J Optim Theory Appl (2016) 170:783–803
DOI 10.1007/s10957-016-0979-x

On Some Basic Results Related to Affine Functions on Riemannian Manifolds

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Received: 9 November 2015 / Accepted: 4 July 2016 / Published online: 18 July 2016

Abstract We study some basic properties related to affine functions on Riemannian manifolds. A characterization for a function to be linear affine is given and a counterexample on Poincaré plane is provided, which, in particular, shows that assertions **(i)** and **(ii)** claimed by Papa Quiroz in (J Convex Anal 16(1):49–69, 2009, Proposition 3.4) are not true, and that the function involved in assertion **(ii)** is indeed not quasi-convex. Furthermore, we discuss the convexity properties of the sub-level sets of the function on Riemannian manifolds with constant sectional curvatures.

Keywords Riemannian manifold · Hadamard manifold · Sectional curvature · Convex function · Quasi-convex function · Linear affine function

Mathematics Subject Classification 52A55 · 53C20 · 90C25

Communicated by Sándor Zoltán Németh.

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1 Introduction

Recent interests are focused on the topic on basic results related to affine functions on Riemannian manifolds; see, e.g., [1–3]. In particular, fixing a point x_0 in a Hadamard manifold and a nonzero vector u_0 in the tangent space at x_0, one considers the special function f_0, generated by the inner product between vector u_0 and the inverse of the exponential map at point x_0, and the associated parallel vector field X_0, formed by the parallel transportation of the given vector u_0 (see (5) and (6) in Sect. 3, respectively). Assertions **(i)** and **(ii)** below were claimed in [2, Proposition 3.4] (without the proof for assertion **(ii)**).

(i) The gradient of f_0 is the associated parallel vector field X_0;

(ii) f_0 is linear affine.

Assertions **(i)** and **(ii)** have been used in [2,4] to study the proximal point algorithm for quasi-convex/convex functions with Bregman distances on Hadamard manifolds, while assertion **(ii)** was also used in [5,6] to establish some existence results of solutions for equilibrium problems and vector optimization problems on Hadamard manifolds, respectively. However, assertion **(ii)** is clearly not true in general because, by [1, p. 299, Theorem 2.1]), any twice differentiable linear affine function on Poincaré plane $\mathbb{H}$ (a two-dimensional Hadamard manifold of constant curvature -1) is constant. Indeed, it has been further shown in [3, Theorem 2.1] that assertion **(ii)** is true for all points x_0 in the Hadamard manifold and vectors u_0 in the corresponding tangent space if and only if the manifold is isometric to the Euclidean space $\mathbb{R}^n$. Furthermore, one can easily check that the function f_0 is even not convex, in general, because, otherwise, one has that both f_0 and $-f_0$ are convex (and so linear affine). This motivates us to consider the following problems:

Problem 1 Is f_0 quasi-convex?

Problem 2 Is assertion **(i)** true?

The first purpose of this paper is to present a characterization for linear affine functions on Hadamard manifolds in terms of assertion **(i)** and parallel transports, and to provide a counterexample on Poincaré plane to illustrate that the answer to each problem is negative. In particular for Problem 2, we show that the vector field X_0 is even not a gradient field.

Our second purpose in the present paper is, in spirit of the negative answer to Problem 1, to study the convexity issue of sub-level sets of the function f_0 mentioned above in Riemannian manifolds with constant sectional curvatures. Our main results provide the exact estimate of the constant c such that the sub-level set L_{c,f_0}, consisting of all points x with value $f_0(x)$ being no more than c, is strongly convex, which in particular improves and extends the corresponding result in [7, Corollary 3.1].

The paper is organized as follows. We review, in Sect. 2, some basic notions, notations, and some classical results of Riemannian geometry that will be needed afterward. The characterization for linear affine functions on Hadamard manifolds and the counterexample on Poincaré plane are presented in Sect. 3. Finally, in Sect. 4, the convexity properties of the sub-level sets of the function f_0 in Riemannian manifolds with constant sectional curvatures are discussed.

2 Notations, Notions, and Preliminaries

In the present section, we present some basic notations, definitions, and properties of Riemannian manifolds. The readers are referred to some textbooks for details, for example [1,8,9].

Let M be a complete, simply connected n-dimensional Riemannian manifold with the Levi-Civita connection ∇ on M. We denote the tangent space at $x \in M$ by $T_x M$, and Let $\mathcal{X}(M)$ denote all (C^∞) vector fields on M. By $\langle \cdot, \cdot \rangle_x$ and $\| \cdot \|_x$, we mean the corresponding Riemannian inner product and the norm on $T_x M$, respectively (where the subscript x is sometimes omitted). For $x, y \in M$, let $\gamma : [0, 1] \to M$ be a piecewise smooth curve joining x to y. Then, the arc-length of γ is defined by $l(\gamma) := \int_0^1 \|\dot{\gamma}(t)\| \mathrm{d}t$, while the Riemannian distance from x to y is defined by $\mathrm{d}(x, y) := \inf_\gamma l(\gamma)$, where the infimum is taken over all piecewise smooth curves $\gamma : [0, 1] \to M$ joining x to y. We use $\mathbb{B}(x, r)$ to denote the open metric ball at x with radius r, that is,

$$\mathbb{B}(x, r) := \{y \in M : \mathrm{d}(x, y) < r\}.$$

For a smooth curve γ, if $\dot{\gamma}$ is parallel along itself, then γ is called a geodesic, that is, a smooth curve γ is a geodesic if and only if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$. A geodesic $\gamma : [0, 1] \to M$ joining x to y is minimal if its arc-length equals its Riemannian distance between x and y. By the Hopf–Rinow theorem [8], (M, d) is a complete metric space, and there is at least one minimal geodesic joining x to y. The set of all geodesics $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ is denoted by Γ_{xy}, that is,

$$\Gamma_{xy} := \{\gamma : [0, 1] \to M : \ \gamma(0) = x, \ \gamma(1) = y \text{ and } \nabla_{\dot{\gamma}} \dot{\gamma} = 0\}.$$

We use $\gamma_{xy} : [0, 1] \to M$ to denote the minimal geodesic satisfying $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$ if it is unique.

Let γ be a geodesic. We use $P_{\gamma,\cdot,\cdot}$ to denote the parallel transport on the tangent bundle TM (defined below) along γ with respect to ∇, which is defined by

$$P_{\gamma,\gamma(b),\gamma(a)} v = X(\gamma(b)) \quad \text{for all } a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)} M, \tag{1}$$

where X is the unique vector field satisfying

$$X(\gamma(a)) = v \quad \text{and} \quad \nabla_{\dot{\gamma}} X = 0. \tag{2}$$

Then, for any $a, b \in \mathbb{R}$, $P_{\gamma,\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)} M$ to $T_{\gamma(b)} M$. We will write $P_{y,x}$ instead of $P_{\gamma,y,x}$ in the case when γ is a minimal geodesic joining x to y and no confusion arises.

The exponential map of M at $x \in M$ is denoted by $\exp_x(\cdot) : T_x M \to M$. For a C^∞ function $f : M \to \mathbb{R}$, grad f and Hess f denote its gradient vector and Hessian, respectively. Let $X, Y \in \mathcal{X}(M)$. The Riemannian connection has the expression in terms of parallel transportation, that is,

$$(\nabla_X Y)(x) = \lim_{t\to 0} \frac{1}{t} \left\{ P_{\gamma,\gamma(0),\gamma(t)} Y(\gamma(t)) - Y(x) \right\} \quad \text{for any } x \in M, \tag{3}$$

where the curve γ with $\gamma(0) = x$ and $\dot{\gamma}(0) = X(x)$ (see, e.g., [9, p. 29 Exercise 5]).

A complete simply connected Riemannian manifold of non-positive sectional curvature is called a *Hadamard manifold*. The following proposition is well known about the Hadamard manifolds; see, e.g, [9, p. 221].

Proposition 2.1 *Suppose that* M *is a Hadamard manifold. Let* $p \in M$*. Then,* $\exp_p : T_pM \to M$ *is a diffeomorphism, and for any two points* $p, q \in M$*, there exists a unique normal geodesic joining* p *to* q*, which is in fact a minimal geodesic.*

The following definition presents the notions of different convexities, where item (a) and (b) are known in [10]; see also [11–13].

