

# Evolutionary Variational Inequality Formulation of the Generalized Nash Equilibrium Problem

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Received: 8 October 2014 / Accepted: 29 December 2015 / Published online: 28 January 2016  
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**Abstract** The formulation of the generalized Nash Equilibrium problem as an evolutionary variational inequality problem is proved in the general setting of quasiconvex decision functions. An existence result for the time-dependent generalized Nash equilibrium problem is deduced, and an application to the dynamic electricity market is also considered.

**Keywords** Generalized Nash equilibrium problem · Evolutionary variational inequality problem · Semistrict quasiconvexity · Sublevel set

**Mathematics Subject Classification** 49J40 · 90C26 · 90B10

## 1 Introduction

Generalized Nash equilibrium problems (GNEPs) are noncooperative games, where the strategy of each player depends on the rival players' strategies. This class of prob-

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Communicated by Dean A. Carlson.

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lems has gained popularity in recent times because of its capacity to model economical systems [1], routing problems in communication networks [2], and engineering applications [3]. A survey on GNEP with much historical information is given in [4]. Here, we aim at studying GNEP in time-dependent context.

In order to study the GNEP and to have an efficient computational process some reformulation of the GNEP have been given in the literature. But the best situation corresponds to the case when the GNEP can be reformulated as a variational inequality problem (VI), thus inheriting all the corresponding theoretical and numerical machinery.

In our present work, we reformulate the time-dependent version of GNEP to a parametric VI in particular an evolutionary variational inequality problem (EVI) where the parameter represents physical time. For further introduction to EVI, the readers are addressed to [5], and to see relationship between the dynamic network equilibrium problems and EVI, please refer to [6].

Our aim in the paper is, through the reformulation of time-dependent GNEP by EVI or evolutionary quasivariational inequalities (EQVI), to prove the existence of a time-dependent equilibrium without assuming any differentiability and, above all, for quasiconvex decision functions, a very classical and natural hypothesis in mathematical economics. The paper is organized as follows. In Sect. 2, we precisely define the time-dependent GNEP that we have in scope and motivate our definition by an application to a dynamic electricity market model. Reformulation by EVI and EQVI are obtained in Sect. 3 and, in the case of semistrictly quasiconvex cost functions, we prove an existence result for time-dependent GNEP in Sect. 4.

## 2 Notation and Motivation Example

### 2.1 General Setting of Time-Dependent GNEP

Assume that there are  $p$  players and each player  $\nu$  controls variable  $x^\nu \in L^2([0, T], \mathbb{R}^{n_\nu})$ . In fact,  $x^\nu(t) \in \mathbb{R}^{n_\nu}$  is a strategy of the player  $\nu$  at time  $t \in [0, T]$ . The full strategy vector  $x$  is thus an element of  $L^2([0, T], \mathbb{R}^n)$ , where  $n = \sum_{\nu=1}^p n_\nu$ , and thus  $x(t)$  is the vector of strategies of all players at a given time  $t \in [0, T]$ . We use the notation  $x^{-\nu} \in L^2([0, T], \mathbb{R}^{n-n_\nu})$ ; so that for any  $t \in [0, T]$ ,  $x^{-\nu}(t) \in \mathbb{R}^{n-n_\nu}$  is the vector formed by all players' decision variables except the player  $\nu$  at time  $t \in [0, T]$ . So, we can also write

$$x = (x^\nu, x^{-\nu}) \subseteq L^2([0, T], \mathbb{R}^n),$$

which is a shortcut (already used in many papers on the subject; see, e.g., [7,8]), to denote  $x = (x^1, \dots, x^{\nu-1}, x^\nu, x^{\nu+1}, \dots, x^p)$ .

Following the pioneering work of Rosen [1], the full strategy vector is chosen in a common subset  $K \subset L^2([0, T], \mathbb{R}^n)$  and thus the admissible strategy set of each player is defined as

$$K_\nu(x^{-\nu}) = \{x^\nu \in L^2([0, T], \mathbb{R}^{n_\nu}) : x = (x^\nu, x^{-\nu}) \in K\}. \tag{1}$$

**Definition 2.1** Let  $\theta_\nu : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$  be the cost (or loss) function for the  $\nu^{th}$ -player. A strategy  $\bar{x} \in K \subseteq L^2([0, T], \mathbb{R}^n)$  is a *time-dependent generalized Nash Equilibrium* if and only if for each player  $\nu$ , we have  $\bar{x}^\nu \in K_\nu(\bar{x}^{-\nu})$  and

$$\theta_\nu(\bar{x}) \leq \theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}), \quad \forall \bar{x}^\nu \in K_\nu(\bar{x}^{-\nu}). \tag{2}$$

In other words this means that  $\bar{x} \in K \subseteq L^2([0, T], \mathbb{R}^n)$  is a time-dependent generalized Nash equilibrium

if for all  $\nu$ ,  $\bar{x}^\nu \in L^2([0, T], \mathbb{R}^{n_\nu})$  solves the following optimization problem

$$P_\nu(\bar{x}^{-\nu}) \quad \min_{x^\nu \in K_\nu(\bar{x}^{-\nu})} \theta_\nu(x^\nu, \bar{x}^{-\nu}). \tag{3}$$

For any given optimal strategy  $\bar{x}^{-\nu}$  of the rival players, the solution set of problem  $P_\nu(\bar{x}^{-\nu})$  is denoted by  $Sol_\nu(\bar{x}^{-\nu})$ .

Let us recall that  $L^2([0, T], \mathbb{R}^n) = L^2([0, T], \mathbb{R}^{n_\nu}) \times L^2([0, T], \mathbb{R}^{n-n_\nu})$ .

Finally for any subset  $C$ ,  $\text{conv}(C)$  denotes the convex hull of  $C$  while  $\text{dom } f$  stands for the domain of the function  $f$ .

### 2.2 A Motivation Example: Dynamic Electricity Market

Let us describe, by a simple example, how the above concept of time-dependent generalized Nash equilibrium can apply to electricity markets. The deregulation and privatization of the electricity market in many countries lead to the development of new models representing them. One classical family of economical models for electricity markets is based on a Cournot-type formulation, a noncooperative game, in which the generators compete only with the energy quantities; see, e.g., [9, 10].

