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Advertising and Quality-Dependent Word-of-Mouth in a Contagion Sales Model

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Abstract In the literature on marketing models, the assumption of mixed word-of-mouth has been limited to the Bass diffusion model. Yet explicit leveraging of the originating factors of such assumption is lacking. Apart from that example, mixed word-of-mouth has been disregarded in contagion sales models. This paper bridges the gap by suggesting a sales model, where both positive and negative word-of-mouth affect the attraction rate of new customers, along with advertising. The difference between positive and negative word-of-mouth is based on the distinction between satisfied and dissatisfied current customers, which is supposed to depend on conformance quality. A primary issue in this paper is to determine how a firm should determine the optimal intertemporal trade-off between investing in advertising-dependent word-of-mouth and quality-dependent word-of-mouth. To address this issue, a contagion sales model is suggested where mixed autonomous word-of-mouth alone can lead to either commercial success or failure of a given brand.

Keywords Sales · Word-of-mouth · Advertising effort · Conformance quality

Mathematics Subject Classification 49N90 · 90B60 · 49N10

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1 Introduction

Mixed word-of-mouth (WOM) for products and services, that is, social transmissions involving both positive and negative evaluations of products and services, has become prevalent in online social networks. Goldenberg et al. [1] show that positive WOM is based on satisfaction of current customers and negative WOM on their dissatisfaction (see also [2]). Anderson [3] suggests that dissatisfied customers spread more WOM than satisfied ones do. Furthermore, negative WOM is more influential than positive WOM for brands with which potential customers are not familiar or have no prior commitment (e.g., [4–6]). The online amplification of these asymmetric patterns has emphasized the challenge for firms to build brands both by capitalizing on positive WOM and counter negative WOM.

The assumption of mixed WOM in the literature on marketing models has been limited to the Bass diffusion model [7,8]. Yet explicit leveraging of the originating factors of such assumption is lacking. Apart from that example, mixed WOM has been disregarded in contagion sales models [9,10]. Rather, such models widely focus on advertising effort and uniformly positive WOM of current customers, i.e., advertising-dependent WOM, and their joint influence on the attraction rate of new customers of a given product. The omission of negative evaluations by current customers and, more importantly, of the originating factors of such negative evaluations may lead to prescription of improper communication policy and therefore poor brand-building strategy.

This paper bridges the gap by suggesting a sales model where both positive and negative WOM affect the attraction rate of new customers, along with advertising. As in [1], the difference between positive and negative WOM is based on the distinction between satisfied and dissatisfied current customers. Considering the case of experience, products, i.e., products whose characteristics can be ascertained only upon consumption [11], the distinction between satisfied and dissatisfied current customers is supposed to depend on "conformance quality." This dimension of quality refers to the extent to which the product conforms to a given design quality standard [12]. In this regard, satisfied current customers are those who experienced a defect-free item, and dissatisfied current customers are those who bought a defective item.

As a marketing instrument dedicated to attracting new customers and, ultimately, brand image building, advertising-dependent WOM may interact with channels of social influence that are not under the firm's direct control. Such channels, termed autonomous WOM, have recently gained much attention in the marketing literature [13–19], notably because of the development of online product ratings.

While advertising-dependent WOM involves satisfied current customers only, autonomous WOM includes both negative WOM of dissatisfied current customers and positive WOM of satisfied current customers. Both influences can significantly influence the effectiveness of advertising-dependent WOM and thus the attraction of new customers. At one extreme, negative autonomous WOM can have such adversarial effects on attraction of new customers that most of the relative effectiveness of advertising-dependent WOM is lost. In this case, the main utility of advertising-dependent WOM is to counter the brand image destruction process implied by negative autonomous WOM of current customers. This case is consistent with the "death-wish

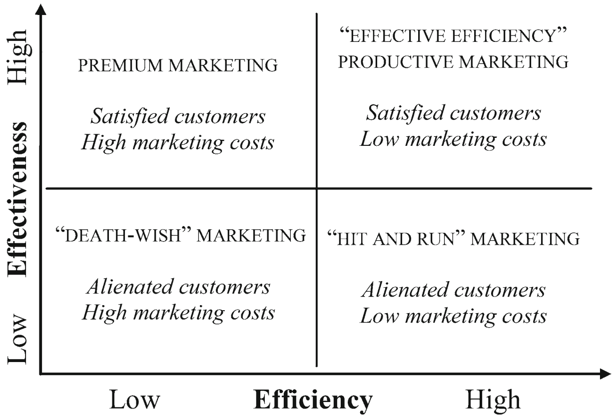

Fig. 1 Marketing efficiency and effectiveness [20]

marketing” case in the typology of Sheth and Sisodia [20], typified by alienated customers and high marketing costs (Fig. 1). At the other extreme, positive autonomous WOM can intensify attraction of new customers so strongly that it may singly enable the building of a brand image, and even exempt the business from having to expend an advertising effort. This case is an idyllic representation of the “productive marketing” case of Sheth and Sisodia [20], where effective efficiency, that is, satisfied customers and low marketing costs, prevails.

Therefore, the trade-off between advertising-dependent and autonomous WOM, which is a contemporary extension of the traditional trade-off between advertising expenditures and customer satisfaction, is not straightforward. Clearly, this trade-off cannot be broken unilaterally by non-trivial marketing improvements in cost–benefit performance, but needs the design of a marketing–operations interface that delivers both lower costs for brand image building and greater customer satisfaction and loyalty. Given the growing interest in marketing productivity [21] and the related need for the marketing function to deliver effective efficiency [20], as well as the traditional emphasis on operations–marketing interface [22,23], the trade-off between advertising-dependent and autonomous WOM deserves thorough examination.

Because it contributes to marketing productivity, conformance quality is, along with advertising, an essential lever that should be included in the economics of customer acquisition, as well as retention. Specifically, conformance quality improvements seek to reduce the number of defects, which has three important effects:

- A direct effect of greater loyalty of current customers,
- An indirect effect of enhanced positive WOM of current customers toward potential customers both through advertising-dependent and autonomous WOM,
- An indirect effect of lower negative WOM from current customers to potential customers through autonomous WOM.

While the direct effect on the number of current customers has been investigated [24–28], the indirect effects are unexplored. A primary issue in this paper is therefore to determine how a firm should determine the optimal intertemporal trade-off between investing in advertising-dependent WOM and quality-dependent WOM. We address the following research questions:

- Should advertising effort be used to counter negative WOM due to insufficient conformance quality, or to support a high level of positive WOM, resulting from a high conformance quality level?
- Does an optimal path leading to effective efficiency always exist? If not, should effectiveness and efficiency be sought, and under which conditions? Finally, are there circumstances under which both dimensions should be sacrificed?

