be dependent on the point $\bar{x}$. Just as shown in [12], even if it is a mere formal question, it is not convenient, and it is better to define u as above in $\mathcal{K}$; see more details in [12,19–21,27,30,31].

Consider a vector-valued function $w : \mathbb{R}^\ell \times \mathbb{R}^m \times \Pi \to \mathbb{R}^\ell$, where Π is a set of parameters to be specified case by case. Throughout this paper, we shall always use the same symbol Π to denote the set of parameters unless otherwise specified. For every $\pi \in \Pi$, the nonnegative level set and the positive level set of the function $w(\bullet; \pi) : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^\ell$ associated with the cone C are, respectively, defined by

$$\operatorname{lev}_C w(\bullet; \pi) := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : w(u, v; \pi) \geq_C 0\}$$

and

$$\operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi) := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : w(u, v; \pi) \geq_{C\setminus\{0\}} 0\}.$$

Next, we recall the following concepts of vector-valued weak separation and regular weak separation functions, respectively.

Definition 2.2 [30,31] The class of all the vector-valued functions $w : \mathbb{R}^\ell \times \mathbb{R}^m \times \Pi \to \mathbb{R}^\ell$ such that

$$\mathcal{H} \subset \mathrm{lev}_C\, w(\bullet; \pi), \quad \forall \pi \in \Pi \tag{1}$$

and

$$\bigcap_{\pi \in \Pi} \mathrm{lev}_{C \setminus \{0\}}\, w(\bullet; \pi) \subset \mathcal{H} \tag{2}$$

is called the class of weak separation functions and is denoted by $\mathcal{W}(\Pi)$.

Definition 2.3 [30,31] The class of all the vector-valued functions $w : \mathbb{R}^\ell \times \mathbb{R}^m \times \Pi \to \mathbb{R}^\ell$ such that

$$\bigcap_{\pi \in \Pi} \mathrm{lev}_{C \setminus \{0\}}\, w(\bullet; \pi) = \mathcal{H} \tag{3}$$

is called the class of regular weak separation functions and is denoted by $\mathcal{W}_R(\Pi)$.

In this paper, we shall pay our attention to establish a Lagrange-type dual problem for (VOP) with applications to exact penalty properties by using the class of regular weak separation functions. To this end, we always assume that the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ enjoys the following two properties:

Assumption $\mathcal{A}$ $w \in \mathcal{W}_R(\Pi)$ satisfies $w(u, 0; \pi) = u$, $\forall \pi \in \Pi$, and

$$\inf_{\pi \in \Pi} w(u, v; \pi) = \begin{cases} u, & \text{if } v \in D, \\ -\infty, & \text{if } v \notin D. \end{cases}$$

Assumption $\mathcal{B}$ $w \in \mathcal{W}_R(\Pi)$ is monotone increasing with respect to the first argument, that is, $w(u^2, v^2; \pi) \leq_C w(u^1, v^1; \pi)$ holds for all $\pi \in \Pi$ and all $(u^i, v^i) \in \mathbb{R}^\ell \times \mathbb{R}^m$, $i = 1, 2$ with $(u^2, v^2) \leq_{C \times D} (u^1, v^1)$.

Remark 2.3 Note that Assumption $\mathcal{A}$ requires the set of all infimum points for the vector-valued function $w(u, v; \bullet)$ on Π to be a single point set. Throughout this paper, we denote the single point set $\{u\}$ by u for simplicity unless otherwise stated.

Next, we consider some important examples for the classes of vector-valued regular weak separation functions which satisfy Assumptions $\mathcal{A}$ and $\mathcal{B}$.

Example 2.1 Recall from [30] that the vector polar of D with respect to C is given by

$$D_C^* := \{M \in \mathbb{R}^{\ell \times m} : Md \geq_C 0, \ \forall d \in D\},$$

where $\mathbb{R}^{\ell \times m}$ denotes the set of all matrices with real entries and with ℓ rows and m columns. It is easy to verify that

$$D_C^* = \{M = (\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}^{\ell \times m} : \alpha_j \in C, \ \forall j = 1, 2, ..., m\},$$

where α_j is the j-th column of M. Moreover, let $J := C \setminus \{0\}$. Then, we have

$$J_C^* = \{M = (\beta_1, \beta_2, ..., \beta_\ell) \in \mathbb{R}^{\ell \times \ell} : \beta_i \in C, \ \forall i = 1, 2, ..., \ell\}$$

and

$$J^*_{C\setminus\{0\}} = \{M = (\beta_1, \beta_2, ..., \beta_\ell) \in \mathbb{R}^{\ell\times\ell} : \beta_i \in C \setminus \{0\},\ \forall i = 1, 2, ..., \ell\},$$

where β_i is the i-th column of M. On the one hand, let $\Pi = (J^*_C \times D^*_C) \setminus \{0\}$ and the vector-valued function $w : \mathbb{R}^\ell \times \mathbb{R}^m \times \Pi \to \mathbb{R}^\ell$ be defined by

$$w(u, v; \Theta, \Lambda) := \Theta u + \Lambda v,\ \ \forall (u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m,\ \forall (\Theta, \Lambda) \in (J^*_C \times D^*_C) \setminus \{0\}.$$

Then, $w \in \mathcal{W}(\Pi)$. On the other hand, let $\Pi = J^*_{C\setminus\{0\}} \times D^*_C$ and the vector-valued function $w : \mathbb{R}^\ell \times \mathbb{R}^m \times J^*_{C\setminus\{0\}} \times D^*_C \to \mathbb{R}^\ell$ be defined by

$$w(u, v; \Theta, \Lambda) := \Theta u + \Lambda v,\ \ \forall (u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m,\ \forall (\Theta, \Lambda) \in J^*_{C\setminus\{0\}} \times D^*_C.$$

Then, we have $w \in \mathcal{W}_R(\Pi)$. We refer to [30] for more details. Obviously, the function w satisfies Assumption $\mathcal{B}$. However, Assumption $\mathcal{A}$ does not hold. In fact, by direct calculating, we have $w(u, 0; \Theta, \Lambda) = \Theta u$ and

$$\inf_{(\Theta, \Lambda)\in\Pi} w(u, v; \Theta, \Lambda) = \begin{cases} 0, & \text{if } u \in C, v \in D, \\ -\infty, & \text{if } u \notin C, v \in D \text{ or } v \notin D. \end{cases}$$

Specially, let $\Pi = E_\ell \times D^*_C$, where E_ℓ denotes the identity matrix of order ℓ, and let the vector-valued function $w : \mathbb{R}^\ell \times \mathbb{R}^m \times E_\ell \times D^*_C \to \mathbb{R}^\ell$ be defined by

$$w(u, v; E_\ell, \Lambda) := u + \Lambda v,\ \ \forall (u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m,\ \forall \Lambda \in D^*_C.$$

Then, $w \in \mathcal{W}_R(\Pi)$. Simultaneously, we also have $w(u, 0; E_\ell, \Lambda) = u$ and

$$\inf_{(E_\ell, \Lambda)\in\Pi} w(u, v; E_\ell, \Lambda) = \begin{cases} u, & \text{if } v \in D, \\ -\infty, & \text{if } v \notin D. \end{cases}$$

In fact, for every $v \in D$, we have $\Lambda v \geq_C 0$ and $w(u, v; E_\ell, \Lambda) = u + \Lambda v \geq_C u$ for all $\Lambda \in D^*_C$. Together with $0 \in D^*_C$ and $w(u, v; E_\ell, 0) = u$, it follows that $u \in \inf_{(E_\ell, \Lambda)\in\Pi} w(u, v; E_\ell, \Lambda)$. Moreover, if $z \in \inf_{(E_\ell, \Lambda)\in\Pi} w(u, v; E_\ell, \Lambda)$, then we have by Definition 2.1 that $w(u, v; E_\ell, \Lambda) \not\leq_{C\setminus\{0\}} z$ for all $\Lambda \in D^*_C$ and that there exists a sequence $w(u, v; E_\ell, \Lambda^k) \to z$ with $\Lambda^k \in D^*_C$. Thus, we get $u = w(u, v; E_\ell, 0) \not\leq_{C\setminus\{0\}} z$. Moreover, since C is closed, and $\Lambda^k v \geq_C 0$ and $w(u, v; E_\ell, \Lambda^k) \geq_C u$ for all $\Lambda^k \in D^*_C$, it follows from $w(u, v; E_\ell, \Lambda^k) \to z$ that $z \geq_C u$. Together with $u \not\leq_{C\setminus\{0\}} z$, we have $z = u$. Thus, we can conclude that

$$\inf_{(E_\ell, \Lambda)\in\Pi} w(u, v; E_\ell, \Lambda) = u,\ \ \forall v \in D.$$

In addition, take arbitrary $v \notin D$. Then, there exists $j_0 \in \{1, 2, ..., m\}$ such that $v_{j_0} < 0$. Let $\Lambda^k = (0, ..., \alpha^k_{j_0}, ..., 0)$ with the j_0-th column $\alpha^k_{j_0} = (k, k, ..., k)^T$ and other columns 0. Then, we have $\Lambda^k \in D_C^*$ for all $k \in \mathbb{N}$. Moreover, it follows that $w(u, v; E_\ell, \Lambda^k) = u + \Lambda^k v = u + v_{j_0}\alpha_{j_0}^k \to -\infty$ as $k \to +\infty$, which implies

$$\inf_{(E_\ell, \Lambda)\in\Pi} w(u, v; E_\ell, \Lambda) = -\infty, \quad \forall v \notin D.$$

Therefore, we can conclude that Assumption $\mathcal{A}$ holds. Simultaneously, for every $(u^i, v^i) \in \mathbb{R}^\ell \times \mathbb{R}^m$, $i = 1, 2$ with $(u^2, v^2) \leq_{C\times D} (u^1, v^1)$. Then, we have $u^2 \leq_C u^1$ and $v^2 \leq_D v^1$, which implies $u^1 - u^2 \in C$, $v^1 - v^2 \in D$ and $\Lambda(v^1 - v^2) \geq_C 0$ for all $\Lambda \in D_C^*$. Thus, we get

$$\begin{aligned} &w(u^1, v^1; E_\ell, \Lambda) - w(u^2, v^2; E_\ell, \Lambda) \\ &\quad = (u^1 - u^2) + \Lambda(v^1 - v^2) \in C + C \subset C, \quad \forall \Lambda \in D_C^*, \end{aligned}$$

that is, $w(u^2, v^2; E_\ell, \Lambda) \leq_C w(u^1, v^1; E_\ell, \Lambda)$ for all $\Lambda \in D_C^*$. Therefore, Assumption $\mathcal{B}$ also holds.

