J Optim Theory Appl (2016) 168:850–863
DOI 10.1007/s10957-015-0802-0

Stability of Set-Valued Optimization Problems with Naturally Quasi-Functions

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Received: 1 December 2014 / Accepted: 24 August 2015 / Published online: 3 September 2015
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Abstract In this paper, we discuss the stability of three kinds of minimal point sets and three kinds of minimizer sets of naturally quasi-functional set-valued optimization problems when the data of the approximate problems converges to the data of the original problems in the sense of Painlevé–Kuratowski. Our main results improve and extend the results of the recent papers.

Keywords Set-valued optimization problems · Stability analysis · Painlevé–Kuratowski convergence · Natural quasi-functions · Minimal point sets · Minimizer sets

Mathematics Subject Classification 49K40 · 90C29 · 90C31

1 Introduction

Stability is widely studied in optimization theory and methodology. In recent years, some papers have appeared, which are devoted to this topic for vector-valued optimization problems; see, e.g., [1–12] and references therein. Attouch and Riahi [1] first studied this topic. Lucchetti and Miglierina [4] discussed stability for a convex vector-valued optimization problem based on the concept of the continuous convergence of the vector-valued mappings. Oppezzi and Rossi [6,7] extended the definition of gamma-convergence from scalar-valued to vector-valued and applied it to this topic

Communicated by Jafar Zafarani.

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for a convex vector-valued problem. Lalitha and Chatterjee [9,10] further discussed this topic for a properly quasi-convex vector-valued problem and improved the main results of [6,7].

Also, set-valued optimization problems have been intensively investigated and applied to various problems, such as game theory, duality principles, robust optimization, and fuzzy optimization; see, e.g., [13–15] and the references therein. However, the studies dealing with the stability of set-valued optimization problems have so far been very limited [2,3]. The line of proof in [2,3] is similar to ones used for vector-valued optimization problems. To the best of our knowledge, the studies of stability based on the other ways for set-valued optimization problems are not available. Maybe, the main reason is that the notion of set-valued mappings is more complicated than vector-valued ones. Therefore, we must employ new analysis tools and techniques, which are different from the ones in [2,3], to discuss the stability of set-valued optimization problems.

The aim of this paper is to use some techniques similar to the ones of [4,6–11] to study the stability for set-valued optimization problems. Following the idea of [4,6–11], we first establish the Painlevé–Kuratowski convergence of three kinds of minimal point sets of naturally quasi-functional set-valued optimization problems. Furthermore, we investigate the Painlevé–Kuratowski set-convergence of three kinds of minimizer sets for set-valued optimization problems. Our results are extensions and improvements in the corresponding ones for vector-valued optimization problems in [9–11].

The paper is organized as follows. In Sect. 2, we recall some concepts and establish fundament results. In Sect. 3, we establish the Painlevé–Kuratowski set-convergence results of three kinds of minimal point sets of naturally quasi-functional set-valued optimization problems. In Sect. 4, we furthermore establish the Painlevé–Kuratowski set-convergence results of three kinds of minimizer sets. In Sect. 5, we provide a conclusion of the work presented.

2 Preliminaries

Throughout this paper, let P be a pointed (i.e., $P \cap (-P) = \{0\}$), closed and convex cone in $\mathbb{R}^l$ with nonempty interior. For a nonempty set D of $\mathbb{R}^l$, let $\operatorname{int} D$, ∂D and D^c denote the interior, boundary and complement of D, respectively. Let $\mathbb{B}(0, r)$ denote the closed ball centered at 0 and with radius r. Some fundamental terminologies are presented as follows.

Definition 2.1 Let D be a nonempty set of $\mathbb{R}^l$. A point $y \in D$ is said to be

(i) a minimal (resp. weak minimal) point of D iff $(D - y) \cap (-P) = \{0\}$ (resp. $(D - y) \cap (-\operatorname{int} P) = \emptyset$), and $\operatorname{Min}_P D$ (resp. $\operatorname{Min}_{\operatorname{int} P} D$) denotes the set of all minimal (resp. weak minimal) points of D;
(ii) a Henig proper minimal point of D iff there exists a convex cone P' with $\operatorname{int} P' \neq \emptyset$ such that $P \backslash \{0\} \subseteq \operatorname{int} P'$ and $(D - y) \cap (-P') = \{0\}$, and $\operatorname{Min}_P^H D$ denotes the set of all Henig properly minimal points of D.

It is easy to check that $\operatorname{Min}_P^H D \subseteq \operatorname{Min}_P D \subseteq \operatorname{Min}_{\operatorname{int} P} D$.

We now consider the following constrained set-valued optimization problem

$$\min_{x\in A} F(x), \qquad (A, F)$$

where $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ is a set-valued mapping and $A \subseteq \mathbb{R}^k$ is a nonempty set.

Based on the above notions of minimality for a set, we denote by $\mathrm{Min}_P F(A)$, $\mathrm{Min}_{\mathrm{int}\, P} F(A)$ and $\mathrm{Min}_P^H F(A)$ the sets of all minimal, weak minimal and Henig proper minimal points, respectively.

Definition 2.2 A pair (x_0, y_0) with $x_0 \in A$ and $y_0 \in F(x_0)$ is said to be a minimizer (resp. weak minimizer, Henig minimizer) of the problem (A, F) iff $y_0 \in \mathrm{Min}_P\, F(A)$ (resp. $y_0 \in \mathrm{Min}_{\mathrm{int}\, P}\, F(A)$, $y_0 \in \mathrm{Min}_P^H\, F(A)$). We denote by $\mathrm{E}_P\,(A, F)$, $\mathrm{E}_{\mathrm{int}\, P}\,(A, F)$ and $\mathrm{E}_P^H\,(A, F)$ the set of all minimizers, weak minimizers and Henig minimizers of the problem (A, F), respectively.

Definition 2.3 (*see* [5]) A sequence of sets $\{D_n\}$ of $\mathbb{R}^k$ converges to a set D in the sense of Painlevé–Kuratowski (for short, P.K.) convergence, denoted by $D_n \stackrel{\text{P.K.}}{\longrightarrow} D$, iff $\limsup_n D_n \subseteq D \subseteq \liminf_n D_n$, where

$$\liminf_n D_n := \{x \in \mathbb{R}^m : x = \lim_{n\to\infty} x_n, x_n \in D_n, \quad \forall n \in \mathbb{N}\} \text{ and}$$
$$\limsup_n D_n := \{x \in \mathbb{R}^m : x = \lim_{k\to\infty} x_k, x_k \in D_{n_k}, \quad \forall k, \{n_k\} \subseteq \mathbb{N}\}.$$

Definition 2.4 (*see* [3,16]) A sequence of set-valued mappings $F_n : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ $(n \in \mathbb{N})$ converges to $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ in the sense of P.K. convergence, denoted by $F_n \stackrel{\text{P.K.}}{\longrightarrow} F$, iff $\mathrm{epi} F_n \stackrel{\text{P.K.}}{\longrightarrow} \mathrm{epi} F$, where $\mathrm{epi} F := \{(x, z) \in \mathbb{R}^k \times \mathbb{R}^l : z \in F(x) + P\}$.