In this model, the electricity market is supposed to be centralized by an independent system operator (ISO). Each agent bids a cost of production, which is a function of time, to the ISO who computes the best response/dispatch in order to minimize the general cost of production.

In order to simplify the model, we consider an electricity spot market in which the agents in the markets are only the producers and the inelastic demand at each node is known. Thus, consider the following electricity network composed of  $N$  nodes ( $\mathfrak{N} = \{1, \dots, N\}$  being the set of nodes) and assume that at each node in the network there is a unique producer, denoted by  $\nu \in \mathfrak{N}$ . The considered time period for production/delivery is  $[0, T]$ . Let us define the following notation:

- $D^\nu \in L^2([0, T], \mathbb{R}_+)$  is the demand function of electricity at node  $\nu \in \mathfrak{N}$ .
- $q^\nu \in L^2([0, T], \mathbb{R}_+)$  represents the production function of electricity of producer  $\nu \in \mathfrak{N}$ .
- $A^\nu(t)q^\nu(t) + B^\nu(t)(q^\nu(t))^2$  is the real cost of power generation for producer  $\nu \in \mathfrak{N}$  at time  $t \in [0, T]$ , where  $A^\nu, B^\nu \in L^2([0, T], \mathbb{R}_+)$  are the real cost coefficient functions.

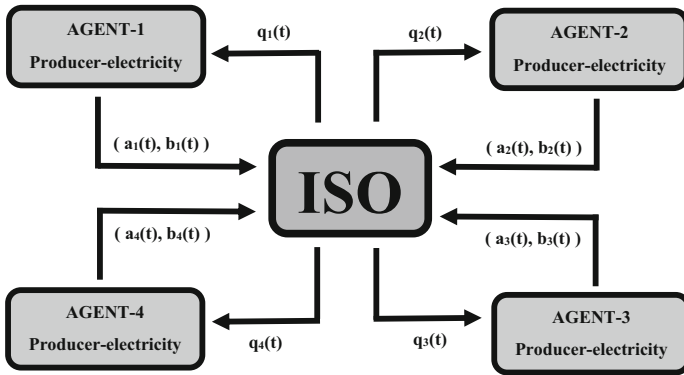


Fig. 1 Market organization with four producers, i.e.,  $p = 4$

In the above notation,  $\mathbb{R}_+$  denotes the set of nonnegative real numbers.

The spot market is regulated by an ISO. The  $\nu^{th}$  agent provides to the ISO a time-dependent quadratic bid function described by the parameters  $a^\nu$  and  $b^\nu$ , elements of  $L^2([0, T], \mathbb{R}_+)$ , and the corresponding bid function is thus  $a^\nu q^\nu + b^\nu (q^\nu)^2$ . Then, based on the agent’s bids, the ISO computes the production function  $q^\nu \in L^2([0, T], \mathbb{R}^n)$ , for each agent  $\nu$ , on the period  $[0, T]$  (see Fig. 1).

Then, for the given time  $t \in [0, T]$ , the ISO computes the set of admissible productions  $q^\nu \in L^2([0, T], \mathbb{R})$ . Compare to [10], the thermal losses are neglected and thus the total demand function on the whole network simply corresponds to  $D \in L^2([0, T], \mathbb{R}_+)$  given by  $D(t) = \sum_\nu D^\nu(t)$ .

Thus, the ISO, knowing the couple of bid vectors  $a(t) = (a^\nu(t))_{\nu \in \mathfrak{N}}$  and  $b(t) = (b^\nu(t))_{\nu \in \mathfrak{N}}$  at  $t \in [0, T]$ , announced by producers, computes a production vector  $q = (q^\nu)_{\nu \in \mathfrak{N}}$  to minimize the total generation cost in the period  $[0, T]$ , that is, to solve the following optimization problem, denoted by  $ISO(a, b)$ .

$$\begin{aligned}
 ISO(a, b) \quad & \min_q \sum_{\nu=1}^N \int_0^T [a^\nu(t) + b^\nu(t)q^\nu(t)] q^\nu(t) dt, \\
 \text{s.t.} \quad & \begin{cases} \sum_{\nu=1}^N q^\nu(t) \geq D(t), \text{ a.e. in } [0, T] \\ q^\nu(t) \geq 0, \quad \forall \nu \in \mathfrak{N}, \text{ a.e. in } [0, T]. \end{cases}
 \end{aligned}$$

The solution set of the above optimization problem is denoted by  $Sol_{ISO}(a, b)$ . Observe that if the  $b^\nu$  are assumed to be positive functions, then the ISO’s problem is a convex program with a strictly convex objective function and admits a unique solution. We denote by

$$S_\nu(a, b) = \{q^\nu : \exists q^{-\nu} \text{ with } (q^\nu, q^{-\nu}) \in Sol_{ISO}(a, b)\}.$$

Clearly the producers cannot act independently from each other in the market, at least because of the finiteness of the demand. In time interval  $[0, T]$ , each pro-

ducer  $\nu$  aims to maximize his profit, defined as the difference between the revenue  $(a^\nu(t) + 2b^\nu(t)q^\nu(t))q^\nu(t)$  and the real generation cost  $A^\nu(t)q^\nu(t) + B^\nu(q^\nu(t))^2$ . But the producer  $\nu$  has to take into account that the production vector  $q(t)$  supplied by the ISO depends on the bids of the other producers, namely of the bid vectors  $a_{-\nu}(t) = (a_j(t))_{j \neq \nu}$  and  $b_{-\nu}(t) = (b_j(t))_{j \neq \nu}$ ,  $t \in [0, T]$ . Therefore, each producer  $\nu$  solves the following optimization problem,  $P_\nu(a^{-\nu}, b^{-\nu})$

$$\begin{aligned} & \max_{a^\nu, b^\nu, q^\nu} \int_0^T [(A^\nu(t) + B^\nu(t)(q^\nu(t))q^\nu(t) - (a^\nu(t) + 2b^\nu(t)q^\nu(t))q^\nu(t)] dt \\ & \text{subject to } \begin{cases} a^\nu(t) \in [\underline{A}^\nu(t), \overline{A}^\nu(t)], \text{ a.e. in } [0, T] \\ b^\nu(t) \in [\underline{B}^\nu(t), \overline{B}^\nu(t)], \text{ a.e. in } [0, T] \\ q^\nu \in S_\nu(a, b), \text{ a.e. in } [0, T], \end{cases} \end{aligned}$$

where  $0 < \underline{A}^\nu(t) \leq \overline{A}^\nu(t)$  and  $0 < \underline{B}^\nu(t) \leq \overline{B}^\nu(t)$  define the feasible range for the bids  $(a^\nu, b^\nu)$ .