To address these questions, a stylized optimal control model is formulated, where the interaction between WOM and advertising in a sales process can be affected by conformance quality. Our model is based on the dynamic sales process originated by Gould [29], where consumer behavior is modeled as a trial-repeat purchase process and the monopolist firm is a price taker. In contrast to the Gould model, we assume that both the abandon rate of dissatisfied customers and their negative WOM on potential customers can be reduced by conformance quality.

Our model differs from dynamic models of perceived quality as in [30,31], wherein WOM is inexistent. It also departs from dynamic models of reputation as in [32], where quality is evaluated by customers as a constant hypothetical price. The current research adds to the literature by suggesting a contagion sales model, where mixed autonomous WOM alone can lead to either commercial success or failure of a given brand.

The paper is organized as follows. In next section, we formulate an optimal control model where a monopolist firm has to determine an optimal strategy in terms of advertising effort and conformance quality. In Sect. 3, we characterize the general solution of the problem. Section 4 investigates the model with numerical means, and Sect. 5 concludes the paper.

2 Model

We assume a monopolistic firm that sells a particular experience product to satisfy market demand. We extend the contagion sales model of Gould [29], introducing two distinguishing features. First, mixed WOM can develop autonomously, such that attraction can vary without advertising effort. Second, both autonomous and advertising-dependent WOMs depend on the firm's ability to reduce the rate of defective items in order to improve the satisfaction of current customers.

The sales process represented is a typical combination of two kinds of influences: an attraction rate of new customers and an abandon rate of current customers. All things being equal, high-quality products favor consumer loyalty more than low-quality products do [33]. Conversely, the abandon rate of current customers can be supposed to depend on the extent to which current customers are dissatisfied with the product's quality. In our framework, dissatisfaction of current customers arises from the consumption experience of defective items. The rate of defective items on the market reflects the level of conformance quality of a product, which refers to the extent to which a product conforms to a given quality standard reflecting the customer's expectations [12]. Conformance quality differs from another key determinant of product quality, namely design quality, which is defined as the set of product attributes or features that enhance the match with the customer's needs [12].[1] In this regard, design quality determines the number of potential customers, whose needs are compatible with a product's attributes.

In our model, the product's conformance quality splits the number of current customers into two categories: those who are fully satisfied because they experienced a non-defective item and those who are dissatisfied because they experienced a defective item. The abandon rate of current customers is therefore expressed as a decreasing function of conformance quality via current customer's dissatisfaction [24–28].

In contrast, design quality is supposed to be initially fixed, which implies a constant number of potential customers. However, the obsolescence of the product's design quality over time leads a fraction of the current customers who are satisfied with the product's conformance quality to forget about the product.

In contrast, the attraction rate of new customers depends on the number of potential customers who are affected by positive and negative WOM, conveyed by satisfied and dissatisfied current customers, respectively [1]. The number of satisfied and dissatisfied current customers and therefore their associated positive and negative WOM are both determined by the product's conformance quality. We assume that positive and negative WOM are conveyed asymmetrically [3–6]. For the firm, to avoid a self-detrimental communication policy or brand image destruction, negative WOM of current customers toward potential customers is supposed to be conveyed only by autonomous channels, that is, channels of social influence that are not under the firm's direct control. In contrast, positive WOM is conveyed to potential customers both through autonomous channels and through the firm's advertising effort.

Overall, while advertising effort plays an important role in connecting satisfied current customers with potential customers to attract new customers [34–36], conformance quality improves the loyalty of current customers and contributes to brand building both by enhancing positive WOM and reducing negative WOM. Conversely, while advertising-dependent WOM can serve to counter adversarial effects of negative WOM linked to imperfect conformance quality, autonomous WOM alone can, depending on the conformance quality level, bring about a product's commercial success or failure. Accounting for all these features, an optimal trade-off between quality improvement efforts and advertising efforts can not only prevent "death-wish marketing," but also contribute to more "productive marketing."

To account for these assumptions, the rate of change of instantaneous sales is formulated as follows:

$$\dot{S}(t) = \left\{ [(\kappa + u(t))\, Q(t) - \beta\,(1 - Q(t))] \left(1 - \frac{S(t)}{N}\right) - [1 - (1-\delta)\, Q(t)] \right\} S(t),$$
$$S(0) = S_0 > 0, \tag{1}$$

where $S(t) \geq 0$ denotes instantaneous sales of a particular product, $u(t) \geq 0$ advertising effort, $Q(t) \in]0, 1]$ conformance quality (i.e., the proportion of non-defective items produced), $N > 0$ the constant potential market, κ, β and δ positive constants,

[1] Design quality is supposed to increase the desirability of the product by enhancing the product's functional performance.

and $0 < Q_0 \leq 1$ the initial conformance quality level. The coefficients κ and β denote the respective coefficients for positive and negative autonomous WOM effectiveness. On the other hand, $0 < \delta < 1$ is the forgetting rate, due to the obsolescence of the product's design quality. Given that $0 < (1-\delta)Q(t) < 1$, the overall number of current customers who decide to abandon the product at each period of time is strictly lower than $S(t)$.

Given that $S(t)/N$ is the sales rate in (1), the difference $[1 - S(t)/N] \geq 0$ is the untapped market potential. This potential is influenced by conformance quality in two ways, that is:

- Positively, by current customers having experienced a defect-free item, who talk about the product positively to potential customers either autonomously or through the firm's advertising effort. The influence of positive autonomous and advertising-dependent WOM on the untapped market potential is, respectively, $\kappa QS\,[1 - S/N] \geq 0$ and $uQS\,[1 - S/N] \geq 0$, where QS represents the number of satisfied current customers.
- Negatively, by dissatisfied current customers who experienced a defective item, and who talk about the product negatively to potential customers. The influence of this negative autonomous WOM on the untapped market potential is denoted by $\beta\,(1-Q)\,S\,[1 - S/N] \geq 0$, where $(1-Q)S$ is the number of dissatisfied current customers.