We introduce a virtual element $+\infty$ in $\mathbb{R}^l$ meaning that for any $y \in \mathbb{R}^l$, $+\infty \in y+P$. Then, we recall the following concept.

Definition 2.5 Let $F, F_n : \mathbb{R}^k \to \mathbb{R}^l (n \in \mathbb{N})$ be set-valued mappings, A, A_n $(n \in \mathbb{N})$ be sets in $\mathbb{R}^k$ and $\{(A_n, F_n) : n \in \mathbb{N}\}$ be the corresponding sequence pair. We say that (A_n, F_n) converges to (A, F) in the sense of P.K. convergence, denoted by $(A_n, F_n) \stackrel{\text{P.K.}}{\longrightarrow} (A, F)$, iff $\bar{F}_n \stackrel{\text{P.K.}}{\longrightarrow} \bar{F}$, where

$$\bar{F}_n(x) = \begin{cases} F_n(x), & x \in A_n; \\ \{+\infty\}, & x \in \mathbb{R}^k \setminus A_n \end{cases} \quad \text{and} \quad \bar{F}(x) = \begin{cases} F(x), & x \in A; \\ \{+\infty\}, & x \in \mathbb{R}^k \setminus A. \end{cases}$$

Definition 2.6 (*see* [17,18]) Let A be a nonempty and convex subset of $\mathbb{R}^k$. A set-valued mapping $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ is said to be:

(i) P-function (for short, P-F) on A iff for every $x_1, x_2 \in A$ and $\lambda \in [0, 1]$, $\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(x_\lambda) + P$, where $x_\lambda := \lambda x_1 + (1-\lambda)x_2$;

(ii) P-like-function (for short, P-LF) on A iff for every $x_1, x_2 \in A$ and $\lambda \in [0, 1]$, there exists $z \in A$ such that $\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(z) + P$;

(iii) properly quasi-P-function (for short, P-PQF) on A iff for every $x_1, x_2 \in A$ and $\lambda \in [0, 1]$, $F(x_1) \subseteq F(x_\lambda) + P$ or $F(x_2) \subseteq F(x_\lambda) + P$;

(iv) naturally quasi-P-function (for short, P-NQF) on A iff for every $x_1, x_2 \in A$, $y_1 \in F(x_1), y_2 \in F(x_2)$ and $\lambda \in [0, 1]$, there exists $\eta \in [0, 1]$ such that $\eta y_1 + (1 - \eta) y_2 \in F(x_\lambda) + P$.

We say F is strictly P-function (for short, P-SF) on A iff for every x_1, $x_2 \in A$, $x_1 \neq x_2$ and $\lambda \in]0, 1[$ such that the inequality of (i) holds whenever P is replaced by int P. Similarly, F is P-SLF, P-SPQF and P-SNQF on A.

Proposition 2.1 (see [17]) *The following statements hold*

(1) P-SF $\Rightarrow$ P-F $\Rightarrow$ P-LF and P-SF $\Rightarrow$ P-SLF $\Rightarrow$ P-LF;
(2) P-SPQF $\Rightarrow$ P-PQF $\Rightarrow$ P-NQF and P-SPQF $\Rightarrow$ P-SNQF $\Rightarrow$ P-NQF.

Proposition 2.2 (see [17]) *A set-valued mapping $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ is P-LF on a nonempty and convex subset A of $\mathbb{R}^k$ if and only if $F(A) + P$ is a convex set.*

Definition 2.7 (see [19]) A set-valued mapping $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ is said to be compact-valued (resp. convex-valued) on a set A in $\mathbb{R}^k$ if for any $x \in A$, $F(x)$ is a compact (resp. convex) subset of $\mathbb{R}^l$.

Lemma 2.1 (see [19]) *Let $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ be a set-valued map and $x_0 \in A$ of $\mathbb{R}^k$ be a given point. If F be compact-valued on A, then F is upper semicontinuous (for short, u.s.c) at $x_0 \in A$ if and only if for any sequence $\{x_n\} \subseteq A$ with $x_n \to x_0$ and for every $y_n \in F(x_n)$, there exist $y_0 \in F(x_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$.*

Lemma 2.2 (see [17]) *Let $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ be u.s.c. and convex-valued on a convex set A in $\mathbb{R}^k$. If F is P-NQF on A, then F is P-FL on A, i.e., $F(A) + P$ is a convex subset.*

Definition 2.8 (*see* [20]) A nonempty and convex subset A of $\mathbb{R}^k$ is said to be rotund if the boundary of A does not contain line segments, i.e., for any $x, x' \in A : x \neq x'$, $]x, x'[\cap (\partial A)^c \neq \emptyset$, where $]x, x'[:= \{\lambda x + (1 - \lambda) x' : \lambda \in]0, 1[\}$.

For the set-valued mapping, we introduce a similar concept as following.

Definition 2.9 A convex-valued mapping $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ is said to be rtound-valued on $A \subseteq \mathbb{R}^k$ iff for any $x \in A$, $F(x)$ is a rotund subset of $\mathbb{R}^l$.

Remark 2.1 If F is a single-valued mapping, then F is both rotund-valued and compact-valued on $A \subseteq \mathbb{R}^k$.

Motivated by the idea of Theorem 4.3 in [21], we get the following result.

Lemma 2.3 *If A is a convex set, F is P-SNQF and F is convex-valued and rotund-valued on A, then* $\mathrm{Min}_{\mathrm{int}\, P} F(A) = \mathrm{Min}_P F(A)$.

Proof It suffices to show that $\mathrm{Min}_{\mathrm{int}\, P} F(A) \subseteq \mathrm{Min}_P F(A)$. Let $y \in \mathrm{Min}_{\mathrm{int}\, P} F(A)$, hence there exists $x \in A$ such that

$$y \in F(x) \quad \text{and} \quad (F(A) - y) \cap (-\mathrm{int}\, P) = \emptyset. \tag{1}$$

By the contradiction, assume $y \notin \mathrm{Min}_P F(A)$, i.e., $(F(A) - y) \cap (-P) \neq \{0\}$. Then there exist $\bar{x} \in A$ and $\bar{y} \in F(\bar{x})$ such that $y \in \bar{y} + P\backslash\{0\}$. The following two cases would be considered: Case 1. If $x = \bar{x}$, then it follows from $y \in \bar{y} + P\backslash\{0\}$ and the rotundity of $F(x)$ that $]y, \bar{y}[\cap (\partial F(x))^c \neq \emptyset$, i.e., $]y, \bar{y}[\cap \mathrm{int}\, F(x) \neq \emptyset$. Hence there exists $y' \in]y, \bar{y}[\cap \mathrm{int}\, F(x)$. We can choose $\epsilon \in \mathrm{int}\, P$ such that $y' - \epsilon \in F(x) \cap (y - \mathrm{int}\, P)$, which contradicts (1).