This leads us to use the following notation

- $x = (a, b, q)$ ,  $x^\nu = (a^\nu, b^\nu, q^\nu)$ ,  $x^{-\nu} = (a^{-\nu}, b^{-\nu}, q^{-\nu})$ ,
- $\theta_\nu(x) = \int_0^T [(A^\nu(t) + B^\nu(t)(q^\nu(t)).q^\nu(t) - (a^\nu(t) + 2b^\nu(t)q^\nu(t))q^\nu(t)] .dt$ ,
- $K_\nu = \{(a^\nu, b^\nu) : \underline{A}^\nu(t) \leq a^\nu(t) \leq \overline{A}^\nu(t) \text{ and } \underline{B}^\nu(t) \leq b^\nu(t) \leq \overline{B}^\nu(t), \text{ a.e. in } [0, T]\}$ ,
- $K_\nu(x^{-\nu}) = K_\nu \times S_\nu(a^\nu, \bar{a}^{-\nu}, b^\nu, \bar{b}^{-\nu})$ .

An equilibrium of the above-described electricity market model is a time-dependent generalized Nash equilibrium in the sense of (3).

### 3 Reformulation of Time-Dependent GNEP

#### 3.1 Preliminary Results and Notation

Evolutionary variational inequalities (EVIs), that are time-dependent variational inequalities, have been extensively used to model mechanical and transportation systems; see, e.g., [5, 11] and the references therein. EVIs are also used in various models of market equilibrium problems and financial equilibrium problems. For more details, please see, [12–16].

In this work we will consider a new and broader class of EVI in which the function is replaced by a set-valued map. Indeed since we will not assume differentiability of the cost functions of the time-dependent GNEPs, set-valued operators will naturally enter into the scope as first-order operator.

Let  $F : L^2([0, T], \mathbb{R}^n) \rightrightarrows L^2([0, T], \mathbb{R}^n)$  be a set-valued map and  $K$  be a nonempty subset of  $L^2([0, T], \mathbb{R}^n)$ . The associated evolutionary variational inequality problem, denoted by  $EVI(F, K)$ , consists of finding an  $x^* \in K$  such that there exists a  $f_{x^*} \in F(x^*)$  for which

$$\langle\langle f_{x^*}, y - x^* \rangle\rangle \geq 0, \quad \forall y \in K, \tag{4}$$

where for  $\phi, \psi \in L^2([0, T], \mathbb{R}^n)$ , we use the notation

$$\langle\langle \phi, \psi \rangle\rangle = \int_0^T \langle \phi(t), \psi(t) \rangle dt,$$

and  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{R}^n$ .

**Lemma 3.1** *Let  $C : [0, T] \rightrightarrows \mathbb{R}^n$  be a set-valued map with nonempty values and define the subset  $K$  of  $L^2([0, T], \mathbb{R}^n)$  by*

$$K = \{x \in L^2([0, T], \mathbb{R}^n) : x(t) \in C(t), \text{ a.e. in } [0, T]\}. \tag{5}$$

*Let  $x^* \in K$ . Then  $x^*$  is a solution of  $EVI(F, K)$  if and only if there exists  $f_{x^*} \in F(x^*)$ , with*

$$\langle f_{x^*}(t), y(t) - x^*(t) \rangle \geq 0, \quad \forall y \in K, \text{ a.e. in } [0, T]. \tag{6}$$

*Proof* Clearly, (4) follows immediately from (6). Next, suppose (6) is not true. Then, for any  $f_{x^*} \in F(x^*)$ , there exist  $\bar{y} \in K$  and a measurable set  $I \subset [0, T]$  with Lebesgue measure  $\mu(I) > 0$  such that

$$\langle f_{x^*}(t), \bar{y}(t) - x^*(t) \rangle < 0, \quad \forall t \in I.$$

Define a function  $\hat{y} : [0, T] \rightarrow \mathbb{R}^n$  as

$$\hat{y}(t) = \begin{cases} x^*(t), & t \in [0, T] \setminus I, \\ \bar{y}(t), & t \in I. \end{cases}$$

Then, since, a.e.  $t \in [0, T]$ ,  $x^*(t)$  and  $\bar{y}(t)$  are elements of  $C(t)$ , one has  $\hat{y} \in K \subseteq L^2([0, T], \mathbb{R}^n)$ , and

$$\begin{aligned} \langle\langle f_{x^*}, \hat{y} - x^* \rangle\rangle &= \int_{t \in [0, T] \setminus I} \langle f_{x^*}(t), x^*(t) - x^*(t) \rangle dt \\ &\quad + \int_{t \in I} \langle f_{x^*}(t), \bar{y}(t) - x^*(t) \rangle dt < 0 \end{aligned}$$

thus completing the proof. □

*Remark 3.1* A simple case of a subset  $K$ , defined by (5), is when for any  $t \in [0, T]$ ,  $C(t) = [\underline{x}(t), \bar{x}(t)]$  with  $\underline{x}$  and  $\bar{x}$  are two given elements of  $L^2([0, T], \mathbb{R}^n)$ . This case has been considered, e.g., in [6].

Let us end this subsection by providing some definitions and related structures associated with quasiconvex functions, which shall play an important role in further discussions.

