

# A Generalization of the Expenditure Function

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**Abstract** We consider a generalized expenditure function and the corresponding Hicksian demand. First, we provide some economic interpretation of the problem at stake. Then, we obtain different properties of the solution: existence, Lipschitz behavior and differential properties. Finally, we provide a Slutsky-type property.

**Keywords** Expenditure function · Slutsky-type property · Lipschitz behavior

**Mathematics Subject Classification** 90C30 · 90C90 · 91B08

## 1 Introduction

In 1947, Samuelson suggested the *dual* approach in consumer theory. Instead of considering a consumer maximizing his utility with respect to a budget constraint, one does consider a consumer minimizing his expenditure with respect to a level of utility he must achieve. In other words, one does consider the reciprocal problem of the *Utility Maximization Problem*. Due to this reciprocity, there exist well-known relations between the Marshallian demand (i.e., the solution of the *Utility Maximization Problem*) and the Hicksian demand (i.e., the solution of the *Expenditure Minimization Problem*). For a discussion concerning these relations, we refer the reader to [1]. For a more general discussion about the concept of *duality* in economics, we refer the reader to [2] and [3]. As Barten and Böhm explained in [1]:

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In certain cases, it provides a more direct analysis of the price sensitivity of demand[...].

For this reason, the so-called *Expenditure Minimization Problem* has been extensively studied during the last decades. For instance, one can mention the early contribution of McKenzie [4], who obtained Slutsky equation in demand theory using expenditure minimization instead of utility maximization. We shall also mention the contribution of Diamond and McFadden [5], who provided three uses of the expenditure function in public finance: deadweight burden of taxation, optimal taxation and optimal investment. For an extensive discussion about the *Expenditure Minimization Problem* and for more references, we refer the reader to [1] and [2].

The purpose of this work is to deal with a generalization of this problem to a multiple constraint case. It is customary to study multiple constraint models in consumer theory. On the other hand, there is little research concerning consumer facing more than one utility constraint. As illustrated in next section, in many relevant situations, the consumer or the economic planner faces more than one utility constraint and this observation calls for such a generalization. Contrary to what one could expect, we do not restrict ourselves to the problem of existence and uniqueness. As a matter of fact, we study the Lipschitz behavior of the solution and we identify the conditions on the parameters (i.e., the utility levels and the price vector) under which this solution is continuously differentiable around some point. Our last contribution is a Slutsky-type property that generalizes the classical one.

The paper is divided as follows. Section 2 states the problem and the assumptions. In addition, we provide three economic motivations for the optimization problem. In Sect. 3, after characterizing the solution by necessary and sufficient first-order conditions, we prove the existence and continuity of the solution. Section 4 studies the classical properties of the generalized expenditure function and states that the generalized Hicksian demand is locally Lipschitz continuous. The proof relies on the result of Cornet and Vial [6] on the Lipschitz behavior of the solution of a mathematical programming problem. In Sect. 5, we show that the generalized Hicksian demand is continuously differentiable if a strict complementary slackness condition holds. Following Fiacco and McCormick [7], this result is a consequence of the classical Implicit Function Theorem. Finally, we obtain a Slutsky-type property for the generalized Hicksian demand. Section 6 presents perspectives of further research, while Sect. 7 summarizes the results.

## 2 Assumptions and Economic Motivations

Let  $u_1, \dots, u_n$  be  $n$  functions defined on  $\mathbb{R}_{++}^\ell$ . The problem with which we shall be concerned throughout this paper is:

$$\max \langle -p \cdot x \rangle \text{ s.t. } u_k(x) \geq v_k, \quad k = 1, \dots, n, \quad x \gg 0 \quad (1)$$

with  $p$  belonging to  $\mathbb{R}_{++}^\ell$  and  $v := (v_k)_{k=1}^n \in \mathbb{R}^n$ . The solution of this problem will be denoted by  $\Delta(p, v)$  and called the *generalized Hicksian/compensated demand*. The aim of the paper is to study the properties of this mapping.

We proceed to posit the assumptions concerning the functions  $(u_k)_{k=1}^n$ .<sup>1</sup>

**Assumption 2.1** For all  $k = 1, \dots, n$ ,

1.  $u_k$  is  $C^2$  on  $\mathbb{R}_{++}^\ell$ ,
2.  $u_k$  is differentially strictly quasi-concave (i.e.,  $D^2u_k(x)$  is negative definite on  $\nabla u_k(x)^\perp$  for all  $x \in \mathbb{R}_{++}^\ell$ ),
3.  $u_k$  is differentially strictly increasing (i.e.,  $\nabla u_k(x) \gg 0, \forall x \in \mathbb{R}_{++}^\ell$ ).

**Assumption 2.2** For all  $k \in \{1, \dots, n\}$ , if a sequence  $(x^v)_{v \geq 0}$  converges to  $x \in \partial \mathbb{R}_{++}^\ell$ , then:

$$\lim_{v \rightarrow +\infty} \frac{\nabla u_k(x^v) \cdot x^v}{\|\nabla u_k(x^v)\|} = 0.$$

The boundary behavior of the preferences is given by Assumption 2.2. Roughly speaking, when the quantity of one good is very small, the consumer basically wants to increase it as explained in [8]. For that reason, this boundary assumption ensures us that the demand is interior. Generally, one considers the usual boundary assumption: For all  $x \in \mathbb{R}_{++}^\ell$ , the closure in  $\mathbb{R}^\ell$  of the set  $\{x' \in \mathbb{R}_{++}^\ell \mid u(x') \geq u(x)\}$  is contained in  $\mathbb{R}_{++}^\ell$  which implies Assumption 2.2. However, both assumptions are not equivalent. For instance, the Expected Utility Function often does not satisfy the classical closure assumption. As an example, the utility function  $u$  defined on  $\mathbb{R}_{++}^2$  by:  $u(x_1, x_2) = \frac{1}{2}\sqrt{x_1} + \frac{1}{2}\sqrt{x_2}$  satisfies Assumption 2.2 but not the classical boundary assumption.<sup>2</sup>

In Problem (1), all individual utility levels are not relevant. To determine the relevant one, we shall define, for  $v \in \mathbb{R}^n$ , the set  $P(v)$  by:

$$P(v) := \left\{ x \in \mathbb{R}_{++}^\ell : u_k(x) \geq v_k, \quad \forall k = 1, \dots, n \right\}.$$

If this set is empty or equal to the whole set, the vector  $v$  is obviously not relevant. This motivates the definition of the set  $\mathcal{V}$  by:

$$\mathcal{V} := \left\{ v \in \mathbb{R}^n : P(v) \neq \emptyset \text{ and } P(v) \neq \mathbb{R}_{++}^\ell \right\}.$$

<sup>1</sup> In the paper, we use the following notations:

- $x \geq y$  means:  $x_h \geq y_h$  for all  $h = 1, \dots, \ell$ .
- $x \gg y$  means:  $x_h > y_h$  for all  $h = 1, \dots, \ell$ .
- $e_h$  denotes the  $h$ -th vector of the canonical basis of  $\mathbb{R}^\ell$ .
- $\mathbf{1}_\ell$  denotes the  $\ell$ -dimensional vector whose coordinates are all equal to one. Similarly,  $\mathbf{1}_n$  denotes the  $n$ -dimensional vector whose coordinates are all equal to one. When there is no confusion, we simply write  $\mathbf{1}$ .
- Let  $v \in \mathbb{R}^\ell, \|v\| := \sum_{h=1}^\ell |v_h|$  denotes the norm of the vector  $v$ .