Case 2. If $x \neq \bar{x}$, by the P-SNQF property of F, for every $\lambda \in]0, 1[$, there exist $\eta \in [0, 1]$ and $y' \in F(\lambda x + (1 - \lambda)\bar{x})$ such that $\eta y + (1 - \eta)\bar{y} \in y' + \mathrm{int}\, P$, which with $y \in \bar{y} + P\backslash\{0\}$ yields that $y \in y' + \mathrm{int}\, P$, which contradicts (1). □

Corollary 2.1 *Assume the conditions of Lemma 2.3 are satisfied. Then we have* $E_{\mathrm{int}\, P}(F, A) = E(F, A)$.

The following example is given to illustrate Lemma 2.3 and Corollary 2.1.

Example 2.1 Let $P = \mathbb{R}^2_+$, $A = \mathbb{R}$ and $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x) = \{(r, s) : x^2 \leq r \leq x^2 + 1, x^2 \leq s \leq x^2 + 1\}, \quad \forall x \in \mathbb{R}.$$

Clearly, we have: (i) F is P-SNQF on A; (ii) $\mathrm{Min}_P\, F(A) = \{(0, 0)\}$ and $\mathrm{Min}_{\mathrm{int}\, P}\, F(A) = \{(0, s) : 0 \leq s \leq 1\} \cup \{(r, 0) : 0 \leq r \leq 1\}$. So, Lemma 2.3 and Corollary 2.1 do not hold in the absence of rotundity of F.

Definition 2.10 For $\alpha \in \mathbb{R}^l$, the sublevel set of $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ on $A \subseteq \mathbb{R}^k$ at height α is $F^\alpha := \{x \in A : \alpha \in F(x) + P\}$.

For any $\alpha \in \mathbb{R}^l$, we have: (i) $F^\alpha = \bar{F}^\alpha$, where $\bar{F} : \mathbb{R}^k \rightrightarrows \mathbb{R}^l$ is defined as in Definition 2.5; (ii) F^α is a convex set whenever F is a P-NQF mapping.

Definition 2.11 (*see* [22]) For a convex set A of $\mathbb{R}^k$, the recession cone of A is the set $0^+(A) := \{d \in \mathbb{R}^m : a + td \in A, \quad \forall a \in A, \quad \forall t \geq 0\}$.

It is known that if A is a closed and convex set in $\mathbb{R}^k$, then $0^+(A) = \{0\}$ iff A is a bounded set. By Theorem 1.1.17 of [5], for any convex and closed set A in $\mathbb{R}^k$, $0^+(A) = \{d \in \mathbb{R}^k : \exists\, a \in A,\, a + td \in A, \quad \forall t \geq 0\}$.

Lemma 2.4 *Let A be a nonempty, closed and convex set in $\mathbb{R}^k$ and F be an u.s.c. P-NQF and compact-valued mapping on A. If $A \cap F^\alpha \neq \emptyset$ for $\alpha \in \mathbb{R}^l$, then $0^+(A \cap F^\alpha) = \{0\}$ if and only if $A \cap F^\alpha$ is bounded.*

Proof If $A \cap F^\alpha \neq \emptyset$ for $\alpha \in \mathbb{R}^l$, then $A \cap F^\alpha$ is obviously a convex set. To apply the remark above, we only need to verify that $A \cap F^\alpha$ is a closed set. Indeed, let $x_n \in A \cap F^\alpha$ with $x_n \to x \in A$ as A is closed. Then there exists $y_n \in F(x_n)$ such that $\alpha \in y_n + P$. By Lemma 2.1, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y \in F(x)$, and so $\alpha \in F(x) + P$. Therefore, $x \in A \cap F^\alpha$, which means that $A \cap F^\alpha$ is closed.

[$\Leftarrow$]. This implication follows from the remark above.

[$\Rightarrow$]. Assume to the contrary that there exists a sequence $\{x_n\}$ in $A \cap F^\alpha$ such that $\|x_n\| \to \infty$. By passing to a subsequence if necessary, we can assume that $\frac{x_n}{\|x_n\|} \to d$ and $\|d\| = 1$. Next we show that in such case $d \in 0^+(A \cap F^\alpha)$. Indeed, let $x \in A \cap F^\alpha$ be arbitrarily given. From the convexity of A, it follows that for any real number $t \geq 0$, $z_n := (1 - \frac{t}{\|x_n\|})x + \frac{t}{\|x_n\|}x_n \in A$ for sufficiently large n. It is easy to see that $z_n \to z := x + td \in A$ as A is closed. Since $x_n, x \in A \cap F^\alpha$, there exist $y_n \in F(x_n)$ and $y \in F(x)$ such that

$$\alpha \in y_n + P \quad \text{and} \quad \alpha \in y + P. \tag{2}$$

As F is P-NQF on A, there is $\eta \in [0, 1]$ such that $\eta y_n + (1 - \eta)y \in F(z_n) + P$, which together with (2) yields that $\alpha \in F(z_n) + P$, i.e., $z_n \in A \cap F^\alpha$. Hence, there exists $w_n \in F(z_n)$ such that $\alpha - w_n \in P$. By Lemma 2.1, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to w \in F(z)$. So, we have $\alpha - w \in P$. This implies that $\alpha \in F(z)+P$, i.e., $z \in F^\alpha$. Thus $z = x+td \in A\cap F^\alpha$ and $0 \neq d \in 0^+(A\cap F^\alpha) = \{0\}$, which is a contradiction. The proof is complete. □

Lemma 2.5 *Let A be a nonempty, closed and convex subset of $\mathbb{R}^k$ and F be an u.s.c., P-NQF and compact-valued mapping on A. If $0^+(A \cap F^\alpha) = \{0\}$ whenever $A \cap F^\alpha \neq \emptyset$ for $\alpha \in \mathbb{R}^l$, then $F(A) + P$ is a closed set.*

