J Optim Theory Appl (2016) 169:424–442
DOI 10.1007/s10957-016-0919-9

Vector Quasi-Equilibrium Problems for the Sum of Two Multivalued Mappings

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Received: 19 March 2015 / Accepted: 4 March 2016 / Published online: 16 March 2016

Abstract In this paper, we study vector quasi-equilibrium problems for the sum of two multivalued bifunctions. The assumptions are required separately on each of these bifunctions. Sufficient conditions for the existence of solutions of such problems are shown in the setting of topological vector spaces. The results in this paper unify, improve and extend some well-known existence theorems from the literature.

Keywords Ky Fan inequality · Browder–Fan mapping · Vector quasi-equilibrium problem · Upper (lower) C-mapping · C-upper (lower) semicontinuous · Multivalued mapping · Generalized C-essentially quasimonotone

Mathematics Subject Classification 49J40 · 47H04 · 47H10

Communicated by Patrice Marcotte.

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1 Introduction

In 1972, Fan [1] established the existence of solutions for an inequality which, along the years, has shown to be a cornerstone result of nonlinear analysis. The problem itself was called *minimax inequality* by Ky Fan, but nowadays it is widely known within the literature as *equilibrium problem*. It plays a very important role in many fields, such as variational inequalities, game theory, mathematical economics, optimization theory, and fixed point theory. As far as we know, the term "equilibrium problem" was coined in 1992 by Muu and Oettli [2], where three standard examples of (EP) were considered: the optimization problems, the variational inequalities and the fixed point problems. Further particular cases, like saddle point (minimax) problems, Nash equilibria problems, convex differentiable optimization and complementarity problems, have been considered in 1994 by Blum and Oettli [3]. Thanks to its wide applications, Ky Fan's result has been generalized in a number of ways (e.g., see [4–15]). In the above-mentioned article, Blum and Oettli proved an interesting result concerning equilibrium problems for the sum of two (scalar) bifunctions.

Recently, equilibrium problems for vector mappings have been considered by many authors (see [9,16–23] and the references therein). Aiming to extend the results of Blum and Oettli [3] from the scalar to the vector case, Tan and Tinh [22] studied the vector equilibrium problems for the sum of two (vector) bifunctions. Their results contain, in particular, the main results of [3]. Very recently, under a new monotonicity assumption (called C-essential quasimonotonicity), Kassay and Miholca [24] obtained some generalizations of the main results of Tan and Tinh [22].

Focusing on multivalued mappings, Fu [18] obtained in 2005 some extensions of the results of [3,22] for multivalued mappings. However, his results don't recover Kassay and Miholca's results since the monotonicity assumption he considered reduces, in case of single-valued vector bifunctions, to the same monotonicity used by Tan and Tinh, and the latter is stronger than the C-essential quasimonotonicity considered by Kassay and Miholca.

In 2003, Ansari and Yao [25] (and approximately at the same time Ansari and Flores Bazan [26]) introduced the so-called *generalized vector quasi-equilibrium problem*, where the feasible set of the problem is provided by a multivalued mapping and the solution should be, in addition, a fixed point of that mapping. Existence results for this problem and its variants can be found for instance in Ansari and Yao [27] and the references therein.

Our concern now is the following *Can we provide a unified extension to the main results of Fu* [18] *and Kassay-Miholca* [24] *to the more general framework of vector quasi-equilibrium problems given by the sum of two multivalued bifunctions?*

In this paper, we give positive answers to this question. Motivated and inspired by the above-mentioned results, we introduce the vector quasi-equilibrium problem (for short, VQEP) for the sum of two multivalued bifunctions. Our main purpose is to provide sufficient conditions and prove the existence of solutions for (VQEP). As a byproduct, our results improve, extend and unify some other well-known existence theorems in [3,18,22,24].

The rest of the paper is organized as follows. In Sect. 2, we collect some definitions and results needed for further investigations. Section 3 deals with the existence of solutions for (VQEP). Our methods are based on a result concerning the existence of maximal elements in the setting of topological vector spaces. In this way, our proof techniques are different from those in [3,18,22,24], since their authors used KKM lemma to obtain the desired results. For a clear understanding of the concepts and to illustrate our results, we give several examples and counterexamples.

2 Preliminaries

Throughout this paper, unless otherwise specified, we assume that X and Y are real Hausdorff topological vector spaces and $C \subset Y$ is a proper convex cone with $\text{int}C \neq \emptyset$, where $\text{int}C$ denotes the topological interior of C. The set of all real numbers is denoted by $\mathbb{R}$. If $T : X \rightrightarrows X$ is a multivalued mapping, the set of fixed points of T is denoted by $\mathcal{F}(T)$, i.e., $\mathcal{F}(T) = \{x \in X : x \in T(x)\}$. If A is a subset of a topological vector space, we shall denote by $\text{conv}A$ the convex hull of A.

Let K be a nonempty subset of X and $f : K \times K \to \mathbb{R}$ a bifunction. Consider the following inequality, which is known as an equilibrium problem (see [2,3]):

$$\text{Find } \bar{x} \in K \text{ such that } f(\bar{x}, y) \geq 0 \ \ \forall y \in K. \tag{1}$$

Note that saddle point problems and Nash equilibrium problems can be formulated as an equilibrium problem but not, in general, as a variational inequality. The very first result concerning the existence of solutions for the inequality (1) is due to Fan [1] (here, we write its dual form).

Theorem 2.1 *Let K be a nonempty compact convex subset of X. Suppose that f: $K \times K \to \mathbb{R}$ satisfies the following conditions:*

(1) $f(x, x) \geq 0$ *for all* $x \in K$*;*
(2) *For each* $x \in K$, $f(x, .)$ *is quasiconvex;*
(3) *For each* $y \in K$, $f(., y)$ *is upper semicontinuous.*

Then there exists $\bar{x} \in K$ *such that* $f(\bar{x}, y) \geq 0 \ \ \forall y \in K$.