<sup>2</sup> Take  $x := \left(\frac{1}{4}, \frac{1}{4}\right)$  and  $\left(x^v := \left(1 + \frac{1}{v}, \frac{1}{v}\right)\right)_{v \geq 1}$  as a counterexample.

As a matter of fact, one can give a more explicit description of  $\mathcal{V} \cdot v \in \mathcal{V}$  means:  $\exists z \in \mathbb{R}_{++}^\ell$  such that  $u_k(z) \geq v_k$  for all  $k = 1, \dots, n$  and  $\exists z' \in \mathbb{R}_{++}^\ell, k_0 \in \{1, \dots, n\}$  such that:  $u_{k_0}(z') < v_{k_0}$ . Clearly, the set  $\mathcal{V}$  is an open<sup>3</sup> subset of  $\mathbb{R}^n$ .

Finally, we proceed to define the generalized expenditure function.

**Definition 2.1** The function  $e$  is defined on  $\mathbb{R}_{++}^\ell \times \mathcal{V}$  by:  $e(p, v) = p \cdot \Delta(p, v)$  and called the *generalized expenditure function*.

Before pursuing the analysis, we present next three applications.

## 2.1 Generalization of the Classical Compensated Demand

If  $n$  is equal to one,  $\Delta(p, v)$  is the so-called compensated demand or Hicksian demand.<sup>4</sup> So  $\Delta(p, v)$  can be viewed as a multi-criterion extension of the Hicksian demand.

## 2.2 Public Goods

The following application concerns economic planning. Consider an economy with  $n$  consumers,  $\ell$  public goods<sup>5</sup> and  $m$  private goods. Suppose that the basket of private goods  $(\xi_k) \in \mathbb{R}_{++}^m$  to be consumed by consumer  $k$  has already been chosen, i.e.,  $u_k(x) := U_k(x, \xi_k)$  where  $U_k$  is the utility function of consumer  $k$ . As usual, consumer  $k$  wishes to achieve an individual level of utility  $v_k \in \mathbb{R}$ . In this situation, the economic planner shall choose the cheapest basket of public goods  $x \in \mathbb{R}_{++}^\ell$  with respect to the price  $p \in \mathbb{R}_{++}^\ell$  given the individual levels of private goods  $(\xi_k)_{k=1}^n$  and the individual levels of utility  $(v_k)_{k=1}^n$ . Therefore, he has to solve Problem (1).

## 2.3 Private Goods and Positive Externalities

Consider an economy with  $n$  consumers and  $r$  private goods. The price of good  $h$  is denoted by  $q_h$ , while the consumption bundle of consumer  $k$  is denoted by  $x_k$ . Suppose that the consumption of every good by another consumer has a positive effect<sup>6</sup> on the utility of consumer  $k$ . Hence, the utility function  $u_k$  of consumer  $k$  is a function of both his consumption bundle  $x_k$  and the consumption bundles of the others  $(x_j)_{j \neq k}$ . Let us write  $\ell := rn$  and denote by  $x := (x_k) \in \mathbb{R}_{++}^\ell$  the concatenation of consumption bundles. In the same way, the vector  $p := (q, \dots, q) \in \mathbb{R}_{++}^\ell$  denotes the  $n$ -replica of the price vector  $p \in \mathbb{R}_{++}^r$ . An economic planner who wants to minimize the

<sup>3</sup> For the sake of completeness, this is proved in “Appendix.”

<sup>4</sup> A presentation of the expenditure minimization problem can be found in any intermediary or advanced microeconomics textbook. For the sake of completeness, we refer the reader to [9].

<sup>5</sup> A good is considered public if its use by one agent does not prevent other agents from using it[...]. (Laffont[10])

<sup>6</sup> Recall the definition in [9]: An externality is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of another agent in the economy. Many economic goods can be considered as positive externalities such as vaccination or network.

expenditure of the society  $p \cdot x = \sum_{k=1}^n q \cdot x_k$  with respect to the individual utility levels  $(v_k)_{k=1}^n$  has<sup>7</sup> to solve Problem (1).

### 3 Existence of the Solution

In order to establish the existence of the solution of Problem (1), we first show that this solution is characterized by first-order conditions. We then introduce an intermediary  $\varepsilon$ -problem and prove that this problem admits a unique solution  $\Delta^\varepsilon(p, v)$  for all  $(p, v) \in \mathbb{R}_{++}^\ell \times \mathcal{V}$ . In addition, the continuity of the function  $\Delta^\varepsilon$  is proved. Finally, we present a characterization of  $\Delta^\varepsilon(p, v)$  by first-order conditions. Combining these results, we show that  $\Delta(p, v)$  is a singleton and that  $\Delta$  defines a continuous function on  $\mathbb{R}_{++}^\ell \times \mathcal{V}$ .

#### 3.1 Characterization of the Generalized Hicksian Demand by First-Order Conditions

**Proposition 3.1** *Let  $p \in \mathbb{R}_{++}^\ell$  and  $v \in \mathcal{V}$ . The two following assertions are equivalent:*

1.  $\bar{x} = \Delta(p, v)$
2. *There exists  $\lambda \in \mathbb{R}_+^n \setminus \{0\}$  such that  $\bar{x}$  is the solution of the system:*

$$\begin{aligned}
 p &= \sum_{k=1}^n \lambda_k \nabla u_k(x) \\
 \lambda_k (u_k(x) - v_k) &= 0, \quad k = 1, \dots, n \\
 u_k(x) &\geq v_k, \quad k = 1, \dots, n \\
 x &\gg 0
 \end{aligned}
 \tag{2}$$

*Proof* We first show that Assertion 1 implies Assertion 2. Since  $v$  belongs to the set  $\mathcal{V}$ , there exists some element  $x \in \mathbb{R}_{++}^\ell$  such that  $u_k(x) \geq v_k$  for all  $k \in \{1, \dots, n\}$ . Hence, by monotony of the functions  $(u_k)_{k=1}^n$ , there exists  $\hat{x}$  such that  $u_k(\hat{x}) > v_k$  for all  $k = 1, \dots, n$ . As a consequence, the first-order conditions are necessary since Slater’s Constraint Qualification holds.<sup>8</sup> The multiplier vector  $\lambda := (\lambda_k)_{k=1}^n$  is necessarily different from zero because the vector  $p$  belongs to  $\mathbb{R}_{++}^\ell$ .