Hence, the first term on the right-hand side of (1) means that the attraction of new customers is the difference between positively influenced potential customers and negatively influenced potential customers. This difference, which represents the net attraction rate, should always be nonnegative with:

$$(\kappa + u(t))Q(t) - \beta(1 - Q(t)) \geq 0, \ \forall t \geq 0.$$

Therefore, a necessary and sufficient condition for a feasible value of the net attraction rate of new customers requires a sufficiently large advertising effort, that is:

$$u(t) \geq \frac{\beta(1 - Q(t))}{Q(t)} - \kappa, \ \forall t \geq 0.$$

The second term on the right-hand side of (1) models the abandon rate, which is composed of all dissatisfied customers (i.e., all customers who experienced a defective item) who stop buying the product, denoted by $(1 - Q(t))S(t)$, along with a constant fraction of satisfied customers who forget about the product, given by $\delta Q(t)S(t)$.

To reduce the abandon rate and negative WOM and enhance positive WOM, we assume that the firm can increase the proportion of non-defective items by investing in efforts to improve its conformance quality. These quality improvement efforts imply an induced learning effect [37]. Therefore, conformance quality is assumed to evolve over time, according to the following equation:

$$\dot{Q}(t) = w(t)\,(1 - Q(t))\,, \quad Q(0) = Q_0 > 0, \tag{2}$$

where $w(t) \geq 0$ is the firm's effort to reduce the rate of defective items $(1 - Q(t))$. In (2), quality is increased by the firm's improvement effort, in proportion to the rate of defectives. There are decreasing marginal returns to improvement efforts. Quality is non-decreasing over time, and in the absence of improvement effort $(w(t) = 0)$, it remains constant. The specification in (2) can be found, e.g., in [38,39].

From (2), it is noteworthy that a sufficient condition that satisfies the condition $u(t) \geq \frac{\beta(1-Q(t))}{Q(t)} - \kappa, \ \forall t \geq 0$, which ensures a feasible net attraction rate of new customers, is a sufficiently large initial conformance quality, i.e., $Q_0 \geq \frac{\beta}{\kappa+\beta}$.

We now turn to defining the firm's objective criterion.

Due to infinitely high adjustment costs, we assume that the sale price is fixed from the outset. In addition, the unit production cost is supposed to be constant. Therefore, at any time, the sales revenue is given by $\pi S(t)$, where $\pi > 0$ is the firm's unit profit margin. Because customers who bought defective items can generally expect to be refunded by the firm [40], we assume that defective items are refunded for their entire value without delay ([27,28,39]). Therefore, the consequence of the fact that customers purchase defective items is expressed in terms of loss in instantaneous revenue, equal to $\pi(1 - Q(t))S(t)$. As a result, the firm's net revenue per unit sold is only a fraction of the profit margin, i.e., $Q(t)\pi$. Clearly, conformance quality improvement efforts should increase the proportion of non-defective items and thus reduce the payment of compensation to customers for quality failures.

Both advertising and quality improvement efforts give rise to an instantaneous cost. However, they differ in that quality improvement efforts correspond to an investment, while advertising efforts do not. Due to decreasing returns to scale, the cost of making an advertising effort, $E(u(t))$, is given by (e.g., [41]):

$$E\left(u(t)\right) := eu(t)^2/2,$$

where $e > 0$ is a cost coefficient of advertising effort. We also assume convex conformance quality improvements costs, $F(w(t))$, which are described by the following function:

$$F(w(t)) := f_0 w(t) + f_1 w(t)^2/2,$$

where $f_0, f_1 > 0$ are marginal cost coefficients of quality improvement effort cost. This function is more general than a purely quadratic cost function because its linear component represents an acquisition cost, which is typical for investment costs. The reason is that the implementation of a conformance quality improvement program requires the coordination of several functions in a company and the involvement of well-trained human resources. It is thus important to invest in interfunctional coordination, on the one hand, and the development of human abilities, on the other hand. Such expenses correspond to acquisition costs.

We consider an infinite planning horizon and assume a positive discount rate, $r > 0$. Therefore, the objective criterion is written as:

$$\underset{u(.),w(.)}{\text{Max}} \ \Pi = \int_0^\infty e^{-rt} \left[Q(t)\pi S(t) - \frac{e}{2}u(t)^2 - w(t)\left(f_0 + \frac{f_1 w(t)}{2}\right)\right] \mathrm{d}t. \quad (3)$$

As will be shown in the subsequent analysis, the control constraint $u(t) \geq \frac{\beta(1-Q(t))}{Q(t)} - \kappa,\ \forall t \geq 0$, that allows for a feasible value of the net attraction rate of new customers is never binding, as long as $1 \geq Q_0 \geq \beta/(\kappa+\beta)$, and we therefore do not explicitly derive the related optimality conditions.

If the firm does not invest in conformance quality, provided $Q(t) = Q_0 \in [\beta/(\kappa+\beta), 1],\ \forall\, t > 0$, we get the Gould model [29] extended to uniformly positive autonomous WOM:

$$\underset{u(.)}{\text{Max}} \int_0^\infty e^{-rt}\left[\bar{\pi} S(t) - \frac{e}{2}u(t)^2\right]\mathrm{d}t, \tag{4}$$

such that:

$$\begin{aligned} \dot{S}(t) &= \left[(\phi + \chi u(t))\left(1 - \frac{S(t)}{N}\right) - \varphi\right] S(t), \\ S(0) &= S_0 > 0, \end{aligned} \tag{5}$$

where $\bar{\pi} = Q_0\pi > 0, \phi = [(\kappa+\beta)\,Q_0 - \beta] \geq 0, \chi = \alpha Q_0 > 0$, and $\varphi = 1 - (1-\delta)\,Q_0 < 1$.

3 Analysis

The current-value Hamiltonian is written as:

$$\begin{aligned} H\,(S, Q, u, w, \lambda, \mu) := Q\pi S - \frac{e}{2}u^2 - w\left(f_0 + \frac{f_1 w}{2}\right) \\ + \lambda\left\{[(\kappa+u)\,Q - \beta(1-Q)]\left(1 - \frac{S}{N}\right)\right. \\ \left. -1 + (1-\delta)\,Q\right\} S + \mu w(1-Q), \end{aligned} \tag{6}$$

and the extended Hamiltonian:

$$L\,(S, Q, u, w, \lambda, \mu, \rho_1, \rho_2) := H\,(S, Q, u, w, \lambda, \mu) + \rho_1 u + \rho_2 w, \tag{7}$$

where $\lambda \equiv \lambda(t)$ and $\mu \equiv \mu(t)$ are the costate variables associated with sales and conformance quality, and ρ_1, ρ_2 are Lagrange multipliers.