Proof Take a sequence $\{y_n\}$ in $F(A) + P$ such that $y_n \to y$. There exists a sequence $\{x_n\}$ in A such that $y_n \in F(x_n) + P$. By $y_n \to y$, for any $\epsilon \in \operatorname{int} P$ sufficiently large n, we have $y + \epsilon \in y_n + P$. Hence, $y + \epsilon \in F(x_n) + P$ for sufficiently large n. So, we have $x_n \in A \cap F^{y+\epsilon}$ and there exists $w_n \in F(x_n)$ such that $y + \epsilon - w_n \in P$. As $0^+(A\cap F^{y+\epsilon}) = \{0\}$, it follows from Lemma 2.4 that $\{x_n\}$ is bounded. By passing to a subsequence if necessary, we can assume that $x_n \to x \in A$ as A is closed. Therefore, from Lemma 2.1 it follows that there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to w \in F(x)$. So, by the arbitrariness of ϵ and the closedness of P, we have $y - w \in P$. This implies that $y \in F(x) + P$ and the proof is complete. □

3 Stability of the Minimal Point Sets

In the sequel, unless otherwise specified, let $F, F_n : \mathbb{R}^k \to \mathbb{R}^l (n \in \mathbb{N})$ be set-valued mappings and $A, A_n(n \in \mathbb{N})$ be nonempty, closed and convex sets in $\mathbb{R}^k$. In this section, we mainly investigate the P.K. convergence of three kinds of minimal point sets of set-valued optimization problems.

Proposition 3.1 *Let $A, A_n(n \in \mathbb{N})$ be nonempty, closed and convex subsets of $\mathbb{R}^k$. Let $F, F_n : \mathbb{R}^k \to \mathbb{R}^l (n \in \mathbb{N})$ be P-NQF mappings on A, A_n, respectively. Assume that: (i) $A_n \xrightarrow{\text{P.K.}} A$; (ii) $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$; (iii) if $A \cap F^\alpha \neq \emptyset$ for some $\alpha \in \mathbb{R}^l$, then $0^+(A \cap F^\alpha) = \{0\}$.*

Then $0^+(A_n \cap F_n^\alpha) = \{0\}$ for sufficiently large n whenever $A_n \cap F_n^\alpha \neq \emptyset$.

Proof Assume to the contrary that for some $\alpha \in \mathbb{R}^l$, we have $A_n \cap F_n^\alpha \neq \emptyset$ and $0^+(A_n \cap F_n^\alpha) \neq \{0\}$ for infinite number of indices n. Then, there exists a subsequence $\{d_k\}$ such that $d_k \in 0^+(A_{n_k} \cap F_{n_k}^\alpha)$, $d_k \to d$, $\|d_k\| = \|d\| = 1$. Let $x \in A$ be fixed, and let $\beta \in F(x)$. Clearly, $A \cap F^\beta \neq \emptyset$. As $\operatorname{int} P \neq \emptyset$, there exit $\epsilon \in \operatorname{int} P$ and $\lambda > 0$ such that $\lambda\epsilon \in \alpha + P$ and $\lambda\epsilon \in \beta + P$. Letting $\gamma = \lambda\epsilon$, we have

$\gamma \in \alpha + P$ and $\gamma \in \beta + P$. Clearly, $x \in A \cap F^{\gamma}$. Since $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, there exists a subsequence $\{(x'_k, \gamma_k)\}$ such that

$$(x'_k, \gamma_k) \to (x, \gamma), \tag{3}$$

$$x'_k \in A_{n_k} \quad \text{and} \quad \gamma_k \in F_{n_k}(x'_k) + P. \tag{4}$$

By (3), we have $\gamma_k \to \gamma$. Then, for any $e \in \operatorname{int} P$ and for sufficiently large k,

$$\gamma + e \in \gamma_k + P. \tag{5}$$

Then it follows from (4) and (5) that for sufficiently large k,$\gamma + e \in F_{n_k}(x'_k) + P$. This means that

$$x'_k \in A_{n_k} \cap F_{n_k}^{\gamma+e}. \tag{6}$$

Now, since $d_k \in 0^+(A_{n_k} \cap F_{n_k}^{\alpha}) \subseteq 0^+(A_{n_k} \cap F_{n_k}^{\gamma+e})$, which together with (6) yields that for any $\mu \geq 0$, $x'_k + \mu d_k \in A_{n_k} \cap F_{n_k}^{\gamma+e}$. By $(A_{n_k}, F_{n_k}) \xrightarrow{\text{P.K.}} (A, F)$, we have that $x'_k + \mu d_k \to x + \mu d \in A \cap F^{\gamma+e}$ for any $\mu \geq 0$. That implies that $0 \neq d \in 0^+(A \cap F^{\gamma+e}) = \{0\}$, which is a contradiction. □

Proposition 3.2 *Assume that all conditions of Proposition 3.1 are satisfied. Then for every $\alpha \in \mathbb{R}^l$ with $A \cap F^{\alpha} \neq \emptyset$ and for every $r > 0$, there exists $k_r \in \mathbb{N}$ such that, $A_n \cap F_n^{\alpha} \subseteq A \cap F^{\alpha} + \mathbb{B}(0, r), \quad \forall n \geq k_r$.*

Proof Suppose to the contrary that there exist $\alpha \in \mathbb{R}^l$, $r > 0$ with $A \cap F^{\alpha} \neq \emptyset$ and the conclusion is not true. Then there exists a subsequence $\{x_k\}$ such that

$$x_k \in A_{n_k} \cap F_{n_k}^{\alpha} \quad \text{and} \quad d(x_k, A \cap F^{\alpha}) > r, \tag{7}$$

where $d(x, A) := \inf_{a \in A} \|x - a\|$. If $\{x_k\}$ is bounded, by passing eventually to a subsequence, we have $x_k \to x_0 \in A$. Then, by $x_k \in A_{n_k} \cap F_{n_k}^{\alpha}$, we have $(x_k, \alpha) \in \operatorname{epi} \bar{F}_{n_k}$ and $(x_k, \alpha) \to (x_0, \alpha)$. From $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$ it follows that $(x_0, \alpha) \in \operatorname{epi} \bar{F}$. This implies that $x_0 \in A \cap F^{\alpha}$, which contradicts (7).

