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Global Uniqueness and Solvability for Tensor Complementarity Problems

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Abstract Recently, the tensor complementarity problem has been investigated in the literature. An important question involving the property of global uniqueness and solvability for a class of tensor complementarity problems was proposed by Song and Qi (J Optim Theory Appl, 165:854–873, 2015). In the present paper, we give an answer to this question by constructing two counterexamples. We also show that the solution set of this class of tensor complementarity problems is nonempty and compact. In particular, we introduce a class of related structured tensors and show that the corresponding tensor complementarity problem has the property of global uniqueness and solvability.

Keywords Tensor complementarity problem · Nonlinear complementarity problem · Global uniqueness and solvability · *P* tensor · Strong *P* tensor

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1 Introduction

As a generalization of the linear complementarity problem [1], the tensor complementarity problem has been introduced and investigated in [2–6]; it is a specific class of nonlinear complementarity problems [7–9]. By using properties of structured tensors, many good results for the tensor complementarity problem have been obtained in the literature.

It is well known that a linear complementarity problem has the property of global uniqueness and solvability (GUS-property), if and only if the matrix involved in the concerned problem is a P-matrix [10]. It is also well known that such a result cannot be generalized to the nonlinear complementarity problem [8,9]. *A natural question is whether such a result can be generalized to the tensor complementarity problem or not*? That is, whether the result that *a tensor complementarity problem has the GUS-property if and only if the tensor involved in the problem is a P tensor* holds or not? Such a question was proposed by Song and Qi (see Question 6.3 in [2]). In this paper, we show that the answer to this question is negative by constructing two counterexamples.

It has been shown that the tensor complementarity problem with a P tensor has a solution; while our counterexample demonstrates that it is possible, such a problem has more than one solution. Thus, a natural question is what more we can say about the solution set of such a complementarity problem; and another natural question is that for which kind of tensor, the corresponding tensor complementarity problem has the GUS-property. For the first question, we will show that the solution set of the tensor complementarity problem is nonempty and compact when the involved tensor is a P tensor; and for the second question, we will introduce a new class of tensors, called the *strong P tensor* and show that the corresponding tensor complementarity problem has the GUS-property. We also show that every strong P tensor is a P tensor, but the converse does not hold; and hence, many results obtained for the case of the P tensor are still satisfied for the case of the strong P tensor.

The rest of this paper is organized as follows. In the next section, we first briefly review some basic concepts and results which are useful in the subsequent analysis. In Sect. 3, we give a negative answer to the result that a tensor complementarity problem has the GUS-property if and only if the tensor involved in the problem is a P tensor, and show that the solution set of the tensor complementarity problem with a P tensor is compact. In Sect. 4, we introduce the concept of the strong P tensor and discuss its related properties. Conclusions are drawn in Sect. 5.

2 Preliminaries

The complementarity problem, denoted by CP(F), consists in finding a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x) \rangle = 0.$$

When $F(x) = Ax + q$ with given $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, CP(F) reduces to the linear complementarity problem (denoted by LCP(q, A)), which consists in finding a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Ax + q \geq 0, \quad \langle x, Ax + q\rangle = 0;$$

and when $F(x) = \mathscr{A}x^{m-1} + q$ with given $\mathscr{A} = (a_{i_1,i_2,\ldots,i_m}) \in T_{m,n}$ (the set of all real mth-order n-dimensional tensors) and $q \in \mathbb{R}^n$, CP(F) reduces to the tensor complementarity problem (denoted by TCP($q, \mathscr{A}$)) [2–6], which consists in finding a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad \mathscr{A}x^{m-1} + q \geq 0, \quad \langle x, \mathscr{A}x^{m-1} + q\rangle = 0,$$

where $\mathscr{A}x^{m-1} \in \mathbb{R}^n$ is defined by

$$(\mathscr{A}x^{m-1})_i := \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2,\ldots,i_m} x_{i_2}, \ldots, x_{i_m}, \quad \forall i \in \{1, 2, \ldots, n\}.$$

It is easy to see that

$$\mathscr{A}x^m = \langle x, \mathscr{A}x^{m-1}\rangle = \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1,i_2,\ldots,i_m} x_{i_1} x_{i_2}, \ldots x_{i_m}.$$