In 1994, Blum and Oettli [3] proved an interesting result for the equilibrium problem (1) in the case

$$f(x, y) = g(x, y) + h(x, y),$$

where $f, g : K \times K \to \mathbb{R}$. The assumptions were required separately on each of these bifunctions. If $g = 0$, the result becomes a variant of Ky Fan inequality [1], whereas for $h = 0$, it becomes a variant of Theorem 3.1 of Mosco [28]. It should be noted that Theorem 3.1 of Mosco is a generalization of Minty's theorem for monotone variational inequalities [29].

As mentioned in Sect. 1, equilibrium problems concerning vector mappings were recently considered by many authors. To formulate this problem, let K be a nonempty subset of X. For a vector-valued function $f : K \times K \to Y$, the weak vector equilibrium problem (see, for example, [9]) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -\mathrm{int}C \;\text{ for all } y \in K, \tag{2}$$

and the strong vector equilibrium problem (see, for example, [16,19]) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -C \setminus \{0\} \;\text{ for all } y \in K. \tag{3}$$

Note that the strong case has not received much attention up to now. Results concerning this case can be found for instance in [16,19,30] and the references therein.

Targeting an extension to the multivalued case of the results of Tan and Tinh [22], Fu [18] considered the following vector equilibrium problems (for short, VEP): let $G, H : K \times K \rightrightarrows Y$ be multivalued mappings with nonempty values;

(**VEP1**) find $\bar{x} \in K$ such that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\mathrm{int}C) \;\text{ for all }\; y \in K, \tag{4}$$

and

(**VEP2**) find $\bar{x} \in K$ such that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus -(C \setminus \{0\}) \;\text{ for all }\; y \in K. \tag{5}$$

A more general form of (2) has been extensively studied in recent years. Ansari and Yao [25] (see also [26]) introduced the generalized vector quasi-equilibrium problem: for $A : K \rightrightarrows K$ and $F : K \times K \rightrightarrows Y$ multivalued mappings with nonempty values, find $\bar{x} \in A(\bar{x})$ such that

$$F(\bar{x}, y) \not\subset -\mathrm{int}C \;\text{ for all } y \in A(\bar{x}). \tag{6}$$

In order to unify and extend the results mentioned above, we introduce the vector quasi-equilibrium problem (for short, VQEP) as follows: let K be a nonempty convex subset of X. For $A : K \rightrightarrows K$ a multivalued mapping with nonempty values and $G, H : K \times K \rightrightarrows Y$ multivalued bifunctions with nonempty values,

(**VQEP**) find $\bar{x} \in A(\bar{x})$ such that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\mathrm{int}C) \text{ for all } y \in A(\bar{x}).$$

This problem includes (4) as a special case. Note that for $F = G + H$, (VQEP) is stronger than (6), in the sense that each solution of (VQEP) is automatically a solution of (6) as well.

Now let us start with some concepts and auxiliary results needed in the sequel.

Let X be a topological space and Y a nonempty set. The mapping $T : X \rightrightarrows Y$ is said to have open lower sections if the inverse mapping $T^{-1} : Y \rightrightarrows X$, defined by

$$T^{-1}(y) = \{x \in X : y \in T(x)\}$$

is open-valued, i.e., for all $y \in Y$, $T^{-1}(y)$ is open in X.

Definition 2.1 Let X be a topological space and Y a topological vector space. A mapping $T : X \rightrightarrows Y$ is called a Browder–Fan mapping iff the following conditions are satisfied:

(1) for each $x \in X$, $T(x)$ is nonempty and convex;
(2) T has open lower sections.

The next statement plays a crucial role in the proof of our main results. It provides the existence of maximal elements for a multivalued mapping. The term "maximal element" is related to economical equilibrium (see [31]).

Lemma 2.1 ([31], Theorem 5.1) *Let K be a nonempty compact convex subset of a Hausdorff topological vector space and $F : K \rightrightarrows K$ a multivalued mapping satisfying the following conditions:*

(1) *For all $x \in K$, $x \notin F(x)$ and $F(x)$ is convex;*
(2) *F has open lower sections.*

Then there exists $\bar{x} \in K$ such that $F(\bar{x}) = \emptyset$.

In this paper, we will consider partial orders on vector spaces induced by cones. We agree that any cone contains the origin, according to the following definition.

Definition 2.2 Let C be a nonempty subset of a vector space Y. The set C is called a cone iff $\lambda x \in C$ for all $x \in C$ and $\lambda \geq 0$. The cone C is pointed iff $C \cap (-C) = \{0\}$; proper iff $C \neq Y$ and $C \neq \{0\}$.

We now recall some concepts of generalized convexity of set-valued mappings. Note that Minh and Tan [32] defined these concepts under the name upper (lower) C-convex (see also Oettli and Schläger [20] for the terminology: left C-convex and right C-convex).

Definition 2.3 Let K be a nonempty and convex subset of a vector space X, C a proper convex cone of a vector space Y and $F : K \rightrightarrows Y$ a multivalued mapping with nonempty values.