Example 2.2 Let $\Pi = D$ and $w : \mathbb{R}^\ell \times \mathbb{R}^m \times \Pi \to \mathbb{R}^\ell$. Given a point $e \in \operatorname{int} C$, let

$$w(u, v; \pi) := u + \Big(\sup_{z\leq_D v}\Big(\langle\pi, z\rangle - r\sigma(z)\Big)\Big)e, \quad \forall (u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m, \ \forall \pi \in D,$$

where $r > 0$ is a real constant and the augmented function $\sigma : \mathbb{R}^m \to \mathbb{R}$ satisfies

$$\arg\min_{z\in\mathbb{R}^m} \sigma(z) = \{0\}, \quad \sigma(0) = 0.$$

Then, we have $w \in \mathcal{W}_R(\Pi)$, and Assumptions $\mathcal{A}$ and $\mathcal{B}$ hold. Next, we first show that the vector-valued function w is a regular weak separation function. On the one hand, for every $(u, v) \in \mathcal{H}$ and $\pi \in D$, we get $u \in C \setminus \{0\}$, $v \in D$ and

$$\sup_{z\leq_D v}\Big(\langle\pi, z\rangle - r\sigma(z)\Big) \geq \langle\pi, 0\rangle - r\sigma(0) = 0.$$

Together with $e \in \operatorname{int} C$, we get

$$w(u, v; \pi) = u + \Big(\sup_{z\leq_D v}\Big(\langle\pi, z\rangle - r\sigma(z)\Big)\Big)e \in C\setminus\{0\} + C \subset C\setminus\{0\}, \quad \forall \pi \in D,$$

which implies

$$\mathcal{H} \subset \bigcap_{\pi\in D} \operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi).$$

On the other hand, take arbitrary

$$(u, v) \in \bigcap_{\pi\in D} \operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi).$$

Then, we have $w(u, v; \pi) \geq_{C\setminus\{0\}} 0$ for all $\pi \in D$. Specially, let $\pi = 0 \in D$. Then, we get

$$w(u, v; 0) = u + \Big(\sup_{z \leq_D v} \big(\langle 0, z\rangle - r\sigma(z)\big)\Big)e = u - r\Big(\inf_{z \leq_D v} \sigma(z)\Big)e \geq_{C\setminus\{0\}} 0.$$

Together with $r > 0$, $\min_{z\in\mathbb{R}^m} \sigma(z) = 0$ and $e \in \operatorname{int} C$, we have

$$u \geq_{C\setminus\{0\}} r\Big(\inf_{z \leq_D v} \sigma(z)\Big)e \geq_C 0,$$

which implies $u \geq_{C\setminus\{0\}} 0$. Moreover, we have $v \in D$. Otherwise, there exists $j_0 \in \{1, 2, ..., m\}$ such that $v_{j_0} < 0$. Since $e \in \operatorname{int} C$, there exists $\rho > 0$ such that $u + \rho v_{j_0} e \leq_{C\setminus\{0\}} 0$. Specially, let $\tilde{\pi} \in \mathbb{R}^m$ with $\tilde{\pi}_{j_0} = \rho$ and $\tilde{\pi}_j = 0$ for every $j \in \{1, 2, ..., m\} \setminus \{j_0\}$. Then, it follows that $\tilde{\pi} \in D$, which implies $w(u, v; \tilde{\pi}) \geq_{C\setminus\{0\}} 0$ by the choice of (u, v). Moreover, we get from $r > 0$, $\min_{z\in\mathbb{R}^m} \sigma(z) = 0$ and $u + \rho v_{j_0} e \leq_{C\setminus\{0\}} 0$ that

$$\sup_{z \leq_D v} \big(\langle \tilde{\pi}, z\rangle - r\sigma(z)\big) \leq \sup_{z \leq_D v} \langle \tilde{\pi}, z\rangle = \sup_{z_{j_0} \leq v_{j_0}} \rho z_{j_0} = \rho v_{j_0}$$

and

$$\begin{aligned} w(u, v; \tilde{\pi}) &= u + \Big(\sup_{z \leq_D v} \big(\langle \tilde{\pi}, z\rangle - r\sigma(z)\big)\Big)e \\ &= u + \rho v_{j_0} e + \Big(\sup_{z \leq_D v} \big(\langle \tilde{\pi}, z\rangle - r\sigma(z)\big) - \rho v_{j_0}\Big)e \\ &\in -C \setminus \{0\} - C \subset -C \setminus \{0\}, \end{aligned}$$

which shows $w(u, v; \tilde{\pi}) \leq_{C\setminus\{0\}} 0$. This contradicts $w(u, v; \tilde{\pi}) \geq_{C\setminus\{0\}} 0$. Then, we have

$$\bigcap_{\pi\in D} \operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi) \subset \mathcal{H}.$$

Thus, we can conclude that

$$\bigcap_{\pi\in D} \operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi) = \mathcal{H}.$$

that is, $w \in \mathcal{W}_R(\Pi)$ is a regular weak separation function. Second, we prove that Assumption $\mathcal{A}$ holds. Note that

$$w(u, 0; \pi) = u + \Big(\sup_{z \leq_D 0} \big(\langle \pi, z\rangle - r\sigma(z)\big)\Big)e, \ \forall \pi \in D.$$

Since $\sigma(0) = 0$, we have $\sup_{z \leq_D 0} \big(\langle \pi, z\rangle - r\sigma(z)\big) \geq \langle \pi, 0\rangle - r\sigma(0) = 0$ for all $\pi \in D$. Moreover, for every $\pi \in D$ and $z \leq_D 0$, we have $\langle \pi, z\rangle \leq 0$. Together with $\sigma(z) \geq 0$ and $r > 0$, we get $\langle \pi, z\rangle - r\sigma(z) \leq 0$ for all $\pi \in D$ and all $z \leq_D 0$, which implies $\sup_{z\leq_D 0}\Big(\langle \pi, z\rangle - r\sigma(z)\Big) \leq 0$ for all $\pi \in D$. Thus, we immediately have $\sup_{z\leq_D 0}\Big(\langle \pi, z\rangle - r\sigma(z)\Big) = 0$ and $w(u, 0; \pi) = u$ for all $\pi \in D$. Take arbitrary $(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m$. If $v \in D$, that is, $0 \leq_D v$, then we get $\sup_{z\leq_D v}\Big(\langle \pi, z\rangle - r\sigma(z)\Big) \geq \langle \pi, 0\rangle - r\sigma(0) = 0$ for all $\pi \in D$. Together with $e \in \operatorname{int} C$, we have

$$w(u, v; \pi) = u + \Big(\sup_{z\leq_D v}\Big(\langle \pi, z\rangle - r\sigma(z)\Big)\Big)e \geq_C u, \quad \forall \pi \in D.$$

Specially, it follows that $w(u, v; 0) \geq_C u$. Furthermore, since $\sigma(z) \geq 0$, $r > 0$ and $e \in \operatorname{int} C$, we get

$$\sup_{z\leq_D v}\Big(\langle 0, z\rangle - r\sigma(z)\Big) \leq 0$$

and

$$w(u, v; 0) = u + \Big(\sup_{z\leq_D v}\Big(\langle 0, z\rangle - r\sigma(z)\Big)\Big)e \leq_C u.$$

Together with $w(u, v; 0) \geq_C u$, we have $w(u, v; 0) = u$. Simultaneously, it follows from $w(u, v; \pi) \geq_C u$ for all $\pi \in D$ that $u \in \inf_{\pi\in D} w(u, v; \pi)$. Moreover, we can conclude by the similar method to Example 2.1 that

$$\inf_{\pi\in D} w(u, v; \pi) = u.$$

Simultaneously, if $v \notin D$, then there exists $j_0 \in \{1, 2, ..., m\}$ such that $v_{j_0} < 0$. Let the sequence $\{\pi^k\} \subset D$ with $\pi^k_{j_0} = k$ and $\pi^k_j = 0$ for every $j \in \{1, 2, ..., m\}$ with $j \neq j_0$. Then, it follows from $\sigma(z) \geq 0$ and $r > 0$ that

$$\sup_{z\leq_D v}\Big(\langle \pi^k, z\rangle - r\sigma(z)\Big) \leq \sup_{z\leq_D v}\langle \pi^k, z\rangle = \sup_{z_{j_0}\leq v_{j_0}} kz_{j_0} = kv_{j_0} \to -\infty \text{ as } k \to +\infty.$$

Together with $e \in \operatorname{int} C$, we have

$$w(u, v; \pi^k) = u + \Big(\sup_{z\leq_D v}\Big(\langle \pi^k, z\rangle - r\sigma(z)\Big)\Big)e \to -\infty \text{ as } k \to +\infty,$$

which implies

$$\inf_{\pi\in D} w(u, v; \pi) = -\infty.$$

Thus, $w \in \mathcal{W}_R(\Pi)$ satisfies Assumption $\mathcal{A}$. Lastly, we show that Assumption $\mathcal{B}$ holds. For all $(u^i, v^i) \in \mathbb{R}^\ell \times \mathbb{R}^m$, $i = 1, 2$ with $(u^2, v^2) \leq_{C\times D} (u^1, v^1)$. Then, we have $u^2 \leq_C u^1$ and $v^2 \leq_D v^1$. Thus, it follows that $\{z \in \mathbb{R}^m : z \leq_D v^2\}$ is a subset of $\{z \in \mathbb{R}^m : z \leq_D v^1\}$ and

$$\sup_{z \le_D v^2} \Big(\langle \pi, z \rangle - r\sigma(z) \Big) \le \sup_{z \le_D v^1} \Big(\langle \pi, z \rangle - r\sigma(z) \Big), \quad \forall \pi \in D.$$

Together with $u^2 \le_C u^1$ and $e \in \operatorname{int} C$, we have

$$\begin{aligned} w(u^2, v^2\pi) &= u^2 + \Big(\sup_{z \le_D v^2} \Big(\langle \pi, z \rangle - r\sigma(z) \Big) \Big) e \\ &\le_C u^1 + \Big(\sup_{z \le_D v^1} \Big(\langle \pi, z \rangle - r\sigma(z) \Big) \Big) e \\ &= w(u^1, v^1\pi), \quad \forall \pi \in D. \end{aligned}$$

Therefore, $w \in \mathcal{W}_R(\Pi)$ satisfies Assumption $\mathcal{B}$.