Definition 2.1 Let Q be a non-empty subset of the Riemannian manifold M. Then, Q is said to be

(a) Weakly convex if, for any $x, y \in Q$, there is a minimal geodesic of M joining x to y and it is in Q;

(b) Strongly convex if, for any $x, y \in Q$, there is just one minimal geodesic of M joining x to y, and it is in Q.

All convexities in a Hadamard manifold coincide and are simply called the convexity. Let $f : M \to \overline{\mathbb{R}}$ be a proper function, and let dom f denote its domain, that is, dom $f := \{x \in M : f(x) \neq \infty\}$. We use Γ^f_{xy} to denote the set of all $\gamma \in \Gamma_{xy}$ such that $\gamma \subseteq$ dom f. In the following definition, item (a) is known in [14,15] and item (b) is an extension of the one in [1, p. 59], which is introduced for the case when dom f is totally convex.

Definition 2.2 Let $f : M \to \overline{\mathbb{R}}$ be a proper function and suppose that dom f is weakly convex. Then, f is said to be

(a) convex if

$$f \circ \gamma(t) \leq (1-t)f(x) + tf(y) \quad \text{for all } x, y \in \text{dom } f,\ \gamma \in \Gamma^f_{xy},\ t \in [0, 1];$$

(b) quasi-convex if

$$f \circ \gamma(t) \leq \max\{f(x), f(y)\} \quad \text{for all } x, y \in \text{dom } f,\ \gamma \in \Gamma^f_{xy},\ t \in [0, 1].$$

Clearly, for a proper function f with a weakly convex domain, the convexity implies the quasi-convexity. Fixing $c \in \mathbb{R}$, we use $L_{c,f}$ to denote the sub-level set of f defined by

$$L_{c,f} := \{x \in M : f(x) \leq c\}.$$

The following proposition describes the relationship between the convexities of a function f and its sub-level sets.

Proposition 2.2 *Let $f : M \to \overline{\mathbb{R}}$ be a proper function with weakly convex domain* dom f. *Then, f is quasi-convex if and only if, for each $c \in \mathbb{R}$, the sub-level set $L_{c,f}$ is totally convex with restricted to* dom f *in the sense that for any $x, y \in L_{c,f}$, if $\gamma \in \Gamma_{xy}^f$ then $\gamma \subseteq L_{c,f}$. In particular, f is quasi-convex if and only if $L_{c,f}$ is strongly convex for each $c \in \mathbb{R}$ in the case when* dom f *is strongly convex.*

Proof We only consider the case when dom f is weakly convex (otherwise when dom f is strongly convex, the result is immediate by definition).

Suppose that f is quasi-convex. Take $c \in \mathbb{R}$. Let $x, y \in L_{c,f} \subseteq \text{dom} f$ and let $\gamma \in \Gamma_{xy}^f$. Then, $f(x) \le c$ and $f(y) \le c$. Noting that f is quasi-convex, it follows that

$$f \circ \gamma(t) \le \max\{f(x), f(y)\} \le c \quad \text{for all} \quad t \in [0, 1].$$

This implies that $\gamma \subseteq L_{c,f}$ and so $L_{c,f}$ is totally convex restricted to dom f since $x, y \in L_{c,f}$ and $\gamma \in \Gamma_{xy}^f$ are arbitrary.

Conversely, suppose that $L_{c,f}$ is totally convex restricted to dom f for each $c \in \mathbb{R}$. Let $x, y \in \text{dom} f$ and let $\gamma \in \Gamma_{xy}^f$. Set $c_0 := \max\{f(x), f(y)\}$. Then, by assumption, $\gamma \subseteq L_{c_0,f}$, that is,

$$f \circ \gamma(t) \le c_0 = \max\{f(x), f(y)\} \quad \text{for all } t \in [0, 1].$$

This implies that f is quasi-convex since $x, y \in \text{dom} f$ and $\gamma \in \Gamma_{xy}^f$ are arbitrary. The proof is complete. □

3 Linear Affine Functions and Counterexampls on Hadamard Manifolds

For the whole section, we assume that M is a Hadamard manifold. Consider a proper convex function $f : M \to \overline{\mathbb{R}}$ on M. The subdifferential of f at $x \in \text{dom} f$ is defined by

$$\partial f(x) := \left\{v \in T_x M : \ f(y) \ge f(x) + \langle v, \dot{\gamma}_{xy}(0)\rangle \quad \text{for all } y \in \text{dom} f.\right\}$$

By [1, p. 74] (see also [15, Proposition 6.2]), $\partial f(x)$ is a non-empty, compact and convex set for any $x \in \text{int(dom} f)$, where int$Q$ denotes the topological interior of a subset Q of M. Let $f : M \to \overline{\mathbb{R}}$ be a proper function with convex domain. Recall that f is linear affine if both f and $-f$ are convex. Furthermore, if f is of C^2 and dom f is open, its Hess f is defined by

$$\text{Hess} f(X, Y) = \langle \nabla_X \text{grad} f, Y\rangle \quad \text{for any } X, Y \in \mathcal{X}(M).$$

Then, by [1, p. 83, remark 6], we have that

$$f \text{ is linear affine} \iff \nabla_X \text{grad} f = 0 \quad \text{for any } X \in \mathcal{X}(M). \tag{4}$$

Let $x_0 \in M$, and let $u_0 \in T_{x_0}M$. Then, the function f_0 and the vector field X_0 considered in the introduction can be formulated as

$$f_0(x) := \langle u_0, \exp_{x_0}^{-1} x\rangle \quad \text{for any} \quad x \in M \tag{5}$$

and

$$X_0(x) := P_{x,x_0}u_0 \qquad \text{for any} \quad x \in M, \tag{6}$$

respectively. Thus, assertions **(i)** and **(ii)** in [2, Proposition 3.4] can be restated as follows:

(i) Grad $f_0 = X_0$.

(ii) f_0 is linear affine on M.

The following theorem presents, in particular, a characterization in Hadamard manifolds for assertion **(ii)** to be true in terms of assertion **(i)** and the parallel transports.

Theorem 3.1 *Let* $f : M \to \overline{\mathbb{R}}$ *be a proper function and suppose that* dom f *is a nonempty open convex subset. If function* f *is linear affine, then, for any* $x_0 \in \text{dom}\, f$, *there exists* $u_0 \in T_{x_0}M$ *such that*

$$P_{x,x_0}u_0 = P_{x,z} \circ P_{z,x_0}u_0 \quad \textit{for any } (z, x) \in \text{dom}\, f \times \text{dom}\, f, \tag{7}$$

$$\text{grad}\, f(x) = P_{x,x_0}u_0 \quad \textit{for any } x \in \text{dom}\, f \tag{8}$$

and

$$f(x) = f(x_0) + \langle u_0, \exp_{x_0}^{-1} x\rangle \quad \textit{for any } x \in \text{dom}\, f. \tag{9}$$

Conversely, if there exist $x_0 \in \text{dom}\, f$ *and* $u_0 \in T_{x_0}M$ *such that* (7) *and* (8) *hold, then* f *is linear affine.*

Proof Assume that f is linear affine. Then, both f and $-f$ are convex. Take $x_0 \in \text{dom}\, f$ and note that dom f is open. It follows that both $\partial f(x_0)$ and $\partial(-f(x_0))$ are non-empty. Thus, one can chose $u_0 \in \partial f(x_0)$ and $u_0' \in \partial(-f(x_0))$, respectively. Then, by definition, we have that, for any $x \in \text{dom}\, f$,

$$f(x) \geq f(x_0) + \left\langle u_0, \exp_{x_0}^{-1} x\right\rangle \quad \text{and} \quad -f(x) \geq -f(x_0) + \left\langle u_0', \exp_{x_0}^{-1} x\right\rangle; \tag{10}$$

hence $\langle u_0 + u_0', \exp_{x_0}^{-1} x\rangle_{x_0} \leq 0$ for any $x \in \text{dom}\, f$. This implies that $u_0 + u_0' = 0$, that is $u_0' = -u_0$ (as dom f is open). Thus, (9) follows from (10). Furthermore, noting that f is of class C^∞ by (9), one then has that Hess $f = 0$ on dom f, that is,

$$\text{Hess}\, f(X, Y) = \langle \nabla_X \text{grad}\, f, Y\rangle = 0 \quad \text{for any } X, Y \in \mathcal{X}(\text{dom}\, f).$$

In particular, one has that

$$\nabla_{\dot{\gamma}_{xz}} \text{grad}\, f = 0 \quad \text{for any } x, z \in \text{dom}\, f,$$

where γ_{xz} is the geodesic joining x and z, which lies in dom f. This, together with the definition of parallel transport (e.g., (1)), implies that

$$\text{grad}\, f(x) = P_{x,z}\text{grad}\, f(z) \quad \text{for any } x, z \in \text{dom}\, f. \tag{11}$$

Note further that, for any $u \in T_{x_0}M$, one has

$$\langle \text{grad}\, f(x_0), u\rangle_{x_0} = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \exp_{x_0} tu \mid_{t=0} = \langle u_0, u\rangle_{x_0}.$$

It follows that grad $f(x_0) = u_0$. This, together with (11), implies that (7) and (8) hold.