Then, an optimal solution (S^*, Q^*, u^*, w^*) satisfies the Hamiltonian maximizing condition:

$$\left(u^*, w^*\right) = \underset{u\geq 0, w\geq 0}{\arg\max}\, H\left(S^*, Q^*, u, w, \lambda, \mu\right), \tag{8}$$

and the costates λ, μ satisfy the adjoint equations:

$$\dot{\lambda} = r\lambda - H_S\left(S^*, Q^*, u^*, w^*, \lambda, \mu\right), \tag{9}$$
$$\dot{\mu} = r\mu - H_Q\left(S^*, Q^*, u^*, w^*, \lambda, \mu\right). \tag{10}$$

The Hessian of $H(S, Q, u, w, \lambda, \mu)$ with respect to (u, w) is negative definite everywhere, so that the Hamiltonian is concave with respect to the control vector (u, w), and this guarantees that any stationary point (u°, w°) of the Hamiltonian is a (local) maximum of the Hamiltonian.

Condition (8) allows us to distinguish four different cases. For each of these cases, we find explicit functions $(u^\circ, w^\circ, \rho_1, \rho_2)$. The costate variables are, respectively, given by:

$$\dot{\lambda} = \lambda \left\{ r - \left[\left(\kappa + u^\circ \right) Q - \beta (1 - Q) \right] \left(1 - \frac{2S}{N} \right) + 1 - (1 - \delta) Q \right\} - Q\pi, \quad (11)$$

$$\dot{\mu} = \mu \left(r + w^\circ \right) - \left\{ \pi + \lambda \left\{ [(\kappa + u) + \beta] \left(1 - \frac{S}{N} \right) + 1 - \delta \right\} \right\} S. \quad (12)$$

In the subsequent paragraphs, we derive the functions $(u^\circ, w^\circ, \rho_1, \rho_2)$.

3.1 Controlled Sales Process

In this case, (8) yields:

$$\left(u^*, w^*, \rho_1, \rho_2 \right) = \left(u^\circ \left(S^*, Q^*, \lambda \right), w^\circ \left(Q^*, \mu \right), 0, 0 \right),$$

with:

$$\left(u^\circ, w^\circ \right)^t := \begin{pmatrix} \frac{\lambda Q S}{e} \left(1 - \frac{S}{N} \right) \\ \frac{\mu (1 - Q) - f_0}{f_1} \end{pmatrix},$$

being the solution of $H_u(S, Q, u^\circ, w^\circ, \lambda, \mu) = 0$ and $H_w(S, Q, u^\circ, w^\circ, \lambda, \mu) = 0$, respectively. Plugging the expressions of u and w in (1)–(2) and (7)–(8), respectively, gives the canonical system:

$$\dot{S} = \left\{ \left[(1 + \beta) Q + \frac{\kappa \lambda Q^2 S}{e} \left(1 - \frac{S}{N} \right) - \beta \right] \left(1 - \frac{S}{N} \right) - 1 + (1 - \delta) Q \right\} S, \quad (13)$$

$$\dot{Q} = [\mu (1 - Q) - f_0] (1 - Q)/f_1, \quad (14)$$

$$\dot{\lambda} = \lambda \left[r + 1 - (1 - \delta) Q - \left\{ [(1 + \beta) Q - \beta] + \frac{\kappa \lambda Q^2 S}{e} \left(1 - \frac{S}{N} \right) \right\} \left(1 - \frac{S}{N} \right) \right] - Q\pi, \quad (15)$$

$$\dot{\mu} = \left[\frac{r f_1 - f_0 + \mu (1 - Q)}{f_1} \right] \mu - \left\{ \pi + \lambda \left[(1 + \beta) \left(1 - \frac{S}{N} \right) + \frac{\kappa \lambda Q S}{e} \left(1 - \frac{S}{N} \right)^2 + 1 - \delta \right] \right\} S. \quad (16)$$

3.2 No Advertising Effort

In this case, the control constraint $u(t) \geq 0$ is active. Using the extended Hamiltonian (7), the maximizing condition (8) yields:

$$\left(u^{*}, w^{*}, \rho_1, \rho_2\right) = \left(0, w^{\circ}\left(Q^{*}, \mu\right), -\frac{\lambda Q S}{e}\left(1-\frac{S}{N}\right), 0\right)$$

where $\rho_1 \leq 0$ and:

$$w^{\circ} = \frac{\mu\left(1-Q\right)-f_0}{f_1},$$

being the solution of $L_u\left(S, Q, 0, w^{\circ}, \lambda, \mu, \rho_1, 0\right) = 0$ and $L_w\left(S, Q, 0, w^{\circ}, \lambda, \mu, \rho_1, 0\right) = 0$, respectively. Plugging the expressions of u and w in (1)–(2) and (7)–(8), respectively, gives:

$$\dot{S} = \left\{\left[\left(\kappa+\beta\right) Q - \beta + \frac{\alpha}{e}\lambda Q^2 S\left(1-\frac{S}{N}\right)\right]\left(1-\frac{S}{N}\right) - 1 + \left(1-\delta\right) Q\right\} S, \tag{17}$$

$$\dot{Q} = \frac{\left[\mu\left(1-Q\right)-f_0\right]\left(1-Q\right)}{f_1}, \tag{18}$$

$$\dot{\lambda} = \lambda\left[r - \left\{(\kappa+\beta)Q - \beta + \frac{\alpha}{e}\lambda Q^2 S\left(1-\frac{S}{N}\right)\right\}\left(1-\frac{S}{N}\right) + 1 - (1-\delta)Q\right] - Q\pi, \tag{19}$$

$$\dot{\mu} = \left[\frac{r f_1 - f_0 + \mu\left(1-Q\right)}{f_1}\right]\mu - \left[\pi + \lambda\left\{\left[\kappa + \beta + \frac{\lambda Q S}{e}\left(1-\frac{S}{N}\right)\right]\left(1-\frac{S}{N}\right) + 1 - \delta\right\}\right] S. \tag{20}$$

3.3 No Quality Improvement Effort

In this case, the control constraint $w(t) \geq 0$ is active. Using (7), condition (8) yields:

$$\left(u^{*}, w^{*}, \rho_1, \rho_2\right) = \left(u^{\circ}\left(S^{*}, Q^{*}, \lambda\right), 0, 0, f_0 - \mu(1-Q)\right)$$

where $\rho_2 \leq 0$, and:

$$u^{\circ} = \frac{\lambda Q S}{e}\left(1-\frac{S}{N}\right),$$

being the solution of $L_u\left(S, Q, u^{\circ}, 0, \lambda, \mu, 0, \rho_2\right) = 0$ and $L_w\left(S, Q, u^{\circ}, 0, \lambda, \mu, 0, \rho_2\right) = 0$, respectively.