Example 2.3 Let $\Pi = D$ and $w : \mathbb{R}^\ell \times \mathbb{R}^m \times \Pi \to \mathbb{R}^\ell$. Given a point $e \in \operatorname{int} C$, then the following functions $w \in \mathcal{W}_R(\Pi)$ satisfy Assumptions $\mathcal{A}$ and $\mathcal{B}$ (see more details in Sect. 4):

$$w(u, v; \pi) := u + \langle \pi, v \rangle e, \quad \forall (u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m, \ \forall \pi \in D;$$
$$w(u, v; \pi) := u + \min\{\pi_1 v_1, \pi_2 v_2, ..., \pi_m v_m\} e, \quad \forall (u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m, \ \forall \pi \in D.$$

3 Lagrange-Type Duality and Zero Duality Gap Property

In this section, we pay our attention to establish a Lagrange-type dual problem for (VOP) and study the zero duality gap property by virtue of $W_R(\Pi)$. Given the class $\mathcal{W}_R(\Pi)$ and a regular weak separation function $w \in \mathcal{W}_R(\Pi)$, we consider the vector-valued function $\mathcal{L}_w : X \times \Pi \to \mathbb{R}^\ell$, defined by

$$\mathcal{L}_w(x, \pi) := w(f(\bar{x}), 0; \pi) - w(f(\bar{x}) - f(x), g(x); \pi), \quad \forall (x, \pi) \in X \times \Pi. \quad (4)$$

The function $\mathcal{L}_w$ will be called a Lagrange-type function for (VOP) with respect to w.

The Lagrange-type dual function, which is a set-valued function from the parameter set Π to $\bar{\mathbb{R}}^\ell$, for (VOP) with respect to w is defined by

$$q_w(\pi) := \inf_{x \in X} \mathcal{L}_w(x, \pi), \quad \forall \pi \in \Pi. \quad (5)$$

The Lagrange-type dual optimization problem (DOP) to the primal problem (VOP) with respect to w is

(DOP) $\qquad \sup_{\pi \in \Pi} q_w(\pi)$.

By symmetry, we are also interested in the set-valued map from X to $\bar{\mathbb{R}}^\ell$, for (VOP) with respect to w, which is defined by

$$r_w(x) := \sup_{\pi \in \Pi} \mathcal{L}_w(x, \pi), \quad \forall x \in X. \quad (6)$$

The primal problem (VOP) related to this map with respect to w is

$(\overline{\text{VOP}})$ $\inf_{x\in X}\ r_w(x).$

Now, we first explain the relationships between the primal problem (VOP) and the related problem $(\overline{\text{VOP}})$ under Assumption $\mathcal{A}$ and then further establish a weak dual relationship for (VOP) and the dual problem (DOP) under Assumptions $\mathcal{A}$ and $\mathcal{B}$.

Lemma 3.1 *Let the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ satisfy Assumption $\mathcal{A}$. Then, the following results hold:*

(i) $r_w(x) = f(x)$ for all feasible point $x \in \mathcal{R}$ and $r_w(x) = +\infty$ otherwise.
(ii) $\hat{x} \in X$ is a strong vmp to (VOP) *if and only if $\hat{x}$ is a strong vmp to* $(\overline{\text{VOP}})$.

Proof (i) It follows from (4), (6) and Assumption $\mathcal{A}$ that

$$\begin{aligned} r_w(x) = \sup_{\pi\in\Pi} \mathcal{L}(x,\pi) &= \text{Sup}\,\{w(f(\bar{x}),0;\pi) - w(f(\bar{x}) - f(x), g(x);\pi) : \pi \in \Pi\} \\ &= f(\bar{x}) + \text{Sup}\,\{-w(f(\bar{x}) - f(x), g(x);\pi) : \pi \in \Pi\} \\ &= f(\bar{x}) - \text{Inf}\,\{w(f(\bar{x}) - f(x), g(x);\pi) : \pi \in \Pi\} \\ &= f(\bar{x}) - \inf_{\pi\in\Pi} w(f(\bar{x}) - f(x), g(x);\pi). \end{aligned}$$

Note that $g(x) \in D$ if $x \in \mathcal{R}$ and $g(x) \notin D$ otherwise. Together with Assumption $\mathcal{A}$, we get $r_w(x) = f(\bar{x}) - \inf_{\pi\in\Pi} w(f(\bar{x}) - f(x), g(x);\pi) = f(x)$ for all feasible point $x \in \mathcal{R}$ and $r_w(x) = +\infty$ otherwise.

(ii) It follows from (i) that $\hat{x} \in X$ is a strong *vmp* to (VOP), that is, $\hat{x} \in \mathcal{R}$ and $f(x) \nleq_{C\setminus\{0\}} f(\hat{x})$ for all $x \in \mathcal{R}$, if and only if $r_w(x) \nleq_{C\setminus\{0\}} r_w(\hat{x})$ for all $x \in X$. This shows that $\hat{x}$ is a strong *vmp* to $(\overline{\text{VOP}})$. This completes the proof. □

Remark 3.1 Clearly, Lemma 3.1 shows that the primal problem (VOP) coincides with the related problem $(\overline{\text{VOP}})$ under Assumption $\mathcal{A}$, in the sense that they have the same feasible sets and the same objective functions. This always holds for the scalar case when $\ell = 1$ and $C = \mathbb{R}_+$.

Theorem 3.1 (Weak duality theorem) *Let the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ satisfy Assumptions $\mathcal{A}$ and $\mathcal{B}$. Then, the following assertions hold:*

(i) For every feasible point $x \in \mathcal{R}$ and every $z \in q_w(\pi)$ with $\pi \in \Pi$, one has $z \ngeq_{C\setminus\{0\}} f(x)$.
(ii) For every $\bar{z} \in \inf_{x\in\mathcal{R}} f(x)$ and every $\tilde{z} \in \sup_{\pi\in\Pi} q_w(\pi)$, one has $\tilde{z} \ngeq_{\text{int C}} \bar{z}$.

Proof (i) Assume that there exist $x_0 \in \mathcal{R}$ and $z_0 \in q_w(\pi_0)$ with $\pi_0 \in \Pi$, such that

$$z_0 \geq_{C\setminus\{0\}} f(x_0). \tag{7}$$

Since $x_0 \in \mathcal{R}$, we get $(f(\bar{x}) - f(x_0), g(x_0)) - (f(\bar{x}) - f(x_0), 0) \in C \times D$. Together with Assumptions $\mathcal{A}$ and $\mathcal{B}$, we have $w(f(\bar{x}) - f(x_0), 0;\pi_0) = f(\bar{x}) - f(x_0)$ and

$$w(f(\bar{x}) - f(x_0), g(x_0);\pi_0) \geq_C w(f(\bar{x}) - f(x_0), 0;\pi_0).$$

This shows $f(x_0) \geq_C f(\bar{x}) - w(f(\bar{x}) - f(x_0), g(x_0); \pi_0)$. Thus, it follows from Assumption $\mathcal{A}$ and (4) that $f(\bar{x}) = w(f(\bar{x}), 0; \pi_0)$ and $f(x_0) \geq_C \mathcal{L}_w(x_0, \pi_0)$. Combining (7), we have $z_0 \geq_{C\setminus\{0\}} \mathcal{L}_w(x_0, \pi_0)$, which implies $z_0 \notin \inf_{x\in X} \mathcal{L}_w(x, \pi_0)$. This is a contradiction to $z_0 \in q_w(\pi_0)$ by (5).

(ii) Suppose that there exist $z_1 \in \inf_{x\in\mathcal{R}} f(x)$ and $z_2 \in \sup_{\pi\in\Pi} q_w(\pi)$, such that

$$z_2 \geq_{\operatorname{int} C} z_1. \tag{8}$$

Then, there exist a sequence $f(x^k) \to z_1$ with $x^k \in \mathcal{R}$, and a sequence $z^k \to z_2$ with $z^k \in q_w(\pi^k)$ and $\pi^k \in \Pi$. Clearly, (i) implies that

$$z^k \not\geq_{C\setminus\{0\}} f(x^k), \quad \forall k \in \mathbb{N}. \tag{9}$$

Note that $z^k - f(x^k) \to z_2 - z_1$. Together with (8), one has $z^k - f(x^k) \in \operatorname{int} C$ for sufficiently large $k \in \mathbb{N}$. This is a contradiction to (9). □

It is worth noting that the conclusion (i) coincides with conclusion (ii) in Theorem 3.1 for the scalar case when $\ell = 1$ and $C = \mathbb{R}_+$. However, they does not coincide for the vector case. Most pertinently, the conclusion (ii) shall not hold if $\operatorname{int} C$ is replaced by $C \setminus \{0\}$. We give the following example to explain the case.