(1) (see also Borwein [33], Definition 1.1.) F is said to be upper C-mapping iff for any pair $x, y \in K$, $\alpha \in [0, 1]$, we have

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(\alpha x + (1 - \alpha)y) + C. \tag{7}$$

(2) F is said to be lower C-mapping iff for any pair $x, y \in K$, $\alpha \in [0, 1]$, we have

$$F(\alpha x + (1 - \alpha)y) \subset \alpha F(x) + (1 - \alpha)F(y) - C. \tag{8}$$

Note that the notion "upper C-mapping" has been used by Fu [18] and named "C-convex," while "lower C-mapping" is equivalent to "$-C$-concave" in the sense of [18]. Observe that in case of single-valued functions, where $\subset$ reduces to $\in$, the concepts given by (7) and (8) become the same: both reduce to the well-known C-convexity for vector functions. If $Y = \mathbb{R}$, F is a (single-valued) real function, and

$C = \mathbb{R}_+ = [0, +\infty[$, then from Definition 2.3 we see that the concept of upper (lower) C-mapping of F is equivalent to the classical convexity of F. Similarly, when $C = \mathbb{R}_- =]-\infty, 0]$, we obtain the classical concavity of F. However, these concepts are different in case of multivalued functions. We illustrate this by the following example.

Example 2.1 Let $X = Y = \mathbb{R}$, $K = [0, 1]$ and $C := \mathbb{R}_+ = [0, +\infty[$.

(1) Let $F : [0, 1] \rightrightarrows \mathbb{R}$ be a mapping defined by

$$F(x) = \begin{cases} [0, x], & x \neq \frac{1}{2}, \\ [0, 1], & x = \frac{1}{2}. \end{cases}$$

Since the right-hand side of (7) is always $[0, +\infty[$, it is clear that this mapping satisfies (7). On the other hand, by taking $x = \frac{1}{5}$, $y = \frac{4}{5}$ and $\alpha = \frac{1}{2}$ in (8), one obtains $[0, 1]$ for the left and $]-\infty, \frac{1}{2}]$ for the right-hand side, hence, (8) is false.

(2) Let $F : [0, 1] \rightrightarrows \mathbb{R}$ given by

$$F(x) = \begin{cases} [0, x], & x \neq \frac{1}{2}, \\ [\frac{1}{4}, \frac{1}{2}], & x = \frac{1}{2}. \end{cases}$$

It is easy to see that for any $x, y, \alpha \in [0, 1]$ one has $F(\alpha x + (1 - \alpha)y) \subset [0, \alpha x + (1 - \alpha)y]$. Moreover, $\alpha F(x) + (1 - \alpha)F(y) - C =]-\infty, \alpha x + (1 - \alpha)y]$ and such, (8) holds. On the other hand, for $x = \frac{1}{5}$, $y = \frac{4}{5}$ and $\alpha = \frac{1}{2}$, we obtain that $\alpha F(x) + (1 - \alpha)F(y) = [0, \frac{1}{2}]$, while $F(\alpha x + (1 - \alpha)y) + C = F(\frac{1}{2}) + C = [\frac{1}{4}, +\infty[$, hence (7) doesn't hold.

We need the following definitions taken from [34,35].

Definition 2.4 Let X be a topological space, Y a topological vector space with a proper convex cone C. Let $F : X \rightrightarrows Y$. We say that

(1) F is C-upper semicontinuous (shortly, C-usc) at $x_0 \in dom F$ iff for any open set V of Y with $F(x_0) \subset V$ there exists a neighborhood U of x_0 such that

$$F(x) \subset V + C \quad \text{for each } x \in dom F \cap U.$$

(2) F is C-lower semicontinuous (shortly, C-lsc) at $x_0 \in dom F$ iff for any open set V of Y with $F(x_0) \cap V \neq \emptyset$ there exists a neighborhood U of x_0 such that

$$F(x) \cap [V + C] \neq \emptyset \quad \text{for each } x \in dom F \cap U.$$

(3) F is C-usc (resp. C-lsc) iff $dom F = X$ and F is C-usc (resp. C-lsc) at each point of $dom F$.

Remark 2.1 It is easy to see that in case of single-valued mappings, the concepts (1) and (2) of Definition 2.4 are the same. Furthermore, if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$ (resp. $C = -\mathbb{R}_+$), F is single-valued and C-usc at x_0, then F is upper semicontinuous (resp. lower semicontinuous) at x_0 in the usual sense.

The extension of monotonicity for multivalued bifunctions has been considered in a natural way (see, for instance, [21]).

Definition 2.5 Let X, Y be vector spaces, K a nonempty convex subset of X and C a proper convex cone in Y. A multivalued mapping $F : K \times K \rightrightarrows Y$ is said to be C-monotone iff for any $x, y \in K$

$$F(x, y) + F(y, x) \subset -C.$$

In order to obtain our results, we need to extend the concept of *C-essential quasi-monotonicity* (see Definition 5 of [24]) for multivalued bifunctions. This generalized monotonicity will play a crucial role in our statements.

Definition 2.6 Let X be a vector space, Y a topological vector space, K a nonempty convex subset of X and C a proper convex cone in Y with $\mathrm{int}C \neq \emptyset$. The bifunction $F : K \times K \rightrightarrows Y$ with nonempty values is said to be *generalized C-essentially quasimonotone* iff for an arbitrary integer $n \geq 1$, for all $x_1, x_2, \ldots, x_n \in K$ and all $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$ it holds that

$$\sum_{i=1}^{n} \lambda_i F\left(x_i, \sum_{j=1}^{n} \lambda_j x_j\right) \cap \mathrm{int}C = \emptyset.$$

The next two statements provide sufficient conditions for generalized C-essential quasimonotonicity.

Lemma 2.2 *Let K be a nonempty convex subset of X and $F : K \times K \rightrightarrows Y$ a bifunction with nonempty values. Suppose that*

(1) *$F(x, x) \subset C$ for all $x \in K$;*
(2) *F is C-monotone and upper C-mapping in its second argument.*

Then F is generalized C-essentially quasimonotone.