Now, suppose that (7) and (8) hold for some $x_0 \in$ dom f and $u_0 \in T_{x_0}M$. Let $x \in$ dom f and $X \in \mathcal{X}(\text{dom}\, f)$. Let $\gamma : [-\varepsilon, \varepsilon] \to$ dom f be the geodesic contained in dom f with $\gamma(0) = x$ and $\dot\gamma(0) = X(x)$. Let $t \in [-\varepsilon, \varepsilon]$. We see from (8) that

$$\text{grad}\, f(x) = P_{x,x_0}u_0, \ \ \text{grad}\, f(\gamma(t)) = P_{\gamma(t),x_0}u_0.$$

In light of (7), it follows that

$$P_{x,\gamma(t)}\text{grad}\, f(\gamma(t)) = P_{x,\gamma(t)} \circ P_{\gamma(t),x_0}u_0 = P_{x,x_0}u_0 = \text{grad}\, f(x).$$

Noting that $P_{x,\gamma(t)} = P_{\gamma,x,\gamma(t)}$, one gets by (3) that

$$(\nabla_X \text{grad}\, f)(x) = \lim_{t\to 0} \frac{1}{t}\{P_{x,\gamma(t)}\text{grad}\, f(\gamma(t)) - \text{grad}\, f(x)\} = 0.$$

Since $X \in \mathcal{X}(\text{dom}\, f)$ and $x \in$ dom f are arbitrary, we conclude that Hess $f = 0$ on dom f, and so f is linear affine. The proof is complete. □

Before considering Problems 1–2 proposed in the introduction which are related to the function f_0 defined by (5), we make a remark regarding the quasi-convexity properties about composite functions. To this end, we have the following theorem, which is due to Udriste [1, p. 101, Theorem 10.9] in the special case when $M = \tilde{M}$. The idea of the proof is the same as that for [1, p. 101, Theorem 10.9] (although the arguments presented there are not completely clear), and so we omit the proof here.

Theorem 3.2 *Let $(\tilde{M}, \tilde{\nabla})$ be a Riemannian manifold with Levi-Civita connection $\tilde{\nabla}$ and $D \subseteq \tilde{M}$ be a totally convex subset. If $\varphi : D \to \mathbb{R}$ is a quasi-convex function on $(\tilde{M}, \tilde{\nabla})$ and $F : \tilde{M} \to M$ is a diffeomorphism, then $\varphi \circ F^{-1}$ is a quasi-convex function on $(M, F_*\tilde{\nabla})$.*

Remark 3.1 Recall that M is with the Levi-Civita connection ∇ and $x_0 \in M$, $u_0 \in T_{x_0}M$. Set $\tilde{M} := T_{x_0}M$, and let $\tilde{M}$ be endowed with the Riemannian metric $\tilde{g}$ given by $\tilde{g}_x(\cdot,\cdot) := \langle\cdot,\cdot\rangle_{x_0}$ for each $x \in \tilde{M}$ (i.e., $\tilde{M}$ is an n-dimensional Hilbert space). Let $\tilde{\nabla}$ be the Levi-Civita connection compatible with the metric. Then, f_0 (defined by (5)) can be written as $f_0 = \varphi \circ F^{-1}$, where $F : \tilde{M} \to M$ and $\varphi : \tilde{M} \to \mathbb{R}$ are defined by $F(\cdot) := \exp_{x_0}(\cdot)$ and

$$\varphi(v) := \langle u_0, v\rangle_{x_0} \quad \text{for any } v \in \tilde{M},$$

respectively. It is evident that φ is linear affine on $\tilde{M}$ and F is a diffeomorphism by Proposition 2.1. Thus, one could apply Theorem 3.2 to conclude that the function f_0 is quasi-convex on M with the connection $F_*\tilde{\nabla}$ but not the Levi-Civita connection ∇. Indeed, we shall see from Example 3.1 below that f_0 is not quasi-convex.

Recalling the equivalence (4), the following problem related to Problems 1-2 is also natural:

Problem 3 Does the vector field X_0 satisfy

$$\nabla_X X_0 = 0 \quad \text{for any } X \in \mathcal{X}(M)?$$

The remainder of this section is to construct a counterexample on Poincaré plane to show that the answer to each of Problems 1–3 is negative. To do this, let

$$M = \mathbb{H} =: \{(t_1, t_2) \in \mathbb{R}^2 : \ t_2 > 0\},$$

be the Poincaré plane endowed with the Riemannian metric, in terms of the natural coordinate system, defined by

$$g_{11} = g_{22} := \frac{1}{t_2^2}, \ g_{12} := 0 \text{ for each } (t_1, t_2) \in \mathbb{H}. \tag{12}$$

The sectional curvature of $\mathbb{H}$ is equal to -1 (see, e.g., [8, p. 160]), and the geodesics on $\mathbb{H}$ are the semilines $\gamma(a; \cdot) := (\gamma^1(a; \cdot), \gamma^2(a; \cdot))$ (through $(a, 1)$), and the semicircles $\gamma(b, r; \cdot) := (\gamma^1(b, r; \cdot), \gamma^2(b, r; \cdot))$ with center at (b, r) and radius r), which admit the following natural parameterizations:

$$\begin{cases} \gamma^1(a; s) = a \\ \gamma^2(a; s) = e^s \end{cases} \quad \text{and} \quad \begin{cases} \gamma^1(b, r; s) = b - r\tanh s \\ \gamma^2(b, r; s) = \frac{r}{\cosh s} \end{cases} \quad \text{for any } s \in \mathbb{R}, \tag{13}$$

respectively; see, e.g., [1, p. 298].

By [1, p. 297], the Riemannian connection ∇ on $\mathbb{H}$ (in terms of the natural coordinate system) has the components:

$$\Gamma_{11}^1 = \Gamma_{22}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = 0, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = -\frac{1}{t_2} \ \text{ and } \ \Gamma_{11}^2 = \frac{1}{t_2}.$$

Hence, noting the expression of the connection ∇ given in [8, p. 51], one has the following formula for the connection $\nabla_{(\cdot)}(\cdot) := \left(\nabla^1_{(\cdot)}(\cdot), \nabla^2_{(\cdot)}(\cdot)\right)$ on $\mathbb{H}$ with

$$\nabla_Y^1 X = Y^1\frac{\partial X^1}{\partial t_1} + Y^2\frac{\partial X^1}{\partial t_2} - \frac{1}{t_2}X^1Y^2 - \frac{1}{t_2}X^2Y^1 \tag{14}$$

and

$$\nabla_Y^2 X = Y^1 \frac{\partial X^2}{\partial t_1} + Y^2 \frac{\partial X^2}{\partial t_2} + \frac{1}{t_2} X^1 Y^1 - \frac{1}{t_2} X^2 Y^2 \tag{15}$$

for any $X := (X^1, X^2)$, $Y := (Y^1, Y^2) \in \mathcal{X}(\mathbb{H})$, where and in sequel, for a differential function ϕ on $\mathbb{H}$, $\frac{\partial \phi}{\partial t_1}$ and $\frac{\partial \phi}{\partial t_2}$ denote the classical partial derivatives of ϕ in $\mathbb{R}^2$ with respect to the first variable t_1 and the second variable t_2, respectively. Consider a differentiable function $f : \mathbb{H} \to \mathbb{R}$. Then, using (12), one concludes that the gradient vector grad f and the differential df of f are, respectively, given by

$$\operatorname{grad} f(x) = t_2^2 \left(\frac{\partial f(x)}{\partial t_1} \frac{\partial}{\partial t_1} + \frac{\partial f(x)}{\partial t_1} \frac{\partial}{\partial t_2} \right) \tag{16}$$

and

$$\mathrm{d} f(x) = \frac{\partial f(x)}{\partial t_1} \mathrm{d} t_1 + \frac{\partial f(x)}{\partial t_2} \mathrm{d} t_2 \tag{17}$$

for any $x = (t_1, t_2) \in \mathbb{H}$; see, e.g., [1, p. 8].