Example 3.1 Consider the following vector optimization problem

$$\min f(x), \qquad \text{s.t.} \quad x \in X,\ g(x) \geq 0,$$

where $X = [-1, 1]$, $f : \mathbb{R} \to \mathbb{R}^2$ with $f(x) = (x + 1, x - 1)^T$ is a vector-valued function and $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x$ is a real-valued function. Obviously, we get $\mathcal{R} = [0, 1]$. Let $\ell = 2, m = 1, C = \mathbb{R}^2_+$ and $D = \mathbb{R}_+$. It follows from Example 2.1 that $D^*_C = \mathbb{R}^2_+$. Let $\Pi = E_2 \times D^*_C$. Then, the vector-valued function $w : \mathbb{R}^2 \times \mathbb{R} \times \Pi \to \mathbb{R}^2$ with $w(u, v; E_2, \Lambda) = u + \Lambda v$, $\forall (u, v) \in \mathbb{R}^2 \times \mathbb{R}$, $\forall \Lambda \in D^*_C$ is a regular weak separation function satisfying Assumptions $\mathcal{A}$ and $\mathcal{B}$. It follows from (4) and (5) that for all $x \in \mathbb{R}$ and all $\Lambda = (\lambda_1, \lambda_2)^T \in \mathbb{R}^2_+$, we have

$$\begin{aligned}\mathcal{L}_w(x; E_2, \Lambda) &= f(x) - \Lambda g(x) \\ &= (x + 1, x - 1)^T - (\lambda_1 x, \lambda_2 x)^T = \Big((1 - \lambda_1)x + 1, (1 - \lambda_2)x - 1\Big)^T\end{aligned}$$

and for all $\Lambda = (\lambda_1, \lambda_2)^T \in \mathbb{R}^2_+$, we have

$$\begin{aligned} q_w(E_2, \Lambda) &= \inf_{x\in X} \mathcal{L}(x; E_2, \Lambda) \\ &= \inf_{x\in[-1,1]} \Big((1 - \lambda_1)x + 1, (1 - \lambda_2)x - 1\Big)^T \\ &= \begin{cases} (\lambda_1, \lambda_2 - 2)^T, & \text{if } 0 \leq \lambda_1, \lambda_2 \leq 1, \\ (2 - \lambda_1, -\lambda_2)^T, & \text{if } \lambda_1 > 1, \lambda_2 > 1, \\ \Big\{\Big((1 - \lambda_1)x + 1, (1 - \lambda_2)x - 1\Big)^T : x \in [-1, 1]\Big\}, & \text{otherwise.} \end{cases}\end{aligned}$$

It is easy to verify that $f(x) \geq_C (1,-1)^T$ for all $x \in \mathcal{R}$, and $z \not\geq_{C\setminus\{0\}} (1,-1)^T$ for all $z \in q_w(E_2, \Lambda)$ and all $\Lambda \in D_C^*$. Thus, for every feasible point $x \in \mathcal{R}$ and every $z \in q_w(E_2, \Lambda)$ with $\Lambda \in D_C^*$, one has $z \not\geq_{C\setminus\{0\}} f(x)$, that is, the conclusion (i) holds. Simultaneously, by direct calculating, we get

$$\inf_{x\in\mathcal{R}} f(x) = (1,-1)^T$$

and

$$\begin{aligned}&\sup_{\Lambda\in D_C^*} q_w(E_2,\Lambda)\\ &\quad= \left\{\Big((1-\lambda_1)x+1, (1-\lambda_2)x-1\Big)^T : x\in[-1,1], 0\leq\lambda_1\leq 1, \lambda_2\geq 1\right\}\\ &\quad\bigcup \left\{\Big((1-\lambda_1)x+1, (1-\lambda_2)x-1\Big)^T : x\in[-1,1], \lambda_1\geq 1, 0\leq\lambda_2\leq 1\right\}.\end{aligned}$$

Clearly, for every $\bar{z} \in \inf_{x\in\mathcal{R}} f(x)$ and every $\tilde{z} \in \sup_{\Lambda\in D_C^*} q_w(E_2,\Lambda)$, one has $\bar{z} = (1,-1)^T$ and $\tilde{z} \not\geq_{\text{int C}} \bar{z}$. Thus, the conclusion (ii) also holds. Note that if we take $\lambda_1 = 0$, $x = 1$ and $\lambda_2 = 1$, then $(2,-1)^T \in \sup_{\Lambda\in D_C^*} q_w(E_2,\Lambda)$. Obviously, we still have $(2,-1)^T \not\geq_{\text{int}\, C} (1,-1)^T$. However, $(2,-1)^T \geq_{C\setminus\{0\}} (1,-1)^T$. This shows that the conclusion (ii) does not hold when we replace $\text{int}\, C$ by $C \setminus \{0\}$.

The following corollary establishes a further relationship between the primal problem (VOP) and the dual problem (DOP).

Corollary 3.1 *Let the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ satisfy Assumptions $\mathcal{A}$ and $\mathcal{B}$. If there exists a feasible point $\hat{x} \in \mathcal{R}$ of* (VOP) *such that $f(\hat{x}) \in \sup_{\pi\in\Pi} q_w(\pi)$, then $\hat{x}$ is a weak vmp to* (VOP).

Proof Assume that $\hat{x}$ is not a weak *vmp* to (VOP). Then, there exists $\bar{x} \in \mathcal{R}$ satisfying $f(\bar{x}) \leq_{\text{int}\, C} f(\hat{x})$. Note that $f(\hat{x}) \in \sup_{\pi\in\Pi} q_w(\pi)$, that is, there exists a sequence $z^k \to f(\hat{x})$ with $z^k \in q_w(\pi^k)$ and $\pi^k \in \Pi$. Thus, we can conclude that $z^k - f(\bar{x}) \to f(\hat{x}) - f(\bar{x})$. Together with $f(\bar{x}) \leq_{\text{int}\, C} f(\hat{x})$, we have $z^k - f(\bar{x}) \in \text{int}\, C$ for sufficiently large $k \in \mathbb{N}$. This is a contradiction from Theorem 3.1(i). □

Next, we focus our attention to investigate some equivalent characterizations of the zero duality gap property between the primal problem (VOP) and the dual problem (DOP). We first follow the classic approach, namely the Lagrange multiplier and the Lagrange saddle point, and then apply the regular separation associated with $\mathcal{K}$ and $\mathcal{H}$ in the IS to discuss the zero duality gap property.

Let Δ_1 be the set of all the infimum values of (VOP) and Δ_2 the set of all the supremum values of (DOP), that is,

$$\Delta_1 = \inf_{x\in\mathcal{R}} f(x) \quad \text{and} \quad \Delta_2 = \sup_{\pi\in\Pi} q_w(\pi).$$

We recall from [30] that $\Delta := \Delta_1 - \Delta_2$ is said to be the duality gap. In the sequel, we call that the zero duality gap property with respect to w holds iff $0 \in \Delta$, that is, $\Delta_1 \cap \Delta_2 \neq \varnothing$.

First of all, we introduce some standard notions associated with the classes of separation functions and the Lagrange-type function for (VOP).

Definition 3.1 [30,31] Given the classes $\mathcal{W}(\Pi)$ and $\mathcal{W}_R(\Pi)$, the sets $\mathcal{K}$ and $\mathcal{H}$ admit a separation with respect to $w \in \mathcal{W}(\Pi)$ and $\hat{\pi} \in \Pi$ iff

$$w(u, v; \hat{\pi}) \not\geq_{C\setminus\{0\}} 0, \quad \forall (u, v) \in \mathcal{K}. \tag{10}$$

Moreover, the separation is said to be regular iff $w \in \mathcal{W}_R(\Pi)$.

We observe from Definitions 2.2 and 2.3 that if either $\mathcal{K}$ or $\mathcal{H}$ admits a separation, and (10) is fulfilled when $C \setminus \{0\}$ is replaced by C, or $\mathcal{K}$ and $\mathcal{H}$ admit a regular separation, then $\mathcal{K} \cap \mathcal{H} = \varnothing$. In addition, let $\bar{x} \in \mathcal{R}$. Then, $\bar{x}$ is a strong *vmp* to (VOP).

Definition 3.2 Given the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ and the Lagrange-type function $\mathcal{L}_w$ for (VOP) with respect to w, then $\hat{\pi} \in \Pi$ is said to be a generalized Lagrange multiplier for (VOP) with respect to w iff

$$f(\mathcal{R}) \cap \inf_{x \in X} \mathcal{L}_w(x, \hat{\pi}) \neq \varnothing.$$

Together with (5), the above inequality is equivalent to

$$f(\mathcal{R}) \cap q_w(\hat{\pi}) \neq \varnothing.$$

Definition 3.3 [30] Given the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ and the Lagrange-type function $\mathcal{L}_w$ for (VOP) with respect to w, the pair $(\hat{x}, \hat{\pi}) \in X \times \Pi$ is said to be a generalized Lagrange saddle point for (VOP) with respect to w iff

$$\mathcal{L}_w(x, \hat{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \hat{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \pi), \quad \forall x \in X, \ \forall \pi \in \Pi.$$

Now, we establish some equivalent characterizations to the zero duality gap property for (VOP) by virtue of the classic approach, namely the Lagrange multiplier and the Lagrange saddle point.

Theorem 3.2 *Given the pair $(\hat{x}, \hat{\pi}) \in X \times \Pi$, and the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ satisfying Assumptions $\mathcal{A}$ and $\mathcal{B}$, then the following assertions are equivalent:*

(i) $(\hat{x}, \hat{\pi}) \in X \times \Pi$ *is a generalized Lagrange saddle point for* (VOP) *with respect to w.*
(ii) $\hat{x} \in \mathcal{R}$ *is a feasible point and* $\hat{\pi} \in \Pi$ *is a generalized Lagrange multiplier for* (VOP) *with respect to w, and moreover,* $f(\hat{x}) \in f(\mathcal{R}) \cap q_w(\hat{\pi})$.

In addition, if any condition (i) *or* (ii) *above fulfills, then* $\hat{x}$ *is a strong vmp of* (VOP) *and* $(\hat{\pi}, f(\hat{x})) \in \operatorname{graph} q_w$ *is a strong vector maximum point of* (DOP) *in the set-valued case, that is,* $f(\hat{x}) \nleq_{C\setminus\{0\}} z$ *for all* $z \in q_w(\pi)$ *with* $\pi \in \Pi$. *Simultaneously, the zero duality gap property with respect to w holds and* $\mathcal{L}_w(\hat{x}, \hat{\pi}) = f(\hat{x}) \in \Delta_1 \cap \Delta_2$. *Moreover, the generalized complementary slackness condition* $w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \hat{\pi}) = f(\bar{x}) - f(\hat{x})$ *also fulfills.*

Proof (i) $\Rightarrow$ (ii) Since $(\hat{x}, \hat{\pi}) \in X \times \Pi$ is a generalized Lagrange saddle point for (VOP) with respect to w, it follows that

$$\mathcal{L}_w(x, \hat{\pi}) \nleq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \hat{\pi}) \nleq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \pi), \quad \forall x \in X, \ \forall \pi \in \Pi. \tag{11}$$

Then, $\mathcal{L}_w(\hat{x}, \hat{\pi}) \nleq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \pi)$ holds for all $\pi \in \Pi$. First, we show $\hat{x} \in \mathcal{R}$, that is, $\hat{x}$ is a feasible point of (VOP). In fact, it follows from Assumption $\mathcal{A}$ and (4) that the above inequality implies

$$w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \pi) \nleq_{C\setminus\{0\}} w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \hat{\pi}), \ \forall \pi \in \Pi. \tag{12}$$