Proof Take $x_1, x_2, \ldots, x_n \in K$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Set $z := \sum_{j=1}^{n} \lambda_j x_j$. Then by the assumptions, we have

$$\sum_{i=1}^{n} \lambda_i F(x_i, z) \subset -C - \sum_{i=1}^{n} \lambda_i F(z, x_i) \subset -C - F(z, z) - C \subset -C.$$

Moreover, we have $\mathrm{int}C \cap (-C) = \emptyset$. Hence

$$\sum_{i=1}^{n} \lambda_i F(x_i, z) \cap \mathrm{int}C = \emptyset.$$

The proof is complete. □

The following result is a generalization of Proposition 1 of Kassay and Miholca [24].

Lemma 2.3 *Let K be a nonempty convex subset of X and $F : K \times K \rightrightarrows Y$ a bifunction with nonempty values. Suppose that F is C-monotone and lower C-mapping in its second argument. Then F is generalized C-essentially quasimonotone.*

Proof Take $x_1, x_2, \ldots, x_n \in K$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Set $z := \sum_{j=1}^{n} \lambda_j x_j$. Then by the assumptions, we have

$$\sum_{i=1}^{n} \lambda_i F(x_i, z) \subset \sum_{i,j=1}^{n} \lambda_i \lambda_j F(x_i, x_j) - C$$
$$= \frac{1}{2} \sum_{i,j=1}^{n} \lambda_i \lambda_j (F(x_i, x_j) + F(x_j, x_i)) - C.$$

Hence

$$\sum_{i=1}^{n} \lambda_i F(x_i, z) \subset -C.$$

The proof is complete. □

The following example shows that a generalized C-essentially quasimonotone bifunction is not necessarily C-monotone, even if it is upper C-mapping in its second argument.

Example 2.2 Let $X = \mathbb{R}$, $K = [0, 1]$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Let $F : [0, 1] \times [0, 1] \rightrightarrows \mathbb{R}^2$ given by

$$F(x, y) = [(0, 0), (|x - y|, 0)] \text{ for every } x, y \in [0, 1],$$

where $[(0, 0), (|x - y|, 0)]$ denotes the line segment joining $(0, 0)$ and $(|x - y|, 0)$. It is easy to see that F is generalized $\mathbb{R}^2_+$-essentially quasimonotone, upper $\mathbb{R}^2_+$-mapping in its second argument, but not $\mathbb{R}^2_+$-monotone, since $F(1, 0) + F(0, 1) = [(0, 0), (2, 0)] \not\subset -\mathbb{R}^2_+$.

3 Vector Quasi-Equilibrium Problems Given by the Sum of Two Multivalued Bifunctions

In this section, using the result on the existence of maximal elements (Lemma 2.1), we give some new existence results for the problem (VQEP). To start, we first need two lemmas, which serve as tools for the proof of the main result. Moreover, the first one can also be seen as an existence result for a special quasi-equilibrium problem, therefore it seems interesting on its own.

Lemma 3.1 *Let D be a nonempty compact convex subset of X. Let $G, H : D \times D \rightrightarrows Y$ be multivalued mappings with nonempty values and $A : D \rightrightarrows D$ a Browder–Fan mapping such that*

$$\mathcal{F}(A) = \{x \in D : x \in A(x)\}$$

is closed in D. Assume that

(1) *$H(x, x) \subset C$ for all $x \in D$;*
(2) *G is generalized C-essentially quasimonotone;*
(3) *G is C-lower semicontinuous in its second argument;*
(4) *H is $-C$-lower semicontinuous in its first argument and upper C-mapping in its second argument.*

Then there exists a point $\bar{x} \in D$ such that

$$\bar{x} \in A(\bar{x}) \text{ and } G(y, \bar{x}) - H(\bar{x}, y) \subset Y \setminus \text{int}C, \tag{9}$$

for all $y \in A(\bar{x})$.

Proof For $x \in D$, we define

$$P(x) = \{y \in D : G(y, x) - H(x, y) \not\subset Y \setminus \text{int}C\}.$$

By Browder–Fan fixed point theorem [31, Theorem 3.3], $\mathcal{F}(A)$ is a nonempty set. Consider the multivalued mapping S from D to itself defined by

$$S(x) = \begin{cases} \text{conv}P(x) \cap A(x), & \text{if } x \in \mathcal{F}(A), \\ A(x), & \text{if } x \in D \setminus \mathcal{F}(A), \end{cases}$$

where the multivalued mapping $\text{conv}P : D \rightrightarrows D$ is defined by $\text{conv}P(x) = \text{conv}(P(x))$. It is easy to see that for any $x \in D$, $S(x)$ is convex and

$$S^{-1}(y) = [(\text{conv}P)^{-1}(y) \cap A^{-1}(y)] \cup [A^{-1}(y) \cap (D \setminus \mathcal{F}(A))].$$

From the assumptions, for any $y \in D$, $A^{-1}(y)$ and $D \setminus \mathcal{F}(A))$ are open in D. Moreover, we have

$$P^{-1}(y) = \{x \in D : G(y, x) - H(x, y) \not\subset Y \setminus \text{int}C\}.$$

For any fixed y, since $G(y, x) - H(x, y)$ is C-lower semicontinuous in x, then by [18, Lemma 2],

$$L = \{x \in D : G(y, x) - H(x, y) \subset Y \setminus \text{int}C\}$$

is closed in D. Therefore

$$P^{-1}(y) = D \setminus L$$

is open in D. Hence $(\text{conv} P)^{-1}(y)$ is open in D by [31, Lemma 5.1], so we have $S^{-1}(y)$ is also open in D.