Now we prove the converse statement. The functions  $(u_k)_{k=1}^n$  are differentiable and quasi-concave and satisfy:  $\nabla u_k(x) \neq 0$  for all  $x \in \mathbb{R}_{++}^\ell$ , while the objective function is linear.<sup>9</sup> This implies that the first-order conditions are sufficient. Thus, Assertion 2 implies Assertion 1. □

<sup>7</sup> The first scalar product  $q \cdot x$  is the one of  $\mathbb{R}^\ell$  where  $\ell = rn$ , while the second scalar product  $p \cdot x_h$  is the one of  $\mathbb{R}^r$ .

<sup>8</sup> See [11].

<sup>9</sup> See [12].

### 3.2 An Intermediary Existence Result

To solve Problem (1), we have to study an intermediary problem. For  $\varepsilon > 0$  and  $(p, v) \in \mathbb{R}_{++}^\ell \times \mathcal{V}$ , we shall consider the following problem:

$$\max \langle -p \cdot x \rangle \text{ s.t. } u_k(x) \geq v_k, \quad k = 1, \dots, n, x_h \geq \varepsilon, h = 1, \dots, \ell \quad (3)$$

We start by an existence result:

**Proposition 3.2** *Let  $\varepsilon > 0$ . The solution of Problem (3) denoted by  $\Delta^\varepsilon(p, v)$  exists and is a singleton.*

*Proof* Since  $v$  belongs to  $\mathcal{V}$ , there exists  $z_0 \in \mathbb{R}_{++}^\ell$  such that  $u_k(z_0) \geq v_k$  for all  $k \in \{1, \dots, n\}$ . Let us choose  $z_1 \in \mathbb{R}^\ell$  such that  $z_{1h} \geq \max\{\varepsilon, z_{0h}\}$  for all  $h \in \{1, \dots, \ell\}$ . We now consider another intermediary problem.

$$\max \langle -p \cdot x \rangle \text{ s.t. } u_k(x) \geq v_k, \quad k = 1, \dots, n, p \cdot x \leq p \cdot z_1, x \geq \varepsilon \mathbf{1}, x \in \mathbb{R}^\ell \quad (4)$$

Note that  $z_1$  is feasible for Problem (4) because  $z_1 \geq z_0$  and because the functions  $(u_k)_{k=1}^n$  are increasing.

We proceed to prove that Problem (4) admits a solution.

The set  $A := \{x \in \mathbb{R}^\ell : x \geq \varepsilon \mathbf{1} \text{ and } p \cdot x \leq p \cdot z_1\}$  is a compact set as a closed and bounded set in a finite-dimensional vector space. Moreover, the function  $x \mapsto -p \cdot x$  is continuous on  $\mathbb{R}_{++}^\ell$ . According to Weierstrass Theorem, this problem admits a solution.

Thanks to Lemma 8.2 proved in “Appendix,” we deduce that Problem (3) also admits a solution. Finally, we show that the set  $\Delta^\varepsilon(p, v)$  is a singleton. Suppose that  $x$  and  $x'$  are distinct solutions of Problem (3). The element  $x'' := \frac{1}{2}(x + x')$  is clearly feasible. By strict quasi-concavity of the functions  $(u_k)_{k=1}^n$ , we have indeed  $u_k(x'') > v_k$  for every  $k = 1, \dots, n$ . On the other hand,  $x'' \geq \varepsilon \mathbf{1}$  obviously holds. Both  $x$  and  $x'$  cannot be equal to  $\varepsilon \mathbf{1}$ . So, at least one of them has a component larger than  $\varepsilon$ . To fix the ideas, suppose that  $x_1$  is larger than  $\varepsilon$ . By continuity of the functions  $(u_k)_{k=1}^n$ , for  $\delta$  positive sufficiently small, we have:  $x'' - \delta e_1 \geq \varepsilon \mathbf{1}$  and  $u_k(x'' - \delta e_1) > v_k$  for every  $k = 1, \dots, n$ . Moreover, the element  $\tilde{x} := x'' - \delta e_1$  satisfies  $-p \cdot \tilde{x} = -p \cdot x'' < -p \cdot x$ . So  $x$  is not a solution. Consequently, one gets a contradiction.  $\square$

### 3.3 Continuity of $\Delta^\varepsilon$

**Proposition 3.3** *Let  $\varepsilon > 0$ . The function  $\Delta^\varepsilon$  is continuous on  $\mathbb{R}_{++}^\ell \times \mathcal{V}$ .*

*Proof* Let  $(\bar{p}, \bar{v}) \in \mathbb{R}_{++}^\ell \times \mathcal{V}$  and a compact neighborhood  $\mathcal{E}$  of  $(\bar{p}, \bar{v})$ .  $\mathcal{E}$  is chosen such that  $v$  belongs to  $\mathcal{V}$  for all  $(p, v) \in \mathcal{E}$ .<sup>10</sup> The compactness of  $\mathcal{E}$  allows us to say that for  $M > \varepsilon$  sufficiently large,  $u := M \mathbf{1}$  belongs to the interior of  $P(v)$  for all

<sup>10</sup> This is possible when the neighborhood is small enough thanks to the openness of  $\mathcal{V}$ .