Throughout this paper, for any positive integer n, we denote $[n] := \{1, 2, \ldots, n\}$ and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$. For any $x \in \mathbb{R}^n$, we denote

$$[x]_+ := (\max\{x_1, 0\}, \ldots, \max\{x_n, 0\})^T.$$

The eigenvalue of tensor is initially studied by Qi [11] and Lim [12]. If there exist a nonzero vector $x \in \mathbb{R}^n$ and a scalar $\lambda \in \mathbb{R}$ such that

$$(\mathscr{A}x^{m-1})_i = \lambda x_i^{m-1}, \quad \forall i \in [n],$$

then λ is called an H-eigenvalue of $\mathscr{A}$ and x is called an H-eigenvector of $\mathscr{A}$ associated with λ; and moreover, if there exist a nonzero vector $x \in \mathbb{R}^n$ and a scalar $\lambda \in \mathbb{R}$ such that

$$\mathscr{A}x^{m-1} = \lambda x, \ \forall i \in [n] \quad \text{and} \quad \langle x, x\rangle = 1,$$

then λ is called a Z-eigenvalue of $\mathscr{A}$ and x is called a Z-eigenvector of $\mathscr{A}$ associated with λ.

Recently, many classes of structured tensors are introduced, and the related properties are studied [2–6,13–19]. In this paper, we need the following concepts of several structured tensors.

Definition 2.1 Let $\mathscr{A} = (a_{i_1,\ldots,i_m}) \in T_{m,n}$. We say that $\mathscr{A}$ is

(1) a **strictly semi-positive tensor** iff for each $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists an index $i \in [n]$ such that $x_i > 0$ and $(\mathscr{A}x^{m-1})_i > 0$;
(2) a P **tensor** iff for each $x \in \mathbb{R}^n \setminus \{0\}$, there exists an index $i \in [n]$ such that $x_i(\mathscr{A}x^{m-1})_i > 0$;
(3) an R**-tensor** iff there is no $(x, t) \in (\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ such that for any $i \in [n]$,

$$\begin{aligned}(\mathscr{A}x^{m-1})_i + t = 0, \ \text{if } x_i > 0,\\ (\mathscr{A}x^{m-1})_i + t \geq 0, \ \text{if } x_i = 0.\end{aligned}$$

Obviously, every P tensor is a strictly semi-positive tensor and an R-tensor. In this paper, we also need the following concepts of functions.

Definition 2.2 ([3,8]) Let mapping $F : K \subseteq \mathbb{R}^n \to \mathbb{R}^n$. We say that F is

(1) a P**-function** iff for all pairs of distinct vectors x and y in K,

$$\max_{i\in[n]} (x_i - y_i)(F_i(x) - F_i(y)) > 0;$$

(2) a **uniform** P**-function** iff there exists a constant $\mu > 0$ such that for all pairs of vectors x and y in K,

$$\max_{i\in[n]} (x_i - y_i)(F_i(x) - F_i(y)) \geq \mu\|x - y\|^2.$$

Obviously, every uniform P-function is a P-function. In addition, it is easy to see from Definitions 2.1 and 2.2 that if the mapping $\mathscr{A}x^{m-1} + q$ with any given $q \in \mathbb{R}^n$ is a P-function, then $\mathscr{A}$ is a P tensor.

Recall that LCP(q, A) is said to have **the GUS-property** if LCP(q, A) has a unique solution for every $q \in \mathbb{R}^n$. Similarly, we say that $TCP(q, \mathscr{A})$ has the GUS-property if TCP$(q, \mathscr{A})$ has a unique solution for every $q \in \mathbb{R}^n$. For the solvability of LCP(q, A), an important result is that LCP(q, A) has the GUS-property if and only if the matrix A is a P-matrix. In fact, the GUS-property has been extensively discussed for various complementarity problems, including nonlinear complementarity problems [20], linear complementarity problems over symmetric cones [21] and Lorentz cone linear complementarity problems in Hilbert spaces [22]. A natural question is given by

Q1 *Whether or not $TCP(q, \mathscr{A})$ has the GUS-property if and only if the tensor $\mathscr{A}$ is a P tensor?*

Such a question was proposed by Song and Qi (see Question 6.3 in [2]). In the next section, we will answer this question.

