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Optimal Control of Investments in Old and New Capital Under Improving Technology

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Abstract An optimal control problem for nonlinear integral equations of special kind is analyzed. It considers a firm's investment into age-dependent capital under improving technology and limited substitutability among capital of different ages. We prove the existence of solutions and analyze their structure. It is shown that the initially bang-bang optimal investment switches to an interior one and eventually converges to a steady-state trajectory that represents balanced economic growth. The obtained analytic outcomes contribute to better understanding of investment policies under technological change.

Keywords Nonlinear optimal control · Existence theorem · Technological change · Constant elasticity of substitution · Bang-bang solution · Balanced growth

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1 Introduction

The paper analyzes a novel optimal control problem with two-dimensional controls, which arises at the intersection of economics and technological change theory with potential applications to biological and environmental sciences. Economics of investments into new technologies has been intensively studied using models with heterogeneous capital or labor, known as the *age-dependent* (or *vintage*) *capital models* [1,2]. Being a part of the abstract optimization theory [3], age-dependent optimization models possess essential specifics and first appeared in population biology [4–10] and, later, in economics [11–18] and environmental sciences [19–21]. Rational management of age-dependent capital leads to nonlinear optimal control of integral equations of special kind [21–25]. The existence and qualitative behavior of solutions to such problems require a special analysis, because no general theory exists. The mathematical complexity comes from economic requirements to reveal and analyze explicit structure of optimal trajectories.

In this paper, we address an important open issue of innovation economics: Why firms invest into old less efficient technologies while newer and better ones are freely available on market. Possible reasons include lower prices of old capital, learning-by-doing and spillovers, prior technology-specific investments, and internal adjustment costs. In the absence of the above factors, the optimal strategy is to invest only in the latest vintage with the highest efficiency [12]. Several papers [16,18,26] study the limited substitutability among vintages, which leads to delayed intra-firm adoption of technology. Chari and Hopenhayn [26] analyze a vintage model with limited substitutability among two (old and new) vintages and establish its bell-shaped investment age profile. Jovanovic and Yatsenko [16] extend this result to the infinite number of vintages and show that their optimization model possesses a steady-state trajectory with notable economic properties. However, neither they obtain a solution to their problem nor prove its existence. The present paper generalizes [16] in several relevant aspects. First, we describe a firm problem with given prices rather than a general equilibrium model. It allows obtaining a richer dynamics of given functions. Most importantly, we analyze the existence and structure of solutions to the optimization problem. We first prove the existence of solutions applying the weak convergence technique and Mazur's theorem of functional analysis [27] in conjunction with a concave structure of the problem. Next, using the Lagrange multipliers, we derive a maximum principle and employ it to reveal how the initially bang-bang investment switches to an interior one and later converges to a steady-state trajectory.

The paper is as follows. Section 2 formulates the nonlinear optimal control problem and provides its applied interpretation. The existence of solutions is proven in Sect. 3. Section 4 studies the structure of solutions, establishes the bang-bang and interior structure of optimal controls, and proves the convergence of solutions to a steady-state trajectory with notable properties. Section 5 discusses new insights into investment policies under technological change, which follow from the obtained analytic outcomes.

2 Statement and Interpretation of the Optimal Control Problem

Let us consider a firm that produces the output $y(t)$ using capital vintages (assets, machines) installed at different times v, $v \le t$. An important issue is whether, when, and how much the firm should invest into buying old vintages in the presence of newer more efficient capital. To address technological change, we assume that newer vintages have larger productivity $g(v, t)$, i.e., g increases in v. Let $U(t)$ and $u(v, t)$ be the firm's investments into new and old vintages correspondingly, $x(v, t)$ be the capital amount of vintage v, and $p(v, t)$ be its price at time t. The firm wishes to find a rational blend of investments into old and new vintages, which maximizes its infinite-horizon discounted profit. It leads to the following nonlinear optimal control problem:

Find $u^*(v, t)$, $U^*(t)$, $x^*(v, t)$, $y^*(t)$, $v \in]-\infty, t]$, $t \in [0, \infty[$, that maximize the objective functional

$$\max_{u,U} I = \max_{u,U} \int_0^\infty e^{-rt} f\left(y(t) - p(t,t)U(t) - \int_{-\infty}^t p(v,t)u(v,t)\mathrm{d}v\right)\mathrm{d}t, \quad (1)$$

subject to constraints-inequalities:

$$0 \le U(t) \le U_{\max}(t), \quad 0 \le u(v,t) \le u_{\max}(v,t), \quad (2)$$

and constraints-equalities:

$$y(t) = \left(\int_{-\infty}^t g(v,t)x^\beta(v,t)\mathrm{d}v\right)^{\alpha/\beta}, \quad (3)$$

$$\frac{\partial x(v,t)}{\partial t} = -\mu x(v,t) + u(v,t), \quad \nu \in]0, t[, t \in [0, \infty[, \quad (4)$$

$$x(t,t) = U(t), \quad t \in [0, \infty[, \quad (5)$$

$$x(v,0) = x_0(v), \quad \nu \in]-\infty, 0], \quad (6)$$

where $r > 0, 0 < \alpha < 1, 0 < \beta < 1, \mu > 0$ and the functions f, $f' \ge 0$, $f'' \le 0$, $g \ge 0$, $p \ge 0$, $x_0 \ge 0$ are given. The unknown U and u are independent controls, while x and y are state variables. The presence of maximal possible investments $U_{\max}$ and $u_{\max}$ in the constraints (2) is caused by capital budgeting and similar economic constraints.

Economic novelty of the optimization model (1)–(6) is in the limited substitutability of different capital vintages, determined by the parameter $0 < \beta < 1$ [16]. The elasticity $1/(1-\beta)$ of substitution among different vintages is higher when β is larger. The parameter $0 < \alpha < 1$ describes decreasing returns to scale due to production and market factors (such as downward sloping inverse demand curve or nonlinear adjustment costs), and $r > 0$ is the industry-wide discount rate.

The dynamics of capital vintages is described by the linear age-dependent population model (4)–(6), where the capital x depreciates at a constant rate $\mu > 0$. The analytic solution to (4)–(6) is

$$x(v,t) = e^{-\mu(t-v)}U(v) + \int_v^t e^{-\mu(t-s)}u(v,s)\mathrm{d}s \quad \text{for } 0 < v \le t, \tag{7}$$

$$x(v,t) = e^{-\mu t}x_0(v) + \int_0^t e^{-\mu(t-s)}u(v,s)\mathrm{d}s \quad \text{for } -\infty < v \le 0, \tag{8}$$

for any given distribution $x_0(v)$ of capital over past vintages $v \in]-\infty, 0]$ at $t = 0$.

3 Existence of Solutions

To prove the existence of optimal solution, we first show the directional differentiability and some necessary estimates of the state equation. Next, we reduce the problem (1)–(6) to a standard form by defining the control set in a proper functional space. Finally, we construct a sequence of admissible controls that converges to an optimal solution.

Let $\hat{u}(a,t) := u(t-a,t)$ and $\hat{x}(a,t) := x(t-a,t)$ for $a \ge 0, t \ge 0$. Then the state problem (4)–(6) with the initial history x_0 is transformed to the following PDE:

$$\left.\begin{array}{l} \frac{\partial \hat{x}(a,t)}{\partial t} + \frac{\partial \hat{x}(a,t)}{\partial a} = -\mu\hat{x}(a,t) + \hat{u}(a,t), \quad a,t \in [0,\infty[, \\ \hat{x}(0,t) = U(t), \quad t \in [0,\infty[, \\ \hat{x}(t,0) = x_0(-a), \quad a \in [0,\infty[. \end{array}\right\}$$

with the solution $\hat{x}(a,t)$ defined by

$$\hat{x}(a,t) = \begin{cases} e^{-\mu a}U(t-a) + \int_{t-a}^t e^{-\mu(t-s)}\hat{u}(s+a-t,s)\mathrm{d}s, & a < t, \\ e^{-\mu t}x_0(t-a) + \int_0^t e^{-\mu(t-s)}\hat{u}(s+a-t,s)\mathrm{d}s, & a \ge t. \end{cases} \tag{9}$$

Let $Q :=]0,\infty[\times]0,\infty[$. Let us define the derivative along the characteristic line (cf. [8]) as

$$D\hat{x}(a,t) := \lim_{h\to 0}\frac{\hat{x}(a+h,t+h) - \hat{x}(a,t)}{h} \quad \text{for } (a,t) \in Q.$$

Following common approach of the optimization theory [3], we first prove the existence of solutions in certain general functional space (in Chapter 3) and, next, demonstrate that the obtained solutions belong to such spaces (in Chapter 4). Specifically, we assume that the endogenous controls $\hat{u} \in L^1 \cap L^\infty(Q)$ and $U \in L^1 \cap L^\infty]0,\infty[$.