- A function  $\psi : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$  is said to be *quasiconvex* if and only if, for any  $x, y \in L^2([0, T], \mathbb{R}^n)$  and  $\lambda \in [0, 1]$ , we have

$$\psi(\lambda x + (1 - \lambda)y) \leq \max \{ \psi(x), \psi(y) \}.$$

- A function  $\psi : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$  is said to be *semistrictly quasiconvex* if and only if it is quasiconvex, and for any  $x, y \in L^2([0, T], \mathbb{R}^n)$ , such that  $\psi(x) \neq \psi(y)$ , and for any  $\lambda \in (0, 1)$ , we have

$$\psi(\lambda x + (1 - \lambda)y) < \max \{\psi(x), \psi(y)\}.$$

- Given a function  $\psi : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$  and  $x \in L^2([0, T], \mathbb{R}^n)$ , the *sublevel set* of  $\psi$  at  $x$  is defined as

$$S_\psi(x) = \{u \in L^2([0, T], \mathbb{R}^n) : \psi(u) \leq \psi(x)\},$$

and the *strict sublevel set* of  $\psi$  at  $x$  by

$$S_\psi^<(x) = \{u \in L^2([0, T], \mathbb{R}^n) : \psi(u) < \psi(x)\}.$$

It is well known that  $\psi$  is quasiconvex over  $L^2([0, T], \mathbb{R}^n)$  if and only if  $S_\psi(x)$  is convex set for any  $x \in L^2([0, T], \mathbb{R}^n)$ .

### 3.2 Reformulation as Evolutionary Variational Inequalities

As we have already mentioned earlier, our aim is to prove the existence of time-dependent Nash equilibrium, and this will be done, in Sect. 4, thanks to a reformulation of the equilibrium problem in an associated evolutionary variational inequality. This reformulation, which can be considered as a kind of first-order condition, cannot be done with classical derivatives since no differentiability assumptions will be made. On the other hand, since we want to obtain sufficient optimality conditions, subdifferentials are not adapted since our decision functions are not supposed to be convex but only quasiconvex. Nevertheless, as shown in [17, 18], the concept of *normal operator* is particularly adapted to this setting.

Given a quasiconvex function  $\psi : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$ , the *normal operator* is the set-valued map  $N_\psi : L^2([0, T], \mathbb{R}^n) \rightrightarrows L^2([0, T], \mathbb{R}^n)$  defined by simply considering the normal cone to the sublevel set of  $\psi$  at the point, that is,

$$N_\psi(x) = \{v \in L^2([0, T], \mathbb{R}^n) : \langle v, y - x \rangle \leq 0, \forall y \in S_\psi(x)\}. \quad (7)$$

As it was shown in [17, 18], this normal operator has very nice properties for semistrictly quasiconvex functions and even for quasiconvex functions with a more general definition of *adjusted normal operator*  $N_\psi^a$ . See, [19], for a state of art on the normal operator in quasiconvex optimization. Along this paper only continuous semistrictly quasiconvex function will be considered, and in this case only the above classical definition  $N_\psi$  is needed. Additionally, the operator  $N_\psi$  inherits the properties of this adjusted normal operator  $N_\psi^a$  since they coincide for continuous semistrictly quasiconvex functions.

The following properties of the normal operator will play a central role in the sequel.

**Proposition 3.1** ([19]) *Let  $X$  be a nonempty subset of  $L^2([0, T], \mathbb{R}^n)$ . Let  $\psi : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$  be any semistrictly quasiconvex function on  $X$ .*

(i)  $N_\psi$  is quasimonotone, that is, for any  $x, y \in X$  and any  $x^* \in N_\psi, y^* \in N_\psi$ ,

$$\langle x^*, y - x \rangle > 0 \implies \langle y^*, y - x \rangle \geq 0.$$

(ii)  $N_\psi$  is nontrivial (i.e.,  $N_\psi(x)$  does not reduce to  $\{0\}$ ) on  $\text{dom } \psi \setminus \arg \min_X \psi$ .

In the context of generalized Nash equilibrium problems, an adaptation of the definition of normal operator has to be considered and followed, mainly on the lines of [20]. Let  $x \in L^2([0, T], \mathbb{R}^n)$ .

- $S_v(x) = S_{\theta_v(\cdot, x^{-v})}(x^v) = \{u^v \in L^2([0, T], \mathbb{R}^{n_v}) : \theta_v(u^v, x^{-v}) \leq \theta_v(x)\}$ ,
- $S_v^<(x) = S_{\theta_v^<(\cdot, x^{-v})}(x^v) = \{u^v \in L^2([0, T], \mathbb{R}^{n_v}) : \theta_v(u^v, x^{-v}) < \theta_v(x)\}$ ,
- $A_v(x^{-v}) = \arg \min_{x^v \in L^2([0, T], \mathbb{R}^{n_v})} \theta_v(x^v, x^{-v})$ .
- $N_{\theta_v} : L^2([0, T], \mathbb{R}^{n_v}) \rightrightarrows L^2([0, T], \mathbb{R}^{n_v})$  stands for the normal operator of the quasiconvex function  $\theta_v(\cdot, x^{-v})$  at  $x^v$ , that is,

$$N_{\theta_v}(x^v) = \{g^v \in L^2([0, T], \mathbb{R}^{n_v}) : \langle g^v, u^v - x^v \rangle \leq 0, \forall u^v \in S_v(x)\}.$$

- Let  $\overline{\mathfrak{B}}_v(0, 1)$  be the closed unit ball in  $L^2([0, T], \mathbb{R}^{n_v})$ . The set-valued map  $F_v$  is defined from  $L^2([0, T], \mathbb{R}^{n_v})$  to  $L^2([0, T], \mathbb{R}^{n_v})$  by

$$F_v(x^v) = \begin{cases} \overline{\mathfrak{B}}_v(0, 1), & \text{if } x^v \in A_v(x^{-v}), \\ N_{\theta_v}(x^v) \setminus \{0\}, & \text{otherwise.} \end{cases}$$

Let us observe that this map is nonempty and convex-valued.

- Define the set-valued map  $N_\theta : L^2([0, T], \mathbb{R}^n) \rightrightarrows L^2([0, T], \mathbb{R}^n)$  such that, for any  $x \in L^2([0, T], \mathbb{R}^n)$ , (given  $\sum_{v=1}^p n_v = n$ ), we have,

$$N_\theta(x) = \prod_{v=1}^p F_v(x^v). \tag{8}$$

Now, following the same line as in Theorem 4.1 of [20], we can obtain a sufficient optimality condition for the GNEP (3) in terms of evolutionary variational inequality.