3 Answer to Q1

First, we construct a TCP$(q, \mathscr{A})$, which has a unique solution for every $q \in \mathbb{R}^2$.

Example 3.1 Let $\mathscr{A} = (a_{i_1,i_2,i_3}) \in T_{3,2}$, where $a_{111} = 1, a_{222} = 1$ and all other $a_{i_1i_2i_3} = 0$. Then,

$$\mathscr{A}x^2 = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}.$$

In this case, TCP$(q, \mathscr{A})$ consists in finding $x \in \mathbb{R}^2$ such that

$$\begin{cases} x_1 \geq 0, \\ x_2 \geq 0, \end{cases} \quad \begin{cases} x_1^2 + q_1 \geq 0, \\ x_2^2 + q_2 \geq 0, \end{cases} \quad \text{and} \quad \begin{cases} x_1(x_1^2 + q_1) = 0, \\ x_2(x_2^2 + q_2) = 0. \end{cases} \tag{1}$$

For any $q \in \mathbb{R}^2$, let $x^q := (x_1^q, x_2^q) \in \mathbb{R}^2$ be given by

$$x_i^q := \begin{cases} 0, & \text{if} \quad q_i \geq 0, \\ \sqrt{-q_i}, & \text{otherwise.} \end{cases}$$

It is easy to see that for every $q \in \mathbb{R}^2$, TCP$(q, \mathscr{A})$ given by (1) has a unique solution x^q. Thus, we obtain that TCP$(q, \mathscr{A})$ given in this example has the GUS-property.

It is proved in [23, Proposition 2.1] that there does not exist an odd-order P tensor; and hence, the tensor given in Example 3.1 is not a P tensor. This, together with Example 3.1, implies that one does not obtain that tensor $\mathscr{A}$ is a P tensor under the assumption that TCP$(q, \mathscr{A})$ has the GUS-property.

Second, we construct the following TCP$(q, \mathscr{A})$, where $\mathscr{A} \in T_{4,2}$ is a P tensor, but it has two distinct solutions for some $q \in \mathbb{R}^2$.

Example 3.2 Let $\mathscr{A} = (a_{i_1,i_2,i_3,i_4}) \in T_{4,2}$, where $a_{1111} = 1, a_{1112} = -2, a_{1122} = 1$, $a_{2222} = 1$ and all other $a_{i_1i_2i_3i_4} = 0$. Then,

$$\mathscr{A}x^3 = \begin{pmatrix} x_1^3 - 2x_1^2x_2 + x_1x_2^2 \\ x_2^3 \end{pmatrix},$$

and

$$x_1(\mathscr{A}x^3)_1 = x_1^4 - 2x_1^3x_2 + x_1^2x_2^2, \quad x_2(\mathscr{A}x^3)_2 = x_2^4.$$

For any $x \in \mathbb{R}^2 \setminus \{0\}$, it is easy to see that

- when $x_2 \neq 0$, it follows that $x_2(\mathscr{A}x^3)_2 > 0$; and
- when $x_2 = 0$, it follows that $x_1 \neq 0$ since $x \neq 0$, and in this case, we have $x_1(\mathscr{A}x^3)_1 = x_1^4 > 0$.

Thus, for any $x \in \mathbb{R}^2 \setminus \{0\}$, there is at least one index $i \in \{1, 2\}$ such that $x_i(\mathscr{A}x^3)_i > 0$. So, we obtain that tensor $\mathscr{A}$ given in this example is a P tensor by Definition 2.1(2).

Taking $q = (0, -1)^T$, we consider TCP$(q, \mathscr{A})$, which consists in finding $x \in \mathbb{R}^2$ such that

$$\begin{cases} x_1 \geq 0, \\ x_2 \geq 0, \end{cases} \quad \begin{cases} x_1^3 - 2x_1^2x_2 + x_1x_2^2 \geq 0, \\ x_2^3 - 1 \geq 0, \end{cases} \quad \text{and} \quad \begin{cases} x_1(x_1^3 - 2x_1^2x_2 + x_1x_2^2) = 0, \\ x_2(x_2^3 - 1) = 0. \end{cases} \tag{2}$$

It is easy to see that both $x = (0, 1)^T$ and $x = (1, 1)^T$ are the solutions to TCP$(q, \mathscr{A})$ given by (2).