Theorem 3.1 (differentiability and estimates of $\hat{x}$) *If $\hat{u} \in L^1 \cap L^\infty(Q)$ and $U \in L^1 \cap L^\infty]0,\infty[$, and $x_0 \in L^1 \cap L^\infty]-\infty,0[$, then $\hat{x} \in L^1 \cap L^\infty(Q)$, $\hat{x}$ is differentiable along the characteristic line through almost every point in $\{0\}\times]0,\infty[\cup]0,\infty[\times\{0\}$ and satisfies*

$$\left.\begin{array}{ll} D\hat{x}(a,t) = -\mu\hat{x}(a,t) + \hat{u}(a,t), & \text{a.e. } (a,t) \in Q, \\ \hat{x}(0,t) := \lim_{h\to 0}\hat{x}(h,t+h) = U(t), & \text{a.e. } t \in [0,\infty[, \\ \hat{x}(a,0) := \lim_{h\to 0}\hat{x}(a+h,h) = x_0(-a), & \text{a.e. } a \in [0,\infty[. \end{array}\right\} \tag{10}$$

Furthermore, the following estimates hold:

$$\left\|\hat{x}\right\|_{L^\infty(Q)} \leq \max\left\{\|U\|_{L^\infty]0,\infty[}, \|x_0\|_{L^\infty]-\infty,0[}\right\} + \frac{1}{\mu}\left\|\hat{u}\right\|_{L^\infty(Q)}, \tag{11}$$

$$\left\|\hat{x}\right\|_{L^1(Q)} \leq \frac{1}{\mu}\left\{\|U\|^+_{L^1]0,\infty[}\left\|\hat{u}\right\|^+_{L^1(Q)}\|x_0\|_{L^1]-\infty,0[}\right\}. \tag{12}$$

In addition, if $\hat{u}(a,t) \geq 0$ *a.e.*$(a,t) \in Q$, $U(t) \geq 0$ *a.e.* $t \in [0,\infty[$, *and* $x_0(v) \geq 0$ *a.e.* $v \in]-\infty, 0[$, *then* $\hat{x}(a,t) \geq 0$ *a.e.* $(a,t) \in Q$.

Proof It can be easily shown that (10) and (11) follow from (9). We shall justify the estimate (12). Changing the variables of integration and using Fubini's theorem, we have

$$\begin{aligned}
\left\|\hat{x}(\cdot,t)\right\|_{L^1]0,\infty[} &= \int_0^t \left|\hat{x}(a,t)\right| \mathrm{d}a + \int_t^\infty \left|\hat{x}(a,t)\right| \mathrm{d}a \\
&= \int_0^t e^{-\mu(t-s)} |U(s)| \mathrm{d}s + \int_0^t \int_{t-s}^\infty e^{-\mu(t-s)} \left|\hat{u}(s+a-t,s)\right| \mathrm{d}a\mathrm{d}s \\
&\quad + \int_{-\infty}^0 e^{-\mu t} |x_0(v)| \mathrm{d}v \\
&= \int_0^t e^{-\mu(t-s)} |U(s)| \mathrm{d}s + \int_0^t \int_0^\infty e^{-\mu(t-s)} \left|\hat{u}(b,s)\right| \mathrm{d}b\mathrm{d}s \\
&\quad + \int_{-\infty}^0 e^{-\mu t} |x_0(v)| \mathrm{d}v.
\end{aligned}$$

Integrating the above and using Fubini's theorem again, we obtain

$$\begin{aligned}
\int_0^\infty \left\|\hat{x}(\cdot,t)\right\|_{L^1]0,\infty[} \mathrm{d}t \leq & \int_0^\infty \left(\int_s^\infty e^{-\mu(t-s)} \mathrm{d}t\right) |U(s)| \mathrm{d}s \\
& + \int_0^\infty \left(\int_s^\infty e^{-\mu(t-s)} \mathrm{d}t\right) \left\|\hat{u}(\cdot,s)\right\|_{L^1]0,\infty[} \mathrm{d}s \\
& + \left(\int_0^\infty e^{-\mu t} \mathrm{d}t\right) \int_{-\infty}^0 |x_0(v)| \mathrm{d}v.
\end{aligned}$$

Since $\int_s^\infty e^{-\mu(t-s)} \mathrm{d}t = \frac{1}{\mu}$, the estimate (12) holds. □

Next, we reduce the problem (1) to a standard form. Let us assume that $U_{\max} \in L^1 \cap L^\infty]0,\infty[$, $u_{\max} \in L^1 \cap L^\infty(\Delta)\Delta := \{(\nu,t) : -\infty < \nu < t,\ t > 0\}$, and there exists a constant $C(u_{\max}) \geq 0$ such that

$$\int_{-\infty}^t u_{\max}(\nu,t)\mathrm{d}\nu \leq C(u_{\max}) \quad \text{for a.e. } t \in]0,\infty[. \tag{13}$$

Let $\hat{u}_{\max}(a,t) := u_{\max}(t-a,t)$. Then $\hat{u}_{\max} \in L^1 \cap L^\infty(Q)$ and we define the *control set* as follows:

$$\mathcal{U} := \left\{ \boldsymbol{U} = (\hat{u}, U) \in L^1 \cap L^\infty(Q) \times L^1 \cap L^\infty]0,\infty[: \begin{array}{l} 0 \le \hat{u}(a,t) \le \hat{u}_{\max}(a,t) \\ 0 \le U(t) \le U_{\max}(t) \end{array} \right\}. \tag{14}$$

Since $L^1 \cap L^\infty(Q) \subset L^2(Q)$and $L^1 \cap L^\infty]0,\infty[\subset L^2]0,\infty[$, the control set $\mathcal{U}$ can be viewed as a subset of $L^2(Q) \times L^2]0,\infty[$. For a control $\boldsymbol{U} = (\hat{u}, U) \in \mathcal{U}$, denote $\hat{x}(a,t)$ by $\hat{x}^{\boldsymbol{U}}(a,t)$ to indicate the dependence on $\boldsymbol{U}$ and put $\hat{y}^{\boldsymbol{U}}(t) := \left(\int_0^\infty g(t,t-a)(\hat{x}^{\boldsymbol{U}})^\beta(a,t)\mathrm{d}a\right)^{\frac{\alpha}{\beta}}$. To prove the meaningfulness of the integral (1), we assume that $g(t,\nu)$ is measurable on $\bar{\Delta}' := \{(t,\nu) : -\infty < \nu \le t,\ t \ge 0\}$ and satisfies

$$\int_{-\infty}^t g(t,v)\mathrm{d}v \le Me^{\omega t} \quad \text{for a.e.} \quad t \in]0,\infty[\tag{15}$$

for some $M \ge 0$ and $\omega \ge 0$. Then, we have

$$0 \le \hat{y}^{\boldsymbol{U}}(t) \le M^{\frac{\alpha}{\beta}} e^{\frac{\alpha}{\beta}\omega t} \left\| \hat{x}^{\boldsymbol{U}} \right\|^\alpha_{L^\infty(Q)} \quad \text{for all } t \in]0,\infty[\tag{16}$$

and $\hat{y}^{\boldsymbol{U}}(t)$ is well defined. Noting that

$$y(t) = \left(\int_{-\infty}^t g(t,v)x^\beta(v,t)\mathrm{d}v\right)^{\frac{\alpha}{\beta}} = \left(\int_0^\infty g(t,t-a)(\hat{x}^{\boldsymbol{U}})^\beta(a,t)\mathrm{d}a\right)^{\frac{\alpha}{\beta}} = \hat{y}^{\boldsymbol{U}}(t),$$

the optimal control problem (1) is reduced to

$$\max_{\boldsymbol{U}\in\mathcal{U}} J(\boldsymbol{U}) = \max_{\boldsymbol{U}\in\mathcal{U}} \int_0^\infty e^{-rt} f\left(\hat{y}^{\boldsymbol{U}}(t) - p(t,t)U(t) - \int_0^\infty p(t-a,t)\hat{u}(a,t)\mathrm{d}a\right)\mathrm{d}t. \tag{17}$$