When $\{x_k\}$ is unbounded, we can assume that $\|x_k\| \to \infty$. Let $x' \in A \cap F^{\alpha}$ be given. By $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, there exists a subsequence $\{(x'_k, \gamma_k)\}$ in $\operatorname{epi} \bar{F}_{n_k}$ such that $(x'_k, \gamma_k) \to (x', \alpha)$. Clearly, for any k,

$$\gamma_k \in F_{n_k}(x'_k) + P. \tag{8}$$

By passing to a subsequence if necessary, we can assume that for any $t \geq 0$,

$$\frac{t}{\|x_k\|} x_k \to td \quad \text{and} \quad \|d\| = 1. \tag{9}$$

From the convexity of A_{n_k} it follows that

$$z_k := \left(1 - \frac{t}{\|x_k\|}\right) x_k' + \frac{t}{\|x_k\|} x_k \in A_{n_k} \text{ for } k \text{ sufficiently large.} \tag{10}$$

By (9), (10) and $A_n \xrightarrow{\text{P.K.}} A$, we have

$$z_k \to z := x' + td \in A. \tag{11}$$

Since $\gamma_k \to \alpha$, for any $\epsilon \in \operatorname{int} C$, there exists $k_\epsilon \in \mathbb{N}$ such that,

$$\alpha + \epsilon \in \gamma_k + P, \quad \forall k \geq k_\epsilon. \tag{12}$$

By (7), (8) and (12), there exist $y_k \in F_{n_k}(x_k)$ and $y_k' \in F_{n_k}(x_k')$ such that

$$\alpha \in y_k + P \quad \text{and} \quad \alpha + \epsilon \in y_k' + P. \tag{13}$$

As F_{n_k} is P-NQF on A_{n_k}, there exists $\eta \in [0, 1]$ such that

$$\eta y_k + (1 - \eta) y_k' \subseteq F_{n_k}(z_k) + P. \tag{14}$$

Combining (13) and (14), we have $\alpha + (1 - \eta)\epsilon \in F_{n_k}(z_k) + P$, which implies that $z_k \in A_{n_k} \cap F_{n_k}^{\alpha+(1-\eta)\epsilon}$. Then from $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$ and (11) it follows that $z \in A \cap F^{\alpha+(1-\eta)\epsilon}$. That is $0 \neq d \in 0^+(A \cap F^{\alpha+(1-\eta)\epsilon}) = \{0\}$, which is a contradiction. Thus, the conclusion holds and the proof is complete. □

Proposition 3.3 *Assume that all conditions of Proposition 3.1 hold and F is an u.s.c. and compact-valued map on A. Then $F_n(A_n) + P \xrightarrow{\text{P.K.}} F(A) + P$.*

Proof Firstly we prove that $F(A)+P \subseteq \liminf_n (F_n(A_n)+P)$. Let $y \in F(A)+P$ be arbitrarily given. Then there exists $x \in A$ such that $y \in F(x)+P$. Since $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, there exists a sequence $\{(x_n, y_n)\}$ in epi $\bar{F}_n$ such that $(x_n, y_n) \to (x, y)$. This means that $y \in \liminf_n (F_n(A_n) + P)$.

Secondly we prove that $\limsup_n (F_n(A_n) + P) \subseteq F(A) + P$. Take y in $\limsup_n (F_n(A_n) + P)$, hence there exists a subsequence $\{y_k\}$ in $F_{n_k}(A_{n_k}) + P$ such that $y_k \to y$. Then we can choose $x_k \in A_{n_k}$ such that $y_k \in F_{n_k}(x_k) + P$. Noting that $y_k \to y$, then for any $\epsilon \in \operatorname{int} P$, there exists $k_\epsilon \in \mathbb{N}$ such that $y + \epsilon \in y_k + P$, $\forall k > k_\epsilon$. So, $y + \epsilon \in F_{n_k}(x_k) + P$, $\forall k > k_\epsilon$. This means that $x_k \in A_{n_k} \cap F_{n_k}^{y+\epsilon}$. In virtue of Lemma 2.4 and Proposition 3.2, we have that $\{x_k\}$ is bounded. By passing to a subsequence if necessary, we assume that $x_k \to x \in A$. From $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, we get $(x_k, y_k) \to (x, y) \in \text{epi}\, F$, that is, $y \in F(x) + P \subseteq F(A) + P$. The proof is completed. □

Lemma 3.1 (see [4]) *Let $D_n (n \in \mathbb{N})$ and D be nonempty, closed and convex subsets of $\mathbb{R}^l$. Assume that $D_n \xrightarrow{\text{P.K.}} D$. Then*

$$(i) \quad \mathrm{Min}_P D \subseteq \liminf_n \mathrm{Min}_P D_n; \; (ii) \quad \mathrm{Min}_P^H D \subseteq \liminf_n \mathrm{Min}_P^H D_n.$$

Theorem 3.1 *Assume all conditions of Proposition 3.1 are satisfied and F, F_n $(n \in \mathbb{N})$ are u.s.c. and compact-valued mappings on A, $A_n (n \in \mathbb{N})$, respectively. Then*

$$(i)\mathrm{Min}_P F(A) \subseteq \liminf_n \mathrm{Min}_P F_n(A_n); \; (ii)\mathrm{Min}_P^H F(A) \subseteq \liminf_n \mathrm{Min}_P^H F_n(A_n).$$

Proof (i) From Proposition 3.3, we have $F_n(A_n) + P \xrightarrow{\text{P.K.}} F(A) + P$. Since F_n is an u.s.c., P-NQF and compact-valued map on A_n, by Lemmas 2.2 and 2.5, $F_n(A_n) + P$ is closed and convex for sufficiently large n. So, by Lemma 3.1 (i) and Proposition 3.3, we have $\mathrm{Min}_P(F(A) + P) \subseteq \liminf_n \mathrm{Min}_P(F_n(A_n) + P)$. As $\mathrm{Min}_P(A + P) = \mathrm{Min}_P A$ for any $A \subseteq \mathbb{R}^k$, $\mathrm{Min}_P F(A) \subseteq \liminf_n \mathrm{Min}_P F_n(A_n)$.

(ii) The proof follows on similar lines by using Lemma 3.1 (ii). □

Corollary 3.1 *Assume that all conditions of Theorem 3.1 hold and F is P-SNQF and rotund-valued on A. Then* $\mathrm{Min}_{\mathrm{int}\, P} F(A) \subseteq \liminf_n \mathrm{Min}_{\mathrm{int}\, P} F_n(A_n)$.

Proof By Lemma 2.3, we have $\mathrm{Min}_{\mathrm{int}\, P}\, F(A) = \mathrm{Min}_P\, F(A)$, which together with Theorem 3.1 yields that $\mathrm{Min}_{\mathrm{int}\, P}\, F(A) \subseteq \liminf_n \mathrm{Min}_P\, F_n(A_n)$. For any set $A_n \subseteq \mathbb{R}^k$, we have $\liminf_n \mathrm{Min}_P\, F_n(A_n) \subseteq \liminf_n \mathrm{Min}_{\mathrm{int}\, P}\, F_n(A_n)$. So, the conclusion is true and the proof is complete. □

Theorem 3.2 *If* $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, $\limsup_n \mathrm{Min}_{\mathrm{int}\, P} F_n(A_n) \subseteq \mathrm{Min}_{\mathrm{int}\, P} F(A)$.