From Example 3.2, we obtain that TCP$(q, \mathscr{A})$ with a P tensor $\mathscr{A}$ does not possess the GUS-property.

Note that TCP$(q, \mathscr{A})$ with a P tensor $\mathscr{A}$ has a solution for every $q \in \mathbb{R}^n$ by [3, Corollary 3.3], and it is possible that this class of complementarity problems has more than one solution from Example 3.2. What more can we say about the solution set of this class of complementarity problems? In the following, we show that the solution set of TCP$(q, \mathscr{A})$ with a P tensor $\mathscr{A}$ is compact.

Theorem 3.1 *For any $q \in \mathbb{R}^n$ and a P tensor $\mathscr{A} \in T_{m,n}$, the solution set of TCP$(q, \mathscr{A})$ is nonempty and compact.*

Proof Since $\mathscr{A}$ is a P tensor, it follows from [3, Corollary 3.3] that TCP$(q, \mathscr{A})$ has a solution for every $q \in \mathbb{R}^n$. So we only need to show that the solution set of TCP$(q, \mathscr{A})$ is compact. We divide the proof into two parts.

Part 1 We show the boundedness of the solution set. To this end, we first show the following result:

R1 If there is a sequence $\{x^k\} \subset \mathbb{R}^n_+$ satisfying

$$\|x^k\| \to \infty \quad \text{and} \quad \frac{[-\mathscr{A}(x^k)^{m-1} - q]_+}{\|x^k\|} \to 0 \quad \text{as } k \to \infty, \tag{3}$$

then there exists an $i \in [n]$ such that $x_i^k[\mathscr{A}(x^k)^{m-1} + q]_i > 0$ holds for some $k \geq 0$.

In the following, we assume that the result **R1** does not hold and derive a contradiction. Given an arbitrary sequence $\{x^k\} \subset \mathbb{R}^n_+$ satisfying (3), then since the result **R1** does not hold, we have that

$$x_i^k[\mathscr{A}(x^k)^{m-1} + q]_i \leq 0, \quad \forall i \in [n],\ \forall k \geq 0. \tag{4}$$

Since the sequence $\{\frac{x^k}{\|x^k\|}\}$ is bounded, without any loss of generality, we can assume $\lim_{k\to\infty} \frac{x^k}{\|x^k\|} = \bar{x} \in \mathbb{R}^n$. From $\{x^k\} \subset \mathbb{R}^n_+$ and $\|x^k\| \to \infty$ as $k \to \infty$, we obtain that

$$\bar{x} \geq 0, \quad \bar{x} \neq 0. \tag{5}$$

If $i \in \{i \in [n] : [\mathscr{A}(x^k)^{m-1} + q]_i \leq 0\}$, then

$$[-(\mathscr{A}(x^k)^{m-1} + q)_i]_+ = -[\mathscr{A}(x^k)^{m-1} + q]_i.$$

Since $\lim_{k\to\infty}\frac{q_i}{\|x^k\|}=0$ for all $i\in[n]$, we have

$$0=\lim_{k\to\infty}\frac{[-(\mathscr{A}(x^k)^{m-1}+q)_i]_+}{\|x^k\|}=\lim_{k\to\infty}\frac{-[\mathscr{A}(x^k)^{m-1}]_i-q_i}{\|x^k\|^{m-1}}$$
$$=\lim_{k\to\infty}\frac{-[\mathscr{A}(x^k)^{m-1}]_i}{\|x^k\|^{m-1}}=-[\mathscr{A}\bar{x}^{m-1}]_i;$$

and moreover, if $i\in\{i\in[n]:[\mathscr{A}(x^k)^{m-1}+q]_i\geq 0\}$, then

$$0\leq\lim_{k\to\infty}\frac{[\mathscr{A}(x^k)^{m-1}+q]_i}{\|x^k\|^{m-1}}=\lim_{k\to\infty}\frac{[\mathscr{A}(x^k)^{m-1}]_i}{\|x^k\|^{m-1}}=[\mathscr{A}\bar{x}^{m-1}]_i.$$