Here, f: IR $\to$ IR is assumed to be a C^2 function satisfying $f' \ge 0, f'' \le 0$; $r > 0$ is fixed, and $p(v,t)$ is a nonnegative bounded measurable function on $\bar{\Delta} := \{(\nu,t) : -\infty < \nu \le t,\ t \ge 0\}$. Then we have

$$\begin{aligned} \int_0^\infty p(t-a,t)\hat{u}(a,t)\mathrm{d}a &\le \bar{p}\int_0^\infty \hat{u}_{\max}(a,t)\mathrm{d}a \\ &= \bar{p}\int_{-\infty}^t u_{\max}(v,t)\mathrm{d}v \le \bar{p}C(u_{\max}), \end{aligned} \tag{18}$$

where $\bar{p}$ denotes the upper bound of $p(v, t)$ and $C(u_{\max})$ appears in (13). Then, by Maclaurin's theorem, we have from (16) and (18) that

$$\begin{aligned} & e^{-rt} f\left(-\bar{p}\,\|U_{\max}\|_{L^\infty(0,\infty)} - \bar{p}C(u_{\max})\right) \\ & \le e^{-rt} f\left(\hat{y}^{\boldsymbol{U}}(t) - p(t,t)U(t) - \int_0^\infty p(t-a,t)\hat{u}(a,t)\mathrm{d}a\right) \\ & \le f(0)e^{-rt} + f'(0)e^{-rt}\hat{y}^{\boldsymbol{U}}(t) \le f(0)e^{-rt} + f'(0)M^{\frac{\alpha}{\beta}} e^{-\left(r-\frac{\alpha}{\beta}\omega\right)t} \left\|\hat{x}^{\boldsymbol{U}}\right\|^{\alpha}_{L^\infty(Q)} \end{aligned} \tag{19}$$

By (19), the integral with respect to t in (17) is meaningful if the parameter ω in assumption (15) satisfies

$$0 \le \omega < \frac{\beta}{\alpha} r. \tag{20}$$

Remark 3.1 In Sect. 4, we take $g(t, \nu) = e^{\gamma\nu}G(t-\nu)$ in (32) with $\gamma > 0$ and $e^{-\gamma a}G(a)$ being integrable over $]0, \infty[$, $G \ge 0$ and assume $\frac{\gamma\alpha}{\beta(1-\alpha)} < r$ in (33). This implies that

$$\int_{-\infty}^t g(t,\nu)\mathrm{d}\nu = \int_{-\infty}^t e^{\gamma\nu}G(t-\nu)\mathrm{d}\nu = e^{\gamma t}\int_0^\infty e^{-\gamma a}G(a)\mathrm{d}a$$

and, hence, the above assumptions (15) and (20) hold with $M = \int_0^\infty e^{-\gamma a}G(a)\mathrm{d}a$ and $\omega = \gamma$. Therefore, solutions constructed later in Sect. 4, satisfy (15) and (20).

Now, let $d = \sup_{\boldsymbol{U}\in\mathcal{U}} J(\boldsymbol{U})$. For each n, take $\boldsymbol{U}_n = (\hat{u}_n, U_n)$ in $\mathcal{U}$ such that $d - \frac{1}{n^2} < J(\boldsymbol{U}_n) \le d$. Since $\{\boldsymbol{U}_n\} \subset \mathcal{U}$ is bounded in $L^2(Q) \times L^2]0, \infty[$, we can extract a subsequence denoted again by $\{\boldsymbol{U}_n\}$ such that $\boldsymbol{U}_n$ converges weakly to some $\boldsymbol{U}_* = (\hat{u}_*, U_*)$ in $L^2(Q) \times L^2]0, \infty[$. By Mazur's theorem ([27, Cor.3.8]), there exists a subsequence $\{\boldsymbol{U}_n\}$ such that

$$\bar{\boldsymbol{U}}_n := \sum_{i=n+1}^{k_n} \lambda_i^n \boldsymbol{U}_i \quad (k_n \ge n+1), \quad \lambda_i^n \ge 0, \quad \sum_{i=n+1}^{k_n} \lambda_i^n = 1$$

and $\bar{\boldsymbol{U}}_n = (\bar{u}_n, \bar{U}_n)$ converges strongly to $\boldsymbol{U}_* = (\hat{u}_*, U_*)$ in $L^2(Q) \times L^2]0, \infty[$, where $\bar{u}_n(a,t) = \sum_{i=n+1}^{k_n} \lambda_i^n \hat{u}_i(a,t)$ and $\bar{U}_n(t) = \sum_{i=n+1}^{k_n} \lambda_i^n U_i(t)$. Then it follows that

$$\lim_{n\to\infty} \bar{u}_n(a,t) = \hat{u}_*(a,t) \quad \text{a.e. } (a,t) \in Q, \tag{21}$$

$$\lim_{n\to\infty} \bar{U}_n(t) = U_*(t) \quad \text{a.e. } t \in]0, \infty[, \tag{22}$$

by taking a subsequence if necessary. Also, we have $\bar{\boldsymbol{U}}_n = (\bar{u}_n, \bar{U}_n) \in \mathcal{U}$ because

$$0 \le \bar{u}_n(a,t) = \sum_{i=n+1}^{k_n} \lambda_i^n \hat{u}_i(a,t) \le \sum_{i=n+1}^{k_n} \lambda_i^n \hat{u}_{\max}(a,t) = \hat{u}_{\max}(a,t),$$
$$0 \le \bar{U}_n(t) = \sum_{i=n+1}^{k_n} \lambda_i^n U_i(t) \le \sum_{i=n+1}^{k_n} \lambda_i^n U_{\max}(t) = U_{\max}(t). \tag{23}$$

From (21)–(23), we find that $\boldsymbol{U}_* = (\hat{u}_*, U_*) \in \mathcal{U}$. Since the state problem is linear, we find that $\hat{x}^{\bar{\boldsymbol{U}}_n} = \sum_{i=n+1}^{k_n} \lambda_i^n \hat{x}^{\boldsymbol{U}_i}$ and $\hat{x}^{\bar{\boldsymbol{U}}_n} - \hat{x}^{\boldsymbol{U}_*} = \hat{x}^{\bar{\boldsymbol{U}}_n - \boldsymbol{U}_*}$ with the initial value $x_0 \equiv 0$. Hence, it follows from (12) that

$$\left\| \hat{x}^{\bar{\boldsymbol{U}}_n} - \hat{x}^{\boldsymbol{U}_*} \right\|_{L^1(Q)} = \left\| \hat{x}^{\bar{\boldsymbol{U}}_n - \boldsymbol{U}_*} \right\|_{L^1(Q)}$$
$$\le \frac{1}{\mu} \left\{ \left\| \bar{U}_n - U_* \right\|_{L^1(0,\infty)} + \|\bar{u}_n - u_*\|_{L^1(Q)} \right\} \to 0 \text{ as } n \to \infty.$$

Therefore, by taking a subsequence if necessary, we obtain

$$\lim_{n\to\infty} \hat{x}^{\bar{\boldsymbol{U}}_n}(a,t) = \hat{x}^{\boldsymbol{U}_*}(a,t), \quad \text{a.e. } (a,t) \in Q. \tag{24}$$

Lemma 3.1 *Let* (13), (15)*, and* (20) *hold. Then, we have*

$$\lim_{n\to\infty} J(\bar{\boldsymbol{U}}_n) = J(\boldsymbol{U}_*), \tag{25}$$

where $J(\boldsymbol{U}) = \int_0^\infty e^{-rt} f\left(\hat{y}^{\boldsymbol{U}}(t) - p(t,t)U(t) - \int_0^\infty p(t-a,t)\hat{u}(a,t)da\right) dt,$

$$\hat{y}^{\boldsymbol{U}}(t) = \left(\int_0^\infty g(t,t-a)(\hat{x}^{\bar{U}})^\beta(a,t)da\right)^{\frac{\alpha}{\beta}} \quad \textit{for } \boldsymbol{U} = (\hat{u}, U) \in \mathcal{U}.$$