**Theorem 3.1** *Let, for any  $v$ , the function  $\theta_v$  be continuous and semistrict quasiconvex with respect to the  $v$ -th variable. Assume that  $K$  is a nonempty subset of  $L^2([0, T], \mathbb{R}^n)$  and that the subsets  $K_v(x^{-v})$  are defined by (1).*

*Then every solution of  $EVI(N_\theta, K)$  is a time-dependent generalized Nash equilibrium of (3).*



*Proof* Let  $x^* \in K \subseteq L^2([0, T], \mathbb{R}^n)$  be a solution of  $EVI(N_\theta, K)$ . If  $x^*$  is not a solution of the GNEP, then there exist  $\bar{\nu}$  and a strategy vector  $\tilde{x}^{\bar{\nu}} \in K_{\bar{\nu}}(x^{*- \bar{\nu}}) \subseteq L^2([0, T], \mathbb{R}^{n_{\bar{\nu}}})$  such that

$$\theta_{\bar{\nu}}(\tilde{x}^{\bar{\nu}}, x^{*- \bar{\nu}}) < \theta_{\bar{\nu}}(x^*).$$

Moreover,  $\theta_{\bar{\nu}}(\cdot, x^{*- \bar{\nu}})$  is continuous and semistrictly quasiconvex, and thus  $\tilde{x}^{\bar{\nu}} (\neq x^{*\bar{\nu}}) \in \text{int}(S_{\bar{\nu}}^<(x^*))$ . Consequently, for any  $\phi^{\bar{\nu}} \in N_{\theta_{\bar{\nu}}}(x^{*\bar{\nu}}) \setminus \{0\}$ , we have

$$\langle \langle \phi^{\bar{\nu}}, \tilde{x}^{\bar{\nu}} - x^{*\bar{\nu}} \rangle \rangle < 0.$$

Define  $\hat{y} : [0, T] \rightarrow \mathbb{R}^n$  such that

$$\hat{y}(t) = \begin{cases} \hat{y}^\nu(t) = x^{*\nu}(t), & \nu \neq \bar{\nu}, \\ \hat{y}^\nu(t) = \tilde{x}^{\bar{\nu}}(t), & \nu = \bar{\nu}. \end{cases}$$

Then,  $\hat{y} \in K \subseteq L^2([0, T], \mathbb{R}^n)$ , and for any  $\phi = (\phi^\nu)_{\nu=1}^p \in N_\theta(x^*)$ , we have

$$\langle \langle \phi, \hat{y} - x^* \rangle \rangle < 0,$$

which contradicts that  $x^* \in K$  is the solution of  $EVI(N_\theta, K)$ . The requisite result follows. □

### 3.3 Reformulation as Evolutionary Quasivariational Inequalities

A natural question is whether a time-dependent generalized Nash equilibrium is also a solution of the associated evolutionary variational inequality problem? Such a necessary condition is proved in this section but using the concept of *evolutionary quasivariational inequality*.

Given two set-valued maps  $F : L^2([0, T], \mathbb{R}^n) \rightrightarrows L^2([0, T], \mathbb{R}^n)$  and  $\mathcal{K} : L^2([0, T], \mathbb{R}^n) \rightrightarrows L^2([0, T], \mathbb{R}^n)$ , an evolutionary quasivariational inequality associated with these maps, denoted by  $EQVI(F, \mathcal{K})$ , consists of finding an  $x^* \in \mathcal{K}(x^*)$  such that there exists a  $f_{x^*} \in F(x^*)$  satisfying

$$\langle \langle f_{x^*}, y - x^* \rangle \rangle \geq 0, \quad \forall y \in \mathcal{K}(x^*), \tag{9}$$

or in other words,

$$\int_0^T \langle f_{x^*}(t), y(t) - x^*(t) \rangle dt \geq 0, \quad \forall y \in \mathcal{K}(x^*).$$

**Theorem 3.2** *Let  $n = \sum_{\nu=1}^p n_\nu$ . Let  $x^* \in K \subseteq L^2([0, T], \mathbb{R}^n)$  be a solution of the time-dependent GNEP (3), and the following assumption holds*

- (i)  $K$  is a nonempty and convex subset of  $L^2([0, T], \mathbb{R}^n)$ ;
- (ii) For any  $\nu$ , the decision function  $\theta_\nu$  are continuous and semistrictly quasiconvex with respect to the  $\nu$ -th variable;
- (iii) the map  $\mathcal{K} : L^2([0, T], \mathbb{R}^n) \rightrightarrows L^2([0, T], \mathbb{R}^n)$  is defined by the Rosen’s law, that is,  $\mathcal{K}(x^*) = \prod_{\nu=1}^p K_\nu(x^{*-\nu})$ , where  $K_\nu(x^{*-\nu}) = \{x^\nu : x = (x^\nu, x^{*-\nu}) \in K\}$ .

Then,  $x^*$  is a solution of the EQVI  $(N_\theta, \mathcal{K})$ .

*Proof* We need to show that there exists a  $f_{x^*} \in N_\theta(x^*)$  such that

$$\langle f_{x^*}, y - x^* \rangle \geq 0, \quad \forall y \in \mathcal{K}(x^*).$$

Now, if  $x^{*\nu} \in A_\nu(x^{*-\nu})$ ,  $\forall \nu$ , then  $0 \in F_\nu(x^*)$ ,  $\forall \nu$ . Hence, considering  $f_{x^*} = 0 \in N_\theta(x^*)$ . Then,  $x^*$  will be the solution of EQVI  $(N_\theta, \mathcal{X})$ .