Suppose that $\hat{x} \notin \mathcal{R}$, that is, $g(\hat{x}) \notin D$. By Assumption $\mathcal{A}$, we get

$$\inf_{\pi \in \Pi} w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \pi) = -\infty,$$

which implies that there exists a sequence $w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \pi^k) \to -\infty$ with $\pi^k \in \Pi$, which implies $w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \pi^k) \leq_{C\setminus\{0\}} w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \hat{\pi})$ for sufficiently large $k \in \mathbb{N}$. This is a contradiction to (12). Thus, we get $\hat{x} \in \mathcal{R}$. Second, we show that

$$f(\hat{x}) = \mathcal{L}_w(\hat{x}, \hat{\pi}). \tag{13}$$

Note that $\mathcal{L}_w(\hat{x}, \hat{\pi}) \nleq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \pi), \ \forall \pi \in \Pi$ implies $\mathcal{L}_w(\hat{x}, \hat{\pi}) \in \sup_{\pi \in \Pi} \mathcal{L}_w(\hat{x}, \pi)$. Together with (6), $\hat{x} \in \mathcal{R}$ and Lemma 3.1 (i), we have

$$\mathcal{L}_w(\hat{x}, \hat{\pi}) \in \sup_{\pi \in \Pi} \mathcal{L}_w(\hat{x}, \pi) = r_w(\hat{x}) = f(\hat{x}),$$

which implies that (13) holds. At last, we prove that $\hat{\pi} \in \Pi$ is a generalized Lagrange multiplier for (VOP) with respect to w and $f(\hat{x}) \in f(\mathcal{R}) \cap q_w(\hat{\pi})$. It follows from (11) that $\mathcal{L}_w(x, \hat{\pi}) \nleq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \hat{\pi})$ holds for all $x \in X$. Together with (5), we get $\mathcal{L}_w(\hat{x}, \hat{\pi}) \in \inf_{x \in X} \mathcal{L}_w(x, \hat{\pi}) = q_w(\hat{\pi})$. Note that $\hat{x} \in \mathcal{R}$ and $f(\hat{x}) = \mathcal{L}_w(\hat{x}, \hat{\pi})$. Then, it follows that $f(\hat{x}) \in f(\mathcal{R}) \cap q_w(\hat{\pi})$ and $\hat{\pi} \in \Pi$ is a generalized Lagrange multiplier for (VOP) with respect to w.

(ii) $\Rightarrow$ (i) Since $\hat{x} \in \mathcal{R}$ is a feasible point, we have $g(\hat{x}) \in \mathbb{R}^m_+$. Together with Assumptions $\mathcal{A}$ and $\mathcal{B}$, we get

$$(f(\bar{x}) - f(\hat{x}), g(\hat{x})) - (f(\bar{x}) - f(\hat{x}), 0) = (0, g(\hat{x})) \in C \times D$$

and

$$w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \hat{\pi}) \geq_C w(f(\bar{x}) - f(\hat{x}), 0; \hat{\pi}) = f(\bar{x}) - f(\hat{x}).$$

Applying (4) and Assumption $\mathcal{A}$, we have

$$f(\hat{x}) \geq_C f(\bar{x}) - w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \hat{\pi}) = \mathcal{L}_w(\hat{x}, \hat{\pi}). \tag{14}$$

In addition, since $\hat{\pi} \in \Pi$ is a generalized Lagrange multiplier for (VOP) with respect to w, and moreover, $f(\hat{x}) \in f(\mathcal{R}) \cap q_w(\hat{\pi})$, it follows from (5) that

$$f(\hat{x}) \in \inf_{x \in X} \mathcal{L}_w(x, \hat{\pi}). \tag{15}$$

Thus, $f(\hat{x}) \not\geq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \hat{\pi})$. Together with (14), we have

$$f(\hat{x}) = \mathcal{L}_w(\hat{x}, \hat{\pi}). \tag{16}$$

On the one hand, combining (15) and (16), we have $\mathcal{L}_w(x, \hat{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \hat{\pi})$ for all $x \in X$. On the other hand, it follows from $\hat{x} \in \mathcal{R}$, Lemma 3.1 (i), (6) and (16) that $\mathcal{L}_w(\hat{x}, \hat{\pi}) = f(\hat{x}) = r_w(\hat{x}) = \sup_{\pi \in \Pi} \mathcal{L}_w(\hat{x}, \pi)$, which implies $\mathcal{L}_w(\hat{x}, \hat{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \pi)$ for all $\pi \in \Pi$. Therefore, we can conclude that

$$\mathcal{L}_w(x, \hat{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \hat{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \pi), \quad \forall x \in X, \ \forall \pi \in \Pi,$$

that is, $(\hat{x}, \hat{\pi}) \in X \times \Pi$ is a generalized Lagrange saddle point for (VOP) with respect to w.

In addition, let any condition (i) or (ii) be fulfilled. We firstly show that $\hat{x} \in \mathcal{R}$ is a strong *vmp* to (VOP). Otherwise, there exists $\tilde{x} \in \mathcal{R}$ such that

$$f(\tilde{x}) \leq_{C\setminus\{0\}} f(\hat{x}). \tag{17}$$

Note that (11) implies $\mathcal{L}_w(\tilde{x}, \hat{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\hat{x}, \hat{\pi})$. Applying $f(\hat{x}) = \mathcal{L}_w(\hat{x}, \hat{\pi})$, Assumption $\mathcal{A}$ and (4), we get

$$f(\bar{x}) - w(f(\bar{x}) - f(\tilde{x}), g(\tilde{x}); \hat{\pi}) \not\leq_{C\setminus\{0\}} f(\hat{x}). \tag{18}$$

Moreover, it follows from $\tilde{x} \in \mathcal{R}$, and Assumptions $\mathcal{A}$ and $\mathcal{B}$, we have

$$(f(\bar{x}) - f(\tilde{x}), g(\tilde{x})) - (f(\bar{x}) - f(\tilde{x}), 0) = (0, g(\tilde{x})) \in C \times D$$

and

$$w(f(\bar{x}) - f(\tilde{x}), g(\tilde{x}); \hat{\pi}) \geq_C w(f(\bar{x}) - f(\tilde{x}), 0; \hat{\pi}) = f(\bar{x}) - f(\tilde{x}).$$

Together with (17), we have

$$f(\bar{x}) - w(f(\bar{x}) - f(\tilde{x}), g(\tilde{x}); \hat{\pi}) \leq_C f(\tilde{x}) \leq_{C\setminus\{0\}} f(\hat{x}),$$

which contradicts (18). Secondly, we show that $(\hat{\pi}, f(\hat{x})) \in \operatorname{graph} q_w$ is a strong vector maximum point of (DOP). Obviously, we have $f(\hat{x}) \in q_w(\hat{\pi})$ from (ii). Moreover, it follows from $\hat{x} \in \mathcal{R}$ and Theorem 3.1(i) (weak duality theorem) that $f(\hat{x}) \nleq_{C\setminus\{0\}} z$ for all $z \in q_w(\pi)$ with $\pi \in \Pi$. Thus, $(\hat{\pi}, f(\hat{x})) \in \operatorname{graph} q_w$ is a strong vector maximum point to (DOP). Lastly, since $\hat{x} \in \mathcal{R}$ is a strong *vmp* to (VOP), we have $f(\hat{x}) \in \inf_{x\in\mathcal{R}} f(x) = \Delta_1$. Furthermore, since $f(\hat{x}) \in q_w(\hat{\pi})$ and $(\hat{\pi}, f(\hat{x})) \in \operatorname{graph} q_w$ is a strong vector maximum point of (DOP), we get $f(\hat{x}) \in \sup_{\pi\in\Pi} q_w(\pi) = \Delta_2$. Combining $\mathcal{L}_w(\hat{x}, \hat{\pi}) = f(\hat{x})$, we can conclude that the zero duality gap property with respect to w holds, and moreover, $\mathcal{L}_w(\hat{x}, \hat{\pi}) = f(\hat{x}) \in \Delta_1 \cap \Delta_2$. Simultaneously, it follows from $\mathcal{L}_w(\hat{x}, \hat{\pi}) = f(\hat{x})$, (4) and Assumption $\mathcal{A}$ that $f(\bar{x}) - w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \hat{\pi}) = f(\hat{x})$, which means that the generalized complementary slackness condition $w(f(\bar{x}) - f(\hat{x}), g(\hat{x}); \hat{\pi}) = f(\bar{x}) - f(\hat{x})$ fulfills. This completes the proof. □

Specially, let $\bar{x} \in \mathcal{R}$. Then, we immediately have the following characterization to the zero duality gap property for (VOP) by means of the regular separation associated with $\mathcal{K}$ and $\mathcal{H}$ in the IS.

Theorem 3.3 *Given the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ satisfying Assumptions $\mathcal{A}$ and $\mathcal{B}$, then the following assertions are equivalent:*

(i) *$\bar{x} \in \mathcal{R}$, and the sets $\mathcal{K}$ and $\mathcal{H}$ admit a regular separation with respect to w and $\bar{\pi} \in \Pi$.*
(ii) *$(\bar{x}, \bar{\pi}) \in X \times \Pi$ is a generalized Lagrange saddle point for* (VOP) *with respect to w.*
(iii) *$\bar{x} \in \mathcal{R}$ is a feasible point and $\bar{\pi} \in \Pi$ is a generalized Lagrange multiplier for* (VOP) *with respect to w, and moreover, $f(\bar{x}) \in f(\mathcal{R}) \cap q_w(\bar{\pi})$.*

In addition, if any condition (i), (ii) *or* (iii) *above fulfills, then $\bar{x}$ is a strong vmp of* (VOP) *and $(\bar{\pi}, f(\bar{x})) \in \operatorname{graph} q_w$ is a strong vector maximum point of* (DOP) *in the set-valued case. Simultaneously, the zero duality gap property with respect to w holds and $\mathcal{L}_w(\bar{x}, \bar{\pi}) = f(\bar{x}) \in \Delta_1 \cap \Delta_2$. Moreover, the generalized complementary slackness condition $w(0, g(\bar{x}); \bar{\pi}) = 0$ also fulfills.*

Proof Obviously, we only need to prove that (i) ⇒ (ii) and (iii) ⇒ (i) from Theorem 3.2.