Proof By (11), $\hat{x}^{\bar{\boldsymbol{U}}_n}$ is bounded by some constant depending on $\|U_{\max}\|_{L^\infty(0,\infty)}$, $\left\|\hat{u}_{\max}\right\|_{L^\infty(Q)}$, $\|x_0\|_{L^\infty(-\infty,0)}$, and, by (15), the function $a \mapsto g(t,t-a) \in L^1]0,\infty[$ for each $t \in]0,\infty[$. Then (24) implies

$$\lim_{n\to\infty} \hat{y}^{\bar{\boldsymbol{U}}_n}(t) = \hat{y}^{\boldsymbol{U}_*}(t) \quad \text{a.e. } t \in]0,\infty[\tag{26}$$

by Lebesgue's dominated convergence theorem. Also, since $\hat{u}_{\max} \in L^1(Q)$ and p is bounded, it follows from (21) and (23) that

$$\lim_{n\to\infty} \int_0^\infty p(t-a,t)\bar{u}_n(a,t)\mathrm{d}a$$
$$= \int_0^\infty p(t-a,t)\hat{u}_*(a,t)\mathrm{d}a, \quad \text{a.e. } t \in]0,\infty[. \tag{27}$$

Then from (26), (21) and (27) combined with the estimate (19), we conclude that (25) holds by Lebesgue's dominated convergence theorem. □

Theorem 3.2 (the existence of the optimal control) *Let* (13), (15), (20) *hold and*

$$0 < \alpha \leq \beta \leq 1. \tag{28}$$

Then, there exists at least one optimal control.

Proof Since both functions $x \mapsto x^{\beta}$, $x \mapsto x^{\alpha/\beta}$ are concave by (28) and $\hat{x}^{\bar{\boldsymbol{U}}_n} = \sum_{i=n+1}^{k_n} \lambda_i^n \hat{x}^{\boldsymbol{U}_i}$, we have the following estimate:

$$\begin{aligned}
\hat{y}^{\bar{\boldsymbol{U}}_n}(t) &= \left(\int_0^{\infty} g(t, t-a) \left(\hat{x}^{\bar{\boldsymbol{U}}_n}\right)^{\beta}(a, t)\mathrm{d}a\right)^{\frac{\alpha}{\beta}} \\
&= \left(\int_0^{\infty} g(t, t-a) \left(\sum_{i=n+1}^{k_n} \lambda_i^n \hat{x}^{\boldsymbol{U}_i}\right)^{\beta}(a, t)\mathrm{d}a\right)^{\frac{\alpha}{\beta}} \\
&\geq \left(\int_0^{\infty} g(t, t-a) \sum_{i=n+1}^{k_n} \lambda_i^n \left(\hat{x}^{\boldsymbol{U}_i}\right)^{\beta}(a, t)\mathrm{d}a\right)^{\frac{\alpha}{\beta}} \\
&\geq \sum_{i=n+1}^{k_n} \lambda_i^n \left(\int_0^{\infty} g(t, t-a) \left(\hat{x}^{\boldsymbol{U}_i}\right)^{\beta}(a, t)\mathrm{d}a\right)^{\frac{\alpha}{\beta}} \\
&= \sum_{i=n+1}^{k_n} \lambda_i^n \hat{y}^{\boldsymbol{U}_i}(t).
\end{aligned}$$

Since $f' \geq 0$ and $f'' \leq 0$, f is nondecreasing and concave, and so, we have

$$\begin{aligned}
& f\left(\hat{y}^{\bar{\boldsymbol{U}}_n}(t) - p(t,t)\bar{U}_n - \int_0^{\infty} p(t-a, t)\bar{u}_n(a, t)\mathrm{d}a\right) \\
&\geq f\left(\sum_{i=n+1}^{k_n} \lambda_i^n \left(\hat{y}^{\boldsymbol{U}_i}(t) - p(t,t)U_i(t) - \int_0^{\infty} p(t-a, t)\hat{u}_i(a, t)\mathrm{d}a\right)\right) \\
&\geq \sum_{i=n+1}^{k_n} \lambda_i^n f\left(\hat{y}^{\boldsymbol{U}_i}(t) - p(t,t)U_i(t) - \int_0^{\infty} p(t-a, t)\hat{u}_i(a, t)\mathrm{d}a\right).
\end{aligned}$$

Then we get $d \geq J\left(\bar{\boldsymbol{U}}_n\right) \geq \sum_{i=n+1}^{k_n} \lambda_i^n J\left(\boldsymbol{U}_i\right) \geq \sum_{i=n+1}^{k_n} \lambda_i^n \left(d - \frac{1}{i^2}\right) = d - \sum_{i=n+1}^{k_n} \lambda_i^n \frac{1}{i^2} \geq d - \sum_{i=n+1}^{\infty} \frac{1}{i^2}$.

Therefore, by Lemma 3.1, we have $J\left(\boldsymbol{U}_*\right) = \lim_{n\to\infty} J\left(\bar{\boldsymbol{U}}_{\boldsymbol{n}}\right) = d$, that shows that $\boldsymbol{U}_*$ is optimal. □

4 Qualitative Behavior and Structure of Solutions

This section analyzes the structure of solutions to the optimal control problem (1)–(6) at the linear $f(x) = x$. To derive extremum conditions, we make the same assumptions about the endogenous controls as in Sect. 3: $u \in L^1 \cap L^\infty(Q)$ and $U \in L^1 \cap L^\infty]0, \infty[$. As we will see, those assumptions will be later satisfied for the solution obtained in Theorem 4.2.

Lemma 4.1 (maximum principle) *Let* $(u^*, U^*, x^*, y*)$ *be a solution to the problem* (1)–(6)*, then*

$$\begin{aligned}
&I'_u(v,t) \le 0 \text{ at } u^*(v,t) = 0,\; I'_u(v,t) = 0 \text{ at} 0 < u^*(v,t) < u_{\max}(v,t),\\
&I'_u(v,t) \ge 0 \text{ at } u^*(v,t) = u_{\max}(v,t),\\
&I'_U(v,t) \le 0 \text{ at } U^*(t) = 0,\; I'_U(v,t) = 0 \text{ at } 0 < U^*(t) < U_{\max}(t),\\
&I'_U(v,t) \ge 0 \text{ at } U^*(t) = U_{\max}(t), 0 \le t < \infty, -\infty < v < t,
\end{aligned} \tag{29}$$

where the Frechet derivatives are

$$\begin{aligned}
I'_u(v,t) &= \alpha \int_t^\infty e^{-rs-\mu(s-t)} g(v,s) y^{\frac{\alpha-\beta}{\alpha}}(s) x^{\beta-1}(v,s) ds\\
&\quad -e^{-rt} p(v,t) \quad \textit{at } 0 < v < t,
\end{aligned} \tag{30}$$

$$\begin{aligned}
I'_U(t) &= \alpha \int_t^\infty e^{-rs-\mu(s-t)} g(t,s) y^{\frac{\alpha-\beta}{\alpha}}(s) x^{\beta-1}(t,s) ds\\
&\quad -e^{-rt} p(t,t), \quad 0 < t < \infty.
\end{aligned} \tag{31}$$

Proof is standard and based on the method of Lagrange multipliers adjusted to the vintage capital models in [13,17,21]. □

Economically, $I'_u(v,t)$ in (29)–(30) describes the future rental value of vintage v at time t. If $I'_u(v,t) < 0$, then the optimal $u(v,t) = 0$ by Lemma 4.1, which means no old vintage v is bought at time t.

Next, we analyze the interior long-term and transition (short-term) dynamics of the problem (1)–(6). To get meaningful applied results, we assume the unitary price p and a special structure of the kernel g:

$$f(x) = x, \quad p(v,t) = 1, \quad g(v,t) = e^{\gamma v} G(t-v), \tag{32}$$

where $\gamma > 0$ is the rate of *exponential technological change* and G describes age-dependent effects such as spillovers, learning-by-doing, forgetting-by-doing, which depend on the age $t - v$ [12,16,18,20].

4.1 Steady-State Analysis

Under the condition (32), the problem (1)–(6) possesses an interior *steady-state solution*, known as the balanced growth path in the economic theory [12,17,21]. It is a solution when all major system characteristics grow with the same exponential rate but given initial conditions are disregarded. Economists commonly use the concept of balanced growth path to analyze behavior of complex models.