For convenience, we also need the expressions of the exponential map $\exp_x^{-1} y$ and the geodesic γ_{xy} joining x to y, which can be found in [16]. To this end, let $x := (t_1, t_2)$ and $y := (s_1, s_2)$ be in $\mathbb{H}$, and set for any x, y with $t_1 \neq s_1$,

$$b_{xy} := \frac{(s_1)^2 + (s_2)^2 - ((t_1)^2 + (t_2)^2)}{2(s_1 - t_1)} \tag{18}$$

and

$$r_{xy} := \sqrt{(s_1 - b_{xy})^2 + (s_2)^2}. \tag{19}$$

For saving of printing space, we use ω to denote the inverse function of the hyperbolic tangent function tanh, that is,

$$\omega(t) := \tanh^{-1} t \quad \text{for all } t \in \mathbb{R}.$$

Then, one has

$$\exp_y^{-1} x = \begin{cases} \left(0, s_2 \ln \frac{t_2}{s_2}\right), & \text{if } t_1 = s_1, \\ \frac{s_2}{r_{xy}} \left(\omega\left(\frac{b_{xy} - s_1}{r_{xy}}\right) - \omega\left(\frac{b_{xy} - t_1}{r_{xy}}\right) \right) (s_2, b_{xy} - s_1), & \text{if } t_1 \neq s_1. \end{cases} \tag{20}$$

and $\gamma_{xy} := \left(\gamma_{xy}^1, \gamma_{xy}^2\right)$ with γ_{xy}^1 and γ_{xy}^2 defined, respectively, by

$$\gamma_{xy}^1(s) := \begin{cases} t_1, & \text{if } t_1 = s_1, \\ b_{xy} - r_{xy} \tanh\left((1-s)\, \omega\left(\frac{b_{xy} - t_1}{r_{xy}}\right) + s\, \omega\left(\frac{b_{xy} - s_1}{r_{xy}}\right) \right), & \text{if } t_1 \neq s_1, \end{cases} \tag{21}$$

and

$$\gamma^2_{xy}(s) := \begin{cases} e^{(1-s)\cdot\ln t_2 + s\cdot\ln s_2}, & \text{if } t_1 = s_1, \\ \frac{r_{xy}}{\cosh\left((1-s)\,\omega(\frac{b_{xy}-t_1}{r_{xy}}) + s\,\omega(\frac{b_{xy}-s_1}{r_{xy}})\right)}, & \text{if } t_1 \neq s_1, \end{cases} \tag{22}$$

for any $s \in [0, 1]$. Now, we are ready to present the counterexample.

Example 3.1 Let $x_0 := (0, 1)$, and let $u_0 := (0, 1) \in T_{x_0}\mathbb{H}$ be a unit vector. Let $f_0 : \mathbb{H} \to \mathbb{R}$ and $X_0 : \mathbb{H} \to T\mathbb{H}$ be the function and the vector field defined by

$$f_0(x) := \langle u_0, \exp^{-1}_{x_0} x\rangle \quad \text{for any } x \in M \tag{23}$$

and

$$X_0(x) := P_{x,x_0} u_0 \qquad \text{for any } x \in M, \tag{24}$$

respectively. We claim that, for each $x = (t_1, t_2) \in \mathbb{H}$,

$$f_0(x) = \begin{cases} \ln t_2, & \text{if } t_1 = 0, \\ \frac{b_x}{r_x}\left(\omega(\frac{b_x}{r_x}) - \omega(\frac{b_x - t_1}{r_x})\right), & \text{if } t_1 \neq 0, \end{cases} \tag{25}$$

and

$$X_0(x) = \begin{cases} (0, t_2), & \text{if } t_1 = 0, \\ \left(\frac{b_x t_2^2 - t_2(b_x - t_1)}{b_x^2+1}, \frac{b_x t_2(b_x - t_1) + t_2^2}{b_x^2+1}\right), & \text{if } t_1 \neq 0, \end{cases} \tag{26}$$

where, for any x with $t_1 \neq 0$,

$$b_x := b_{xx_0} = \frac{t_1^2 + t_2^2 - 1}{2t_1} \quad \text{and} \quad r_x := r_{xx_0} = \sqrt{b_x^2 + 1}. \tag{27}$$

Indeed, let $x = (t_1, t_2) \in \mathbb{H}$. Then, by (20), we get that

$$\exp^{-1}_{x_0} x = \begin{cases} (0, \ln t_2), & \text{if } t_1 = 0, \\ \frac{1}{r_x}\left(\omega\left(\frac{b_x}{r_x}\right) - \omega\left(\frac{b_x - t_1}{r_x}\right)\right)(1, b_x), & \text{if } t_1 \neq 0; \end{cases}$$

thus (25) follows immediately from definition. To check (26), let γ be the geodesic through x and x_0. By the definition of X_0 and thanks to (2), we have to show $\nabla_{\dot\gamma} X_0 = 0$. To do this, write $X_0 := (X_0^1, X_0^2)$ and $\gamma := (\gamma^1, \gamma^2)$. Then,

$$X_0^1(x) = \begin{cases} 0, & \text{if } t_1 = 0, \\ \frac{b_x t_2^2 - t_2(b_x - t_1)}{b_x^2+1}, & \text{if } t_1 \neq 0, \end{cases} \tag{28}$$

and

$$X_0^2(x) = \begin{cases} t_2, & \text{if } t_1 = 0, \\ \frac{b_x t_2(b_x - t_1) + t_2^2}{b_x^2+1}, & \text{if } t_1 \neq 0. \end{cases} \tag{29}$$

In expression of the differential equations (see, e.g., [8, p. 53]), we only need to verify that X_0 and γ satisfy

$$\frac{\mathrm{d}\left(X_0^1\circ\gamma\right)}{\mathrm{d}s}-\frac{X_0^1\circ\gamma}{\gamma^2}\frac{\mathrm{d}\gamma^2}{\mathrm{d}s}-\frac{X_0^2\circ\gamma}{\gamma^2}\frac{\mathrm{d}\gamma^1}{\mathrm{d}s}=0,$$
$$\frac{\mathrm{d}\left(X_0^2\circ\gamma\right)}{\mathrm{d}s}+\frac{X_0^1\circ\gamma}{\gamma^2}\frac{\mathrm{d}\gamma^1}{\mathrm{d}s}-\frac{X_0^2\circ\gamma}{\gamma^2}\frac{\mathrm{d}\gamma^2}{\mathrm{d}s}=0. \tag{30}$$

Without loss of generality, we assume that $t_1\neq 0$ and adopt the expression (13) of the geodesic, that is, $(\gamma^1(\cdot),\gamma^2(\cdot))=(\gamma^1(b_x,r_x;\cdot),\gamma^2(b_x,r_x;\cdot))$ with

$$\gamma^1(b_x,r_x;s)=b_x-r_x\tanh s \quad\text{and}\quad \gamma^2(b_x,r_x;s)=\frac{r_x}{\cosh s} \tag{31}$$

for any $s\in\mathbb{R}$ $\left(\text{noting } x_0=\gamma\left(b_x,r_x;\omega(\frac{b_x}{r_x})\right)\text{ and } x=\gamma\left(b_x,r_x;\omega(\frac{b_x-t_1}{r_x})\right)\right)$, where b_x and r_x are defined by (27). Thus, using (31), one conclude that, for each $s\in\mathbb{R}$,

$$X_0^1\circ\gamma(b_x,r_x;s)=\frac{1}{b_x^2+1}\left(\frac{b_xr_x^2}{\cosh^2 s}-\frac{r_x^2\sinh s}{\cosh^2 s}\right),$$
$$X_0^2\circ\gamma(b_x,r_x;s)=\frac{1}{b_x^2+1}\left(\frac{b_xr_x^2\sinh s}{\cosh^2 s}+\frac{r_x^2}{\cosh^2 s}\right),$$

and so

$$\frac{\mathrm{d}X_0^1\circ\gamma(b_x,r_x;s)}{\mathrm{d}s}=\frac{1}{b_x^2+1}\left(-\frac{2b_xr_x^2\sinh s}{\cosh^3 s}-\frac{r_x^2(1-\sinh^2 s)}{\cosh^3 s}\right),$$
$$\frac{\mathrm{d}X_0^2\circ\gamma(b_x,r_x;s)}{\mathrm{d}s}=\frac{1}{b_x^2+1}\left(\frac{b_xr_x^2(1-\sinh^2 s)}{\cosh^3 s}-\frac{2r_x^2\sinh s}{\cosh^3 s}\right).$$