Further, we claim that for all $x \in D$, $x \notin S(x)$. Indeed, suppose to the contrary that there exists a point $z \in D$ such that $z \in S(z)$. If $z \in D \setminus \mathcal{F}(A)$ then $z \in A(z)$ which is a contradiction. So $z \in \mathcal{F}(A)$ and we have $z \in S(z) = \text{conv} P(z) \cap A(z)$. We deduce that there exist $\{y_1, y_2, \ldots, y_n\} \subset P(z)$ such that $z = \sum_{i=1}^n \lambda_i y_i$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$. By the definition of P we can see that

$$G(y_i, z) - H(z, y_i) \not\subset Y \setminus \text{int} C \text{ for all } i = 1, 2, \ldots, n.$$

Then, there exist $a_i \in G(y_i, z)$, $b_i \in H(z, y_i)$ such that $a_i - b_i \in \text{int} C$. Thus,

$$\sum_{i=1}^n \lambda_i (a_i - b_i) \in \text{int} C. \tag{10}$$

By the assumption (2), G is generalized C-essentially quasimonotone, therefore we have

$$\sum_{i=1}^n \lambda_i G(y_i, z) \cap \text{int} C = \emptyset. \tag{11}$$

Since $H(x, y)$ is upper C-mapping in y, we get

$$\sum_{i=1}^n \lambda_i H(z, y_i) \subset H(z, z) + C \subset C + C = C.$$

and therefore

$$-\sum_{i=1}^n \lambda_i H(z, y_i) \subset -C. \tag{12}$$

By (11), (12) it follows that

$$\sum_{i=1}^n \lambda_i (a_i - b_i) \notin \text{int} C, \tag{13}$$

which contradicts (10). Applying Lemma 2.1, we conclude that there exists a point $\bar{x} \in D$ with $S(\bar{x}) = \emptyset$. If $\bar{x} \in D \setminus \mathcal{F}(A))$, then $S(\bar{x}) = A(\bar{x}) = \emptyset$, contradicting the fact that A has nonempty values. Therefore, $\bar{x} \in \mathcal{F}(A)$ and $\text{conv} P(\bar{x}) \cap A(\bar{x}) = \emptyset$. This clearly implies that $P(\bar{x}) \cap A(\bar{x}) = \emptyset$, hence, for all $y \in A(\bar{x})$ one has $y \notin P(\bar{x})$, i.e.,

$$\bar{x} \in A(\bar{x}) \text{ and } G(y, \bar{x}) - H(\bar{x}, y) \subset Y \setminus \text{int} C,$$

for all $y \in A(\bar{x})$. Thus, the result holds and the proof is complete. □

Remark 3.1 Observe that Lemma 3.1 fails to hold if the closedness assumption on the set $\mathcal{F}(A)$ is violated.

This remark is illustrated by the following example.

Example 3.1 Let $X = \mathbb{R}$, $D = [0, 1]$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$.

(1) Let $G : D \times D \rightrightarrows \mathbb{R}^2$ given by $G(x, y) = \{(|x - y|, 0)\}$ for every $x, y \in D$. By Example 1 in [24], F is upper $\mathbb{R}^2_+$-mapping in its second argument, generalized $\mathbb{R}^2_+$-essentially quasimonotone but not C-monotone.
(2) Let $H : D \times D \rightrightarrows \mathbb{R}^2$ given by $H(x, y) = \{(0, x - y)\}$ for every $x, y \in D$.
(3) Let $A : D \rightrightarrows D$ be the mapping defined by

$$A(x) = \begin{cases} 0 & \text{if} \quad x = 1, \\ [0, 1] & \text{if} \quad x \in]0, 1[, \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Then $A(x)$ is a nonempty convex subset of D and $A^{-1}(y)$ is open in D for all $x, y \in D$. Therefore, A is a Browder–Fan mapping. Moreover, the set $\mathcal{F}(A) =]0, 1[$ is open in D.

It is easy to see that each of conditions (1), (2), (3), (4) of Lemma 3.1 is satisfied. However, (9) has no solution. Indeed, if $\bar{x}$ is a solution of (9) then $\bar{x} \in]0, 1[$ and

$$\begin{aligned} G(y, \bar{x}) - H(\bar{x}, y) \subset Y \setminus \operatorname{int} C, \\ \Longleftrightarrow (|\bar{x} - y|, y - \bar{x}) \notin \operatorname{int} C, \end{aligned}$$

for all $y \in A(\bar{x}) = [0, 1]$, which is impossible.

Corollary 3.1 ([18], Lemma 5) *Let D be a nonempty compact convex subset of X and C be a pointed, closed convex cone of Y with* $\operatorname{int} C \neq \emptyset$. *Let $G, H : D \times D \rightrightarrows Y$ be multivalued mappings with nonempty values such that*

(1) $0 \in G(x, x) \subset C$ *and* $0 \in H(x, x) \subset C$ *for all* $x \in D$;
(2) *G is C-monotone;*
(3) *G is C-lower semicontinuous and upper C-mapping in its second argument;*
(4) *H is $-C$-lower semicontinuous in its first argument and upper C-mapping in its second argument;*

Then there exists a point $\bar{x} \in D$ such that

$$G(y, \bar{x}) - H(\bar{x}, y) \subset Y \setminus \operatorname{int} C,$$

for all $y \in D$.

Proof From the assumptions (1), (2), (3) and Lemma 2.2, we infer that G is generalized C-essentially quasimonotone. The conclusion follows from Lemma 3.1 by letting $A(x) = D$ for all $x \in D$. □

Remark 3.2 Let us underline the following items.