(i) ⇒ (ii) Since $\bar{x} \in \mathcal{R}$, and the sets $\mathcal{K}$ and $\mathcal{H}$ admit a regular separation with respect to w and $\bar{\pi} \in \Pi$, we have $w(f(\bar{x}) - f(x), g(x); \bar{\pi}) \ngeq_{C\setminus\{0\}} 0$ for all $x \in X$. This shows $f(\bar{x}) - w(f(\bar{x}) - f(x), g(x); \bar{\pi}) \nleq_{C\setminus\{0\}} f(\bar{x})$ for all $x \in X$. Together with (4) and Assumption $\mathcal{A}$, we get

$$\mathcal{L}_w(x, \bar{\pi}) \nleq_{C\setminus\{0\}} f(\bar{x}), \quad \forall x \in X. \tag{19}$$

Take specially $x = \bar{x}$. Then, we have

$$\mathcal{L}_w(\bar{x}, \bar{\pi}) \nleq_{C\setminus\{0\}} f(\bar{x}). \tag{20}$$

Moreover, since $\bar{x} \in \mathcal{R}$, that is, $g(\bar{x}) \in D$, it follows from Assumptions $\mathcal{A}$ and $\mathcal{B}$ that $(0, g(\bar{x})) \in C \times D$ and $w(0, g(\bar{x}); \bar{\pi}) \geq_C w(0, 0; \bar{\pi}) = 0$. Together with (4) and

Assumption $\mathcal{A}$, we get $f(\bar{x}) \geq_C f(\bar{x}) - w(0, g(\bar{x}); \bar{\pi}) = \mathcal{L}_w(\bar{x}, \bar{\pi})$. Applying (20), it follows that

$$f(\bar{x}) = \mathcal{L}_w(\bar{x}, \bar{\pi}). \tag{21}$$

On the one hand, combining (19) and (21), we get $\mathcal{L}_w(x, \bar{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\bar{x}, \bar{\pi})$ for all $x \in X$. On the other hand, since $\bar{x} \in \mathcal{R}$, it follows from Lemma 3.1 (i), (6) and (21) that $\mathcal{L}_w(\bar{x}, \bar{\pi}) = f(\bar{x}) = r_w(\bar{x}) = \sup_{\pi \in \Pi} \mathcal{L}_w(\bar{x}, \pi)$, which implies $\mathcal{L}_w(\bar{x}, \bar{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\bar{x}, \pi)$ for all $\pi \in \Pi$. Therefore, we can conclude that $(\bar{x}, \bar{\pi}) \in X \times \Pi$ is a generalized Lagrange saddle point for (VOP) with respect to w.

(iii) $\Rightarrow$ (i) Since $\bar{x} \in \mathcal{R}$ is a feasible point, and $\bar{\pi} \in \Pi$ is a generalized Lagrange multiplier for (VOP) with respect to w and $f(\bar{x}) \in f(\mathcal{R}) \cap q_w(\bar{\pi})$, it follows that $f(\bar{x}) \in \inf_{x \in X} \mathcal{L}_w(x, \bar{\pi})$, which implies $\mathcal{L}_w(x, \bar{\pi}) \not\leq_{C\setminus\{0\}} f(\bar{x})$ for all $x \in X$. Together with (4) and Assumption $\mathcal{A}$, we can conclude that for all $x \in X$, one has $f(\bar{x}) - w(f(\bar{x}) - f(x), g(x); \bar{\pi}) \not\leq_{C\setminus\{0\}} f(\bar{x})$, namely $w(f(\bar{x}) - f(x), g(x); \bar{\pi}) \not\geq_{C\setminus\{0\}} 0$. Thus, the sets $\mathcal{K}$ and $\mathcal{H}$ admit a regular separation with respect to w and $\bar{\pi} \in \Pi$. ◻

To end this section, we give the following example to explain Theorem 3.3.

Example 3.2 Consider Example 3.1. Let $\bar{x} = 0$ and $\bar{\pi} = (E_2, \bar{\Lambda}) \in \Pi$ with $\bar{\Lambda} = (1, 1)^T$. Then, it follows that $\bar{x} \in \mathcal{R}$, $f(\bar{x}) = (1, -1)^T$ and

$$\mathcal{K} = \left\{(u, v) \in \mathbb{R}^2 \times \mathbb{R} : u = f(\bar{x}) - f(x) = \begin{pmatrix} -x \\ -x \end{pmatrix}, v = g(x) = x, x \in X\right\}.$$

Thus, we have

$$w(u, v; E_2, \bar{\Lambda}) = u + \bar{\Lambda} v = \begin{pmatrix} -x \\ -x \end{pmatrix} + x \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \not\geq_{C\setminus\{0\}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall (u, v) \in \mathcal{K},$$

which implies that the sets $\mathcal{K}$ and $\mathcal{H}$ admit a regular separation with respect to w and $\bar{\pi} \in \Pi$. Note that the Lagrange-type function

$$\mathcal{L}_w(x, \pi) = f(x) - \Lambda g(x) = \begin{pmatrix} (1 - \lambda_1)x + 1 \\ (1 - \lambda_2)x - 1 \end{pmatrix}, \quad \forall x \in \mathbb{R}, \ \forall \pi = (E_2, \Lambda) \in \Pi,$$

where $\Lambda = (\lambda_1, \lambda_2)^T \in \mathbb{R}^2_+$. Then, we get

$$\mathcal{L}_w(\bar{x}, \pi) = \mathcal{L}_w(x, \bar{\pi}) = \mathcal{L}_w(\bar{x}, \bar{\pi}) = f(\bar{x}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \forall x \in X, \ \forall \pi \in \Pi.$$

Obviously, we have

$$\mathcal{L}_w(x, \bar{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\bar{x}, \bar{\pi}) \not\leq_{C\setminus\{0\}} \mathcal{L}_w(\bar{x}, \pi), \quad \forall x \in X, \ \forall \pi \in \Pi,$$

that is, $(\bar{x}, \bar{\pi}) \in X \times \Pi$ is a generalized Lagrange saddle point for (VOP) with respect to w. Moreover, for all $\pi = (E_2, \Lambda) \in \Pi$, we have

$$\begin{aligned} q_w(\pi) &= \inf_{x \in X} \mathcal{L}(x, \pi) \\ &= \begin{cases} (\lambda_1, \lambda_2 - 2)^T, & \text{if } 0 \leq \lambda_1, \lambda_2 \leq 1, \\ (2 - \lambda_1, -\lambda_2)^T, & \text{if } \lambda_1 > 1, \lambda_2 > 1, \\ \left\{ \Big((1 - \lambda_1)x + 1, (1 - \lambda_2)x - 1 \Big)^T : x \in [-1, 1] \right\}, & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, we get $f(\bar{x}) = (1, -1)^T \in f(\mathcal{R}) \cap q_w(\bar{\pi})$. Thus, $\bar{\pi} \in \Pi$ is a generalized Lagrange multiplier for (VOP) with respect to w. In addition, it is easy to verify that $\bar{x}$ is a strong *vmp* of (VOP) and $(\bar{\pi}, f(\bar{x})) \in \operatorname{graph} q_w$ is a strong vector maximum point of (DOP). Simultaneously, the zero duality gap property with respect to w holds and $\mathcal{L}_w(\bar{x}, \bar{\pi}) = f(\bar{x}) \in \Delta_1 \cap \Delta_2$. Moreover, the generalized complementary slackness condition $w(0, g(\bar{x}); \bar{\pi}) = \bar{\Lambda} g(\bar{x}) = (0, 0)^T$ also fulfills.

4 Exact Penalization

In this section, we first consider a special class of regular weak separation functions, which are separable with respect to the objective and the constraints. Subsequently, we focus on establishing a local exact penalization result for (VOP) by virtue of the special regular separation functions and a local image regularity condition in the IS.

Let the parameter set $\Pi = D$ and $\pi \in \Pi$. The vector-valued function $w : \mathbb{R}^\ell \times \mathbb{R}^m \times \Pi \to \mathbb{R}^\ell$, which is separable with respect to the objective and the constraints, is defined by

$$w(u, v; \pi) := u + \underline{w}(v, \pi)e, \quad \forall (u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m, \ \forall \pi \in D, \tag{a}$$

where $e \in \operatorname{int} C$ is a fixed element and the function $\underline{w} : \mathbb{R}^m \times D \to \mathbb{R}$ must be such that

$$\bigcap_{\pi \in D} \operatorname{lev}_{\geq 0} \underline{w}(\bullet, \pi) = D, \tag{b}$$

$$\forall \pi \in D, \quad \underline{w}(0, \pi) = 0, \tag{c}$$

$$\forall \pi \in D, \ \forall \theta \in \mathbb{R}_+, \ \exists \pi_\theta \in D, \quad \text{s.t. } \theta \underline{w}(v, \pi) = \underline{w}(v, \pi_\theta), \ \forall v \in \mathbb{R}^m, \tag{d}$$

$$\forall v^1, v^2 \in \mathbb{R}^m, \ v^1 \geq_D v^2 \quad \Rightarrow \quad \underline{w}(v^1, \pi) \geq \underline{w}(v^2, \pi), \quad \forall \pi \in D. \tag{e}$$

Clearly, it follows from (b) and (d) that

$$\forall v \notin D, \ \exists \hat{\pi} \in D, \quad \text{s.t. } \underline{w}(v, \hat{\pi}) < 0, \tag{f}$$

$$\exists \bar{\pi} \in D \ \text{ s.t. } \ \underline{w}(v, \bar{\pi}) = 0, \ \forall v \in \mathbb{R}^m. \tag{g}$$

Theorem 4.1 *The vector-valued functions given by conditions* (a–e) *are regular weak separation functions satisfying Assumptions $\mathcal{A}$ and $\mathcal{B}$.*

Proof We firstly prove that the vector-valued functions given by conditions (a–e) are regular weak separation functions. It is easy to verify that

$$\mathcal{H} \subset \bigcap_{\pi \in D} \operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi).$$

In fact, for every $(u, v) \in \mathcal{H}$ and every $\pi \in D$, we get $u \in C \setminus \{0\}$, and $v \in D$. Moreover, since $e \in \operatorname{int} C$, it follows from (b) that $\underline{w}(v, \pi) \geq 0$ and $\underline{w}(v, \pi)e \in C$. Then, we have $w(u, v; \pi) = u + \underline{w}(v, \pi)e \in C \setminus \{0\} + C \subset C \setminus \{0\}$, which implies $(u, v) \in \operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi)$. Next, we prove the converse inclusion

$$\bigcap_{\pi \in D} \operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi) \subset \mathcal{H}.$$