Plugging the expressions of u^* and w^* in (1)–(2) and (7)–(8), respectively, gives:

$$\dot{S} = \left\{ \left[(\kappa + \beta)\, Q_0 - \beta + \frac{Q_0^2}{e} \lambda S \left(1 - \frac{S}{N}\right) \right] \left(1 - \frac{S}{N}\right) - 1 + (1 - \delta)\, Q_0 \right\} S,$$

$$\dot{\lambda} = \lambda \left[r - \left\{ (\kappa + \beta)\, Q_0 - \beta + \frac{Q_0^2}{e} \lambda S \left(1 - \frac{S}{N}\right) \right\} \left(1 - \frac{S}{N}\right) + 1 - (1 - \delta)\, Q_0 \right] - Q_0 \pi.$$

3.4 Uncontrolled Sales Process

Here, both control constraints are active so that the sales process is solely driven by spontaneous WOM. Using the extended Hamiltonian, the maximizing condition (8) yields:

$$\left(u^*, w^*, \rho_1, \rho_2\right) = \left(0, 0, -\frac{\lambda Q S}{e} \left(1 - \frac{S}{N}\right), f_0 - \mu\,(1 - Q)\right)$$

where $\rho_1, \rho_2 \leq 0$. Plugging the expressions of u^* and w^* in (1)–(2) gives:

$$\dot{S} = \left\{ [(\kappa + \beta)\, Q_0 - \beta] \left(1 - \frac{S}{N}\right) - 1 + (1 - \delta)\, Q_0 \right\} S,$$

which has two steady states, that is, $S_1^\infty = 0$, and $S_2^\infty = N\left[1 - \frac{1-(1-\delta)Q_0}{(\kappa+\beta)Q_0-\beta}\right]$. If $\delta < \kappa$ and $\frac{1+\beta}{1+\beta+\kappa-\delta} < Q_0 \leq 1$, then $S_2^\infty > 0$. If $\delta \geq \kappa$, then the two steady states coincide with $S_1^\infty = S_2^\infty = 0$, whatever $1 \geq Q_0 \geq \beta/(\kappa + \beta)$.

The stability of the sales evolution is given by the condition:

$$\dot{S}_S \underset{>}{\overset{<}{=}} 0 \;\Big|\; [(\kappa + \beta)\, Q_0 - \beta] \left(1 - \frac{2S}{N}\right) - 1 + (1 - \delta)\, Q_0 \underset{>}{\overset{<}{=}} 0\,.$$

Therefore, whatever the magnitude of the initial sales, if $\delta \geq \kappa$, $S_1^\infty = 0$ is the only steady state and it is asymptotically stable. An illustration of this case is given in Fig. 2a. Conversely, if $\delta < \kappa$ and $Q_0 \in]\frac{1+\beta}{1+\beta+\kappa-\delta}, 1]$, then $S_2^\infty = N\left[1 - \frac{1-(1-\delta)Q_0}{(\kappa+\beta)Q_0-\beta}\right] > 0$ is the only stable solution (see the illustration of this case in Fig. 2b). In other words, if the marginal effectiveness of positive autonomous WOM on the untapped market potential is greater than the fraction of current customers satisfied with the product's conformance quality who forget about the product, then a sufficiently but not very high quality level will result in the nil steady state, while a very high quality level will result in the positive steady state being reached whatever the initial sales level $S_0 > 0$. Therefore, powerful enough positive autonomous WOM can singly ensure the commercial success of a given product if its conformance quality level is sufficiently

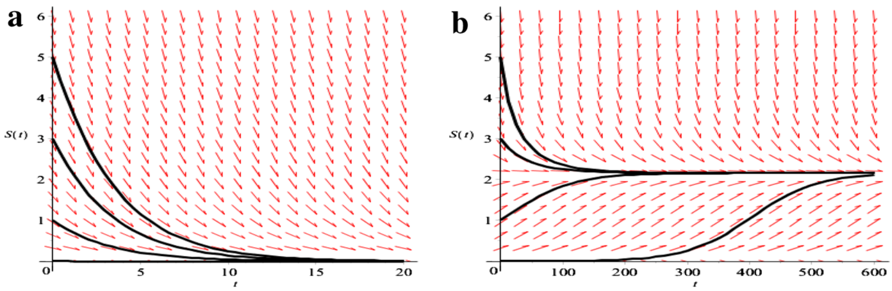

Fig. 2 Uncontrolled sales process with commercial failure and success. **a** Commercial failure ($\delta \geq \kappa$). **b** Commercial success ($1 \geq Q_0 > \frac{1+\beta}{1+\beta+\kappa-\delta}$)

Table 1 Base case parameter values

r	κ	β	δ	N	π	f_0	f_1
0.03	0.035	0.5	0.05	10	1	17	1

high. This possibility, which illustrates the idyllic case of effective efficiency with satisfied customers at no marketing cost, is absent from the existing contagion sales models, where only advertising-dependent WOM can, at cost, allow for such result from a sufficiently high initial sales level. Conversely, overly relatively weak positive autonomous WOM leads to commercial failure whatever the brand's quality level. While this scenario is also plausible, it is not reflected in the existing contagion sales models.

4 Numerical Study

To analyze our model numerically, we choose the following parameter values (Table 1).

The choice of the parameter values is based on two main assumptions:

– Asymmetry between positive and negative WOM ($\kappa < \beta$). This assumption is consistent with the literature [3–6].
– Lower effectiveness of positive autonomous WOM than the forgetting rate ($\delta > \kappa$), to prevent the idyllic configuration where no advertising effort is needed in the long run.

The value of the discounting rate is chosen to reflect the approximate market interest in normal time. For simplicity, the firm's unit profit margin is normalized to 1. To keep the focus on the issue of marketing effective efficiency [20], we investigate the sensitivity of the system to different advertising costs by considering three scenarios of small, intermediate, and high values of e, that is, $e \in (1, 800, 1400)$.