Moreover, we also have that

$$\frac{\mathrm{d}\gamma^1(b_x,r_x;s)}{\mathrm{d}s}=-\frac{r_x}{\cosh^2 s}\quad\text{and}\quad\frac{\mathrm{d}\gamma^2(b_x,r_x;s)}{\mathrm{d}s}=-\frac{r_x\sinh s}{\cosh^2 s}$$

for any $s\in\mathbb{R}$. Thus, (30) is seen to hold. Hence, $\nabla_{\dot\gamma}X_0=0$, and (26) is checked.

Below we show the following assertions:

(i) f_0 is not quasi-convex.

(ii) Grad $f_0\neq X_0$.

(iii) $\nabla_{\frac{\partial}{\partial t_1}}X_0\neq 0$.

(iv) X_0 is not a gradient vector field.

To show assertion (i), take $x=(\frac{1}{2},\frac{1}{2})$, $y=(-\frac{1}{2},\frac{1}{2})\in\mathbb{H}$, and let $c_0:=-0.4$. Then, $x,y\in L_{c_0,f_0}$ because, by (25) and (27),

$$f_0(x)=f_0(y)=\frac{1}{\sqrt5}\left(\omega\left(\frac{2}{\sqrt5}\right)-\omega\left(\frac{1}{\sqrt5}\right)\right)=-0.4304\cdots<-0.4.$$

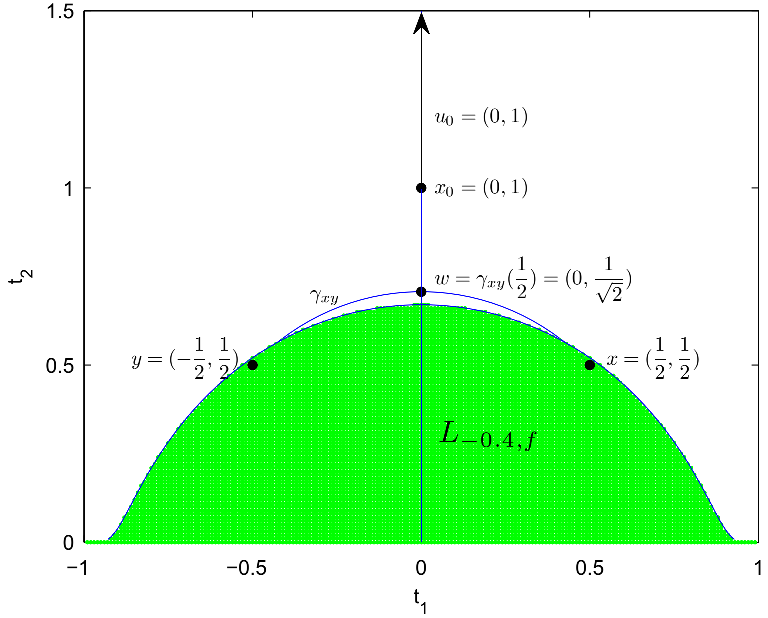

Fig. 1 Non-convexity of the sub-level set $L_{-0.4,f}$

Let $\gamma_{xy} = (\gamma^1_{xy}, \gamma^2_{xy})$ be the geodesic segment joining x to y. Then, for any $s \in [0, 1]$,

$$\gamma^1_{xy}(s) := -\frac{1}{\sqrt{2}} \tanh\left((2s-1)\omega\left(\frac{1}{\sqrt{2}}\right)\right)$$

and

$$\gamma^2_{xy}(s) := \frac{1}{\sqrt{2}\cosh\left((2s-1)\omega\left(\frac{1}{\sqrt{2}}\right)\right)}$$

thanks to (18), (19), (21), and (22). Hence, $\gamma_{xy}(\frac{1}{2}) = (0, \frac{1}{\sqrt{2}})$, and

$$f_0\left(\gamma_{xy}\left(\frac{1}{2}\right)\right) = \ln\frac{1}{\sqrt{2}} = -0.3465\cdots > -0.4$$

by (25). This means that $\gamma_{xy}(\frac{1}{2}) \notin L_{c,f_0}$, and so L_{c,f_0} is not convex; see Fig. 1. In view of Proposition 2.2, we see that f_0 is not quasi-convex, and assertion (i) holds.

To show assertion (ii), take $z := (2, 1)$. Then,

$$b_z = 1 \quad \text{and} \quad r_z = \sqrt{2} \tag{32}$$

(see (27)). Therefore, we have by (26) that $X_0(z) = (1, 0)$. On the other hand, we get from (16) that

$$\text{grad } f_0(z) = \left(\frac{\partial f_0}{\partial t_1}, \frac{\partial f_0}{\partial t_2}\right)$$

where $\frac{\partial f_0}{\partial t_1}$ and $\frac{\partial f_0}{\partial t_2}$ are classical partial derivatives in $\mathbb{R}^2$. Then, using (25) and (27), one calculates

$$\text{grad } f_0(z) = \left(\frac{\sqrt{2}}{8}\ln(3+2\sqrt{2}) + \frac{1}{2}, \frac{\sqrt{2}}{8}\ln(3+2\sqrt{2}) - \frac{1}{2}\right).$$

Therefore, grad $f_0(z) \neq X_0(z)$, and assertion (ii) is checked. We further have that

$$\nabla_{\frac{\partial}{\partial t_1}} X_0(z) \neq 0. \tag{33}$$

Granting this, assertion (iii) is also checked. To show (33), we get from (14) and (15) that

$$\nabla_{\frac{\partial}{\partial t_1}} X_0 = \left(\frac{\partial X_0^1}{\partial t_1} - \frac{1}{t_2}X_0^2, \frac{\partial X_0^2}{\partial t_1} + \frac{1}{t_2}X_0^1\right). \tag{34}$$

(noting that $\frac{\partial}{\partial t_1} = (1, 0)$ for any $x \in \mathbb{H}$). Recalling that X_0^1 and X_0^2 are given by (28), (29), and $z := (2, 1)$, we have that $X_0^1(z) = 1$ and $X_0^2(z) = 0$ (noting (32)). Furthermore, by elemental calculus, we can calculate the partial derivatives

$$\frac{\partial X_0^1}{\partial t_1}\ |_z = 0 \quad \text{and} \quad \frac{\partial X_0^2}{\partial t_1}\ |_z = -\frac{1}{2}.$$

Thus, we conclude from (34) that $\nabla_{\frac{\partial}{\partial t_1}} X_0 \ |_z = (0, \frac{1}{2}) \neq 0$, as desired to show.

For assertion (iv), we suppose on the contrary that there exists a C^∞ function f such that $X_0 = \text{grad } f$. Then, $\mathrm{d} \circ \mathrm{d} f = 0$ by the fundamental property (see, e.g., [9, p. 17]). To proceed, note that $X_0 = X_0^1 \frac{\partial}{\partial t_1} + X_0^2 \frac{\partial}{\partial t_2}$, where X_0^1 and X_0^2 are defined by (28) and(29), respectively. Then, we calculate by elementary calculus that

$$\left.\left(\frac{\partial\left(\frac{1}{t_2^2}X_0^1\right)}{\partial t_2} - \frac{\partial\left(\frac{1}{t_2^2}X_0^2\right)}{\partial t_1}\right)\right|_{x=(2,1)} = \frac{1}{2} \neq 0. \tag{35}$$

Furthermore, by (16) and (17), one has that

$$\mathrm{d} f = \frac{1}{t_2^2}X_0^1 \mathrm{d}t_1 + \frac{1}{t_2^2}X_0^2 \mathrm{d}t_2,$$

and so the exterior differentiation

$$\mathrm{d}\circ\mathrm{d}f = \left(\frac{\partial\left(\frac{1}{t_2^2}X_0^2\right)}{\partial t_1} - \frac{\partial\left(\frac{1}{t_2^2}X_0^1\right)}{\partial t_2} \right) \mathrm{d}t_1 \wedge \mathrm{d}t_2,$$

where $\wedge$ is the exterior product; see, e.g., [9, p. 17]. This, together with (35), means that $\mathrm{d}\circ\mathrm{d}f \neq 0$, and so assertion (iv) is shown.