Proof For any $y \in \limsup_n \mathrm{Min}_{\mathrm{int}\, P} F_n(A_n)$, there exists a subsequence $\{(x_k, y_k)\}$ in $\mathrm{E}_{\mathrm{int}\, P}\,(A_{n_k}, F_{n_k})$ such that $y_k \to y$. Suppose $y \notin \mathrm{Min}_{\mathrm{int}\, P}\, F(A)$. Then there exist $x_0 \in A$ and $y_0 \in F(x_0)$ such that $y_0 - y \in -\mathrm{int}\, P$. Let $\epsilon := y - y_0 \in \mathrm{int}\, P$. Since $y_k \to y$, there exists $k_\epsilon \in \mathbb{N}$ such that $k \geq k_\epsilon$,

$$y_k \in y - \frac{\epsilon}{4} + \mathrm{int}\, P. \tag{15}$$

As $(x_0, y_0) \in \mathrm{epi}\bar{F}$ and $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, there exists a sequence $\{(u_n, v_n)\}$ in $\mathrm{epi}\, \bar{F}_n$ such that $(u_n, v_n) \to (x_0, y_0)$. Then there exists $k'_\epsilon \geq k_\epsilon$ such that for any $n \geq k'_\epsilon$,

$$v_n \in y_0 + \frac{\epsilon}{2} - \mathrm{int}\, P = y - \frac{\epsilon}{2} - \mathrm{int}\, P. \tag{16}$$

Combining (15) and (16), we have for any $k \geq k'_\epsilon$,

$$v_k \in y_k - \frac{\epsilon}{4} - \mathrm{int}\, P \in y_k - \mathrm{int}\, P. \tag{17}$$

As $(u_k, v_k) \in \operatorname{epi} F_{n_k}$, there exists $v'_k \in F_{n_k}(u_k)$ such that $v'_k \in v_k - P$, which together with (17) yields that $v'_k \in y_k - \operatorname{int} P$. This contradicts the fact $(x_k, y_k) \in \mathrm{E}_{\operatorname{int} P}(A_{n_k}, F_{n_k})$. Therefore, $y \in \operatorname{Min}_{\operatorname{int} P} F(A)$ and the proof is complete. □

Corollary 3.2 *Suppose that* $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$ *and* F *is* P*-SNQF and rotund-valued on* A*. Then, we have* $\limsup_n \operatorname{Min}_P F_n(A_n) \subseteq \operatorname{Min}_P F(A)$.

Proof By Theorem 3.2 and Lemma 2.3, $\limsup_n \operatorname{Min}_{\operatorname{int} P} F_n(A_n) \subseteq \operatorname{Min}_P F(A)$. Then, we have $\limsup_n \operatorname{Min}_{\operatorname{int} P} F_n(A_n) \subseteq \operatorname{Min}_P F(A)$, which together with the fact $\limsup_n \operatorname{Min}_P F_n(A_n) \subseteq \limsup_n \operatorname{Min}_{\operatorname{int} P} F_n(A_n)$, $\forall A_n \subseteq \mathbb{R}^k$ yields that $\limsup_n \operatorname{Min}_P F_n(A_n) \subseteq \operatorname{Min}_P F(A)$. So, the proof is complete. □

By strengthening the conditions of Theorem 3.2, we can establish the upper part of convergence of Henig minimal point sets for perturbed problems.

Theorem 3.3 *Assume that all conditions of Proposition 3.1 are satisfied. Let* F, F_n $(n \in \mathbb{N})$ *be u.s.c.,* P*-SPQF and compact-valued maps on* A, $A_n (n \in \mathbb{N})$*, respectively. Furthermore, assume that* $\operatorname{epi} F$ *is a closed set. Then, we have*

$$\limsup_n \operatorname{Min}_P^H F_n(A_n) \subseteq \operatorname{Min}_P^H F(A).$$

Proof Let $y \in \limsup_n \operatorname{Min}_P^H F_n(A_n)$ be any given. By Theorem 3.2, we have $y \in \limsup_n \operatorname{Min}_P^H F_n(A_n) \subseteq \operatorname{Min}_{\operatorname{int} P} F(A) \subseteq F(A)$. Hence there exists $x \in A$ such that $y \in F(x)$. We claim that $y \in \operatorname{Min}_P^H F(A)$. By the contradiction, we suppose that for any pointed convex cone P_1 with $P \backslash \{0\} \subseteq \operatorname{int} P_1$, there exist $x_0 \in A$ and $y_0 \in F(x_0)$ such that

$$0 \neq z := y_0 - y \in -P_1. \tag{18}$$

As $y \in \limsup_n \operatorname{Min}_P^H F_n(A_n)$, there exists a sequence $\{y_k\}$ in $\operatorname{Min}_P^H F_{n_k}(A_{n_k})$ such that $y_k \to y$. Then there exits $x_k \in A_{n_k}$ such that $y_k \in F_{n_k}(x_k)$. Let $\epsilon \in \operatorname{int} P$ be an arbitrary element. As $y_k \to y$, it follows that $y + \epsilon \in F_{n_k}(x_k) + P$ for sufficiently large k. This means that $x_k \in A_{n_k} \cap F_{n_k}^{y+\epsilon}$. By Lemma 2.4 and Proposition 3.2, the sequence $\{x_k\}$ is bounded and has a convergent subsequence. Without loss of generality, we assume that $x_k \to \hat{x} \in A$. Since $(x_0, y_0) \in \operatorname{epi} \bar{F}$ and $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, there exists a sequence $\{(u_k, v_k)\}$ in $\operatorname{epi} \bar{F}_{n_k}$ such that $(u_k, v_k) \to (x_0, y_0)$. Then for any $\epsilon \in \operatorname{int} P$, there exists $k_\epsilon \in \mathbb{N}$ such that

$$y_0 + \epsilon \in F_{n_k}(u_k) + P, \text{ for any } k \geq k_\epsilon. \tag{19}$$

For any $k > 1$, set $s_k := \frac{1}{k} u_k + (1 - \frac{1}{k}) x_k$, which together with the convexity of A_{n_k} yields that $s_k \in A_{n_k}$. Noting that $A_n \xrightarrow{\text{P.K.}} A$, we have that $s_k \to \hat{x} \in A$. If $x_k = u_k$ for any k, then

$$\hat{x} = x_0 \text{ and } F(\hat{x}) = F(x_0) \ni y_0. \tag{20}$$

If $x_k \neq u_k$ for some k, then the P-SPQF property of F_{n_k} implies that, either $F_{n_k}(x_k) \subseteq F_{n_k}(s_k) + \text{int}\, P$ or $F_{n_k}(u_k) \subseteq F_{n_k}(s_k) + \text{int}\, P$. Since $y_k \in F_{n_k}(x_k)$ and $y_k \in \text{Min}_P^H F_{n_k}(A_{n_k})$, $F_{n_k}(x_k) \subseteq F_{n_k}(s_k) + \text{int}\, P$ does not hold. Therefore, we have $F_{n_k}(u_k) \subseteq F_{n_k}(s_k) + \text{int}\, P$ for any k. Then from (19) it follows that $y_0 + \epsilon \in F_{n_k}(s_k) + \text{int}\, P$, for any $k \geq k_\epsilon$. Then, $(s_k, y_0 + \epsilon) \in \text{epi}F_{n_k}$ and $(s_k, y_0 + \epsilon) \to (\hat{x}, y_0 + \epsilon)$. By the fact $(F_n, A_n) \xrightarrow{\text{P.K.}} (F, A)$, we have that $(\hat{x}, y_0 + \epsilon) \in \text{epi}F$. This implies that