The numerical calculation used roughly consists of two steps. First the canonical system is analyzed to determine possible limit sets of the solution path candidates. In general, these limit sets are steady states, limit cycles or, as in the actual case, a manifold of steady states. In next step, the stability information of the limit sets is used to formulate an appropriate boundary value problem (BVP). To solve the BVP,

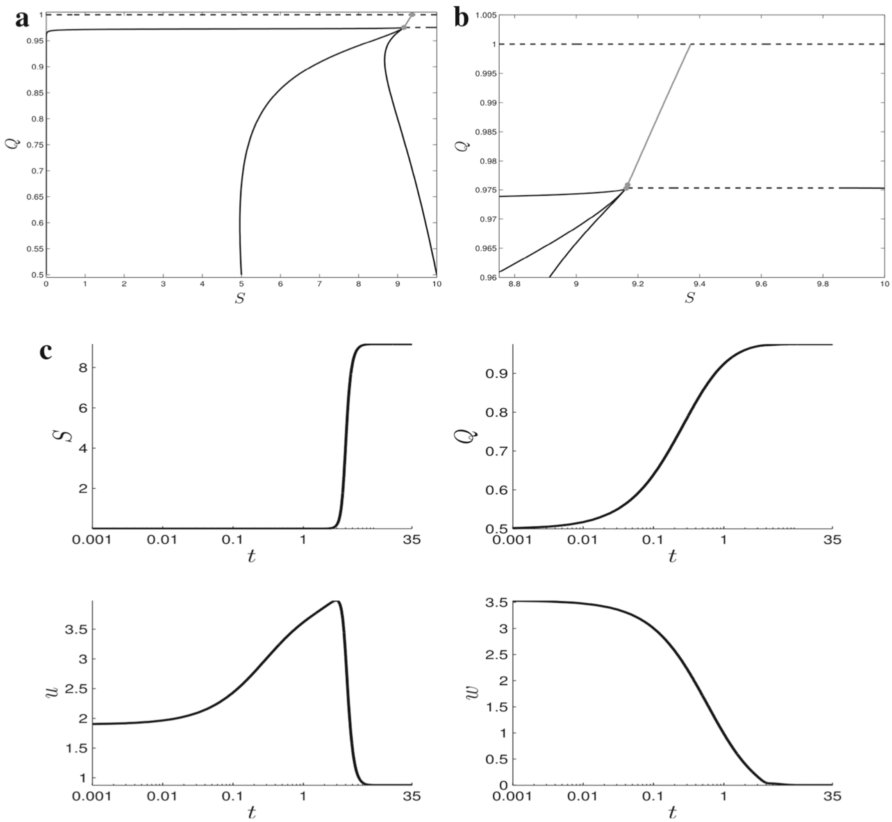

Fig. 3 **a** The complete state-state diagram in the relevant region in case of low advertising cost ($e = 1$). **b** The state-state diagram close to the equilibria (*gray line* is a continuum of equilibria). **c** The time paths in case of low advertising cost ($e = 1$) when both S and Q start from a low level. **d** The time paths in case of low advertising cost ($e = 1$) when S start from a high level and Q start from a low level

a path-following (continuation) technique is used that is implemented in a MATLAB toolbox OCMat (see [42,43]).

The results are depicted in Figs. 3, 4 and 5, where we distinguish between scenarios of advertising being inexpensive ($e = 1$, in Fig. 3), moderately expensive ($e = 800$, in Fig. 4) and expensive ($e = 1400$, in Fig. 5).

The case of inexpensive advertising depicted in Fig. 3 suggests a bright picture for the firm in the long run. First, we see that in the long run the firm conquers almost the whole potential market, i.e., the sales level converges to a level in the neighborhood of $N = 10$. Second, the firm invests so much in conformance quality that, in the long run, there are hardly any defective products, because Q converges to a level close to 1. Fig. 3b shows that the steady state the firm converges to is not unique. In fact, there is a curve of equilibria; however, all equilibria have in common that S is close to 10 and Q close to 1. The reason is that quality improvement effort w is expensive. This results in firm policies where, when the initial conformance quality level is large, and the firm carries out no quality improvement effort at all. Therefore, conformance