4 Convexity Properties of Sub-Level Sets on Riemannian Manifolds

Throughout this section, let $\kappa \in \mathbb{R}$ and assume that M is a n-dimensional complete, simply connected Riemannian manifold of constant sectional curvature κ. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{S}^n_\rho$ be the n-dimensional sphere of radius $\frac{1}{\sqrt{\rho}}$ in $\mathbb{R}^{n+1}$ and $\mathbb{H}^n_\rho$ be the manifold obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the Riemannian metric by the positive constant $\frac{1}{\rho} > 0$. Then, M is isometric to

(a) $\mathbb{H}^n_\kappa$, if $\kappa < 0$,
(b) $\mathbb{S}^n_\kappa$, if $\kappa > 0$,
(c) $\mathbb{R}^n$, if $\kappa = 0$;

see, e.g., [9, p. 135]. As usual, define $D_\kappa := \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ and $D_\kappa := +\infty$ otherwise. The proposition below is a direct consequence of [17, Propositions 1.4 and 1.7]; see also [15, Proposition 4.1].

Proposition 4.1 *Let $x, y \in M$. We have the following assertions:*
(i) *If $\mathrm{d}(x, y) < D_\kappa$, then Γ_{xy} contains a unique minimal geodesic γ_{xy}.*
(ii) *Any open ball $\mathbb{B}(x, r)$ with $r \leq \frac{D_\kappa}{2}$ is strongly convex.*

Let $x_0 \in M$ and $u_0 \in T_{x_0}M \setminus \{0\}$. Consider the following function $f_0 : M \to \overline{\mathbb{R}}$ defined by

$$f_0(x) = \begin{cases} \langle u_0, \dot{\gamma}_{x_0x}(0)\rangle, & \text{if } x \in \mathbb{B}(x_0, \frac{D_\kappa}{2}), \\ +\infty, & \text{otherwise}, \end{cases} \tag{36}$$

where $\gamma_{x_0x}(0) \in \Gamma_{x_0x}$ is the unique minimal geodesic lying in $\mathbb{B}(x_0, \frac{D_\kappa}{2})$. It is clear that $\mathrm{dom}\, f_0 = \mathbb{B}(x_0, \frac{D_\kappa}{2})$ is strongly convex. If M is a Hadamard manifold, function (36) is reduced to the function defined by (23), that is,

$$f_0(x) := \left\langle u_0, \exp_{x_0}^{-1} x \right\rangle \quad \text{for any} \quad x \in M. \tag{37}$$

For any $c \in \mathbb{R}$, the sub-level set of f_0 is denoted by $L_{c,f_0}(c \in \mathbb{R})$ and defined by

$$L_{c,f_0} := \{x \in M : f_0(x) \leq c\}.$$

Note by Example 3.1 that L_{c,f_0} is not strongly convex in general. This section is devoted to estimating the constant c such that the sub-level set L_{c,f_0} is strongly convex. For this purpose, we first recall that a geodesic triangle $\triangle(p_1 p_2 p_3)$ in M is a figure consisting of three points p_1, p_2, p_3 (the vertices of $\triangle(p_1 p_2 p_3)$) and three minimal geodesic segments γ_i (the edges of $\triangle(p_1 p_2 p_3)$) such that $\gamma_i(0) = p_{i-1}$ and $\gamma_i(1) = p_{i+1}$ with $i = 1, 2, 3 \pmod 3$. For each $i = 1, 2, 3 \pmod 3$, the inner angle of $\triangle(p_1 p_2 p_3)$ at p_i is denoted by $\angle p_i$, which equals the angle between the tangent vectors $\dot{\gamma}_{i+1}(0)$ and $-\dot{\gamma}_{i-1}(1)$. The following proposition (i.e., comparison theorem for triangles) follows immediately from [9, p.161 Theorem 4.2 (ii), p. 138 Low of Cosines and p. 167 Remark 4.6].

Proposition 4.2 *Let $\triangle(p_1 p_2 p_3)$ be a geodesic triangle in M of the perimeter less than $2D_\kappa$. Set $l_i = \mathrm{d}(p_{i+1}, p_{i-1})$ for each $i = 1, 2, 3$. Then, the following relations hold:*

$$l_i^2 < l_{i-1}^2 + l_{i+1}^2 - 2l_{i-1}l_{i+1}\cos\angle p_i \quad \text{if } \kappa > 0, \tag{38}$$

and

$$l_i^2 > l_{i-1}^2 + l_{i+1}^2 - 2l_{i-1}l_{i+1}\cos\angle p_i \quad \text{if } \kappa < 0. \tag{39}$$

Recall from [18, p. 104] that a k-dimensional submanifold $N \subset M$ is totally geodesic if any geodesic in N is also a geodesic in M. Another property for a complete, simply connected Riemannian manifolds of constant curvature, which will be used in sequel, is the axiom of plane described as follows: (see, e.g., [9, p. 137]):

Proposition 4.3 *Let $x \in M$, and let W be a m-dimensional subspace of $T_x M$ ($m \geq 2$). Then, the submanifold $N := \exp_x W$ is a m-dimensional totally geodesic and complete simply connected Riemannian manifold of constant curvature κ, that is, N is isometric to $\mathbb{H}_\kappa^m$ if $\kappa < 0$ and to $\mathbb{S}_\kappa^m$ if $\kappa > 0$.*

The following lemma, taken from [19, Theorem 3.1 and Remark 3.6], plays a very key role in our study afterward.

Lemma 4.1 *Let $\triangle(ypq)$ be a geodesic triangle in M of the perimeter less than $2D_\kappa$. Let $\triangle(\tilde{y}\tilde{p}\tilde{q})$ be a triangle in $\mathbb{R}^2$ such that*

$$\mathrm{d}(y, p) = \|\overrightarrow{\tilde{y}\tilde{p}}\|, \quad \mathrm{d}(y, q) = \|\overrightarrow{\tilde{y}\tilde{q}}\| \quad \text{and} \quad \angle pyq = \angle \tilde{p}\tilde{y}\tilde{q}. \tag{40}$$

Let x be in the minimal geodesic joining p to q, and $\tilde{x}$ be the corresponding point in the interval $[\tilde{p}, \tilde{q}]$ satisfying

$$\angle pyx = \angle \tilde{p}\tilde{y}\tilde{x} \quad \text{and} \quad \angle qyx = \angle \tilde{q}\tilde{y}\tilde{x} \tag{41}$$

(*see Fig.* 2). *Then, the following assertions hold:*

$$\mathrm{d}(y, x) \geq \|\overrightarrow{\tilde{y}\tilde{x}}\| \ \text{ if } \kappa \geq 0 \quad \text{and} \quad \mathrm{d}(y, x) \leq \|\overrightarrow{\tilde{y}\tilde{x}}\| \ \text{ if } \kappa \leq 0. \tag{42}$$

Recall that γ_{xy} is the unique minimal geodesic joining x to y for any $x, y \in M$ with $\mathrm{d}(x, y) < D_\kappa$ (see Proposition 4.1(i)).