Combining these two situations together, we have

$$[\mathscr{A}\bar{x}^{m-1}]_i\geq 0,\quad\forall i\in[n].\tag{6}$$

In addition, by using (4), we have

$$\bar{x}_i[\mathscr{A}\bar{x}^{m-1}]_i=\lim_{k\to\infty}\frac{x_i^k}{\|x^k\|}\frac{[\mathscr{A}(x^k)^{m-1}]_i}{\|x^k\|^{m-1}}=\lim_{k\to\infty}\frac{x_i^k}{\|x^k\|}\frac{[\mathscr{A}(x^k)^{m-1}+q]_i}{\|x^k\|^{m-1}}\leq 0.$$

By using (5) and (6), we have $\bar{x}_i[\mathscr{A}\bar{x}^{m-1}]_i\geq 0$. This, together with the above inequality, implies

$$\bar{x}_i[\mathscr{A}\bar{x}^{m-1}]_i=0,\quad\forall i\in[n].\tag{7}$$

Furthermore, by combining (5) with (6) and (7), we obtain that $\bar{x}$ is a nonzero solution of TCP(0, $\mathscr{A}$). However, since every P tensor is a strictly semi-positive tensor, while if $\mathscr{A}$ is strictly semi-positive, then TCP(0, $\mathscr{A}$) has a unique solution 0 (see [4, Theorem 3.2]). This derives a contradiction. So the result **R1** holds.

Now, suppose that the solution set of TCP($q,\mathscr{A}$) is unbounded. Then, there exists an unbounded solution sequence $\{x^k\}$ of TCP($q,\mathscr{A}$) such that $\|x^k\|\to\infty$ as $k\to\infty$, and for all $i\in[n]$, $k\geq 0$, it follows that

$$x^k\geq 0,\quad \mathscr{A}(x^k)^{m-1}+q\geq 0,\quad \langle x^k,\mathscr{A}(x^k)^{m-1}+q\rangle=0.\tag{8}$$

Obviously, $[\mathscr{A}(x^k)^{m-1}+q]_i\geq 0$ implies that

$$\frac{[-(\mathscr{A}(x^k)^{m-1}+q)_i]_+}{\|x^k\|}\to 0\quad\text{as } k\to\infty.$$

Thus, the solution sequence $\{x^k\}$ satisfies (3); and furthermore, by using the result **R1**, there exist an index $i_0\in[n]$ and an integer $k^*>0$ such that $x_{i_0}^{k^*}[\mathscr{A}(x^{k^*})^{m-1}+q]_{i_0}>0$, which is contrary to that $x_i^k[\mathscr{A}(x^k)^{m-1}+q]_i=0$ for all $i\in[n]$ and $k\geq 0$. So the solution set of TCP($q,\mathscr{A}$) is bounded.

Part 2 We now show that the solution set of TCP$(q, \mathscr{A})$ is closed. Suppose that $\{x^k\}$ is a solution sequence of TCP$(q, \mathscr{A})$ and

$$\lim_{k\to\infty} x_k = \bar{x}, \tag{9}$$

we need to show that $\bar{x}$ solves TCP$(q, \mathscr{A})$. Since $\mathscr{A}x^{m-1} + q$ is continuous, by (9) we have

$$\lim_{k\to\infty} [(\mathscr{A}(x_k)^{m-1} + q] = \mathscr{A}\left(\lim_{k\to\infty} x_k\right)^{m-1} + q = \mathscr{A}\bar{x}^{m-1} + q. \tag{10}$$

Since $\{x^k\}$ is a solution sequence of TCP$(q, \mathscr{A})$, we have that (8) holds. Furthermore, by using (8), (9) and (10), we can obtain that

$$\bar{x} \geq 0, \quad \mathscr{A}\bar{x}^{m-1} + q \geq 0, \quad \langle \bar{x}, \mathscr{A}\bar{x}^{m-1} + q\rangle = 0.$$

So $\bar{x}$ is a solution of TCP$(q, \mathscr{A})$. Therefore, we obtain that the solution set is closed.