$(p, v) \in \mathcal{E}$ . Take such a number  $M$ . The budget sets<sup>11</sup>  $B(p, p \cdot u)$  are all contained in a compact set  $K$  since  $p$  varies in a compact set contained in  $\mathbb{R}_{++}^\ell$  when  $(p, v)$  belongs to  $\mathcal{E}$ . For  $\bar{M} > 0$  large enough,  $\bar{M}\mathbf{1}$  does not belong to the compact set  $K$ . Proceed now to define the correspondence  $C^\varepsilon$ , for  $(p, v) \in \mathcal{E}$ , by:

$C^\varepsilon(p, v) := \{x \in \mathbb{R}_{++}^\ell : u_k(x) \geq v_k, \forall k = 1, \dots, n \text{ and } \varepsilon\mathbf{1} \leq x \leq \bar{M}\mathbf{1}\}$  and observe that, by construction, for all  $(p, v) \in \mathcal{E}$ ,  $\Delta^\varepsilon(p, v)$  is the solution of the following problem:

$$\max\langle -p \cdot x \rangle \text{ s.t. } x \in C^\varepsilon(p, v).$$

On  $\mathcal{E}$ , the interior of  $C^\varepsilon(p, v)$  is nonempty since  $\mathcal{V}$  is an open set. We now prove that the function  $(p, v) \mapsto \Delta^\varepsilon(p, v)$  is continuous on  $\mathcal{E}$ . This is a consequence of Berge’s Theorem [13]. We have to prove that the correspondence  $C^\varepsilon$  is both upper semi-continuous and lower semi-continuous on  $\mathcal{E}$ . First, we show that  $C^\varepsilon$  is upper semi-continuous. On  $\mathcal{E}$ , the set  $C^\varepsilon(p, v)$  remains in a fixed compact set. Hence, the upper semi-continuity of  $C^\varepsilon$  is equivalent to the closedness of its graph, which is a consequence of the continuity of the functions  $(u_k)_{k=1}^n$ .

We now have to show that the correspondence  $C^\varepsilon$  is lower semi-continuous. We proceed to define the correspondence  $\hat{C}^\varepsilon$  on  $\mathcal{E}$  by:

$\hat{C}^\varepsilon(p, v) := \{x \in \mathbb{R}_{++}^\ell : u_k(x) > v_k, \forall k = 1, \dots, n \text{ and } \varepsilon\mathbf{1} \ll x \ll \bar{M}\mathbf{1}\}$ . The correspondence  $\hat{C}^\varepsilon$  has an open graph by the continuity of the functions  $(u_k)_{k=1}^n$ . So  $\hat{C}^\varepsilon$  is lower semi-continuous. Note that  $\hat{C}^\varepsilon(p, v)$  is nonempty for every  $(p, v) \in \mathcal{E}$  since  $\bar{M}\mathbf{1}$  belongs to the interior of  $P(v)$ .

Moreover, the closure of  $\hat{C}^\varepsilon(p, v)$  is  $C^\varepsilon(p, v)$ . Let  $x \in C^\varepsilon(p, v)$ . We have to show that  $x$  is the limit of a sequence of elements of  $\hat{C}^\varepsilon(p, v)$ . We choose  $y \in \hat{C}^\varepsilon(p, v)$  and observe that for all  $\lambda \in ]0, 1[$ ,  $(1 - \lambda)x + \lambda y$  belongs to  $\hat{C}^\varepsilon(p, v)$  since the functions  $(u_k)_{k=1}^n$  are strictly quasi-concave. To conclude,  $x$  is the limit of the sequence  $\left(x^v := \left(1 - \frac{1}{v}\right)x + \frac{1}{v}y\right)_{v \geq 1}$ . Moreover, one remarks that  $x^v$  belongs to

$\hat{C}^\varepsilon(p, v)$  for all  $v \geq 1$  and the result follows.

We deduce that the correspondence  $C^\varepsilon$  is lower semi-continuous since the closure of a lower semi-continuous correspondence is lower semi-continuous.<sup>12</sup> Berge’s Theorem implies that the function  $\Delta^\varepsilon$  is continuous on the set  $\mathcal{E}$ . Since  $(\bar{p}, \bar{v})$  was arbitrary chosen, the function  $\Delta^\varepsilon$  is continuous on  $\mathbb{R}_{++}^\ell \times \mathcal{V}$  as required.  $\square$

### 3.4 Characterization of $\Delta^\varepsilon(p, v)$ by First-Order Conditions

Let  $\varepsilon > 0$ . The first-order conditions corresponding to Problem 3 are: there exists  $\lambda^\varepsilon \in \mathbb{R}_+^n$  and  $\mu^\varepsilon \in \mathbb{R}_+^\ell$  such that  $\Delta^\varepsilon(p, v)$  is the solution of the system:

<sup>11</sup> Like in standard microeconomics, for  $p \in \mathbb{R}_{++}^\ell$  and  $w > 0$ , the budget set  $B(p, w)$  is defined by:  $B(p, w) := \{x \in \mathbb{R}_{++}^\ell : p \cdot x \leq w\}$ .

<sup>12</sup> See the appendix of [14].

$$\begin{aligned}
 p &= \sum_{k=1}^n \lambda_k^\varepsilon \nabla u_k(x) + \mu^\varepsilon \\
 \lambda_k^\varepsilon (u_k(x) - v_k) &= 0, \quad k = 1, \dots, n \\
 u_k(x) &\geq v_k, \quad k = 1, \dots, n \\
 \mu_h^\varepsilon (\varepsilon - x_h) &= 0, \quad h = 1, \dots, \ell \\
 x_h &\geq \varepsilon, \quad h = 1, \dots, \ell
 \end{aligned} \tag{5}$$

As before, the first-order conditions are necessary since Slater’s Constraint Qualification holds. These are sufficient since the objective function is linear, the functions  $(u_k)_{k=1}^n$  are quasi-concave functions satisfying  $\nabla u_k(x) \neq 0$  for all  $x \in \mathbb{R}_{++}^\ell$ , and the  $\ell$  additional constraints are affine.

### 3.5 Existence and Continuity of the Solution of Problem 1

In this subsection, we show the main result of the section:

**Proposition 3.4** *For  $(p, v) \in \mathbb{R}_{++}^\ell \times \mathcal{V}$ ,  $\Delta(p, v)$  is a singleton. Moreover, the function  $\Delta$  is continuous on  $\mathbb{R}_{++}^\ell \times \mathcal{V}$ .*

*Proof* Let  $(p, v) \in \mathbb{R}_{++}^\ell \times \mathcal{V}$  and a compact neighborhood  $\mathcal{E}$  of  $(p, v)$ . As before, the set  $\mathcal{E}$  is chosen such that  $v'$  belongs to  $\mathcal{V}$  for all  $(p', v') \in \mathcal{E}$ . Our goal is to show that there exists  $\bar{\varepsilon} > 0$  such that the multipliers  $\mu^\varepsilon$  corresponding to the additional constraints are equal to zero for all  $(p', v')$  in  $\mathcal{E}$ . We reason by contradiction. Otherwise, there would exist a decreasing sequence  $(\varepsilon_q)_{q \geq 0}$  that converges to zero and a sequence of  $\mathcal{E}$  denoted by  $(p_q, v_q)_{q \geq 0}$  such that  $\mu^{\varepsilon_q} := \mu^{\varepsilon_q}(p_q, v_q) \neq 0$  for all  $q \in \mathbb{N}$ .