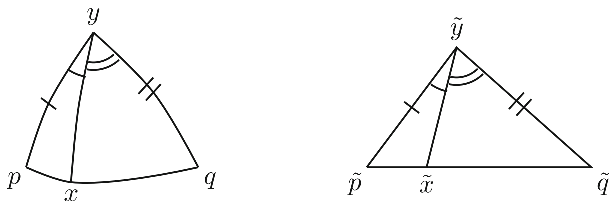

Fig. 2 Geodesic triangle $\triangle ypq$ in M and its comparison triangle $\triangle\tilde{y}\tilde{p}\tilde{q}$ in $\mathbb{R}^2$ in Lemma 4.1 and 4.2

Lemma 4.2 *Let $\triangle(ypq)$ be a geodesic triangle in M of the perimeter less than $2D_\kappa$. Let $\gamma := \gamma_{pq} : [0,1] \to M$ be the unique minimal geodesic joining p to q. Then, for each $t \in]0,1[$, there exist two positive numbers a_t and b_t satisfying*

$$a_t + b_t \begin{cases} \geq 1, & \text{if } \kappa \geq 0, \\ \leq 1, & \text{if } \kappa \leq 0, \end{cases} \tag{43}$$

such that

$$\dot{\gamma}_{y\gamma(t)}(0) = a_t \dot{\gamma}_{yp}(0) + b_t \dot{\gamma}_{yq}(0). \tag{44}$$

Proof Set $N := \exp_y\{\text{span}\{\dot{\gamma}_{yp}(0), \dot{\gamma}_{yq}(0)\}\}$. By assumption, we see from Proposition 4.3 that $\gamma \subset N$, and so

$$\dot{\gamma}_{y\gamma(t)}(0) \in T_y N = \text{span}\{\dot{\gamma}_{yp}(0), \dot{\gamma}_{yq}(0)\} \quad \text{for any} \quad t \in [0,1]. \tag{45}$$

Thus, there exist some $a_t, b_t \in \mathbb{R}$ such that (44) holds (see Fig. 2).

Below, we show that a_t, b_t are positive and satisfy (43). To this end, as in Lemma 4.1 (see Fig. 2), set $x = \gamma(t)$, and let $\triangle(\tilde{y}\tilde{p}\tilde{q})$ be the corresponding triangle of $\triangle(ypq)$ in $\mathbb{R}^2$ satisfying (40) and $\tilde{x}$ be the corresponding point in the interval $[\tilde{p}, \tilde{q}]$ satisfying (41). Without loss of generality, we may assume by (40) that $\overrightarrow{\tilde{y}\tilde{p}} = \dot{\gamma}_{yp}(0)$ and $\overrightarrow{\tilde{y}\tilde{q}} = \dot{\gamma}_{yq}(0)$. Note, by (45), that the vectors $\overrightarrow{\tilde{y}\tilde{x}}$ and $\dot{\gamma}_{yx}(0)$ are in the same 2-dimensional Euclidean plane. It follows from (41), together with (44), that there exists some $\lambda > 0$ such that

$$\lambda\overrightarrow{\tilde{y}\tilde{x}} = \dot{\gamma}_{yx}(0) = a_t \dot{\gamma}_{yp}(0) + b_t \dot{\gamma}_{yq}(0). \tag{46}$$

Note that $\tilde{x}$ lies actually in the open interval $(\tilde{p}, \tilde{q})$ in $\mathbb{R}^2$ (as $0 < t < 1$ and so $\angle\tilde{p}\tilde{y}\tilde{x} > 0$, $\angle\tilde{q}\tilde{y}\tilde{x} > 0$ by (41)). It follows from (46) that

$$a_t > 0, \quad b_t > \quad \text{and} \quad \frac{a_t + b_t}{\lambda} = 1. \tag{47}$$

Furthermore, in view of (42), we see that $\lambda \leq 1$ if $\kappa \geq 0$ and $\lambda \geq 1$ if $\kappa \leq 0$. This, together with (47), implies that (43) holds and the proof is complete. □

Recall that M is assumed to be of constant sectional curvature κ. Now, we are ready to verify the first theorem in the present section.

Theorem 4.1 *Suppose $\kappa > 0$, and let f_0 be the function defined by* (36)*. Then, the sub-level set L_{c,f_0} is strongly convex if and only if either $c \le 0$ or $c \ge \frac{\|u_0\|D_\kappa}{2}$.*

Proof We first show the sufficiency part. To do this, suppose that $c \le 0$ or $c \ge \frac{\|u_0\|D_\kappa}{2}$. Note that if $c \ge \frac{\|u_0\|D_\kappa}{2}$, then $L_{c,f_0} = \mathbb{B}(x_0, \frac{D_\kappa}{2})$ is strongly convex because

$$f_0(x) = \langle u_0, \dot{\gamma}_{x_0x}(0)\rangle \le \|u_0\| \cdot \|\dot{\gamma}_{x_0x}(0)\| \le \frac{\|u_0\|D_\kappa}{2} \le c.$$

holds for all $x \in \mathbb{B}(x_0, \frac{D_\kappa}{2})$. Thus, we need only to consider the case when $c \le 0$. To proceed, fix $c \le 0$, and let $p, q \in L_{c,f_0}$, that is,

$$\langle u_0, \dot{\gamma}_{x_0p}(0)\rangle \le c \quad \text{and} \quad \langle u_0, \dot{\gamma}_{x_0q}(0)\rangle \le c. \tag{48}$$

Then, $p, q \in \mathbb{B}(x_0, \frac{D_\kappa}{2})$ and the geodesic triangle $\triangle(x_0pq)$ is well defined with perimeter less than $2D_\kappa$. Let $t \in [0, 1]$. By assumption, Lemma 4.2 is applicable to concluding that there exist two positive numbers a_t and b_t satisfying with $a_t + b_t \ge 1$ such that

$$\dot{\gamma}_{x_0\gamma(t)}(0) = a_t\dot{\gamma}_{x_0p}(0) + b_t\dot{\gamma}_{x_0q}(0),$$

where $\gamma := \gamma_{pq} : [0, 1] \to M$ is the unique minimal geodesic joining p and q. It follows from (36) and (48) that

$$f_0(\gamma(t)) = a_t\langle u_0, \dot{\gamma}_{x_0p}(0)\rangle + b_t\langle u_0, \dot{\gamma}_{x_0q}(0)\rangle \le c(a_t + b_t) \le c$$

(note that $c < 0$). This means that $\gamma_{p,q}(t) = \gamma(t) \in L_{c,f_0}$ for all $t \in [0, 1]$, and so L_{c,f_0} is strongly convex as desired to show. The proof for the sufficiency part is complete.

To show the necessity part, without loss of generality, we may assume that $\|u_0\| = 1$. Let $0 < c < \frac{D_\kappa}{2}$. It suffices to verify that L_{c,f_0} is not strongly convex, or equivalently, to construct two points p, q and a number $\bar{t} \in]0, 1[$ such that

$$p, q \in L_{c,f_0} \quad \text{and} \quad \bar{z} := \gamma_{pq}(\bar{t}) \notin L_{c,f_0}. \tag{49}$$

To do this, consider the geodesic $\gamma : [0, \frac{D_\kappa}{2}) \to M$ defined by

$$\gamma(t) := \exp_{x_0} tu_0 \quad \text{for each} \quad t \in \left[0, \frac{D_\kappa}{2}\right[. \tag{50}$$

Clearly, it is contained in $\mathbb{B}\left(x_0, \frac{D_\kappa}{2}\right)$. Since $\mathbb{B}\left(x_0, \frac{D_\kappa}{2}\right)$ is strongly convex, we see that, for each $t \in [0, \frac{D_\kappa}{2}[$, the unique minimal geodesic joining x_0 and $\gamma(t)$ can be expressed as

$$\gamma_{x_0\gamma(t)}(s) = \exp_{x_0} s(tu_0) \quad \text{for each } s \in [0, 1].$$

This in particular implies that, for each $t \in [0, \frac{D_\kappa}{2}[$, $\dot{\gamma}_{x_0\gamma(t)}(0) = tu_0$ and so

$$f_0(\gamma(t)) = \langle u_0, \dot{\gamma}_{x_0\gamma(t)}(0)\rangle = \langle u_0, tu_0\rangle = t.$$

Hence,

$$\gamma(t) \in L_{c,f_0} \text{ for all } t \in [0, c] \quad \text{and} \quad \gamma(t) \notin L_{c,f_0} \text{ for all } t \in \left]c, \frac{D_\kappa}{2}\right[\tag{51}$$