Combining **Part 1** with **Part 2**, we obtain that the solution set of TCP$(q, \mathscr{A})$ is compact. This completes the proof. □

In our proof of **Part 1**, we use the property of the strictly semi-positive tensor. When we revise this paper, we are informed that two new papers [24,25] studied the boundedness of the solution set of two related classes of TCP$(q, \mathscr{A})$. It is easy to see that our approach is different from those in these two papers. Recall that a tensor $\mathscr{A} \in T_{m,n}$ is called an R_0-tensor, if and only if there exists no $x \in \mathbb{R}^n_+ \setminus \{0\}$ such that

$$(\mathscr{A}x^{m-1})_i = 0, \text{ if } \ x_i > 0,$$
$$(\mathscr{A}x^{m-1})_i \geq 0, \text{ if } \ x_i = 0.$$

Obviously, every R-tensor is an R_0-tensor, and every strictly semi-positive tensor is also an R_0-tensor. Thus, every P tensor is certainly an R_0-tensor. In fact, in **Part 1**, we demonstrate that, if $\mathscr{A}$ is an R_0-tensor, then **R1** holds; and if **R1** holds and the solution set of the corresponding TCP$(q, \mathscr{A})$ is nonempty, then it is bounded. Thus, the result of **Part 1** can be also obtained by using the property of the R_0-tensor. In particular, by using our method given in **Part 1**, some better results can be easily obtained (see a new paper [26]).

4 Strong P Tensor and Related Properties

In this section, we consider the question: for which kind of tensor, TCP$(q, \mathscr{A})$ has the GUS-property. For this purpose, we introduce a new class of tensors, called the *strong P tensor*, which is defined as follows.

Definition 4.1 Let $\mathscr{A} = (a_{i_1,\ldots,i_m}) \in T_{m,n}$. We say that $\mathscr{A}$ is a strong P tensor iff $\mathscr{A}x^{m-1}$ is a P-function.

It is well known that a matrix $A \in \mathbb{R}^{n\times n}$ is a P-matrix if and only if $F(x) = Ax+q$ is a P-function for any $q \in \mathbb{R}^n$. Note that $\mathscr{A}x^{m-1}$ is a P-function if and only if $\mathscr{A}x^{m-1}+q$ is a P-function for any $q \in \mathbb{R}^n$. Thus, the strong P tensor is a generalization of the P-matrix from matrix to tensor. From the definitions of the P tensor and the strong P tensor, it is easy to see that every strong P tensor must be a P tensor. The strong P tensor is defined with the help of the P-function, so an advantage of this way is that the related results and methods associated with the P-function can be applied to study this class of tensors.

Now we are going to give a simple example of the strong P tensor.

Example 4.1 Given a tensor $\mathscr{A} = (a_{i_1,i_2,\ldots,i_m}) \in T_{m,n}$, where m is an even number, $a_{ii,\cdots,i} > 0$ for all $i \in [n]$ and all other $a_{i_1,i_2,\ldots,i_m} = 0$. Obviously,

$$\mathscr{A}x^{m-1} = \begin{pmatrix} a_{11,\ldots,1}x_1^{m-1} \\ a_{22,\ldots,2}x_2^{m-1} \\ \cdots \\ a_{nn,\ldots n}x_n^{m-1} \end{pmatrix}.$$

Then, for all pairs of distinct vectors x and y in $\mathbb{R}^n$, we have

$$\begin{aligned}(x_i - y_i)\left[(\mathscr{A}x^{m-1})_i - (\mathscr{A}y^{m-1})_i\right] &= (x_i - y_i)(a_{ii,\ldots,i}x_i^{m-1} - a_{ii,\ldots,i}y_i^{m-1}) \\ &= a_{ii,\ldots,i}(x_i - y_i)(x_i^{m-1} - y_i^{m-1}).\end{aligned}$$

Since x and y are two different vectors, $a_{ii,\ldots,i} > 0$ for all $i \in [n]$ and m is an even number, there exists at least one index $j \in [n]$ such that $x_j \neq y_j$, and thus

$$a_{jj,\ldots,j}(x_j - y_j)(x_j^{m-1} - y_j^{m-1}) > 0.$$

That is to say,

$$\max_{i\in[n]} (x_i - y_i)((\mathscr{A}x^{m-1})_i - (\mathscr{A}y^{m-1})_i) > 0.$$

So $\mathscr{A}x^{m-1}$ is a P-function and thus $\mathscr{A}$ is a strong P tensor.

The following result comes from [27, Theorem 2.3].