(1) Lemma 3.1 extends Lemma 5 of Fu [18] in the following aspects:
  (a) Concerns on the more general multivalued vector quasi-equilibrium problems instead of multivalued vector equilibrium problems;
  (b) The assumptions on G to be C-monotone and C-mapping on its second argument are replaced by the generalized C-essential quasimonotonicity, the latter being a weaker assumption in view of Lemma 2.2 and Example 2.2.
  (c) It is not required for the cone C to be closed and pointed.
(2) Lemma 3.1 improves and extends Lemma 2 of Blum and Oettli [3] and Lemma 3.2 of Tan and Tinh [22] in the following aspects:
  (a) From (single-valued) vector equilibrium problems to multivalued vector quasi-equilibrium problems;
  (b) The assumptions are weaker: the monotonicity and convexity of G are replaced by the generalized C-essential quasimonotonicity.
(3) Extends Lemma 3 of Kassay and Miholca [24] from (single-valued) vector equilibrium problems to multivalued vector quasi-equilibrium problems.
(4) In the proof of Lemma 3.1 we used a result on the existence of maximal elements, while the authors of [3,18,22,24] used KKM lemma to prove their result. Hence, our proof techniques are different.

The next lemma makes the connection between the special equilibrium problem considered in Lemma 3.1 and the equilibrium problem we are interested in.

Lemma 3.2 *Let D be a nonempty closed convex subset of X. Let $G, H : D \times D \rightrightarrows Y$ be multivalued mappings with nonempty values and $A : D \rightrightarrows D$ a multivalued mapping with nonempty convex values. Assume that*

(1) *$G(x, x) \subset C$ and $0 \in H(x, x)$ for all $x \in D$;*
(2) *for all $x, y \in D$, the mapping $g : [0, 1] \rightrightarrows Y$ defined by*

$$g(t) := G(ty + (1 - t)x, y)$$

is $-C$-lower semicontinuous at $t = 0$;
(3) *G, H are upper C-mappings in their second argument.*

If there exists a point $\bar{x} \in D$ such that

$$\bar{x} \in A(\bar{x}) \text{ and } G(y, \bar{x}) - H(\bar{x}, y) \subset Y \setminus \mathrm{int}C \text{ for all } y \in A(\bar{x})$$

then

$$\bar{x} \in A(\bar{x}) \text{ and } G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\mathrm{int}C) \text{ for all } y \in A(\bar{x}).$$

Proof Let $\bar{x} \in D$ be such that

$$\bar{x} \in A(\bar{x}) \text{ and } G(y, \bar{x}) - H(\bar{x}, y) \subset Y \setminus \mathrm{int}C \text{ for all } y \in A(\bar{x}).$$

We set $x_t := ty + (1-t)\bar{x}$, $t \in [0,1]$. It is clear that $x_t \in A(\bar{x})$ for all $t \in [0,1]$ and therefore, we have

$$\bar{x} \in A(\bar{x}) \text{ and } G(x_t, \bar{x}) - H(\bar{x}, x_t) \subset Y \setminus \mathrm{int}C. \tag{14}$$

By the assumptions (1) and (3) we have

$$tG(x_t, y) + (1-t)G(x_t, \bar{x}) \subset G(x_t, x_t) + C \subset C + C = C, \tag{15}$$

$$tH(\bar{x}, y) \subset tH(\bar{x}, y) + (1-t)H(\bar{x}, \bar{x}) \subset H(\bar{x}, x_t) + C. \tag{16}$$

By (15) and (16), we obtain

$$tG(x_t, y) + t(1-t)H(\bar{x}, y) \subset -(1-t)G(x_t, \bar{x}) + (1-t)H(\bar{x}, x_t) + C. \tag{17}$$

We claim that

$$G(x_t, y) + (1-t)H(\bar{x}, y) \subset Y \setminus (-\mathrm{int}C) \ \ \forall t \in]0,1]. \tag{18}$$

Indeed, if (18) is false, then there exist some $t \in]0,1]$ and some $a \in G(x_t, y)$, $b \in H(\bar{x}, y)$ such that

$$a + (1-t)b \in -\mathrm{int}C. \tag{19}$$

By (17), there exist $z \in G(x_t, \bar{x})$, $w \in H(\bar{x}, x_t)$ and $\bar{c} \in C$ such that

$$t[a + (1-t)b] = -(1-t)(z-w) + \bar{c}.$$

By (19), we have

$$(1-t)(z-w) = -t[a + (1-t)b] + \bar{c} \in \mathrm{int}C + \bar{c} \subset \mathrm{int}C.$$

Hence, $z - w \in \mathrm{int}C$, which contradicts (14). Let $h(t) = G(x_t, y) + (1-t)H(\bar{x}, y)$, $t \in [0,1]$. Suppose that $h(0) \not\subset Y \setminus (-\mathrm{int}C)$, then there is a point $v \in h(0)$ such that $v \in -\mathrm{int}C$. By the assumption (2), $h(t)$ is $-C$-l.s.c at $t = 0$, then there is a $\delta \in]0,1[$ such that for all $t \in [0, \delta]$, $h(t) \cap (-\mathrm{int}C - C) = h(t) \cap (-\mathrm{int}C) \neq \emptyset$. This contradicts (18). Thus we obtain $h(0) \subset Y \setminus (-\mathrm{int}C)$, that is,

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\mathrm{int}C) \ \ \forall y \in A(\bar{x}).$$

This completes the proof of the lemma. □

The next concept plays an important role in our results when dealing with noncompact sets.

Definition 3.1 ([3]) Let K and D be nonempty convex subsets in X with $D \subset K$. Then $\text{core}_K D$, the core of D relative K, is defined through $a \in \text{core}_K D$ iff $a \in D$ and $D \cap]a, y] \neq \emptyset$ for all $y \in K \setminus D$, where

$$]a, y] = \{x \in X : x = \alpha a + (1-\alpha)y \text{ for all } \alpha \in [0, 1[\}.$$

We are now in a position to prove our main result.