In particular, $z := \gamma(c) \in L_{c,f_0}$ with

$$\mathrm{d}(x_0, z) = c < \frac{D_\kappa}{2} \tag{52}$$

by the choice of c. Take $u \in T_z M$ such that $u \perp \dot{\gamma}(c)$. Then, by (52), there exists some $\varepsilon > 0$ such that the geodesic $\tau : [-\varepsilon, \varepsilon] \to M$, determined by

$$\tau(0) = z \quad \text{and} \quad \dot{\tau}(0) = u, \tag{53}$$

is contained in $\mathbb{B}\left(x_0, \frac{D_\kappa}{2}\right) \cap \mathbb{B}\left(z, \frac{D_\kappa}{2}\right)$. Set $p_\varepsilon := \tau(\varepsilon)$ and $q_\varepsilon := \tau(-\varepsilon)$. Then,

$$p_\varepsilon,\ q_\varepsilon \in \mathbb{B}\left(x_0, \frac{D_\kappa}{2}\right) \cap \mathbb{B}\left(z, \frac{D_\kappa}{2}\right). \tag{54}$$

Below, we shall show that

$$p_\varepsilon,\ q_\varepsilon \in L_{c,f_0} \text{ with } f_0(p_\varepsilon) < c \text{ and } f_0(q_\varepsilon) < c. \tag{55}$$

Consider the geodesic triangle $\triangle(x_0 z p_\varepsilon)$. Then, its perimeter is less than $2D_\kappa$ thanks to (52) and (54). Thus, Proposition 4.2 is applicable, and using (39), we have that

$$\begin{aligned} \mathrm{d}^2(x_0, p_\varepsilon) &< \mathrm{d}^2(x_0, z) + \mathrm{d}^2(z, p_\varepsilon) - 2\mathrm{d}(x_0, z)\mathrm{d}(z, p_\varepsilon)\cos\angle p_\varepsilon z x_0 \\ &= \mathrm{d}^2(x_0, z) + \mathrm{d}^2(z, p_\varepsilon) \end{aligned}$$

(noting that $\angle p_\varepsilon z x_0 = \frac{\pi}{2}$ as $\dot{\tau}(0) \perp \dot{\gamma}(c)$), and

$$\mathrm{d}^2(z, p_\varepsilon) < \mathrm{d}^2(x_0, z) + \mathrm{d}^2(x_0, p_\varepsilon) - 2\mathrm{d}(x_0, z)\mathrm{d}(x_0, p_\varepsilon)\cos\angle p_\varepsilon x_0 z.$$

Combining these two inequalities, we get that

$$\mathrm{d}(x_0, p_\varepsilon)\cos\angle p_\varepsilon x_0 z < \mathrm{d}(x_0, z).$$

Thus,

$$f_0(p_\varepsilon) = \mathrm{d}(x_0, p_\varepsilon) \cdot \|u_0\| \cdot \cos\angle p_\varepsilon x_0 z = \mathrm{d}(x_0, p_\varepsilon)\cos\angle p_\varepsilon x_0 z < \mathrm{d}(x_0, z) = c,$$

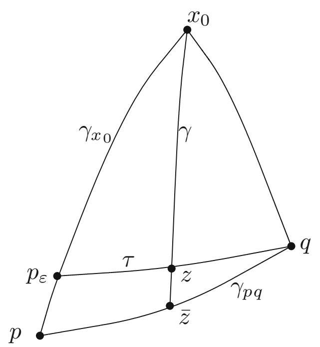

Fig. 3 Non-convexity of the sub-level set $L_{c,f}$ for some $0 < c < \frac{D_\kappa}{2}$ in Theorem 4.1

where the last equality holds because of (52). Similarly, we have $f_0(q_\varepsilon) < c$ and (55) is shown.

Let $\gamma_{x_0} : [0, \infty) \to M$ be the geodesic satisfying $\gamma_{x_0} |_{[0,1]} = \gamma_{x_0 p_\varepsilon}$. In light of (54) and (55), we get by the continuity of f_0 that there exists $t_0 > 1$ such that $\gamma_{x_0} |_{[0,t_0]}$ is the minimal geodesic joining x_0 and $\gamma_{x_0}(t_0)$ and $\gamma_{x_0}(t_0) \in L_{c,f_0}$. Set $p := \gamma_{x_0}(t_0)$ and $q := q_\varepsilon$. Then, $p,\ q \in L_{c,f_0}$ $\left(\text{and so } p, q \in \mathbb{B}\left(x_0, \frac{D_\kappa}{2}\right)\right)$. Since $\mathbb{B}\left(x_0, \frac{D_\kappa}{2}\right)$ is strongly convex by Proposition 4.1(ii), it follows that $\gamma_{pq} \subset \mathbb{B}(x_0, \frac{D_\kappa}{2})$. We further show that

$$\bar{z} := \gamma_{pq}(\bar{t}) \notin L_{c,f_0} \ \text{ for some } \bar{t} \in]0, 1[. \tag{56}$$

Granting this, (49) is established. To show (56), write

$$N := \exp_z\{\text{span}\{\dot{\gamma}(c), \dot{\tau}(0)\}\}$$

where τ and γ are geodesics defined by (53) and (50), respectively. Then, N is isometric to $\mathbb{S}^2_\kappa$ by Proposition 4.3, and we assume that $N = \mathbb{S}^2_\kappa$ without loss of generality. Noting that $x_0 = \gamma(0), z = \gamma(c) = \tau(0), p_\varepsilon = \tau(\varepsilon)$ and $q = \tau(-\varepsilon)$, we have that x_0, q, p_ε, z lie in $\mathbb{S}^2_\kappa$. Recall that γ_{x_0} is the geodesic passing through x_0 and p_ε. It follows that $p = \gamma_{x_0}(t_0)$ is also in $\mathbb{S}^2_\kappa$. By the definition of the geodesic γ and the choice of the points $p,\ q$ in the two-dimensional sphere $\mathbb{S}^2_\kappa$, one checks that γ_{pq} must meet γ at some point $\bar{z} := \gamma_{pq}(\bar{t}) = \gamma(c_0)$ with $\bar{t} \in (0, 1)$ and $c_0 > c$ (see Fig. 3). Hence, $\bar{z} \notin L_{c,f_0}$ thanks to(51). Thus, (56) is shown, and the proof is complete. □

Our second theorem in this section is Theorem 4.2 below, which is an analogue of Theorem 4.1 on Hadamard manifold of constant sectional curvature. In particular, Theorem 4.2 improves and extends the corresponding result in [7, Corollary 3.1], where it was shown that the sub-level sets L_{c,f_0} are convex in the special case when $c = 0$. The proof of Theorem 4.2 is quite similar to that we did for Theorem 4.1 and so we omit it here.

Theorem 4.2 *Suppose $\kappa < 0$, and let f_0 be the function defined by* (37). *Then, L_{c,f_0} is convex if and only if $c \geq 0$.*

As a direct consequence of Theorems 4.1 and 4.2, together with Proposition 2.2, we have the following corollary which shows that the function defined by (36) is not quasi-convex in general.

Corollary 4.1 *Suppose that M is of nonzero constant sectional curvature. Let* $x_0 \in M$ *and* $u_0 \in T_{x_0}M \setminus \{0\}$. *Then, the functions defined by* (36) *are not quasi-convex.*

5 Conclusions

The function $f_0 : M \to \mathbb{R}$ defined by (36) is widely used in equilibrium problems, vector optimization problems, and the proximal point algorithm in Riemannian manifolds. Such class of functions is clearly linear affine in Euclidean spaces; we obtain some basic results related to the function f_0, and our results show that it is not quasi-convex even in Poincaré plane. Moreover, we estimate the constant c such that the sub-level set L_{c,f_0} is strongly convex in Riemannian manifolds of constant curvatures. The results could be used to study some existence results in equilibrium problems and vector optimization problems in Riemannian manifolds of constant curvatures. However, it remains open to estimate the constant c such that the sub-level set L_{c,f_0} is strongly convex in general Riemannian manifolds, or in Riemannian manifolds of bounded constant curvatures, and this is one possible direction for our future work.

Acknowledgments The authors are grateful to both anonymous reviewers for their valuable suggestions and remarks. Research of the first author is supported in part by the Scientific Research Projects of Guizhou University (Grant 201406). Research of the second author is supported in part by the National Natural Science Foundation of China (Grants 11571308, 11371325). Research of the third author is supported in part by the Grant MOST 102-2115-M-039-003-MY3.

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