Lemma 4.1 *Let $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ be a P-function; then, the corresponding CP(F) has no more than one solution.*

With the help of Lemma 4.1, we show the following result.

Theorem 4.1 *Suppose that $\mathscr{A} \in T_{m,n}$ is a strong P tensor; then, TCP$(q, \mathscr{A})$ has the GUS-property.*

Proof Since $\mathscr{A}$ is a strong P tensor, it follows that $\mathscr{A}$ is a P tensor. Furthermore, it follows from [3, Corollary 3.3] that TCP$(q, \mathscr{A})$ has a solution for every $q \in \mathbb{R}^n$. Moreover, since $\mathscr{A}$ is a strong P tensor, it follows that $\mathscr{A}x^{m-1} + q$ is a P-function; and hence, from Lemma 4.1 it follows that TCP$(q, \mathscr{A})$ has no more than one solution. Therefore, TCP$(q, \mathscr{A})$ has a unique solution for every $q \in \mathbb{R}^n$, i.e., TCP$(q, \mathscr{A})$ has the GUS-property. □

Corollary 4.1 *Given $\mathscr{A} \in T_{m,n}$ and $q \in \mathbb{R}^n$. Suppose that $F(x) = \mathscr{A}x^{m-1} + q$ is a P-function; then, the corresponding CP(F) has the GUS-property; and m must be an even number.*

In the theory of nonlinear complementarity problems, when the involved function F is a P-function, one can only obtain that the corresponding CP(F) has no more than one solution (see Lemma 4.1); when the involved function F is a uniform P-function, one can obtain that the corresponding CP(F) has the GUS-property (see [28, Corollary 3.2]). From the first result of Corollary 4.1, we see that CP(F) has the GUS-property when $F(x) = \mathscr{A}x^{m-1} + q$ is a P-function. In addition, from the second result of Corollary 4.1, we obtain that a class of functions (i.e., $F(x) = \mathscr{A}x^{m-1} + q$ with m being odd) cannot be in the class of the P-functions.

In the following, we investigate the relationship between the P tensor and the strong P tensor. Recall that every strong P tensor is a P tensor. The following example demonstrates that the converse does not hold.

Example 4.2 Suppose $\mathscr{A} = (a_{i_1i_2i_3i_4}) \in T_{4,2}$, where $a_{1111} = 1, a_{1222} = -1, a_{1122} = 1, a_{2222} = 1, a_{2111} = -1, a_{2211} = 1$ and all other $a_{i_1i_2i_3i_4} = 0$. Obviously,

$$\mathscr{A}x^3 = \begin{pmatrix} x_1^3 - x_2^3 + x_1x_2^2 \\ x_2^3 - x_1^3 + x_2x_1^2 \end{pmatrix};$$

then, $x_1(\mathscr{A}x^3)_1 = x_1^4 - x_1x_2^3 + x_1^2x_2^2$ and $x_2(\mathscr{A}x^3)_2 = x_2^4 - x_2x_1^3 + x_1^2x_2^2$. We consider the following several cases:

- when $x_1 = x_2 \neq 0$, we have $x_1(\mathscr{A}x^3)_1 = x_1^4 - x_1^4 + x_1^4 = x_1^4 > 0$;
- when only one $x_i = 0$ for any $i \in \{1, 2\}$, we have $x_j(\mathscr{A}x^3)_j = x_j^4 > 0$ for $j \neq i$;
- when $x_1 > x_2 > 0$, we have $x_1(\mathscr{A}x^3)_1 = x_1(x_1^3 - x_2^3) + x_1^2x_2^2 > 0$;
- when $x_2 > x_1 > 0$, we have $x_2(\mathscr{A}x^3)_2 = x_2(x_2^3 - x_1^3) + x_1^2x_2^2 > 0$;
- when $0 > x_1 > x_2$, we have $x_2(\mathscr{A}x^3)_2 = x_2(x_2^3 - x_1^3) + x_1^2x_2^2 > 0$;
- when $0 > x_2 > x_1$, we have $x_1(\mathscr{A}x^3)_1 = x_1(x_1^3 - x_2^3) + x_1^2x_2^2 > 0$;
- when $x_1 > 0 > x_2$, we have $-x_2x_1^3 > 0$, and so $x_2(\mathscr{A}x^3)_2 > 0$;
- when $x_2 > 0 > x_1$, we have $-x_1x_2^3 > 0$, and so $x_1(\mathscr{A}x^3)_1 > 0$.