Lemma 4.2 *(balanced growth path) If* (32) *and*

$$\gamma\alpha/\left(\beta(1-\alpha)\right) < r \tag{33}$$

hold, then the problem (1)–(6) *has a unique steady-state (balanced growth) trajectory* $\tilde{x}$, $\tilde{y}$, $\tilde{U}$, $\tilde{u}$:

$$\tilde{x}(v,t) = e^{\zeta t}\chi(t-v), \tag{34}$$

$$\tilde{y}(t) = \begin{cases} e^{\zeta t}\bar{k}^{\frac{\alpha(1-\beta)}{\alpha-\beta}}\left(\frac{\alpha}{r+\mu}\right)^{-\frac{\alpha}{\alpha-\beta}} & \text{at} \quad \alpha \neq \beta, \\ e^{\zeta t}\bar{k}^{\beta}\int_0^\infty e^{-\frac{\gamma}{1-\beta}s}G^{\frac{1}{1-\beta}}(s)\mathrm{d}s & \text{at} \quad \alpha = \beta, \end{cases} \tag{35}$$

$$\tilde{u}(v,t) = e^{\zeta t}\omega(t-v), \quad \tilde{U}(t) = \bar{k}e^{\zeta t}G^{\frac{1}{1-\beta}}(0), \quad -\infty < v < t, 0 < t < \infty, \tag{36}$$

where the balanced growth rate ζ *is*

$$\zeta = \frac{\gamma\alpha}{\beta(1-\alpha)}, \quad \textit{and} \tag{37}$$

$$\chi(a) = \bar{k}e^{-\frac{\gamma}{1-\beta}a}G^{\frac{1}{1-\beta}}(a), \quad a = t - v, \tag{38}$$

$$\bar{k} = \left(\frac{\alpha}{r+\mu}\right)^{\frac{1}{1-\alpha}}\left(\int_0^\infty e^{-\frac{\gamma}{1-\beta}s}G^{\frac{1}{1-\beta}}(s)ds\right)^{\frac{\alpha-\beta}{\beta(1-\alpha)}}, \tag{39}$$

$$\omega(a) = \bar{k}e^{-\frac{\gamma}{1-\beta}a}\left[\left(\mu + \frac{\alpha-\beta}{\beta(1-\alpha)(1-\beta)}\gamma\right)G^{\frac{1}{1-\beta}}(a) + \frac{\mathrm{d}}{\mathrm{d}a}\left(G^{\frac{1}{1-\beta}}(a)\right)\right]. \tag{40}$$

Proof To find the balanced growth path, we assume that the optimality condition (29) holds for all vintages $-\infty < v < t$. By Lemma 4.1, substituting (32) to (30), setting it to zero for $-\infty < v < t, 0 < t < \infty$, and differentiating it lead to the first-order extremum condition for *an interior solution* $x(v,t) > 0$, $y(t) > 0$

$$\alpha e^{\gamma v}y^{(\alpha-\beta)/\alpha}(t)G(t-v)x^{\beta-1}(v,t) = r + \mu. \tag{41}$$

Next, assuming a balanced capital $\tilde{x}(v,t)$ of the form (34), we solve the nonlinear system (3), (7), (41) and obtain the solution (35)–(40). Finally, the inequality (33) obtained from (35), (36), and (1) secures convergence of the improper integral in (1). □

Remark 4.1 The balanced growth path (34)–(40) in the problem (1)–(6) does not satisfy the initial condition (6) and instead assumes that $x(v,0) = \tilde{x}(v,0)$ for $v \in]-\infty, 0[$. So, it does not represent a real solution to the problem (1)–(6), but appears to play an important role in complete dynamics of the solution analyzed in Sects. 4.2 and 4.2. Moreover, the optimal solution converges to the balanced growth path.

To explore restrictions on the key parameters α, β, γ, and μ, let us assume the age-independent vintage productivity $G(v,t) \equiv G$ in (32). Then by (35)–(40), we have an exponential steady-state trajectory:

$$\tilde{u}(v,t) = \bar{k}\left(\mu + \frac{\gamma(\alpha-\beta)}{\beta(1-\alpha)(1-\beta)}\right) e^{\frac{\gamma\alpha}{\beta(1-\alpha)}t} e^{-\frac{\gamma}{1-\beta}(t-v)} G^{\frac{1}{1-\beta}}, \tag{42}$$

$$\tilde{U}(t) = \bar{k}e^{\frac{\gamma\alpha}{\beta(1-\alpha)}t} G^{\frac{1}{1-\beta}}, \quad \tilde{y}(t) = \bar{k}^{\frac{\alpha(1-\beta)}{\alpha-\beta}} \left(\frac{\alpha}{r+\mu}\right)^{-\frac{\alpha}{\alpha-\beta}} e^{\frac{\gamma\alpha}{\beta(1-\alpha)}t}, \tag{43}$$

$$\tilde{x}(v,t) = \bar{k}e^{\frac{\gamma\alpha}{\beta(1-\alpha)}t} e^{-\frac{\gamma}{1-\beta}(t-v)} G^{\frac{1}{1-\beta}}, \quad \bar{k}^{1-\alpha} = \frac{\alpha}{r+\mu}\left(\frac{\gamma}{1-\beta}\right)^{\frac{\beta-\alpha}{\beta}} G^{\frac{\alpha-\beta}{\beta(1-\beta)}}. \tag{44}$$

Thus, at $G(v,t) \equiv G$, the balanced long–term dynamics (42) of the investment $\tilde{u}(v,t)$ into old vintages resembles the motion (44) of the vintage capital stock $\tilde{x}(v,t)$. Let us consider three cases:

- $\alpha > \beta$, then $\tilde{u}(v,t)$ and $\tilde{x}(v,t)$ are both positive and increase in t for any fixed v. It means that the firm always buys *more* capital *of all vintages* and the actual amount of a vintage v exponentially increases in time t. Such dynamics is not sustainable, which is consistent with the existence Theorem 3.2.
- $\alpha = \beta$, then the amount of capital $\tilde{x}(v,t)$ of a vintage v remains constant in time t. This case does not look very natural for real economies, but is formally possible in the model and is used to reveal the solution structure in Sect. 4.2.
- $\alpha < \beta$, then $\tilde{x}(v,t)$, $\tilde{y}(t)$, and $\tilde{U}(t)$ are positive, but $\tilde{u}(v,t) > 0$ if and only if $\mu(1-\alpha) > \gamma$ and

$$\beta < \beta_{cr}, \quad \beta_{cr} = \frac{\mu(1-\alpha) - \gamma + \sqrt{(\mu(1-\alpha)-\gamma)^2 + 4\gamma\alpha(1-\alpha)\mu}}{2\mu(1-\alpha)} < 1 \quad \text{at } \alpha < 1. \tag{45}$$

So, $\tilde{u}(v,t) > 0$ at $\alpha < \beta < \beta_{\mathrm{cr}}$, see also [16]. Then $\tilde{u}(v,t)$ for every old vintage $-\infty < v < t$ decreases exponentially in time t because of (42). i.e., the firm buys *less* capital *of older vintages*. The capital $x(v,t)$ of a specific vintage v decreases in time t because the investment into the vintage v does not compensate deterioration. The optimal $\tilde{u}(v,t) \to 0$ when $\beta \to \beta_{cr}$. At $\beta \geq \beta_{\mathrm{cr}}$, the firm does not buy old vintages at all: $\tilde{u}(v,t) = 0$. So, the control $\tilde{u}$ becomes non-essential and can be removed from the model. Thus, other vintage models with investment only in the latest technology, e.g., [1,11,12,21], can be used at $\beta \geq \beta_{\mathrm{cr}}$.

The impact of changing asset price. Similar analysis was provided for exponential price $p(v,t)$ in (32). The major finding is that evolving capital prices are less relevant for the optimal distribution of investments into old capital than the substitutability among vintages. Specifically, at the infinite elasticity of substitution $\beta = 1$, the problem (1)–(6) cannot possess a balanced growth path (42)–(44) with positive investments into old vintages at *any* price dynamics.