$$y_0 + \epsilon \in F(\hat{x}) + P. \tag{21}$$

So, from (20) and (21) we always have $y_0 + \epsilon \in F(\hat{x}) + P$. By the arbitrariness of ϵ and the closedness of epi F, we have $y_0 \in F(\hat{x}) + P$, that is, there exists $p_0 \in P$ such that $y_0 - p_0 \in F(\hat{x})$.

The following two cases would be considered:

Case 1. If $x = \hat{x}$, then $y \in F(x) = F(\hat{x})$. We claim that $y \neq y_0 - p_0$. Otherwise, we have

$$z = y_0 - y = p_0 \in P \subseteq P_1. \tag{22}$$

Hence, from (18) and (22) it follows that $0 \neq z \in P_1 \cap (-P_1)$. As P_1 is a pointed cone, $0 \neq z \in P_1 \cap (-P_1) = \{0\}$, which is a contradiction. Thus $y \neq y_0 - p_0$, which with the rotundity of $F(\hat{x})$ yields that $]y, y_0 - p_0[\cap (\partial F(\hat{x}))^c \neq \emptyset$, that is, $]y, y_0 - p_0[\cap \text{int}\, F(\hat{x}) \neq \emptyset$. Hence there exists $y' \in]y, y_0 - p_0[\cap \text{int}\, F(\hat{x})$. Now we can choose $\epsilon \in \text{int}\, P$ such that $y' - \epsilon \in F(\hat{x}) \cap (y - \text{int}\, P)$, which contradicts the fact $y \in \text{Min}_{\text{int}\, P} F(A)$.

Case 2. If $x \neq \hat{x}$, then P-SPQF property of F implies that for any $\lambda \in]0, 1[$, either $F(x) \subseteq F(\lambda\hat{x} + (1-\lambda)x) + \text{int}\, P$ or $F(\hat{x}) \subseteq F(\lambda\hat{x} + (1-\lambda)x) + \text{int}\, P$. As $y \in F(x)$ and $y \in \text{Min}_{\text{int}\, P} F(A)$, $F(x) \subseteq F(\lambda\hat{x} + (1-\lambda)x) + \text{int}\, P$ does not hold. Hence, for any $\lambda \in]0, 1[$, $F(\hat{x}) \subseteq F(\lambda\hat{x} + (1-\lambda)x) + \text{int}\, P$, which together with $y_0 \in F(\hat{x}) + P$ yields that $y_0 \in F(\lambda\hat{x} + (1-\lambda)x) + \text{int}\, P$. Since epi F is closed, by taking limit as $\lambda \to 0_+$, we have $(x, y_0) \in \text{epi}\, F$. Then there exists $p_0' \in P$ such that $y_0 - p_0' \in F(x)$. Similar to the Case 1, we can show that $y \neq y_0 - p_0'$. By the rotundity of $F(x)$, we have $]y, y_0 - p_0'[\cap \text{int}\, F(x) \neq \emptyset$. Hence there exists $y' \in]y, y_0 - p_0'[\cap \text{int}\, F(x)$. Now we can choose $\epsilon \in \text{int}\, P$ such that $y' - \epsilon \in F(x) \cap (y - \text{int}\, P)$, which contradicts the fact $y \in \text{Min}_{\text{int}\, P} F(A)$.

So, from Cases 1 and 2 we can see that $y \in \text{Min}_P^H F(A)$. □

Summarizing Theorems 3.1 (i), 3.2 and Corollaries 3.1, 3.2, we have established the P.K. convergence of the (weak) minimal point sets.

Theorem 3.4 *Let A, $A_n (n \in \mathbb{N})$ be nonempty, closed and convex subsets of $\mathbb{R}^k$. Let $F : \mathbb{R}^k \to \mathbb{R}^l$ be an u.s.c., P-SNQF, compact-valued and rotund-valued mapping on A and $F_n : \mathbb{R}^k \to \mathbb{R}^l (n \in \mathbb{N})$ be u.s.c., P-NQF and compact-valued mappings on $A_n (n \in \mathbb{N})$. Assume that: (i) $A_n \xrightarrow{\text{P.K.}} A$; (ii) $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$; (iii) if $A \cap F^\alpha \neq \emptyset$*

for some $\alpha \in \mathbb{R}^l$, *then* $0^+(A \cap F^\alpha) = \{0\}$. *Then, we have*

$$(a)\quad \mathrm{Min}_P F_n(A_n) \xrightarrow{\text{P.K.}} \mathrm{Min}_P F(A);\quad (b)\quad \mathrm{Min}_{\mathrm{int}\, P} F_n(A_n) \xrightarrow{\text{P.K.}} \mathrm{Min}_{\mathrm{int}\, P} F(A).$$

Remark 3.1 Theorem 3.4 extends the corresponding ones of [4,6–11] in the following aspects: (i) Theorem 3.4 extends the corresponding ones of [4,6–11] from the vector-valued optimization problem to the set-valued case; (ii) The convexity of objective mappings F, $F_n (n \in \mathbb{N})$ in [4,6–9,11] are weakened to P-NQF; (iii) The convergence of objective mappings $F_n (n \in \mathbb{N})$ in [4,6,7,9,10] is weakened to P.K. convergence (see, Examples 3.2 and 3.3 in [11]).

The following example is given to illustrate Remark 3.1 (i) and (ii).

Example 3.1 Let $A_n = A = [0, 1]$ and $P = \mathbb{R}^2_+$. Let F_n, $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be respectively defined as $F_n(x) = (x^3, 1 - x^2 + \frac{1}{n})$ and $F(x) = (x^3, 1 - x^2)$. Indeed, we have $\mathrm{Min}_{\mathrm{int}\, P} F(A) = \mathrm{Min}_P F(A) = \{(x^3, 1 - x^2) : x \in [0, 1]\}$ and $\mathrm{Min}_{\mathrm{int}\, P} F_n(A_n) = \mathrm{Min}_P F_n(A_n) = \{(x^3, 1 - x^2 + \frac{1}{n}) : x \in [0, 1]\}$. Therefore, $\mathrm{Min}_{\mathrm{int}\, P} F_n(A_n) = \mathrm{Min}_P F_n(A_n) \xrightarrow{\text{P.K.}} \mathrm{WMin}_{\mathrm{int}\, P} F(A) = \mathrm{Min}_P F(A)$ and Theorem 3.4 is applicable.