Thus, for any $x \in \mathbb{R}^2 \setminus \{0\}$, there exists an index $i \in \{1, 2\}$ such that $x_i(\mathscr{A}x^3)_i > 0$. So $\mathscr{A}$ is a P tensor. However, $\mathscr{A}$ is not a strong P tensor. In fact, if we take $x = (2.1, -1.9)^T$ and $y = (2, -2)^T$, then we have

$$(x_1 - y_1)((\mathscr{A}x^3)_1 - (\mathscr{A}y^3)_1) = -0.0299 < 0$$

and

$$(x_2 - y_2)((\mathscr{A}x^3)_2 - (\mathscr{A}y^3)_2) = -0.0499 < 0.$$

Thus, by Definition 4.1 we obtain that $\mathscr{A}$ is not a strong P tensor.

Many properties of the P tensor have been obtained in the literature. Since every strong P tensor is a P tensor, we may easily obtain the following properties of strong P tensor:

Proposition 4.1 *If $\mathscr{A} \in T_{m,n}$ is a strong P tensor; then,*

(1) $\mathscr{A}$ must be strictly semi-positive;
(2) $\mathscr{A}$ must be an R-tensor;
(3) all of its H-eigenvalues and Z-eigenvalues are positive;
(4) all the diagonal entries of $\mathscr{A}$ are positive;
(5) every principal sub-tensor of $\mathscr{A}$ is still a strong P tensor.

Proof Since we already know that a strong P tensor is a P tensor, the first four results can be easily obtained from [2,3]. Now we prove the result (5). Let an arbitrary principal sub-tensor $\mathscr{A}_r^J \in T_{m,r}$ of the strong P tensor $\mathscr{A} \in T_{m,n}$ be given. We choose any $x = (x_{j_1}, x_{j_2}, \ldots, x_{j_r}) \in \mathbb{R}^r \setminus \{0\}$ and $y = (y_{j_1}, y_{j_2}, \ldots, y_{j_r}) \in \mathbb{R}^r \setminus \{0\}$ with $x \neq y$. Then, let $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in \mathbb{R}^n$, where $\bar{x}_i = x_{j_i}$ for $i \in J$ and $\bar{x}_i = 0$ for $i \notin J$. In a similar way, let $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \in \mathbb{R}^n$, where $\bar{y}_i = y_{j_i}$ for $i \in J$ and $\bar{y}_i = 0$ for $i \notin J$. Since $\mathscr{A}$ is a strong P tensor, there exists an index $k \in [n]$ such that

$$\begin{aligned} 0 &< \max_{k \in [n]} (\bar{x}_k - \bar{y}_k)((\mathscr{A}\bar{x}^{m-1})_k - (\mathscr{A}\bar{y}^{m-1})_k) \\ &= \max_{k \in J} (x_k - y_k)((\mathscr{A}_r^J x^{m-1})_k - (\mathscr{A}_r^J y^{m-1})_k). \end{aligned}$$

Thus, $\mathscr{A}_r^J$ is a strong P tensor. □

5 Conclusions

By constructing two counterexamples, we proved that it is possible that TCP(q, $\mathscr{A}$) has no the GUS-property when $\mathscr{A}$ is a P tensor; and that it is also possible that $\mathscr{A}$ is not a P tensor when TCP(q, $\mathscr{A}$) has the GUS-property. These gave a negative answer to Question 6.3 proposed in [2]. We also showed that the solution set of TCP(q, $\mathscr{A}$) is nonempty and compact when $\mathscr{A}$ is a P tensor.

In order to investigate that for which kind of tensor, the tensor complementarity problem has the GUS-property, we introduced the concept of the strong P tensor, and showed that TCP(q, $\mathscr{A}$) has the GUS-property when $\mathscr{A}$ is a strong P tensor. Since every strong P tensor is a P tensor, many known results associated with the P tensor are still satisfied for the strong P tensor.

Note that the strong P tensor is defined by using the P-function, we believe that more properties related to the strong P tensor can be further studied with the help of known methods and results for the P-function.

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