4.2 Solution at $\alpha = \beta$

As follows from Sect. 4.1, the inequality $\alpha \le \beta$ represents a key restriction of the model (1)–(6). To get an informal insight into the structure of solutions, we start with the special case $\alpha = \beta$, which appears to be simpler. Then, the steady-state capital $\tilde{x}$ found from (41) does not depend on $\tilde{y}$:

$$\tilde{x}(v,t) = \left(\alpha(t-v)e^{\gamma v}/(r+\mu)\right)^{1/(1-\beta)} \tag{46}$$

for all vintages $-\infty < v < t$. Hence, the solution (u^*, U^*, x^*, y^*) for vintages installed at $v > 0$ coincides with the steady state $(\tilde{U}, \tilde{u}, \tilde{x}, \tilde{y})$ of Sect. 4.2. Indeed, $x^*(v,t) = \tilde{x}(v,t)$ follows the Eq. (34) and, thus, leads to the same formulas (36) for $U^*(t) = \tilde{U}(t)$ and $u^*(v,t) = \tilde{u}(v,t)$.

The solution structure is different for older vintages installed on the prehistory $]-\infty, 0]$. It involves a transition from the non-optimal initial state $x_0(v)$ of capital x to the best possible balanced trajectory $\tilde{x}$. For clarity, let $U_{\max}(t)$ and $u_{\max}(v,t)$ be constant, $U_{\max}(t) = U_{\max}$ and $u_{\max}(v,t) = u_{\max}$.

Theorem 4.1 *Let $\alpha = \beta$, $\mu(1-\alpha) > \gamma$, and $U_{\max} \ge \|\tilde{u}\|$. Then, the problem (1)–(6) has a solution:*

$$u^*(v,t) = \tilde{u}(v,t),\ U^*(t) = \tilde{U}(t),\ x^*(v,t) = \tilde{x}(v,t), \quad y^*(t) = \tilde{y}(t)$$
$$\textit{for} \quad 0 < v < t, \quad 0 < t < \infty.$$

where $\tilde{x}$, $\tilde{y}$, $\tilde{U}$, $\tilde{u}$ are given by (34)–(40). At $-\infty < v \le 0$:

(a) *If $x_0(v) = \tilde{x}(v,0)$, then $u^*(v,t) = \tilde{u}(v,t)$ for $0 < t < \infty$.*

(b) *If $x_0(v) < \tilde{x}(v,0)$, then* $u^*(v,t) = \begin{cases} u_{max} & \textit{for } 0 < t \le \bar{t}(v), \\ \tilde{u}(v,t) & \textit{for } \bar{t}(v) < t < \infty, \end{cases}$

where the instant $\bar{t}(v)$, $-\infty < v < 0$, is found from the nonlinear equation

$$\frac{u_{\max}}{\mu}e^{\mu\bar{t}} - e^{\mu\bar{t}}\left(\frac{\alpha}{r+\mu}e^{\gamma v}\,G(\bar{t}-v)\right)^{\frac{1}{1-\beta}} = \frac{u_{\max}}{\mu} - x_0(v), \tag{47}$$

that has solution at

$$u_{\max} > \mu\left(\alpha e^{\gamma v} \max_{[0,\infty)} G(v)/(r+\mu)\right)^{\frac{1}{1-\beta}}. \tag{48}$$

(c) *If $x_0(v) > \tilde{x}(v,0)$, then* $u^*(v,t) = \begin{cases} 0 & \textit{for } 0 < t \le \bar{t}(v), \\ \tilde{u}(v,t) & \textit{for } \bar{t}(v) < t < \infty, \end{cases}$

where $\bar{t}(v) > 0$, $-\infty < v < 0$, *is found from the equation*

$$e^{\mu\bar{t}}\left(\alpha e^{\gamma v}G(\bar{t}-v)/(r+\mu)\right)^{\frac{1}{1-\beta}} = x_0(v), \tag{49}$$

that has a solution at $\min\limits_{[0,\infty)} G'(v) > -\mu/(1-\beta)$. *The corresponding* x^* *is*

$$x^*(v,t) = \begin{cases} e^{-\mu t}x_0(v) + \int_0^t e^{-\mu(t-s)}u^*(v,s)ds & \textit{for } 0 < t < \bar{t}(v), \\ \tilde{x}(v,t) & \textit{for } \bar{t}(v) \le t < \infty. \end{cases} \tag{50}$$

Proof For each fixed $v < 0$, let us construct the feasible control $u(v,t)$ as

(a) $u(v,t) = \tilde{u}(v,t)$, if ; $x_o(v) = \tilde{x}(v,0)$;
(b)

$$u(v,t) = \begin{cases} u\text{max} & \text{at } x(v,t) < \tilde{x}(v,t), \\ \tilde{u}(v,t) & \text{at } x(v,t) = \tilde{x}(v,t), \end{cases} \quad \text{if } x_0(v) < \tilde{x}(v,0); \tag{51}$$

(c) $u(v,t) = \begin{cases} 0 & \text{at } x(v,t) > \tilde{x}(v,t), \\ \tilde{u}(v,t) & \text{at } x(v,t) = \tilde{x}(v,t), \end{cases}$ if $x_0(v) > \tilde{x}(v,0)$; $0 < t < \infty$,

and prove that the control u satisfies the maximum principle of Lemma 4.1 in all three cases (a)–(c).

Case (a) Since $x_0(v) = \tilde{x}(v,0)$, we can choose $x^*(v,t) = \tilde{x}(v,t)$, $0 < t < \infty$, and $I'_u(x;v,t) = 0$, by construction of $\tilde{x}$ in Lemma 4.2. The optimal $u^*(v,t)$ is found from the VIE-I (8) at the given x^*. The Eq. (8) at $x_0(v) = \tilde{x}(v,0)$ is the same as the equation for $\tilde{u}$, so its solution is $u^*(v,t) = \tilde{u}(v,t)$.

Case (b) $x_0(v) < \tilde{x}(v,0)$. Let us recall that $I'_u(\tilde{x};v,t) = 0$ by construction of $\tilde{x}$. Then, by (30),

$$\begin{aligned} I'_u(x;v,t) &= I'_u(x;v,t) - I'_u(\tilde{x};v,t) \\ &= \alpha\int_t^\infty e^{-rs-\mu(s-t)}g(v,s)\left[x^{\beta-1}(v,s) - \tilde{x}^{\beta-1}(v,s)\right]\mathrm{d}s \ge 0, \end{aligned} \tag{52}$$

because $\beta < 1$ and $x(v,t) \le \tilde{x}(v,t)$ by (51). Moreover, $I'_u(x;v,0) > 0$ because $x(v,0) = x_0(v) < \tilde{x}(v,0)$. Since $I'_u(x;v,t)$ is continuous in x by (52) and $x(v,t)$ is continuous in t by (8), then $I'_u(x;v,t) > 0$ on some, possibly small, interval $[0,\bar{t}[$. Hence, $u(v,t) = u_{\max}$, $t \in [0,\bar{t}[$ by (29). Then, by (7),

$$x(v,t) = e^{-\mu t}x_0(v) + u_{\max}\int_0^t e^{-\mu(t-s)}\mathrm{d}s = e^{-\mu t}x_0(v) + \frac{1-e^{-\mu t}}{\mu}u_{\max}.$$

Thus, for a fixed $v < 0$, the optimal control $u^*(v,t) = u_{\max}$ on $[0,\bar{t}[$ until the instant $\bar{t}$ is such that $x(v,\bar{t}) = \tilde{x}(v,\bar{t})$. Substituting $\tilde{x}(v,t)$ from (34), we obtain the Eq. (47) for $\bar{t}$. It has at least one solution $0 < \bar{t} < \infty$ if $u_{\max}$ satisfies (48). Next, $x(v,\bar{t}) = \tilde{x}(v,\bar{t})$ that lead to Case (a) on $[\bar{t},\infty[$.