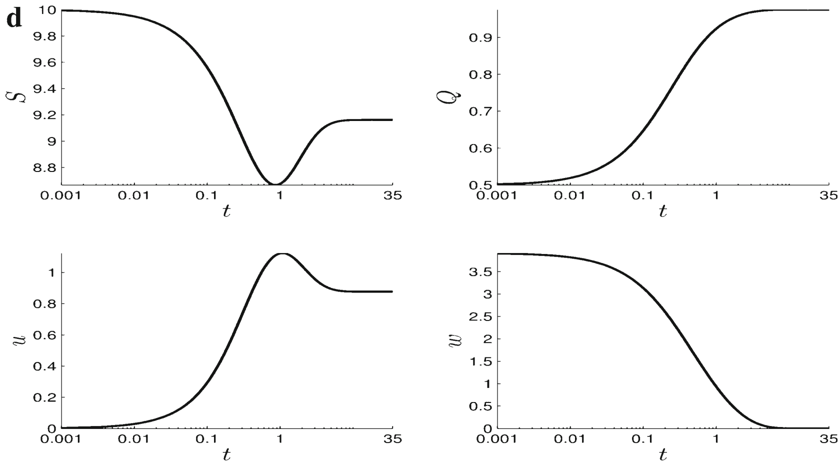

Fig. 3 continued

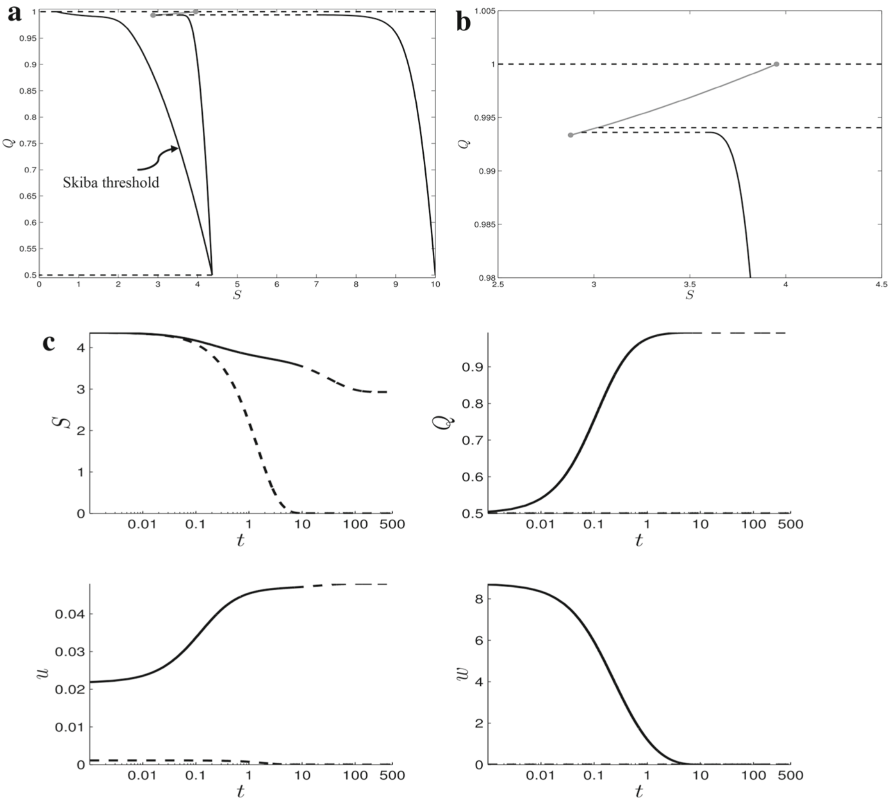

Fig. 4 **a** The complete state-state diagram in the relevant region in case of intermediate advertising cost ($e = 800$) **b** The state-state diagram close to the equilibria (*gray line* is a continuum of equilibria). **c** The time paths in case of intermediate advertising cost ($e = 800$) when S and Q start close to the Skiba curve

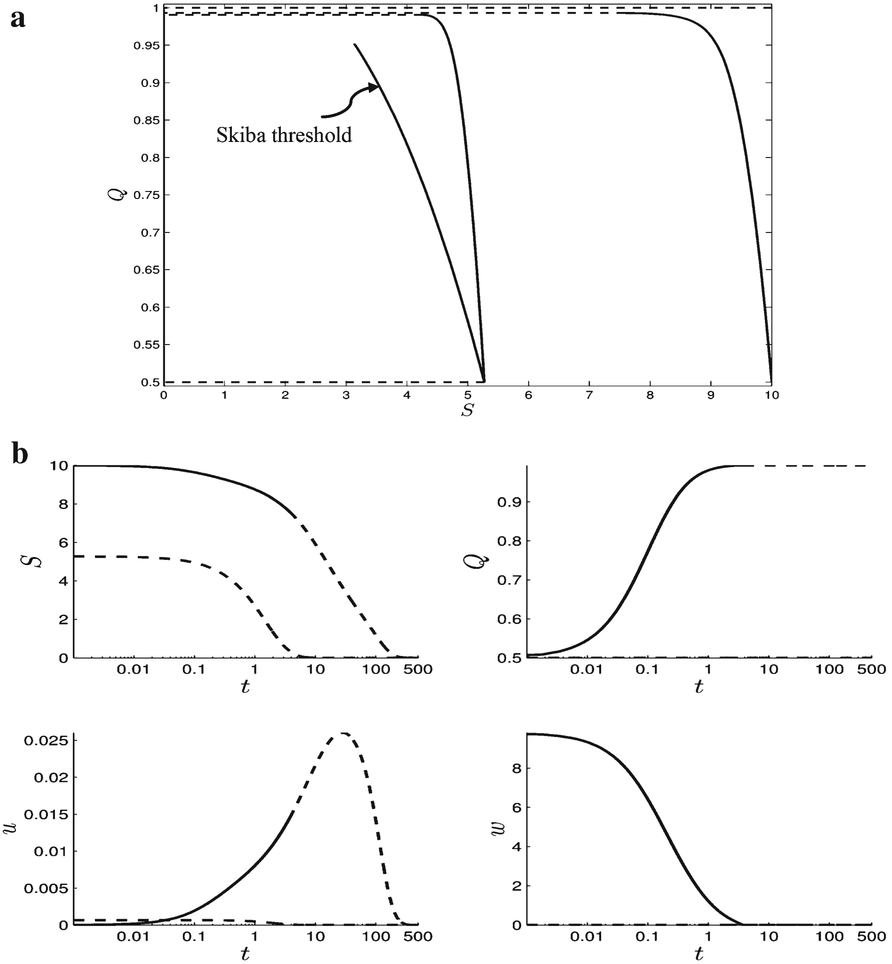

Fig. 5 **a**The state-state diagram in the relevant region in case of high advertising cost ($e = 1400$). **b** The time paths in case of high advertising cost, $e = 1400$, when S and Q start to the right of the Skiba curve; S approaches zero, while Q approaches a high level.

quality level Q is constant and remains at its initial level from time zero on. This initial conformance quality level then automatically becomes the long-run steady-state level of Q, and the long-run steady-state sales level adjusts to that.

In the case where the initial conformance quality level is small, it is optimal for the firm to improve quality from the start. Quality improvement effort is large in the beginning, but decreases as the quality level increases. This is because quality investment reduces the number of defective products, and therefore, its efficiency is proportional to $1 - Q$. As a result, the quality level steadily increases over time and converges to the lowest point on the curve with equilibria, where the quality level is still close to 1 and sales are close to 10. Sales increase by WOM and the firm advertises to strengthen this effect. Of course, WOM raises sales even more when customers are satisfied, i.e., when the conformance quality level is high. For WOM to take off, some customers must be able "to spread the word," which implies that they must already have bought the product. However, for WOM to be effective, there should still be room for sales to increase. This implies that effective WOM requires that sales should not be too low or too high. The firm enhances WOM by advertising, which explains why the effectiveness of u is measured by $QS(1 - S/N)$. The above reasoning accounts for the development of advertising on the different solution paths.

In Fig. 3c, advertising effort is increasing during an initial time period, because both quality and sales start increasing from a level of zero on. This continues until sales pass the level of 5 (note that $S(1 - S/N)$ is maximal for $S = 5$, because $N = 10$). After that, u starts to decrease until it reaches its steady-state level $u = 1$.

Figure 3d depicts the solution path, where the initial sales level is high and the initial quality is low. Low quality implies that customers are not attracted by the product due to predominant negative autonomous WOM, so sales decrease. The firm increasingly invests in advertising-dependent WOM to slow the decrease in sales and, simultaneously, in conformance quality to reduce the abandon rate of dissatisfied current customers and counter the related negative WOM. As soon as conformance quality has risen enough, the loyalty of current customers increases and positive WOM becomes predominant, thus halting the decrease in sales. The firm can then reduce its advertising effort, while sales eventually start to increase until they reach their steady-state level.

Therefore, when advertising effort is relatively inexpensive and conformance quality is low, whatever the initial level of sales, advertising-dependent WOM precedes positive autonomous WOM to compensate for the adversarial effects of negative autonomous WOM. This strategy implies that advertising effort and quality improvement effort are substitutes for each other: Advertising effort increases and quality improvement effort decreases over time. The reason is that the advertising effort has an immediate effect, while the effect of quality improvement effort is delayed. The cost of this strategy is higher when initial sales are low because high initial advertising effort is then required. In all cases, despite a high steady-state conformance quality level, advertising effort remains positive in the long run in order to take advantage of the efficiency of the high level of positive WOM. Overall, advertising effort is deployed first to counteract negative WOM and then to capitalize on positive WOM. If advertising effort is inexpensive, then this strategy ensures for any initial configuration an optimal transition to the "productive marketing" case.