Otherwise, since  $x^*$  is a solution of the GNEP (2), then for each  $\nu$ , we have

$$\theta_\nu(x^*) \leq \theta_\nu(x^\nu, x^{*-\nu}), \quad \forall x^\nu \in K_\nu(x^{*-\nu}). \tag{10}$$

Note that under assumption (ii), for each  $\nu$ , the subset  $S_\nu(x^*)$  is closed and convex with a nonempty interior (since  $x^{*\nu} \in A_\nu(x^{*-\nu})$  and  $\theta_\nu(\cdot, x^{-\nu})$  is continuous) where,

$$\text{int}(S_\nu(x^*)) = \{u^\nu \in L^2([0, T], \mathbb{R}^{n_\nu}) : \theta_\nu(u^\nu(t), x^{*-\nu}) < \theta_\nu(x^{*\nu}, x^{*-\nu})\}.$$

Hence, since  $x^*$  is a solution of the GNEP (2), (10) yields

$$\text{int}(S_\nu(x^*)) \cap K_\nu(x^{*-\nu}) = \emptyset. \tag{11}$$

And because of (i),  $K_\nu(x^{*-\nu}) = \{x^\nu : (x^\nu, x^{*-\nu}) \in K\}$  is a convex set. Therefore, by the separation theorem in the Hilbert space  $L^2([0, T], \mathbb{R}^{n_\nu})$  (see, e.g., [21, Theorem 5.67]), there exists a non zero  $h^\nu \in L^2([0, T], \mathbb{R}^{n_\nu})$  such that

$$\inf_{y^\nu \in K_\nu(x^{*-\nu})} \langle h^\nu, y^\nu \rangle \geq \sup_{x^\nu \in \text{int}(S_\nu(x^*))} \langle h^\nu, x^\nu \rangle,$$

which implies

$$\langle h^\nu, x^\nu - y^\nu \rangle \leq 0, \quad \forall y^\nu \in K_\nu(x^{*-\nu}), \forall x^\nu \in \text{int}(S_\nu(x^*)). \tag{12}$$

Note that the sets involved in (11) depend on  $x^*$ . So, the separating function  $h^\nu$  also depends on  $x^*$ .

Let  $y^\nu = x^{*\nu} \in K_\nu(x^{*-\nu})$ . We have,

$$\langle h^\nu, x^\nu - x^{*\nu} \rangle \leq 0, \quad \forall x^\nu \in \text{int}(S_\nu(x^*)).$$

Since  $S_\nu(x^*)$  is convex, the same inequality holds for any  $x^\nu$  in  $S_\nu(x^*)$ . Thus  $h^\nu$  is an element of  $N_{\theta_\nu}(x^{*\nu})$ , and consequently it is an element of  $F_\nu(x^*)$ .

Again from (12), for any  $v$ , we have

$$\langle \langle h^v, y^v - x^v \rangle \rangle \geq 0, \quad \forall y^v \in K_v(x^{*-v}), \quad \forall x^v \in \text{int}(S_v(x^*)).$$

By continuity of inner product and the fact that  $x^{*v} \in S_v(x^*)$ , for any  $v$ , we have

$$\langle \langle h^v, y^v - x^{*v} \rangle \rangle \geq 0, \quad \forall y^v \in K_v(x^{*-v}).$$

Let us now construct  $h = (h^v)_{v=1}^p \in N_\theta(x^*)$ . Then, for any  $y \in \mathcal{K}(x^*)$ , we have

$$\langle \langle h, y - x^* \rangle \rangle = \sum_{v=1}^p \langle \langle h^v, y^v - x^{*v} \rangle \rangle \geq 0,$$

yielding  $x^*$  is a solution of  $EQVI(N_\theta, \mathcal{K})$ .

## 4 Existence of Time-Dependent Equilibrium

Although the decision functions  $\theta_v$  in our electricity market example is differentiable, it is not always the case in general. We can find time-dependent generalized Nash equilibrium where the decision functions are not differentiable. For example, this may happen when a single producer owns several generation units with different marginal costs thus generating a discontinuous overall marginal cost function; this case has been presented by Anderson and Xu in [22]. One can see some other examples on non differentiable decision functions (e.g., the thesis of Martinez [23]) for further references.

Note also that there is no reason for the objective functions of the players to be convex or pseudoconvex. As shown in [10], the difference of the quadratic convex functions can eventually be proved to be (semistrictly) quasiconvex but under quite restrictive assumptions.

Our aim in this section is to prove the existence of a time-dependent generalized Nash equilibrium for problem (3) under semistrictly quasiconvex hypothesis on the decision functions  $\theta_v$  without assuming any differentiability of the cost functions. This will be obtained as a consequence of an existence result for quasimonotone evolutionary variational inequality (forthcoming Theorem 4.1) combined with the reformulation statement of Sect. 3.2 (Proposition 3.1).

Let us first recall some definitions for set-valued map. A set-valued map  $F : L^2([0, T], \mathbb{R}^n) \rightrightarrows L^2([0, T], \mathbb{R}^n)$  is said to be:

- *quasimonotone* if for any  $f_x \in F(x)$

$$\langle \langle f_x, y - x \rangle \rangle > 0 \Rightarrow \langle \langle f_y, y - x \rangle \rangle \geq 0, \quad \forall f_y \in F(y).$$

- *upper semicontinuous* at the point  $y \in K \subset L^2([0, T], \mathbb{R}^n)$  if, for any open neighborhood  $W$  of  $F(y)$ , there exists a neighborhood  $U$  of  $y$  such that, for all  $x \in U$ , we have  $F(x)$  is a subset of  $W$ .

- *locally upper semicontinuous* on a subset  $K$  of  $L^2([0, T], \mathbb{R}^n)$  if, for any  $u \in K$ , there exists an open neighborhood  $V_u$  of  $u$  and a set-valued map  $S_u : V_u \rightrightarrows L^2([0, T], \mathbb{R}^n)$  which is upper semicontinuous on  $V_u$  with nonempty convex and  $w^*$ -compact values and satisfying  $S_u(v) \subset F(v)$ , for all  $v \in V_u$ .

**Theorem 4.1** *Let  $K \subseteq L^2([0, T], \mathbb{R}^n)$  be a nonempty compact and convex set defined by (5). Let the set-valued map  $F : L^2([0, T], \mathbb{R}^n) \rightrightarrows L^2([0, T], \mathbb{R}^n)$  be quasi-monotone, convex-valued, and locally upper semicontinuous on  $K$ . Then,  $EVI(F, K)$  has a solution.*

*Proof* The following proof is inspired from the one presented in Theorem 2.1 in [24]. Define  $\phi : K \times K \rightarrow \mathbb{R}$  such that

$$\phi(x, y) = \inf_{f \in F(x)} \langle \langle f, x - y \rangle \rangle, \quad \forall x, y \in K.$$

Observe  $\phi(x, \cdot)$  is a continuous and concave function relative to  $y$ . Moreover, for all  $x \in K$ , the subset  $\{y \in K : \phi(x, y) \geq 0\}$  is compact since  $K$  is compact.