Case (c) $x_0(v) > \tilde{x}(v, 0)$. Then, as in Case (b), $I'_u(x; v, t) \leq 0$ because $\beta < 1$ and $x(v, t) \geq \tilde{x}(v, t)$. Moreover, $I'_u(x; v, 0) < 0$ because $x(v, 0) = x_0(v) > \tilde{x}(v, 0)$. Therefore $I'_u(x; v, t) < 0$ and, by (29), $u(v, t) = 0$ on some interval $[0, \bar{t}]$. Then, by (8), $x(v, t) = e^{-\mu t} x_0(v)$ exponentially decreases in t and becomes $x(v, \bar{t}) = \tilde{x}(v, \bar{t})$ at the instant $\bar{t}$. Thus, for a fixed $v < 0$, the optimal $u(v, t) = 0$ on $[0, \bar{t}[$ until the instant $\bar{t}$ such that $e^{-\mu \bar{t}} x_0(v) = \tilde{x}(v, \bar{t})$. Using $\tilde{x}(v, \bar{t})$ from (34), we obtain (49) for $\bar{t}$. In (49), $e^{\mu \bar{t}}$ increases and $G(a)$ can decrease, so (49) has a solution $0 < \bar{t} < \infty$, at least, at $\min_{[0,\infty)} G'(v) > -\mu/(1-\beta)$. If (49) has no solution, then $I'_u(x; v, t) < 0$ and $u^*(v, t) = 0$ for $0 < t < \infty$. Next, $x(v, \bar{t}) = \tilde{x}(v, \bar{t})$, and we get Case (a) on $[\bar{t}, \infty[$. Thus, the constructed solution satisfies the maximum principle (29). □

Interpretation Following common economic terminology [11,12,16,17,21], we refer to the solution on the interval $[0, \bar{t}(v)]$ as the *transition dynamics* of vintage v. By Theorem 4.2, an *"ideal" initial distribution of capital* on the prehistory $]-\infty, 0]$ appears to be $x_0(v) = \tilde{x}(v, 0)$ for all vintages $v < 0$. Then, there is *no transition dynamics at all* and the optimal control is $u^*(v, t) = \tilde{u}(v, t)$.

By Theorem 4.1, the optimal investment $u^*(v, t)$ in a general case is bang-bang for each fixed old vintage $-\infty < v < 0$. Indeed, if $x_0(v) < \tilde{x}(v, 0)$, then u^* is initially maximal possible until some time $\bar{t}(v)$ when $x(v, \bar{t})$ reaches $\tilde{x}(v, \bar{t})$. At $x_0(v) > \tilde{x}(v, 0)$, the optimal investment u^* is initially $u^*(v, t) = 0$ until the capital decreases to the optimal level $\tilde{x}(v, t)$ because of deterioration. Solving the nonlinear Eqs. (47) or (49) for $\bar{t}$, we obtain the *length of transition dynamics* $\bar{t}$ for each vintage $-\infty < v < 0$, installed on the prehistory. If that equation has no solution, then the transition dynamics never ends for that specific vintage v. Such an unlikely situation may occur if a vintage is largely overfunded or $u_{\max}$ is not large enough.

Let us denote the *maximal length of transition dynamics* as $\bar{t}_{\max} = \max_{-\infty<v\leq 0} \bar{t}(v)$. As shown in Theorem 4.1, the solution coincides with the steady-state trajectory at $t \geq \bar{t}_{\max}$ for all vintages $-\infty < v < t$.

4.3 Solution at $\alpha < \beta$

In the general case $\alpha \neq \beta$, the solution structure is more challenging, because the optimal capital

$$x^*(v, t) = \left[\alpha e^{\gamma v} y^{*^{(\alpha-\beta)/\alpha}}(t) G(t - v)/(r + \mu)\right]^{\frac{1}{1-\beta}} \tag{53}$$

depends on the output $y^*(t)$ by the extremum condition (41). Correspondingly, $y^*(t)$ is different from $\tilde{y}(t)$ during the transition dynamics. The ideal initial distribution of capital

$$\breve{x}(v, 0) \stackrel{df}{=} \left[\alpha e^{\gamma v} y(0)^{(\alpha-\beta)/\alpha} G(-v)/(r + \mu)\right]^{\frac{1}{1-\beta}}, \tag{54}$$

on the prehistory $]-\infty, 0]$ is different from $\tilde{x}(v, 0)$. Fortunately, arising mathematical complications do not essentially affect the qualitative behavior of solutions, which is described by the following statement.

Theorem 4.2 (structure of solutions) *Let $\alpha < \beta$ and $\mu(1-\alpha) > \gamma$. Then, the optimal control problem* (1)–(6) *has a solution* (U^*, u^*, x^*, y^*) *with the following structure:*

$$U^*(t) = \begin{cases} \breve{U}(t) \text{ for } 0 \le t < t_{\max} \\ \tilde{U}(t) \textit{ for } t_{\max} < t < \infty \end{cases}, \quad y^*(t) = \begin{cases} \breve{y}(t) \text{ for } 0 \le t < t_{\max}, \\ \tilde{y}(t) \textit{ for } t_{\max} < t < \infty, \end{cases} \tag{55}$$

where $\tilde{x}$, $\tilde{y}$, $\tilde{U}$, $\tilde{u}$ are given by (34)–(40), *$\breve{y}$ is a unique function, and $t_{\max}$ is determined by* (64) *below. At $0 < v < \infty$:*

$$x^*(v,t) = \begin{cases} \breve{x}(v,t) \text{ for } 0 \le t < t_{\max}, \\ \tilde{x}(v,t) \text{ for } t_{\max} < t < \infty, \end{cases}$$

$$u^*(v,t) = \begin{cases} \breve{u}(v,t) \text{ for } 0 \le t < t_{\max}, \\ \tilde{u}(v,t) \text{ for } t_{\max} < t < \infty, \end{cases} \tag{56}$$

$$\text{where } \breve{x}(v,t) = \left[\alpha \breve{y} \frac{\alpha-\beta}{\alpha}(t) e^{\gamma v} G(t-v)/(r+\mu)\right]^{\frac{1}{1-\beta}},$$

$$\breve{u}(v,t) = \mu \breve{x}(v,t) + \partial \breve{x}(v,t)/\partial t. \tag{57}$$

At $-\infty < v \le 0$:

(a)

$$\textit{If } x_0(v) = \breve{x}(v,0), \quad \text{then } u^*(v,t) = \breve{u}(v,t) \quad \text{for } 0 < t < \infty. \tag{58}$$

(b)

$$\textit{If } x_0(v) < \breve{x}(v,0), \text{ then } u^*(v,t) = \begin{cases} u_{\max} & \text{for } 0 < t \le \bar{t}(v), \\ \breve{u}(v,t) & \text{for } \bar{t}(v) \le t < t_{\max}, \\ \tilde{u}(v,t) & \text{for } t_{\max} < t < \infty, \end{cases} \tag{59}$$

where the instant $\bar{t}(v)$, $-\infty < v < 0$, is found from the nonlinear equation

$$u_{\max} e^{\mu \bar{t}}/\mu - e^{\mu \bar{t}} \breve{x}(v,\bar{t}) = u_{\max}/\mu - x_0(v), \tag{60}$$

that has a solution, at least, at $u_{\max}/\mu >> 1$.

(c)

$$\textit{If } x_0(v) > \breve{x}(v,0), \text{ then } u^*(v,t) = \begin{cases} 0 & \text{for } 0 < t \le \bar{t}(v), \\ \breve{u}(v,t) & \text{for } \bar{t}(v) \le t < t_{\max}, \\ \tilde{u}(v,t) & \text{for } t_{\max} < t < \infty, \end{cases} \tag{61}$$

where $\bar{t}(v) > 0$, $-\infty < v < 0$, is found from the equation

$$e^{\mu \bar{t}} \breve{x}(v,\bar{t}) = x_0(v). \tag{62}$$

If (62) *has no solution, then* $u^*(v,t) = 0$ *at* $0 < t < \infty$. *The corresponding* x *is*

$$x^*(v,t) = \begin{cases} e^{-\mu t} x_0(v) + \int_0^t e^{-\mu(t-s)} u(v,s) ds & \text{for } 0 < t < \bar{t}(v), \\ \breve{x}(v,t) & \text{for } \bar{t}(v) \le t < t_{\max}, \\ \tilde{x}(v,t) & \text{for } t_{\max} \le t < \infty, \end{cases} \quad (63)$$

$$\text{where } \bar{t}_{\max} = \max_{-\infty < v \le 0} \bar{t}(v). \quad (64)$$

Proof Let us assume that a unique optimal state variable y^* exists (it will be determined from the Eqs. (68)–(69) below). Then, we can construct the following feasible control u:

(a)

$$u(v,t) = \breve{u}(v,t),\ 0 < t < \infty, \quad \text{if } x_0(v) = \breve{x}(v,0); \quad (65)$$

(b)

$$u(v,t) = \begin{cases} u_{\max} & \text{at } x(v,t) < \breve{x}(v,t), \\ \tilde{u}(v,t) & \text{at } x(v,t) = \breve{x}(v,t), \end{cases} \quad \text{if } x_0(v) < \breve{x}(v,0); \quad (66)$$

(c)

$$u(v,t) = \begin{cases} 0 & \text{at } x(v,t) > \breve{x}(v,t), \\ \breve{u}(v,t) & \text{at } x(v,t) = \breve{x}(v,t), \end{cases} \quad \text{if } x_0(v) > \breve{x}(v,0); \quad (67)$$

for each fixed vintage $-\infty < v < 0$, where $\breve{x}(v,t)$ is determined from the Eq. (53) as (56) and the interior control $\breve{u}(v,t)$ is determined from the VIE-I (7) as (57).