Take arbitrary $(u, v) \in \bigcap_{\pi \in D} \operatorname{lev}_{C\setminus\{0\}} w(\bullet; \pi)$, that is,

$$w(u, v; \pi) = u + \underline{w}(v, \pi)e \in C \setminus \{0\}, \quad \forall \pi \in D. \tag{22}$$

Then, it follows from (g) that there exists $\bar{\pi} \in \Pi$ such that

$$w(u, v; \bar{\pi}) = u + \underline{w}(v, \bar{\pi})e = u \in C \setminus \{0\},$$

that is, $u \in C \setminus \{0\}$. Moreover, we get $v \in D$. Otherwise, it follows from (f) that there exists $\hat{\pi} \in D$ satisfying $\underline{w}(v, \hat{\pi}) < 0$. Let $\theta > 0$ and $\theta \to +\infty$. Then, we get from (d) that there exists $\hat{\pi}_\theta \in D$ such that $\underline{w}(v, \hat{\pi}_\theta) = \theta \underline{w}(v, \hat{\pi}) \to -\infty$. Together with $e \in \operatorname{int} C$, it follows that $w(u, v; \hat{\pi}_\theta) = u + \underline{w}(v, \hat{\pi}_\theta)e \notin C \setminus \{0\}$ for sufficiently large $\theta > 0$. This is a contradiction to (22). Thus, we can conclude that $u \in C \setminus \{0\}$ and $v \in D$, that is, $(u, v) \in \mathcal{H}$. This shows that the vector-valued functions given by conditions (a–e) are regular weak separation functions. Secondly, we show that Assumption $\mathcal{A}$ holds. On the one hand, it follows from (c) and (e) that

$$w(u, 0; \pi) = u + \underline{w}(0, \pi)e = u, \quad \forall \pi \in D \tag{23}$$

and $\underline{w}(v, \pi) \geq \underline{w}(0, \pi) = 0, \ \forall v \in D, \ \forall \pi \in D$. Together with $e \in \operatorname{int} C$, we have

$$w(u, v; \pi) = u + \underline{w}(v, \pi)e \geq_C u, \quad \forall v \in D, \ \forall \pi \in D. \tag{24}$$

Furthermore, (g) implies that there exists $\bar{\pi} \in \pi$ such that

$$w(u, v; \bar{\pi}) = u + \underline{w}(v, \bar{\pi})e = u, \quad \forall v \in D. \tag{25}$$

Thus, we have $u \in \inf_{\pi \in \mathbb{R}^m_+} w(u, v; \pi)$ for all $v \in D$. Moreover, we can conclude that

$$\inf_{\pi \in D} w(u, v; \pi) = u, \ \ \forall v \in D. \tag{26}$$

In fact, take arbitrary $v \in D$ and $z \in \inf_{\pi \in D} w(u, v; \pi)$. Then, it follows that

$$w(u, v; \pi) \nleq_{C \setminus \{0\}} z, \ \ \forall \pi \in D, \tag{27}$$

and there exists a sequence $w(u, v; \pi^k) \to z$ with $\pi^k \in D$. Together with (24), we get $w(u, v; \pi^k) \geq_C u$ and $w(u, v; \pi^k) - u \to z - u$, which implies $z - u \in C$, i.e., $z \geq_C u$, since C is closed. Simultaneously, it follows from (25) and (27) that $z \ngeq_{C \setminus \{0\}} w(u, v; \bar{\pi}) = u$. Thus, we have $z = u$. This shows that (26) holds since $v \in D$ and $z \in \inf_{\pi \in D} w(u, v; \pi)$ are arbitrary. On the other hand, take arbitrary $v \notin D$. Then, it follows from (f) that there exists $\hat{\pi} \in D$ satisfying $\underline{w}(v, \hat{\pi}) < 0$. Let $\theta > 0$ and $\theta \to +\infty$. Then, we get from (d) that there exists $\hat{\pi}_\theta \in D$ such that $\underline{w}(v, \hat{\pi}_\theta) = \theta \underline{w}(v, \hat{\pi}) \to -\infty$. Note that $e \in \operatorname{int} C$. Thus, we have $w(u, v; \hat{\pi}_\theta) = u + \underline{w}(v, \hat{\pi}_\theta) e \to -\infty$, which implies

$$\inf_{\pi \in D} w(u, v; \pi) = -\infty, \ \ \forall v \notin D.$$

Together with (23) and (26), it follows that Assumption $\mathcal{A}$ holds. Lastly, we prove that Assumption $\mathcal{B}$ also holds. For all $(u^i, v^i) \in \mathbb{R}^\ell \times \mathbb{R}^m$, $i = 1, 2$ with $(u^1, v^1) \geq_{C \times D} (u^2, v^2)$, that is, $u^1 \geq_C u^2$ and $v^1 \geq_D v^2$, it follows from $e \in \operatorname{int} C$ and (e) that $\underline{w}(v^1, \pi) \geq \underline{w}(v^2, \pi)$ and

$$w(u^1, v^1; \pi) - w(u^2, v^2; \pi) = (u^1 - u^2) + \Big(\underline{w}(v^1, \pi) - \underline{w}(v^2, \pi)\Big) e \in C + C \subset C,$$

which implies $w(u^1, v^1; \pi) \geq_C w(u^2, v^2; \pi)$ for all $\pi \in D$. Therefore, Assumption $\mathcal{B}$ holds. This completes the proof. ◻

Next, we consider the following penalization optimization problem

(POP) $\quad \min f(x), \quad$ s.t. $\ x \in X, [g_j(x)]_- \geq 0, j = 1, 2, ..., m,$

where $[g_j(x)]_- := \min\{0, g_j(x)\}$. Note that (POP) is equivalent to (VOP) in the sense that (POP) and (VOP) have the same feasible sets and the same objective functions. For simplicity, let $[g(x)]_- := \Big([g_1(x)]_-, [g_2(x)]_-, ..., [g_m(x)]_-\Big)$ for all $x \in X$. In the sequel, we always assume that $X \subset \mathbb{R}^n$ is nonempty and closed, and moreover, every component function $f_i : X \to \mathbb{R}$, $i = 1, 2, ..., \ell$ of the vector-valued function $f = (f_1, f_2, ..., f_\ell) : X \to \mathbb{R}^\ell$ is continuous. It is easy to verify that the feasible set $\mathcal{R} = \{x \in X : g_j(x) \geq 0, j = 1, 2, ..., m\}$ is closed under the assumption that g_j is continuous or upper semi-continuous for each $j \in \{1, 2, ..., m\}$.

Given the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ with conditions (a–e), then the Lagrange-type function $\mathcal{L}^\diamond_w : X \times \Pi \to \mathbb{R}^\ell$ for (POP) with respect to w has the form

$$\mathcal{L}^\diamond_w(x, \pi) := w(f(\bar{x}), 0; \pi) - w(f(\bar{x}) - f(x), [g(x)]_-; \pi), \ \ \forall (x, \pi) \in X \times \Pi.$$

As usual, $\mathcal{L}_w^\diamond$ is said to be the Lagrange-type penalty function and the following optimization problem
(VPP) $\min_{x\in X} \mathcal{L}_w^\diamond(x,\pi)$
is called the vector penalty problem for (VOP) with respect to w. Clearly, it follows from Theorem 4.1 and (a) that

$$\mathcal{L}_w^\diamond(x,\pi) = f(x) - \underline{w}([g(x)]_-, \pi), \ \ \forall (x,\pi) \in X \times \Pi.$$

Now, we are in the position to extend the local image regularity condition from scalar case to (VOP) in the IS. Let $\bar{x} \in \mathcal{R}$ and $\delta > 0$. The localization of the image $\mathcal{K}$ for (VOP) with respect to $\bar{x}$ is defined by

$$\mathcal{K}^\delta := \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u = f(\bar{x}) - f(x), v = g(x), x \in X \cap \mathcal{B}(\bar{x},\delta)\}.$$

Definition 4.1 Suppose that $\bar{x} \in \mathcal{R}$ is a local strong *vmp* to (VOP). The local image regularity condition holds at $\bar{x}$ for (VOP) iff there exists $\delta > 0$ such that

$$\mathcal{H}_u \cap \mathrm{cl\,cone}\,(\mathcal{K}^\delta - \mathrm{cl}\,\mathcal{H}) = \varnothing.$$

Remark 4.1 Specially, the local image regularity condition at $\bar{x} \in \mathcal{R}$ for (VOP) coincides with the local image regularity condition in [27,28] for the scalar case when $\ell = 1$ and $C = \mathbb{R}_+$.

In the sequel, we discuss the existence of a class of exact penalty functions for (VOP) by virtue of the local image regularity condition in the IS. To this end, we shall consider the regular weak separation function $w \in \mathcal{W}_R(\Pi)$ given by conditions (*a–e*) satisfying the following assumption:

Assumption $\mathcal{C}$ There exists some $\alpha > 0$ such that

$$\underline{w}(v,\pi) \le \min\left\{\alpha\pi_1 v_1, \alpha\pi_2 v_2, ..., \alpha\pi_m v_m\right\}$$

holds for all $v = (v_1, v_2, ..., v_m)^T \in -D$ and all $\pi = (\pi_1, \pi_2, ..., \pi_m)^T \in D$.

Example 4.1 Let $\Pi = D$ and $e \in \mathrm{int}\, C$. It is easy to verify that the following vector-valued functions $w : \mathbb{R}^\ell \times \mathbb{R}^m \times \Pi \to \mathbb{R}^\ell$ satisfy conditions (a–e) and Assumption $\mathcal{C}$:

(i) $w(u, v; \pi) = u + \langle \pi, v\rangle e, \ \forall (u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m, \ \forall \pi \in D$ where $\underline{w}(v,\pi) = \langle \pi, v\rangle$;
(ii) $w(u, v; \pi) = u + \min\{\pi_1 v_1, \pi_2 v_2, ..., \pi_m v_m\}e, \ \forall (u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m, \ \forall \pi \in D$ where $\underline{w}(v,\pi) = \min\{\pi_1 v_1, \pi_2 v_2, ..., \pi_m v_m\}$.