Case 1: If  $\phi$  is a KKM-application on  $K$  (that is, for all  $x_1, x_2, \dots, x_n \in K$  and for all  $x \in \text{conv}\{x_1, x_2, \dots, x_n\}$ , there exists  $i \in \{1, 2, \dots, n\}$  such that  $\phi(x_i, x) \geq 0$ ) then, using a KKM Lemma, there exists  $x^* \in K$  such that

$$\phi(x, x^*) \geq 0, \quad \forall x \in K.$$

Therefore, for all  $f \in F(x)$ , we have

$$\langle \langle f, x - x^* \rangle \rangle \geq 0, \quad \forall x \in K.$$

By Lemma 3.1, we have

$$\langle f(t), x(t) - x^*(t) \rangle \geq 0, \quad \forall x \in K, \quad \text{a.e. in } [0, T]. \tag{13}$$

Case 2: Suppose  $\phi$  is not a KKM-application. Then, there exist a family  $x_1, x_2, \dots, x_n \in K$  and an element  $x^*$  of  $\text{conv}\{x_1, x_2, \dots, x_n\}$  such that

$$\phi(x_i, x^*) < 0, \quad \forall i = 1, \dots, n,$$

that is, for all  $i = 1, \dots, n$ , there exists  $f_{x_i} \in F(x_i)$ , such that

$$\langle \langle f_{x_i}, x^* - x_i \rangle \rangle > 0.$$

By Lemma 3.1, we have  $\langle f_{x_i}(t), x^*(t) - x_i(t) \rangle > 0$  a.e. in  $[0, T]$ . Hence, for some  $\rho > 0$  and for all  $z \in B(x^*, \rho) \cap K$ , we have

$$\langle f_{x_i}(t), z(t) - x_i(t) \rangle > 0 \quad \text{a.e. in } [0, T].$$

By quasimonotonicity of  $F(x_i)$ , for all  $z \in B(x^*, \rho) \cap K$ , and for all  $f_z \in F(z)$ , we have

$$\langle f_z(t), z(t) - x_i(t) \rangle \geq 0, \quad \text{a.e. in } [0, T], \quad \forall i = 1, \dots, n.$$

Since  $x^* \in \text{conv}\{x_1, x_2, \dots, x_n\}$ , therefore

$$\langle f_z(t), z(t) - x^*(t) \rangle \geq 0, \quad \text{a.e. in } [0, T]. \tag{14}$$

Combining (13) and (14), for all  $z \in B(x^*, \rho) \cap K$  and for all  $f_z \in F(z)$ ,

$$\langle f_z(t), z(t) - x^*(t) \rangle \geq 0, \quad \text{a.e. in } [0, T].$$

Let  $x \in K$  be such that  $x \neq x^*$ . Then, there exists a  $\lambda_0 > 0$  such that  $z_\lambda = x^* + \lambda(x - x^*) \in B(x^*, \rho) \cap K$ , for all  $\lambda \in (0, \lambda_0)$ , so that, for all  $f_{z_\lambda} \in F(z_\lambda)$ , we have

$$\langle f_{z_\lambda}(t), x(t) - x^*(t) \rangle \geq 0, \quad \forall \lambda \in (0, \lambda_0), \text{ a.e. in } [0, T]. \tag{15}$$

Let us now define an open (convex) set

$$W_x = \{g \in L^2([0, T], \mathbb{R}^n) : \langle g(t), x(t) - x^*(t) \rangle < 0, \text{ a.e. in } [0, T]\}.$$

If  $F(x^*) \subset W_x$ , then by locally upper semicontinuity of  $F$ , for  $x^* \in K$ , there exists an open neighborhood  $V_{x^*}$  of  $x^*$  and a map  $S_{x^*} : V_{x^*} \rightrightarrows L^2([0, T], \mathbb{R}^n)$ , which is upper semicontinuous on  $V_{x^*}$ , such that  $S_{x^*}(v) \subset W_x$  for every  $v \in V_{x^*}$ . Consequently, for any  $\lambda$  sufficiently close to 0, we have  $S_{x^*}(z_\lambda) \subset W_x$ , which contradicts (15). Hence, for any  $x \in K$ , there exist  $f_{x^*} \in S_{x^*}(x^*)$  and an open interval  $I \subset [0, T]$  with Lebesgue measure  $\mu(I) > 0$  such that

$$\langle f_{x^*}(t), x(t) - x^*(t) \rangle \geq 0, \quad \forall t \in I.$$

Therefore, for any  $x \in K$ ,

$$\sup_{f_{x^*} \in S_{x^*}(x^*)} \langle f_{x^*}(t), x(t) - x^*(t) \rangle \geq 0, \quad \forall t \in I,$$

which implies

$$\min_{x \in K} \sup_{f_{x^*} \in S_{x^*}(x^*)} \langle f_{x^*}(t), x(t) - x^*(t) \rangle \geq 0, \quad \forall t \in I.$$

Taking into account that  $K$  is compact and convex subset of  $L^2([0, T], \mathbb{R}^n)$  and  $S_{x^*}(x^*)$  is convex set, by the classical Sion minimax theorem

$$\sup_{f_{x^*} \in S_{x^*}(x^*)} \min_{x \in K} \langle f_{x^*}(t), x(t) - x^*(t) \rangle \geq 0, \quad \forall t \in I.$$

Since,  $S_{x^*}(x^*)$  is compact, there exists  $f_{x^*} \in F(x^*)$  (independent of the element  $x \in K$ ) such that

$$\min_{x \in K} \langle f_{x^*}(t), x(t) - x^*(t) \rangle \geq 0, \quad \forall t \in I.$$

This yields  $\langle f_{x^*}(t), x(t) - x^*(t) \rangle \geq 0, \forall x \in K, t \in I$ . Define a function  $\bar{x} : [0, T] \rightarrow \mathbb{R}^n$  as

$$\bar{x}(t) = \begin{cases} x(t), & t \in [0, T] \setminus I, \\ x^*(t), & t \in I. \end{cases}$$