Necessarily,  $(x_q := \Delta^{\varepsilon_q}(p_q, v_q))_{q \geq 0}$  is bounded. Observe that for all  $q \in \mathbb{N}$ ,  $x_q \gg 0$  and  $a \cdot x_q \leq p_q \cdot x_q \leq p_0 \cdot x_0$  where the vector  $a$  is defined by  $a_h := \min\{p'_h : (p', v') \in \mathcal{E}\}$  for  $h = 1, \dots, \ell$ . The vector  $a$  is well defined and belongs to  $\mathbb{R}_{++}^\ell$  thanks to the compactness of  $\mathcal{E}$ . Therefore, the sequence  $(x_q)_{q \geq 0}$  converges, up to a subsequence, to an element  $\hat{x}$  belonging to the boundary of  $\mathbb{R}_{++}^\ell$ , and the sequence  $(p_q, v_q)_{q \geq 0}$  converges, up to a subsequence, to some element  $(\hat{p}, \hat{v}) \in \mathcal{E}$  since  $\mathcal{E}$  is a compact set. In particular, remark that  $\hat{v}$  belongs to  $\mathcal{V}$  and that  $\hat{p}$  is necessarily different from zero. With a slight abuse of notation, we denote the converging subsequences as the original sequences.

Observe that the sequence  $(\mu^{\varepsilon_q})_{q \geq 0}$  is also a bounded sequence thanks to the first equation of (5) and to the compactness of  $\mathcal{E}$ . From the same equation, recalling that we consider the 1-norm, we have,<sup>13</sup> for all  $k \in \{1, \dots, n\}$  and all  $q \in \mathbb{N}$ :

$$\lambda_k^{\varepsilon_q} \|\nabla u_k(x_q)\| \leq \|p_q\|.$$

<sup>13</sup> We write:  $\lambda_k^{\varepsilon_q} := \lambda_k^{\varepsilon_q}(p_q, v_q)$  for  $k \in \{1, \dots, n\}$  and  $q \in \mathbb{N}$  to simplify the notation.



Therefore, for all  $k \in \{1, \dots, n\}$  and all  $q \in \mathbb{N}$ , we get:

$$0 \leq \lambda_k^{\varepsilon_q} \frac{\nabla u_k(x_q) \cdot x_q}{\|p_q\|} \leq \frac{\nabla u_k(x_q) \cdot x_q}{\|\nabla u_k(x_q)\|}.$$

Thanks to Assumption 2.2 and in light of the previous inequalities, for all  $k \in \{1, \dots, n\}$ , we get:

$$\lim_{q \rightarrow +\infty} \lambda_k^{\varepsilon_q} \frac{\nabla u_k(x_q) \cdot x_q}{\|p_q\|} = 0.$$

For  $q \in \mathbb{N}$ , doing an inner product with  $x_q$  and dividing by  $\|p_q\|$  in the first equation of (5), we find:

$$\frac{p_q \cdot x_q}{\|p_q\|} = \sum_{k=1}^n \lambda_k^{\varepsilon_q} \frac{\nabla u_k(x_q) \cdot x_q}{\|p_q\|} + \frac{1}{\|p_q\|} \mu^{\varepsilon_q} \cdot x_q.$$

In view of (5), therefore,  $\mu^{\varepsilon_q} \cdot x_q = \varepsilon_q \mu^{\varepsilon_q} \cdot \mathbf{1} = \varepsilon_q \|\mu^{\varepsilon_q}\|$  converges to zero. So the right-hand side goes to zero. Therefore, the left-hand side goes to zero. Since  $(\hat{p}, \hat{v})$  belongs to  $\mathcal{E}$ ,  $\hat{p} \gg 0$  and the limit of the sequence  $(x_q)_{q \geq 0}$  is necessarily zero.

Let  $\bar{x} \in \mathbb{R}_{++}^\ell$ . We show that  $\bar{x}$  belongs to  $P(\hat{v})$ . For  $q$  sufficiently large, one has:  $\bar{x} \gg x_q$  and  $\bar{x} \gg \varepsilon_q \mathbf{1}$ . Thus, by monotony of the functions  $(u_k)_{k=1}^n$ , for all  $k \in \{1, \dots, n\}$ ,  $u_k(\bar{x}) > u_k(x_q) \geq v_{kq}$ . Hence,  $\bar{x}$  belongs to  $P(v_q)$  for  $q$  large enough. By continuity of the functions  $(u_k)_{k=1}^n$ ,  $\bar{x}$  belongs to  $P(\hat{v})$ . Since  $\bar{x}$  was arbitrarily chosen, we have:  $P(\hat{v}) = \mathbb{R}_{++}^\ell$ , which contradicts  $\hat{v} \in \mathcal{V}$ .

Consequently, there exists  $\bar{\varepsilon} > 0$  such that  $\mu^{\bar{\varepsilon}} = 0$ . Thus,  $\Delta^{\bar{\varepsilon}}(p', v')$  satisfies the necessary and sufficient conditions corresponding to Problem 1 for all  $(p', v')$  in  $\mathcal{E}$ . So  $\Delta = \Delta^{\bar{\varepsilon}}$  on  $\mathcal{E}$  and the continuity of  $\Delta$  follows. □

### 4 Properties of $\Delta$ and $e$ .

In Sect. 4.1, we study the properties of the function  $e$ . Section 4.2 concerns the Lipschitz behavior of  $\Delta$  with respect to  $(p, v)$ .

#### 4.1 Properties of $e$

- Proposition 4.1** 1. *The function  $e$  is concave in  $p$ .*  
 2. *The function  $e$  is twice differentiable a.e., and  $D^2e(p, v)$  is semi-definite negative when defined.*  
 3.  *$D_p e(p, v) = \Delta(p, v)$  and  $D_p^2 e(p, v) = D_p \Delta(p, v)$  when defined.*

*Proof* The proof is essentially borrowed from Rader [15]. The function  $-e$  is convex in  $p$  as a maximum of linear functions. For instance,  $e(p, v)$  can be defined by  $-e(p, v) = \max\{-p \cdot y : y \in \mathbb{R}_{++}^\ell, u_k(y) \geq v_k, k = 1, \dots, n\}$  for all  $(p, v) \in \mathbb{R}_{++}^\ell \times \mathcal{V}$ . So the function  $e$  is concave. By Alexandroff’s Theorem, the function  $e$  is twice differentiable a.e. in  $p$ , and its second derivative is semi-definite negative.