In Fig. 4, advertising costs are higher than in Fig. 3, $e = 800$. The implication is that advertising is not a strong enough instrument to enhance WOM. This results in the fact that now the curve of equilibria occurs for lower sales levels, while equilibrium quality levels are still close to 1 (see Fig. 4b). Moreover, the initial situation of the firm needs to be suitable enough to reach such long-run equilibrium. When initial quality level and sales are both low, there is no profitable way for the firm to stimulate the WOM effect to start sales growth. In addition, because the quality increase is only beneficial for a limited amount of products when sales are low, it is therefore not profitable to invest in quality in an attempt to increase it. In such a case, the firm refrains from any investments and stops operating in finite time. Solution paths converge to the vertical axis where sales are ultimately zero. This case, described in Sect. 3.4, is consistent with the "hit and run marketing" case (Fig. 1), typified by alienated customers and low marketing costs.

Knowing that investments in quality and advertising do not pay off when both sales and quality levels are low, a Skiba curve with negative slope exists (black curve in Fig. 4a), distinguishing initial levels of sales and quality that converge to different steady states. To the right of this curve, the solution paths are converging to a steady state with positive sales; see also the upper (black/dashed black) curves in Fig. 4c. To the left of the Skiba threshold, we have horizontal solution paths on which quality is constant and sales are decreasing to zero, such that the firm stops operating in finite time; see the lower (dashed black) curves in Fig. 4c. The upper and lower curves are distinguished only by small differences in the initial values of S and Q.

Therefore, the scenario of moderately expensive advertising effort allows for a switch to the "productive marketing" case only if initial sales are sufficiently high. That is, in spite of a high initial quality level, the sequential strategy that consists of first countering negative WOM and then capitalizing on positive WOM is not sustainable for an initial level of WOM below the Skiba threshold. Given that in this case, quality improvement efforts alone are not sufficient for an optimal switch to productive marketing; the "hit and run marketing" case (Fig. 1), though ephemeral, is an optimal option.

A similar Skiba threshold exists for the case of expensive advertising, $e = 1400$ (Fig. 5a), but now the firm always ceases operations in the long run because maintaining a positive sales level by advertising is too expensive. Now, on the left side of the Skiba curve the firm does not invest at all in quality, while on the right side investments in quality occur such that in the end there are almost no defective products.

In Fig. 5b, we see that the solution paths on the right of the Skiba curve also converge to a long-run sales level that is equal to zero, while quality settles at a high level; see the upper curves in black/dashed black. This is done because before the firm stops being active (in the long run), the sales level remains high for a reasonable amount of time, so that investing in quality still pays off. As long as the firm is active, on both sides of the Skiba curve the firm invests little in advertising: Advertising costs are purely quadratic, so marginal advertising costs are zero for $u = 0$; see the south/west panel of Fig. 5b. In the upper curve, representing a situation starting to the right of the Skiba curve, u is first increasing because quality also goes up, strengthening the WOM effect. After that, sales start to drop, implying that there are fewer customers being active in WOM, and this results in a drop in advertising expenses. Figure 5b also depicts a situation where sales and quality start to the left of the Skiba curve, where investments in quality are not profitable; see the lower curves in all panels (depicted in dashed black).

Overall, the scenario of expensive advertising effort precludes the possibility of "productive marketing" because loyalty of current customers cannot be ensured in the long run. Instead, depending on the initial conditions, only two ephemeral optimal options are possible, namely the "hit and run marketing" option and the "premium marketing" option (Fig. 1). In the first option, the firm makes almost no effort to acquire new customers or to retain current customers. In the second option, in spite of efforts in advertising and quality improvement, the firm can only delay the decline in its sales. Though the customers are satisfied with the high conformance quality, the cost of advertising efforts needed to acquire new customers to replace the current customers lost is not sustainable. Finally, there are no circumstances under which expensive advertising effort would lead the firm to sacrifice both effectiveness and efficiency.

5 Conclusions

The paper considers a firm operating in a market where WOM effects play a prominent role. These effects can be positive or negative. Negative effects are induced by customers being dissatisfied after consuming the product, typically because they have bought an item of insufficient quality. The firm can reduce the number of these defectives by investing in quality. Customers will thus be more satisfied with the product. This enhances the positive WOM effect, which can be heightened by advertising.

Investing in quality to decrease the rate of defective items pays off, especially if sales are high. Similarly, only when there are enough customers to generate considerable WOM does it pay off to enhance this effect by advertising. From a mathematical viewpoint, it is noteworthy that for high initial quality levels, there is a continuum of equilibria with high quality and positive sales. These characteristics automatically lead to history-dependent solutions, i.e., achieving a high long-run sales level requires that the initial sales level must already be high enough. Otherwise the firm will cease operations in the long run. The paper finds that such history-dependent solutions, where initial levels of sales and quality leading to different long-run optimal solutions are separated by a Skiba curve, indeed occur for moderately expensive advertising effort. In addition, commercial success is always reached if advertising costs are low, while expensive advertising effort will result in commercial failure.

A main marketing implication of our study is that an optimal path to "productive marketing" (effective efficiency) systematically exists only for inexpensive advertising effort. In the case of moderately expensive advertising effort, a firm may have to choose between "productive marketing" and "hit and run marketing," which consists of alienating the diminishing number of current customers to save on advertising and operational expenditures. In the case of expensive advertising effort, a firm may even have to choose between "hit and run marketing" and "premium marketing," which consists of providing full satisfaction to a declining number of current customers at a high cost. In this case, the trade-off between advertising-dependent and autonomous WOM may lead a firm to choose between "doing things right" (efficiency) or "doing the right things" (effectiveness). The good news is that "death-wish marketing" is never an optimal option.

The effect of advertising effort can be modeled independently from the number of satisfied customers as well, as in the Vidale–Wolfe model [41]. In addition, such a framework could be employed by considering a second advertising instrument seeking to attract new customers without involving satisfied customers. Future research could also consider adding price as a control variable. Accordingly, the initial sales level could be stimulated by setting a low price level, which could ensure a firm's long-run viability. Another interesting research direction could involve adding competition on the market by developing a differential game model with multiple firms.

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