Note that  $\bar{x} \in K \subseteq L^2([0, T], \mathbb{R}^n)$ . Then, for any  $f_{\bar{x}} \in F(\bar{x})$ , we have

$$\langle \langle f_{\bar{x}}, x - \bar{x} \rangle \rangle = \int_{t \in I} \langle f_{x^*}(t), x(t) - x^*(t) \rangle dt \geq 0, \quad \forall x \in K.$$

Hence  $\bar{x} \in K \subseteq L^2([0, T], \mathbb{R}^n)$  is solution of  $EVI(F, K)$ . □

Let us now end this section with an existence result for our original problem, that is, the time-dependent generalized Nash equilibrium problem (3). This last existence result, the forthcoming Corollary 4.1, is actually a consequence of the combination of the above Theorem 4.1 with the following local continuity property of the normal operator map  $N_\theta$  defined in (7) for any quasiconvex function and in (8) for a collection of decision functions  $(\theta_1, \dots, \theta_p)$  of a time-dependent GNEP.

- Proposition 4.1** (i) *If  $\psi : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$  is a continuous semistrictly quasiconvex function and  $K$  is a set of  $L^2([0, T], \mathbb{R}^n)$  such that  $K \cap \arg \min_{L^2([0, T], \mathbb{R}^n)} \psi = \emptyset$ , then  $N_\psi$  is locally upper semicontinuous on  $K$ .*
- (ii) *Assume that all the decision functions  $\theta_v : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$  are continuous and semistrictly quasiconvex in the  $v$ -variable  $x^v$ , that  $K$  is a nonempty convex and compact subset of  $L^2([0, T], \mathbb{R}^n)$  and that the constraint sets  $K_v$  are defined by (1). If*

$$\forall x \in K, \forall v, \quad K_v(x^{-v}) \cap \arg \min_{L^2([0, T], \mathbb{R}^n)} \theta_v(\cdot, x^{-v}) = \emptyset.$$

*Then, the normal operator  $N_\theta$  is locally upper semicontinuous on  $K$*

*Remark 4.1* Even if it is developed for static (time-independent) models of electricity market, it is interesting to notice that the assumption (ii) (semistrict quasiconvexity of the objective function and convexity and compactness of the constraint set) in the above proposition has been proved to be verified for a spot market model developed in [10] and the adjustment market treated in [25, 26]. Moreover, the fact that the constraint map  $K$  defined by (1) comes, in electricity market models, from the fact that the quantity vector  $q = (q^1, \dots, q^p)$  is solution of the ISO’ problem shared by all producers.

- Proof* (i) Since  $\psi$  is continuous and  $K \cap \arg \min_X \psi = \emptyset$  then, for every  $x \in K$  there exists  $y \in X$  such that  $\psi(y) < \psi(x)$  and  $\text{int}(S_{\psi(y)}) \neq \emptyset$ . Again, since  $K \cap \arg \min_X \psi = \emptyset$  and  $\psi$  is semistrictly quasiconvex,  $N_{\psi}$  and its adjusted operator  $N_{\psi}^a$  coincide on  $K$ . Therefore, according to Proposition 3.5 of [17],  $N_{\psi}$  is norm-to-weak cone upper semicontinuous on  $K$  (see precise definition in [17]). Now combining with Proposition 2.2 of [17],  $N_{\psi}$  is locally upper semicontinuous on  $K$ , since this map has nonempty cone convex values. Actually from the proof of [17, Prop.3.5] one can choose the suboperator map  $S_u$  of  $N_{\psi}$  such that, for any  $u$  and any  $v$  in a neighborhood of  $u$ ,  $S_u(v)$  is a compact base of the cone  $N_{\psi}(v)$ .
- (ii) The locally upper semicontinuity of the set-valued map  $N_{\theta}$  on the space  $L^2([0, T], \mathbb{R}^n) = \prod_{i=1}^p L^2([0, T], \mathbb{R}^{n_i})$  is a consequence of (i) and the calculus properties of upper semicontinuous maps, see, e.g., [27]. □

Now combining Theorem 4.1, Proposition 3.1 and Proposition 4.1, we have the following existence result for time-dependent generalized Nash equilibrium problem (3).

**Corollary 4.1** *Assume that each decision functions  $\theta_v : L^2([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}$  is continuous and semistrictly quasiconvex in the  $v$ -variable and that  $K$  is a nonempty convex and compact subset of  $L^2([0, T], \mathbb{R}^n)$  defined by (5) and that the constraint sets  $K_v$  are defined by (1). If*

$$\forall x \in K, \forall v, \quad K_v(x^{-v}) \cap \arg \min_{L^2([0, T], \mathbb{R}^n)} \theta_v(\cdot, x^{-v}) = \emptyset.$$

*Then, the time-dependent generalized Nash equilibrium problem (3) admits at least a solution.*

Let us observe that in [6] the existence result for time-dependent GNEP has been obtained under the assumption that the cost functions  $\theta_v$  are pseudoconvex and continuously differentiable.

## 5 Conclusions

In this work, we defined, in  $L^2([0, T], \mathbb{R}^n)$ , a concept of the time-dependent generalized Nash equilibrium problem. Using normal operator techniques, a reformulation in terms of an evolutionary variational inequality is proposed and an existence result for the time-dependent GNEP is proved with quasiconvex cost functions.

Our proofs use the properties of the Hilbert space  $L^2([0, T], \mathbb{R}^n)$  in which the decision variables are chosen. This space allows us to consider step cost functions as well as piecewise linear or quadratic cost functions thus covering the classical cases for the bid functions in electricity markets. But one can also define a similar concept as in (3) for time-dependent generalized Nash equilibrium with decision variables in  $L^\infty([0, T], \mathbb{R}^n)$ . Nevertheless, the proofs should have to be reconsidered since there is no simple separation theorem in such a space, an important tool for our reformulation.

**Acknowledgments** The second author acknowledges the Council of Scientific Research (CSIR), India, and IIT-ParisTech Scholarship-2012 by ParisTech Foundation, France, for providing financial assistance for this research. The research was conducted while the second author visited University of Perpignan Via Domitia, France. The author also thanks the University of Perpignan Via Domitia, France.

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