By Theorem 4(iii) of Rader [15],  $D_p e(p, v) = \Delta(p, v)$ . □

### 4.2 Lipschitz Behavior of $\Delta$

Firstly, for all  $(p, v) \in \mathbb{R}_{++}^\ell \times \mathcal{V}$ , we define the set  $M(p, v)$  by:

$$M(p, v) := \{k \in \{1, \dots, n\} : u_k(x) = v_k\}.$$

Secondly, we define the set  $\Pi$  by:

$$\Pi := \left\{ (p, v) \in \mathbb{R}_{++}^\ell \times \mathcal{V} : (\nabla u_k(\Delta(p, v)))_{k \in M(p, v)} \text{ are independent.} \right\}.$$

This set is an open subset of  $\mathbb{R}^\ell \times \mathbb{R}^n$  thanks to the continuity of  $\Delta$  on the open subset  $\mathbb{R}_{++}^\ell \times \mathcal{V}$ .

For  $(p, v) \in \Pi$ , the constraints of the optimization problem satisfy the Linear Independence Constraint Qualification(LICQ).<sup>14</sup> Thus, the multipliers are unique, and the following definition makes sense. For  $(p, v) \in \Pi$ , we proceed to define the set  $K(p, v)$  by:

$$K(p, v) := \{k \in \{1, \dots, n\} : \lambda_k(p, v) > 0\}.$$

From Proposition 3.1, one deduces that this set is nonempty. The cardinal of  $K(p, v)$  is denoted by  $\kappa(p, v)$ .

Finally, the set  $\tilde{\Pi}$  is defined as follows:

$$\tilde{\Pi} := \{(p, v) \in \Pi : K(p, v) = M(p, v)\}.$$

**Proposition 4.2** *The function  $\Delta$  and the multipliers  $(\lambda_k)_{k=1}^n$  are locally Lipschitz continuous on  $\Pi$ . Hence, the function  $\Delta$  is differentiable almost everywhere on  $\Pi$ .*

*Proof* This proposition happens to be a consequence of Corollary 2.3 in Cornet and Vial [6]. Actually, we prove that the function  $\Delta$  is locally Lipschitz on  $\Pi$  by verifying that Assumptions (A.0), (C.1) and (C.2) of Corollary 2.3. of [6] are satisfied. We define on  $\mathbb{R}_{++}^\ell \times \Pi$  the following functions:

- $f(x, p, v) := p \cdot x$ ,
- $g_k(x, p, v) := v_k - u_k(x)$  for  $k \in \{1, \dots, n\}$ .

For  $(p, v) \in \Pi$ ,  $\Delta(p, v)$  is the solution of the problem:

$$\begin{aligned} & \min f(x, p, v) \\ & \text{s.t.} \\ & g_k(x, p, v) \leq 0, \quad k = 1, \dots, n \end{aligned} \tag{6}$$

$x$  is the variable and  $(p, v)$  are the parameters.

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<sup>14</sup> See [11].

Assumptions (A.0) are satisfied. Indeed, we take  $U = \mathbb{R}_{++}^\ell$  and  $P = \Pi$ . The set  $U$  is an open set, and the set  $P$  is obviously a metric space. So Assumption (A.0) (i) is satisfied. Assumptions (A.0) (ii), (iii), (iv) and (v) are satisfied because the functions at stake are  $C^2$  on the set  $U \times P$ . Assumption (A.0) (vi) is satisfied with  $Q = C = -\mathbb{R}_+^n$ .

Assumption (C.1) is satisfied. This is an immediate consequence of the definition of  $\Pi$ .

Assumption (C.2) is satisfied. Let  $x$  be a solution of Problem (6) with a corresponding multiplier  $\lambda := (\lambda_k)_{k=1}^n$ . We shall verify that for all  $h \in \mathbb{R}^\ell, h \neq 0$  such that:  $\nabla f(x, p, v) \cdot h = 0$  and  $\nabla g_k(x, p, v) \cdot h = 0$  for  $k \in K(p, v)$ , we have:<sup>15</sup>

$$\left[ D^2 f(x, p, v) + \sum_{k \in K(p, v)} \lambda_k D^2 g_k(x, p, v) \right] h \cdot h > 0.$$

Observing that  $D^2 f \equiv 0$ , it remains to show that:

$$\sum_{k \in K(p, v)} \lambda_k D^2 g_k(x, p, v) h \cdot h > 0.$$

whence

$$- \sum_{k \in K(p, v)} \lambda_k D^2 u^k(x) h \cdot h > 0$$

which is true because of Assumption 2.1 and because  $\nabla u^k(x) \cdot h = 0$  for  $k \in K(p, v)$ .<sup>16</sup>

According to Corollary 2.3 in [6], the function  $\Delta$  is locally Lipschitz on  $\Pi$ . Thanks to Rademacher’s Theorem, the function  $\Delta$  is almost everywhere differentiable on  $\Pi$ . □

## 5 Differential Properties of $\Delta$

In this section, we interest ourselves in the continuous differentiability of  $\Delta$ . We conclude by a Slutsky-type property.

### 5.1 Continuous Differentiability of $\Delta$

**Proposition 5.1** *If  $(\bar{p}, \bar{v}) \in \tilde{\Pi}$ ,  $\Delta$  is continuously differentiable on a neighborhood of  $(\bar{p}, \bar{v})$ .*

*Proof* This proof is essentially an application of the Implicit Function Theorem and is quite standard borrowing ideas from Fiacco and McCormick[7]. Without loss of generality, suppose that:  $M(\bar{p}, \bar{v}) = \{1, \dots, r\}$ . Observe that in light of the continuity

<sup>15</sup> Thanks to Proposition 3.1,  $K(p, v)$  is nonempty.