Theorem 3.1 *Let K be a nonempty closed convex subset of X and D a nonempty compact convex subset of K. Let $G, H : K \times K \rightrightarrows Y$ be multivalued bifunctions with nonempty values and $A : D \rightrightarrows K$ a Browder–Fan mapping such that $B(x) := A(x) \cap D \neq \emptyset$ for all $x \in D$ and*

$$\mathcal{F}(A) = \{x \in D : x \in A(x)\}$$

is closed in D. Assume that

(1) *$G(x, x) \subset C$, $G(x, x) \cap (-C) \neq \emptyset$ and $0 \in H(x, x) \subset C$ for all $x \in K$;*
(2) *G is generalized C-essentially quasimonotone;*
(3) *For all $x, y \in K$, the mapping $g : [0, 1] \rightrightarrows Y$ defined by*

$$g(t) := G(ty + (1-t)x, y)$$

is $-C$-lower semicontinuous at $t = 0$;
(4) *G is upper C-mapping and C-lower semicontinuous in its second argument;*
(5) *H is $-C$-lower semicontinuous in its first argument and upper C-mapping in its second argument;*
(6) *Suppose that for any $x \in B(x) \setminus \text{core}_{A(x)} B(x)$, one can find a point $a \in \text{core}_{A(x)} B(x)$ such that*

$$G(x, a) + H(x, a) \not\subset Y \setminus (-C).$$

Then there exists a point $\bar{x} \in B(\bar{x})$ such that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\text{int}C),$$

for all $y \in A(\bar{x})$.

Proof The multivalued mapping $B : D \rightrightarrows D$ satisfies $B^{-1}(y) = A^{-1}(y)$ for all $y \in D$ and $\mathcal{F}(B) = \mathcal{F}(A)$. Therefore, B is also a Browder–Fan mapping and its fixed point set is nonempty and closed in D. By Lemma 3.1, there exists $\bar{x} \in B(\bar{x})$ such that

$$G(y, \bar{x}) - H(\bar{x}, y) \subset Y \setminus \text{int}C,$$

for all $y \in B(\bar{x})$. Applying Lemma 3.2, we get

$$\bar{x} \in B(\bar{x}) \text{ and } G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\text{int}C), \tag{20}$$

for all $y \in B(\bar{x})$. Further, we define the multivalued mapping $\Phi : K \rightrightarrows Y$ by

$$\Phi(y) = G(\bar{x}, y) + H(\bar{x}, y), \quad y \in K.$$

Assumptions (4) and (5) show that Φ is upper C-mapping and it follows from (20) that

$$\Phi(y) \subset Y \setminus (-\text{int}C) \text{ for all } y \in B(\bar{x}).$$

If $\bar{x} \in \text{core}_{A(\bar{x})} B(\bar{x})$ then we set $x_0 = \bar{x}$, otherwise, since $\bar{x} \in B(\bar{x})$ we set $x_0 = a$, where a is from the assumption (6). Then we always have $\Phi(x_0) \not\subset Y \setminus (-C)$. Using [18, Lemma 6] we conclude that

$$\Phi(y) \subset Y \setminus (-\text{int}C) \text{ for all } y \in A(\bar{x}).$$

It follows that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\text{int}C) \text{ for all } y \in A(\bar{x}).$$

This completes the proof. □

The next result is a special case of Theorem 3.1 for vector equilibrium problems.

Corollary 3.2 *Let K be a nonempty closed convex subset of X. Let $G, H : K \times K \rightrightarrows Y$ be multivalued mappings with nonempty values. Assume that*

(1) *$G(x, x) \subset C$, $G(x, x) \cap (-C) \neq \emptyset$ and $0 \in H(x, x) \subset C$ for all $x \in K$;*
(2) *G is generalized C-essentially quasimonotone;*
(3) *For all $x, y \in K$, the mapping $g : [0, 1] \rightrightarrows Y$ defined by*

$$g(t) := G(ty + (1 - t)x, y)$$

is $-C$-lower semicontinuous at $t = 0$;
(4) *G is upper C-mapping and C-lower semicontinuous in its second argument;*
(5) *H is $-C$-lower semicontinuous in its first argument and upper C-mapping in its second argument;*
(6) *There is a nonempty compact convex subset $D \subset K$ such that for any $x \in K \setminus \text{core}_K D$, one can find a point $a \in \text{core}_K D$ such that*

$$G(x, a) + H(x, a) \not\subset Y \setminus (-C).$$

Then there exists a point $\bar{x} \in D$ such that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\text{int}C),$$

for all $y \in K$.

Proof The conclusion follows from Theorem 3.1 by letting $A(x) = K$ for all $x \in D$ (hence $B(x) = D$ for all $x \in D$). □

With respect to Theorem 3.1 we may observe the following if we argue as in Remark 3.2.

Remark 3.3 (1) If K is compact, we consider $D = K$ and such $B(x) = A(x) \cap K = A(x)$ for each $x \in K$, and then assumption (6) is vacuously satisfied.

(2) Theorem 3.1 extends Theorem 1 of Fu [18] (see Corollary 3.3 below), Theorem 1 of Blum and Oettli [3], Theorem 3.1 of Tan and Tinh [22], and Theorem 1 of Kassay and Miholca [24], in the aspects already mentioned in Remark 3.2. Furthermore, our coercivity condition is more general than those in [3,18,22,24]. Indeed, the latter can be reobtained as special cases when $A(x) = K$ for all $x \in D$.