Analogously to Theorem 4.1, we prove that the control u defined by (65)–(67) satisfies the maximum principle of Lemma 4.1 in all Cases (a)–(c) and the time $\bar{t}(v) > 0$ is determined from (60) or (62) for $-\infty < v < 0$. At $0 < t < \bar{t}(v)$, the optimal $u^*(v,t)$ is zero or $u_{\max}$. The *transition dynamics* for the every past vintage $-\infty < v < 0$ ends at $t = \bar{t}(v)$ and $x^*(v,t) = \breve{x}(v,t)$ at $t > \bar{t}(v)$.

To complete the proof, let us determine the optimal y^* by combining the model Eqs. (3), (7), (8) with (53), (59), (61). Namely, substituting x from (8) and (66) into (3), leads to

$$y(t) = \left(\int_{-\infty}^{t} e^{\gamma v} G(t-v) x^{\beta}(y; v, t) dv \right)^{\alpha/\beta} \quad (68)$$

for the optimal $y = y^*$, where

$$x(y; v, t) = \begin{cases} e^{-\mu t} x_0(v) + \bar{u} \int_0^t e^{-\mu(t-s)} ds & \text{for } 0 < t < \bar{t}(v), \\ \left[\alpha y^{(\alpha-\beta)/\alpha}(t) e^{\gamma v} G(t-v)/(r+\mu) \right]^{\frac{1}{1-\beta}} & \text{for } \bar{t}(v) \le t < \infty, \end{cases}$$
$$\bar{u} = 0 \text{ or } u_{\max}. \quad (69)$$

Let us introduce the new unknown $z(t) = y^{\beta/\alpha}(t)$. Then the Eqs. (68)–(69) is equivalent to

$$z(t) = \int_{-\infty}^{t} e^{\gamma v} G(t-v) x^{\beta}(z; v, t) \mathrm{d}v, \tag{70}$$

$$x(z; v, t) = \begin{cases} h(v, t) & \text{for } 0 < t < \bar{t}(v), \\ z^{\frac{\alpha-\beta}{\beta(1-\beta)}}(t) \left[\alpha e^{\gamma v} G(t-v)/(r+\mu)\right]^{\frac{1}{1-\beta}} & \text{for } \bar{t}(v) \leq t < \infty, \end{cases} \tag{71}$$

where the function $h(v, t) = e^{-\mu t} x_0(v) + \bar{u} \int_0^t e^{-\mu(t-s)} \mathrm{d}s$ does not depend on the unknown z.

By construction, $\bar{t}(v) = 0$ at $v > 0$. Let us split $]-\infty, t]$ into two sets $\Delta_1(t) \cup \Delta_2(t) =]-\infty, t]$ as

$$\Delta_1(t) = \left\{-\infty < v < t : \bar{t}(v) \geq t\right\}, \ \Delta_2(t) = \left\{-\infty < v < 0 : \bar{t}(v) < t\right\}.$$

Then, the Eqs. (70)–(71) become

$$z(t) = [\alpha/(r+\mu)]^{\frac{\beta}{1-\beta}} z^{\frac{\alpha-\beta}{1-\beta}}(t) \int_{\Delta_1(\mathrm{t})} \left[e^{\gamma v} G(t-v)\right]^{\frac{1}{1-\beta}} \mathrm{d}v + \int_{\Delta_2(\mathrm{t})} e^{\gamma v} G(t-v) h^{\beta}(v, t) \mathrm{d}v$$

$$\text{or} \quad z(t) = A_1(t) z^{\frac{\alpha-\beta}{1-\beta}}(t) + A_2(t), \quad \text{where } A_1(t) > 0, \quad A_2(t) > 0. \tag{72}$$

The nonlinear Eq. (72) at $\alpha \leq \beta$ has a unique solution $z(t)$ for any $0 < t < \infty$, because the curves $f_1(z) = A_1 z^{-\theta}$, $\theta > 0$, and $f_2(z) = z - A_2$ have a unique interception point $z^* > 0$. □

Interpretation The purpose of the transition dynamics is to move from the non-optimal capital distribution $x_0(v)$ of pre-installed vintages at $t = 0$ to the optimal $\tilde{x}(v, t)$ for *each past vintage* $v < 0$. The *length of bang-bang dynamics* $\bar{t}(v) > 0$ for each vintage $-\infty < v \leq 0$ is determined from the nonlinear Eqs. (47) or (49). The *total length of transition dynamics* is $t_{\max} = \max_{-\infty < v \leq 0} \bar{t}(v)$. The investment dynamics for the past vintages $v < 0$ is bang-bang on $[0, \bar{t}(v)[$ and is interior on $[\bar{t}(v), t_{\max}[$ and $[t_{\max}, \infty[$. The optimal investment u and capital distribution x are interior on the interval $[\bar{t}(v), t_{\max}]$ but do not coincide with the steady-state trajectory until t reaches $t_{\max}$.

The solution structure is simpler for the "newer" vintages $0 < v \leq t$, installed during the planning horizon. Then, at any instant $t > 0$, we invest the optimal amount $\breve{U}(t)$ into "brand new" vintages $v = t$ and $\breve{u}(v, t)$ into the older vintages $0 \leq v < t$ and obtain the optimal amount of capital.

The solution after $t = t_{\max}$ follows the steady-state trajectory $\tilde{u}, \tilde{U}, \tilde{x}, \tilde{y}$. It implies that the steady-state balanced growth trajectory is *globally stable* at natural conditions. The convergence speed is finite and the length of transition period is determined by the misbalance $|x_0(v) - \tilde{x}(v, 0)|$ of the most unbalanced vintage $\hat{v} = \arg\max_{-\infty < v \leq 0} \bar{t}(v)$.

5 Conclusions

The optimal control of a nonlinear heterogeneous dynamic population with age-and-time-dependent controls is analyzed. The optimization model (1)–(6) describes capital modernization policies in a firm/economy with age-dependent capital under technological change and limited substitutability among capital vintages of different ages [16]. The objective is to maximize utility/profit over the future infinite horizon. Possible model extensions include other production factors (energy, R&D, or human capital) to account for environmental pollutions and restrictions [1,2,18–20]. We prove the existence of solutions, construct explicit solutions, and analyze their structure. The obtained analytic outcomes shed new light on fundamental links between evolving economy and technology.

First of all, the model under exponential technological change possesses a steady-state optimal trajectory (balanced growth) with remarkable properties. In particular, the balanced growth involves positive investments into old vintages only when the elasticity of substitution among vintages is smaller than a certain threshold value.

Next, the optimal investment strategy appears to be very different for vintages installed before and during the planning horizon $[0, \infty[$. Time dynamics of the investment u and capital amount x for newer vintages (installed during the planning horizon) is interior and converges to the balanced growth trajectory. The optimal dynamics for the old vintages (installed before the planning horizon) is richer and includes a *transition* period. The goal of this transition is to switch from an initial non-optimal distribution of pre-installed vintages to the optimal balanced trajectory in the most effective way. As a result, the optimal investment for each old vintage is initially *bang-bang* and depends whether the vintage was initially overfunded or underfunded. For initially underfunded vintages, the optimal investment is maximal until the vintage capital increases to the optimal level. For initially overfunded vintages, the optimal policy is to wait until the vintage capital decreases to the optimal level because of deterioration.

After the *transition dynamics* for every old vintage ends, the investment policy for all vintages is interior and coincides with the long-term balanced growth trajectory. Similar convergence properties were earlier proven for vintage capital models with only one-dimensional investment controls in [13,17].

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