<sup>16</sup> Indeed,  $K(p, v)$  is a subset of  $M(p, v)$

of both  $\Delta$  and the utility functions  $(u_k)_{k=1}^n$ , we can neglect the nonbinding constraints. Moreover, by continuity of the positive multipliers, one has:  $M(p, v) = M(\bar{p}, \bar{v})$  on a neighborhood of  $(\bar{p}, \bar{v})$ . As shown above, since the first-order optimality conditions are necessary and sufficient and in light of the continuity of the functions  $(u_k)_{k=1}^n$ , the element  $\Delta(p, v)$  and the corresponding multipliers  $\lambda(p, v)$  are solution of the equation  $G(x, \lambda, p, v) = 0$  where  $G$  is defined by:

$$G(x, \lambda, p, v) = \begin{cases} p - \sum_{k=1}^r \lambda_k \nabla u_k(x) \\ u_k(x) - v_k, \quad k = 1, \dots, r \end{cases} \tag{7}$$

To show that the function  $\Delta$  and the multipliers are continuously differentiable on a neighborhood of  $(\bar{p}, \bar{v})$ , from the Implicit Function Theorem, it suffices to show that the partial Jacobian matrix of  $G$  with respect to  $(x, \lambda)$  has full column rank<sup>17</sup> at  $\bar{x} := \Delta(\bar{p}, \bar{v})$ . This matrix is equal to:

$$M := \begin{bmatrix} -\sum_{k=1}^n \bar{\lambda}_k D^2 u^k(\bar{x}) - \nabla u_1(\bar{x}) & \dots & \dots & \dots & -\nabla u_r(\bar{x}) \\ \nabla u_1(\bar{x})^T & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \nabla u_r(\bar{x})^T & 0 & \dots & \dots & 0 \end{bmatrix}.$$

It is sufficient to prove that  $M \begin{pmatrix} \Delta x \\ \Delta \lambda \end{pmatrix} = 0$  implies:  $\Delta x = 0$  and  $\Delta \lambda = 0$ .  $\Delta x$  is a column vector of dimension  $\ell$ , and  $\Delta \lambda$  is a column vector of dimension  $r$ . We have to solve the system:

$$\begin{aligned} -\sum_{k=1}^r \bar{\lambda}_k D^2 u_k(\bar{x}) \Delta x - \sum_{k=1}^r \Delta \lambda_k \nabla u_k(\bar{x}) &= 0 \\ \nabla u_k(\bar{x}) \cdot \Delta x &= 0, \quad \forall k = 1, \dots, r \end{aligned}$$

Multiplying the first line by  $(\Delta x)^T$ , one has that:

$$\begin{aligned} -\sum_{k=1}^r \bar{\lambda}_k (\Delta x)^T D^2 u_k(\bar{x}) \Delta x - \sum_{k=1}^r \Delta \lambda_k \nabla u_k(\bar{x}) \cdot \Delta x &= 0 \\ \nabla u_k(\bar{x}) \cdot \Delta x &= 0, \quad \forall k = 1, \dots, r \end{aligned}$$

<sup>17</sup> The vectors are, by convention, column vectors, and the transpose of a vector  $x$  is denoted by  $x^T$ . We use the notation:  $\bar{\lambda} := \lambda(\bar{p}, \bar{v})$ .

whence

$$\begin{aligned}
 & - \sum_{k=1}^r \bar{\lambda}_k (\Delta x)^T D^2 u_k(\bar{x}) \Delta x = 0 \\
 & \nabla u_k(\bar{x}) \cdot \Delta x = 0, \quad \forall k \in \{1, \dots, r\}
 \end{aligned}$$

For all  $k \in \{1, \dots, r\}$ ,  $D^2 u_k(\bar{x})$  is negative definite on  $\nabla u_k(\bar{x})^\perp$  and since  $\Delta x \in \nabla u_k(\bar{x})^\perp$ , we find:  $\Delta x = 0$ . Hence, the first equation becomes:

$$- \sum_{k=1}^r \Delta \lambda_k \nabla u_k(\bar{x}) = 0$$

and we conclude that  $\Delta \lambda = 0$  since  $(\bar{p}, \bar{v}) \in \Pi$ . □

### 5.2 Slutsky-Type Property

The next result is a generalization of the well-known result about the negative definiteness of the Slutsky matrix.

**Proposition 5.2** *Suppose that  $(\bar{p}, \bar{v}) \in \tilde{\Pi}$ . The matrix  $D_p \Delta(\bar{p}, \bar{v})$  has rank  $\ell - \kappa(\bar{p}, \bar{v})$ , and its kernel is the linear space  $\mathcal{L}(\nabla u^k(\bar{x}), k \in M(\bar{p}, \bar{v}))$  spanned by the family  $(\nabla u_k(\bar{x}))_{k \in M(\bar{p}, \bar{v})}$  where  $\bar{x} := \Delta(\bar{p}, \bar{v})$ .*

*Proof* According to Proposition 4.1,  $D_p \Delta(\bar{p}, \bar{v})$  defines a symmetric negative semi-definite bilinear form. Observe that by continuity of  $\Delta$ , we can neglect the nonbinding constraints and, by continuity of the positive multipliers,  $M(p, v) = M(\bar{p}, \bar{v})$  on a neighborhood of  $(\bar{p}, \bar{v})$ .

Without loss of generality, suppose that  $M(\bar{p}, \bar{v}) = \{1, \dots, r\}$ . For  $p \in \mathbb{R}_{++}^\ell$  sufficiently near from  $\bar{p}$ ,  $\Delta(p, \bar{v})$  is characterized by the first-order conditions:<sup>18</sup>

$$\begin{aligned}
 & u_k(\Delta(p, \bar{v})) = \bar{v}^k, \quad k = 1, \dots, r \\
 & p = \sum_{k=1}^n \alpha_k(p) \nabla u_k(\Delta(p, \bar{v})) \text{ with } \alpha_k(p) > 0, \quad k = 1, \dots, r \tag{8}
 \end{aligned}$$

We proceed to differentiate the first condition with respect to  $p$  and readily obtain at  $\bar{p}$  for all  $q \in \mathbb{R}^\ell$ :

$$\nabla u_k(\Delta(\bar{p}, \bar{v})) \cdot D_p \Delta(\bar{p}, \bar{v})(q) = \nabla u_k(\bar{x}) \cdot D_p \Delta(\bar{p}, \bar{v})(q) = 0, \quad \forall k = 1, \dots, r.$$

From these equalities, we readily deduce that the image of  $D_p \Delta(\bar{p}, \bar{v})$  is contained in the linear subspace  $\cap_{k=1}^r \nabla u_k(\bar{x})^\perp$  of dimension  $\ell - r$ , recalling that  $(\bar{p}, \bar{v})$  belongs to  $\Pi$ . Furthermore, since  $D_p \Delta(\bar{p}, \bar{v})$  defines a symmetric negative semi-definite bilinear

<sup>18</sup> To simplify the notation, for all  $k = 1, \dots, r$ , we write  $\alpha_k(p) := \lambda(p, \bar{v})$ .

form,  $\nabla u_k(\bar{x})$  belongs to the kernel of  $D_p\Delta(\bar{p}, \bar{v})$  for all  $k = 1, \dots, r$ . Thus, the dimension of the image of  $D_p\Delta(\bar{p}, \bar{v})$  is at most  $\ell - r$ . Differentiating the second condition with respect to  $p$ , for  $q \in \mathbb{R}^\ell$ , we find:

$$q = \sum_{k=1}^r \alpha_k(p) D^2 u_k(\Delta(p, \bar{v})) D_p \Delta(p, \bar{v})(q) + \sum_{k=1}^n (\nabla \alpha_k(p) \cdot q) \nabla u_k(\Delta(p, \bar{v})).$$