Thus, all the functions defined by (i) or (ii) are regular weak separation functions and satisfy Assumptions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

Theorem 4.2 (Exact penalization) *Let the regular weak separation function* $w \in \mathcal{W}_R(\Pi)$ *given by conditions* (*a–e*) *satisfy Assumption* $\mathcal{C}$ *and let* $\bar{x} \in \mathcal{R}$ *be a local strong vmp to* (VOP). *If the local image regularity condition holds at* $\bar{x}$ *for* (VOP), *then the Lagrange-type penalty function* $\mathcal{L}_w^\diamond(x,\pi)$ *is an exact penalty function for*

(VOP)*, that is, there exists* $\tilde{\pi} \in D$ *such that* $\bar{x}$ *is a local strong vmp to* (VPP) *with respect to* w *for all* $\pi \in D$ *satisfying* $\pi \geq_D \tilde{\pi}$.

Proof Assume that the Lagrange-type penalty function $\mathcal{L}_w^{\diamond}(x, \pi)$ is not an exact penalty function for (VOP). Then, for all $\pi^k = (k, k, ..., k) \in D$ with $k \in \mathbb{N}$, there exist $\bar{\pi} \in D$ satisfying $\bar{\pi} \geq_D \pi^k$ and $x^k \in \mathcal{B}(\bar{x}, \frac{1}{k}) \cap X$ such that

$$\mathcal{L}_w^{\diamond}(x^k, \bar{\pi}) \leq_{C\setminus\{0\}} \mathcal{L}_w^{\diamond}(\bar{x}, \bar{\pi}),$$

which implies $f(x^k) - \underline{w}([g(x^k)]_-, \bar{\pi})e \in f(\bar{x}) - \underline{w}([g(\bar{x})]_-, \bar{\pi})e - C \setminus \{0\}$. Since $\bar{x} \in \mathcal{R}$, that is, $g_j(\bar{x}) \geq 0$ for all $j \in \{1, 2, ..., m\}$, we get $[g(\bar{x})]_- = 0$ and $\underline{w}([g(\bar{x})]_-, \bar{\pi}) = 0$ by (c). Thus, we have

$$f(x^k) - \underline{w}([g(x^k)]_-, \bar{\pi})e \in f(\bar{x}) - C \setminus \{0\}, \quad \forall k \in \mathbb{N}. \tag{28}$$

Let $w_j^k = [g_j(x^k)]_-$, $j = 1, 2, ..., m$. Then, $w^k = (w_1^k, w_2^k, ..., w_m^k)^T = [g(x^k)]_-$ and $w^k \in -D$. It follows from Assumption $\mathcal{C}$ that there exists $\alpha > 0$ such that

$$\underline{w}([g(x^k)]_-, \bar{\pi}) = \underline{w}(w^k, \bar{\pi}) \leq \min\{\alpha\bar{\pi}_1 w_1^k, \alpha\bar{\pi}_2 w_2^k, ..., \alpha\bar{\pi}_m w_m^k\}.$$

Together with $\bar{\pi} \geq_D \pi^k$, we get $\bar{\pi}_j \geq k$ for all $j \in \{1, 2, ..., m\}$ and

$$-\underline{w}([g(x^k)]_-, \bar{\pi}) \geq \alpha k \max\{|w_1^k|, |w_2^k|, ..., |w_m^k|\}.$$

Since $\|w^k\| = \sqrt{|w_1^k|^2 + |w_2^k|^2 + ... + |w_m^k|^2} \leq \sqrt{m(\max\{|w_1^k|, |w_2^k|, ..., |w_m^k|\})^2}$, it follows that

$$\max\{|w_1^k|, |w_2^k|, ..., |w_m^k|\} \geq \frac{\|w^k\|}{\sqrt{m}}.$$

This shows $-\underline{w}([g(x^k)]_-, \bar{\pi}) \geq \alpha k \frac{\|w^k\|}{\sqrt{m}}$. Together with $e \in \operatorname{int} C$, we have

$$\left(\underline{w}([g(x^k)]_-, \bar{\pi}) + \frac{\alpha k}{\sqrt{m}}\|w^k\|\right)e \in -C, \quad \forall k \in \mathbb{N}. \tag{29}$$

Therefore, we can conclude from (28) and (29) that

$$\begin{aligned} f(x^k) + \frac{\alpha k}{\sqrt{m}}\|w^k\|e &= f(x^k) - \underline{w}([g(x^k)]_-, \bar{\pi})e + \left(\underline{w}([g(x^k)]_-, \bar{\pi}) + \frac{\alpha k}{\sqrt{m}}\|w^k\|\right)e \\ &\in f(\bar{x}) - C \setminus \{0\} - C, \\ &\subset f(\bar{x}) - C \setminus \{0\}, \quad \forall k \in \mathbb{N}, \end{aligned}$$

that is,

$$f(\bar{x}) - f(x^k) \in \frac{\alpha k}{\sqrt{m}}\|w^k\|e + C \setminus \{0\}, \quad \forall k \in \mathbb{N}. \tag{30}$$

Together with $e \in \operatorname{int} C$, it follows that $\frac{\alpha k}{\sqrt{m}}\|w^k\|e \in C$. Simultaneously, we have $f(\bar{x}) - f(x^k) \in C + C \setminus \{0\} \subset C \setminus \{0\}$, which implies $f(\bar{x}) \neq f(x^k)$ for all $k \in \mathbb{N}$. Let $t^k := \|f(\bar{x}) - f(x^k)\|$. Then, we have $t^k > 0$ and $t^k \to 0$ since $x^k \to \bar{x}$ as $k \to +\infty$ and $f_i, i = 1, 2, ..., \ell$ are continuous. Note that $\{\frac{f(\bar{x})-f(x^k)}{t^k}\} \subset C \setminus \{0\}$ since C is a cone and $\|\frac{f(\bar{x})-f(x^k)}{t^k}\| = 1$. Therefore, there exists $\tilde{u} \in C \setminus \{0\}$ with $\|\tilde{u}\| = 1$ and a subsequence of $\frac{f(\bar{x})-f(x^k)}{t^k}$ such that it converges to $\tilde{u}$. Without loss of generality, we may suppose that

$$\frac{f(\bar{x}) - f(x^k)}{t^k} \to \tilde{u}. \tag{31}$$

Moreover, it follows from (30) that $f(x^k) - f(\bar{x}) \in -\frac{\alpha k}{\sqrt{m}}\|w^k\|e - C \setminus \{0\}$. Since C is a cone, we have

$$\frac{f(x^k) - f(\bar{x})}{t^k} \in -\frac{\alpha k}{\sqrt{m}}\left\|\frac{w^k}{t^k}\right\|e - C \setminus \{0\}, \quad \forall k \in \mathbb{N}.$$

Applying Lemma 2.1 (nonlinear scalar function), we get

$$\xi_e\left(\frac{f(x^k) - f(\bar{x})}{t^k}\right) \leq -\frac{\alpha k}{\sqrt{m}}\left\|\frac{w^k}{t^k}\right\|, \quad \forall k \in \mathbb{N},$$

which implies

$$\left\|\frac{w^k}{t^k}\right\| \leq -\frac{\sqrt{m}}{\alpha}\xi_e\left(\frac{f(x^k) - f(\bar{x})}{kt^k}\right), \quad \forall k \in \mathbb{N},$$

since ξ_e is nonnegative homogeneous. Note that $\left\|\frac{f(x^k)-f(\bar{x})}{kt^k}\right\| = \frac{1}{k} \to 0$ and ξ_e is continuous. Then, we have $\xi_e\left(\frac{f(x^k)-f(\bar{x})}{kt^k}\right) \to \xi_e(0) = 0$ and $\left\|\frac{w^k}{t^k}\right\| \to 0$. Together with (31), it follows that

$$\frac{1}{t^k}\left(f(\bar{x}) - f(x^k), w^k\right) \to (\tilde{u}, 0). \tag{32}$$

Note that $w_j^k = [g_j(x^k)]_-$, $j = 1, 2, ..., m$. Then, it follows that $g_j(x^k) \geq w_j^k$ for all $j \in \{1, 2, ..., m\}$. Thus, we have $w^k \in g(x^k) - D$, and from $x^k \in X \cap \mathcal{B}(\bar{x}, \frac{1}{k})$ and $\operatorname{cl}\mathcal{H} = C \times D$ that

$$\frac{1}{t^k}\left(f(\bar{x}) - f(x^k), w^k\right) \in \operatorname{cone}(\mathcal{K}^{\frac{1}{k}} - \operatorname{cl}\mathcal{H}), \quad \forall k \in \mathbb{N}.$$

Take arbitrary $\delta > 0$. Then, we get

$$\frac{1}{t^k}\left(f(\bar{x}) - f(x^k), w^k\right) \in \operatorname{cone}(\mathcal{K}^{\delta} - \operatorname{cl}\mathcal{H})$$

for all sufficiently large $k \in \mathbb{N}$. Together with (32) and $\tilde{u} \in C \setminus \{0\}$, we can conclude that $(\tilde{u}, 0) \in \mathcal{H}_u \cap \operatorname{cl}\operatorname{cone}(\mathcal{K}^\delta - \operatorname{cl}\mathcal{H})$, which is a contradiction to the local image regularity condition at $\bar{x}$ for (VOP). This completes the proof. □

Remark 4.2 Note that Theorem 4.4 in [26] also established an existence result for a exact penalty function, where the separable function $\underline{w}$ was particularly defined by a distance function, for the scalar case when $\ell = 1$ and $C = \mathbb{R}_+$. In Theorem 4.2, we extend the results to vector cases by virtue of a more general class of regular weak separation functions.

5 Conclusions

In this paper, we established a Lagrange-type duality for a constrained nonconvex vector optimization problem by virtue of the class of vector-valued regular weak separation functions in the IS. Specially, we obtained some equivalent characterizations to the zero duality gap property including the Lagrange multiplier, the Lagrange saddle point and the regular separation. Moreover, we also established a local exact penalization result by means of a local image regularity condition and a class of particular regular weak separation functions in the IS.

Acknowledgements The author is grateful to the anonymous referees and the Editor-in-Chief for their valuable comments and suggestions, which help to improve the paper. This research was supported by the National Natural Science Foundation of China (Grants: 11526165 and 11601437), the Scientific Research Fund of Sichuan Provincial Science and Technology Department (Grant: 2015JY0237) and the Fundamental Research Funds for the Central Universities (Grant: JBK160129).

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