(3) Our results are different from the results in [25,27] in the sense that none of them can be deduced from the other. Indeed, in these results, comparing with ours, at least one of the conditions about convexity, continuity and coercivity is different. Moreover, our results are proved for the sum of two multivalued bifunctions, each enjoying different properties, while the other results deal with one bifunction.

(4) With respect to the existence results in [28,36–38] for quasi-equilibrium problems in the scalar single-valued case, we observe that:

(a) If K is compact then our Theorem 3.1 with $H = \{\psi\}$ extends Theorem 3.1 of Mosco [28] to multivalued (scalar) quasi-equilibrium problems;

(b) Our results, in general, are different from the results in [36–38]. To see this, we first reiterate the fact that we concern on the sum of two bifunctions, each enjoying different properties while the other results deal with one bifunction. Furthermore, observe that in [36], the results are formulated in the setting of a finite dimensional Euclidean space and either the feasible set is compact or the coercivity is different from the one in ours. In [37] the framework is a finite dimensional Euclidean space, the feasible set is compact (and such, no coercivity condition is needed), while the convexity assumptions are different from ours. In [38] the framework is infinite dimensional, but the feasible set is still compact and the authors limit themselves to the scalar equilibrium problem.

Corollary 3.3 ([18], Theorem 1) *Let K be a nonempty closed convex subset of X and C be a pointed, closed convex cone of Y with* $\mathrm{int}C \neq \emptyset$. *Let $G, H : K \times K \rightrightarrows Y$ be multivalued mappings with nonempty values. Assume that*

(1) $0 \in G(x, x) \subset C$ *and* $0 \in H(x, x) \subset C$ *for all* $x \in K$*;*

(2) *G is C-monotone;*

(3) *For all $x, y \in K$, the mapping $g : [0, 1] \rightrightarrows Y$ defined by*

$$g(t) := G(ty + (1 - t)x, y)$$

is $-C$-lower semicontinuous at $t = 0$;

(4) *G is upper C-mapping and C-lower semicontinuous in its second argument;*

(5) *H is $-C$-lower semicontinuous in its first argument and upper C-mapping in its second argument;*

(6) *There is a nonempty compact convex subset* $D \subset K$ *such that for any* $x \in K \setminus \mathrm{core}_K D$*, one can find a point* $a \in \mathrm{core}_K D$ *such that*

$$G(x, a) + H(x, a) \not\subset Y \setminus (-C).$$

Then there exists a point $\bar{x} \in D$ *such that*

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\mathrm{int} C),$$

for all $y \in K$.

Proof From the assumptions (1), (2), (4) and Lemma 2.2, we infer that G is generalized C-essentially quasimonotone. The conclusion follows from Theorem 3.1 by letting $A(x) = K$ for all $x \in D$ (hence $B(x) = D$ for all $x \in D$). □

Remark 3.4 As indicated by Fu in [18] (Theorem 1), if the hypotheses of Corollary 3.3 are satisfied, and, if in addition, there is a pointed closed convex cone $\widehat{C}$ such that

$$C \setminus \{0\} \subset \mathrm{int}\widehat{C},$$

then there exists a point $\bar{x} \in D$ such that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-C \setminus \{0\}),$$

for all $y \in K$. This shows that the strong vector equilibrium problem has a solution. Observe that this additional result can easily be obtained since $D, K, \widehat{C}, G$ and H satisfy all the assumptions of Corollary 3.3, thus we obtain that there exists a point $\bar{x} \in D$ such that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-\mathrm{int}\widehat{C}),$$

for all $y \in K$. Since $C \setminus \{0\} \subset \mathrm{int}\widehat{C}$, it follows that

$$G(\bar{x}, y) + H(\bar{x}, y) \subset Y \setminus (-C \setminus \{0\}).$$

However, this argument cannot be applied for Theorem 3.1 since the generalized C-essential quasimonotonicity is not inherited if we replace C by $\widehat{C}$, as happens in the case of C-monotonicity. Therefore, obtaining a result for strong vector (quasi)equilibrium problem in the settings of Theorem 3.1 seems to be more delicate and further investigations are necessary.

If G, H are vector single-valued mappings, then Corollary 3.2 becomes Theorem 1 of Kassay and Miholca [24].

4 Conclusions

Equilibrium problems for the sum of two bifunctions initiated by Blum and Oettli [3] provide a unified model for equilibrium problems with the monotonicity assumption (see, for instance, [28]) and equilibrium problems without the monotonicity assumption (see, [1]) in real topological vector spaces. In this paper, we used the essential quasimonotonicity for vector multivalued bifunctions to obtain new existence results which improve, extend and unify the main results of Blum and Oettli [3], Fu [18], Tan and Tinh [22] and Kassay and Miholca [24]. It should be noted that our methods are based on a result concerning the existence of maximal elements in the setting of topological vector spaces. In this way, our proof techniques are different from those in [3,18,22,24], since their authors used KKM lemma to obtain the desired results. As mentioned in Sect. 2, the problem we studied here (VQEP) is stronger than (6) when we concern on the sum of two bifunctions. Thus, our results provide solutions also for the latter. However, it is desirable to obtain existence results for (6) under weaker assumptions. This will be a topic for future research. It is still an open question whether Theorem 3.1 remains valid or not if we replace the essential quasimonotonicity for vector multivalued bifunctions by pseudomonotonicity considered in Oettli and Schläger [20].

Acknowledgments The research of the first two authors was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, Project Number PN-II-ID-PCE-2011-3-0024. The third author is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 101.01-2014.17. The work of the third author on this paper is dedicated to Prof. Pham Huu Sach, in celebration of his 75th birthday. The authors wish to thank the anonymous referees for their useful comments, which helped them to improve the presentation of the paper.

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