In view of the previous equation, for all  $q \in \cap_{k=1}^r \nabla u_k(\bar{x})^\perp$ , we get:

$$q = \left[ \sum_{k=1}^r \alpha_k(\bar{p}) D^2 u_k(\bar{x}) \right] D_p \Delta(\bar{p}, \bar{v})(q).$$

Thus, we have for  $q \in \cap_{k=1}^r \nabla u_k(\bar{x})^\perp$ :

$$D_p \Delta(\bar{p}, \bar{v})(q) = 0 \implies q = 0.$$

So the kernel of the restriction on  $\cap_{k=1}^r \nabla u_k(\bar{x})^\perp$  of  $D_p\Delta(\bar{p}, \bar{v})$  is reduced to zero. As a consequence, the rank of  $D_p\Delta(\bar{p}, \bar{v})$  is at least  $\ell - r$ . Finally, the rank of  $D_p\Delta(\bar{p}, \bar{v})$  is equal to  $\ell - r$ , and the kernel of  $D_p\Delta(\bar{p}, \bar{v})$  is equal to  $\mathcal{L}(\nabla u_k(\bar{x}), k = 1, \dots, r)$ . □

### 6 Perspectives

Two important questions concern, respectively, the sets  $\Pi$  and  $\tilde{\Pi}$ . We wish to know under which conditions on the functions  $(u_k)_{k=1}^n$ , the set  $\Pi$  is “big” from a topological or measure-theoretical point of view. We already know that this set is an open set and a legitimate question would be under which conditions this set is dense and under which conditions it has full Lebesgue measure. When the number of constraints  $n$  is equal to 1, we obviously have:  $\Pi = \mathbb{R}_{++}^\ell \times \mathcal{V}$ . In view of the applications, we provide another framework in which the equality  $\Pi = \mathbb{R}_{++}^\ell \times \mathcal{V}$  holds true. Suppose that for  $k := 1, \dots, n$ ,

$$u_k(x) := \sum_{h=1}^\ell a_{kh} b_h(x_h)$$

where the functions  $(b_h)_{h=1}^\ell$  are twice continuously real-valued functions such that:  $b'_h > 0$  and  $b''_h < 0$ ,  $\lim_{x_h \rightarrow 0} b'(x_h) = 0$  and  $\sum_{h=1}^\ell a_h = 1$  for all  $k \in \{1, \dots, n\}$ . As it was shown in [16], the only requirement is that  $A := (a_{kh})_{1 \leq k \leq n, 1 \leq h \leq \ell}$  has full row rank. Such a framework can be related to decision theory through the Expected Utility Function (if  $b_h := b$  for all  $h \in \{1, \dots, \ell\}$ ) or to the separable preferences in microeconomics. We refer to [9] for a discussion about the Expected Utility Function and to [1] for a presentation of separable preferences. As a consequence, in these cases, the generalized Hicksian demand  $\Delta$  is locally Lipschitz on the whole set  $\mathbb{R}_{++}^\ell \times \mathcal{V}$ .

Similarly, we wish to find conditions under which the set  $\tilde{\Pi}$  is dense in  $\Pi$ . Under these conditions, the generalized Hicksian demand  $\Delta$  would be locally Lipschitz on the

open set  $\Pi$  and continuously differentiable on its open dense subset of full Lebesgue measure  $\tilde{\Pi}$ .

### 7 Conclusions

Existence and uniqueness of the solution of Problem (1) are established under mild assumptions. Without any additional assumption, the continuity of the solution with respect to the parameter is obtained. On the other hand, the Lipschitz behavior only requires (LICQ). We shall point out that this analysis can be carried out because we restricted ourselves to the relevant levels of utility. If Strict Complementary Slackness holds in addition at a point, the solution is continuously differentiable on a neighborhood of this point. Strangely enough, a Slutsky-type property is obtained in this case.

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### Appendix

**Lemma 8.1** *The set  $\mathcal{V}$  is an open set.*

*Proof* Let  $v^0 \in \mathcal{V}$ . We want to construct a neighborhood of  $v^0$  contained in  $\mathcal{V}$ . There exists  $x^0 \in \mathbb{R}^{\ell}_{++}$  such that:  $u_k(x^0) \geq v_k^0$  for all  $k = 1, \dots, n$ , and there exists  $y^0 \in \mathbb{R}^{\ell}_{++}$  and  $k_0 \in \{1, \dots, n\}$  such that:  $u_{k_0}(y^0) < v_{k_0}^0$ . Proceed to define  $\underline{v}$  by  $\underline{v} := v^0 - \mathbf{1}_n$  and  $\bar{v}$  by  $\bar{v}_k := u_k(x^0 + \mathbf{1}_\ell)$  for  $k = 1, \dots, n$ . Finally, we define the sets  $A := \{v \in \mathbb{R}^n : u_{k_0}(y^0) < v_{k_0}\}$  and  $B := \prod_{k=1}^n ]v_k, \bar{v}_k[$ . By construction, the set  $A \cap B$  is a nonempty open neighborhood of  $v^0$  contained in  $\mathcal{V}$ . Since  $v^0$  was arbitrarily chosen, one concludes that the set  $\mathcal{V}$  is an open set. □

**Lemma 8.2** *The sets of solutions of Problem (3) and Problem (4) coincide.*

*Proof* We first prove that the set of solutions of Problem (3) is a subset of the one of Problem (4). Let  $y$  be a solution of Problem (3). Since  $z_1$  is feasible for Problem (3), one finds:  $p \cdot y \leq p \cdot z_1$ . So  $y$  is feasible for Problem (4) and  $y$  is clearly a solution to Problem (4). In fact, the set of feasible points of Problem (4) is obviously contained in the set of feasible points of Problem (3).

Let  $y$  be a solution of Problem (4), and  $y$  is feasible for Problem (3) by construction. Let  $z$  be feasible for Problem (3), either  $-p \cdot z < -p \cdot z_1$  or  $-p \cdot z \geq -p \cdot z_1$ . In the first case, obviously,  $-p \cdot z \leq -p \cdot y$ . In the second case,  $z$  is feasible for Problem (4). Thus,  $-p \cdot z \leq -p \cdot z_1 \leq -p \cdot y$ . Since  $z$  was arbitrarily chosen, we conclude that  $y$  is a solution to Problem (3). □

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