However, the corresponding ones in [4,6–11] are not applicable. The main reasons are that F_n and F are neither P-PQF nor P-F. So, Theorems 3.4 is an improvement of the corresponding ones in [4,6–11].

Combining Proposition 2.1 (2), Theorems 3.1 (ii) and 3.3, we have established the P.K. convergence of the Henig minimal sets.

Theorem 3.5 *Let* A, A_n $(n \in \mathbb{N})$ *be nonempty, closed and convex subsets of* $\mathbb{R}^k$. *Let* F *be an u.s.c.,* P*-SPQF and compact-valued mapping on* A *and* F_n $(n \in \mathbb{N})$ *be* P*-SPQF and compact-valued maps on* $A_n (n \in \mathbb{N})$. *Assume that: (i)* $A_n \xrightarrow{\text{P.K.}} A$*; (ii)* $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$*; (iii) if* $A \cap F^\alpha \neq \emptyset$ *for some* $\alpha \in \mathbb{R}^l$, *then* $0^+(A \cap F^\alpha) = \{0\}$*; (iv) epi* F *is closed. Then, we have*

$$\mathrm{Min}^H_P F_n(A_n) \xrightarrow{\text{P.K.}} \mathrm{Min}^H_P F(A).$$

4 Stability of the Minimizer Sets

In this section, we investigate the P.K. convergence of three kinds of minimizer sets in the given space.

Theorem 4.1 *Suppose that all conditions of Theorem 3.4 hold. Then we have*

$$(a)\quad \mathrm{E}_{\mathrm{int}\, P}(A_n, F_n) \xrightarrow{\text{P.K.}} \mathrm{E}_{\mathrm{int}\, P}(A, F);\quad (b)\, \mathrm{E}_P(A_n, F_n) \xrightarrow{\text{P.K.}} \mathrm{E}_P(A, F).$$

Proof (a) Firstly, we show $\limsup_n \mathrm{E}_{\mathrm{int}\, P}(A_n, F_n) \subseteq \mathrm{E}_{\mathrm{int}\, P}(A, F)$. Let (x, y) in $\limsup_n \mathrm{E}_{\mathrm{int}\, P}(A_n, F_n)$, hence there exists a subsequence $\{(x_k, y_k)\}$ in $\mathrm{E}_{\mathrm{int}\, P}(A_{n_k}, F_{n_k})$ such that $(x_k, y_k) \to (x, y)$. We claim that $(x, y) \in \mathrm{E}_{\mathrm{int}\, P}(A, F)$. Otherwise, there exist $x' \in A$ and $y' \in F(x')$ such that $0 \neq y' - y \in -\text{int}\, P$. Then we can choose $\epsilon \in \text{int}\, P$ such that

$$y' - y + 2\epsilon \in -\text{int}\, P. \tag{23}$$

By the fact $y' \in F(x')$ and $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, there exists a sequence $(x'_k, y'_k) \in \text{epi}\, F_{n_k}$ such that $(x'_k, y'_k) \to (x', y')$. As $y_k \to y$ and $y'_k \to y'$, then for sufficiently large k and the above ϵ, we have $y_k \in y - \epsilon + \text{int}\, P$ and $y'_k \in y' + \epsilon - \text{int}\, P$, which together with (23) yields that $y'_k - y_k \in -\text{int}\, P$. As $(x'_k, y'_k) \in \text{epi}\, F_{n_k}$ for any k, there exists $w_k \in F_{n_k}(x'_k)$ such that $w_k \in y'_k - P$. So, $w_k - y_k \in -\text{int}\, P$, which contradicts $(x_k, y_k) \in \text{E}_{\text{int}\, P}(A_{n_k}, F_{n_k})$.

Secondly, we prove $\text{E}_{\text{int}\, P}(A, F) \subseteq \liminf_n \text{E}_{\text{int}\, P}(A_n, F_n)$. Taking (x, y) in $\text{E}_{\text{int}\, P}(A, F)$, we have $y \in F(x)$ and $y \in \text{Min}_{\text{int}\, P} F(A)$. By Theorem 3.2, there exists a sequence $\{y_n\}$ in $\text{Min}_{\text{int}\, P} F_n(A_n)$ such that $y_n \to y$. We can choose a sequence $\{x_n\}$ in A_n such that $y_n \in F_n(x_n)$ and $(x_n, y_n) \in \text{E}_{\text{int}\, P}(A_n, F_n)$. Let $\epsilon \in \text{int}\, P$ be any element. From the fact $F_n(x_n) \ni y_n \to y$, it follows that $y + \epsilon \in y_n + P \subseteq F_n(x_n) + P$ for sufficiently large n. This means that $x_n \in A_n \cap F_n^{y+\epsilon}$. By Lemma 2.4 and Proposition 3.2, the sequence $\{x_n\}$ is bounded and has a convergent subsequence. Without loss of generality, we assume that $x_n \to \hat{x} \in A$. Since $(A_n, F_n) \xrightarrow{\text{P.K.}} (A, F)$, we have that $\limsup_n \text{epi}\bar{F}_n \subseteq \text{epi}\bar{F}$. According to $(x_n, y_n) \in \text{epi}\bar{F}_n$ and $(x_n, y_n) \to (\hat{x}, y)$, we get that $(\hat{x}, y) \in \text{epi}\bar{F}$. That is, there exists $\hat{y} \in F(\hat{x})$ such that $y \in \hat{y} + P$.

Now we show that $x = \hat{x}$. Suppose to the contrary that $x \neq \hat{x}$. As F is P-SNQF on A and $y \in F(x)$ and $\hat{y} \in F(\hat{x})$, for any $\lambda \in]0, 1[$, there exists $\eta \in [0, 1]$ such that $\eta\hat{y} + (1 - \eta)y \in F(\lambda x + (1 - \lambda)\hat{x}) + \text{int}\, P$ which together with the fact $y \in \hat{y} + P$ yields $y \in F(\lambda x + (1 - \lambda)\hat{x}) + \text{int}\, P$. This contradicts the fact $(x, y) \in \text{E}_{\text{int}\, P}(A, F)$. Thus, every possible convergent subsequence of $\{x_n\}$ converges to x and hence the entire sequence $\{x_n\}$ converges to x. So, $(x_n, y_n) \in \text{E}_{\text{int}\, P}(A_n, F_n)$, $(x_n, y_n) \to (x, y)$ and $\text{E}_{\text{int}\, P}(A, F) \subseteq \liminf_n \text{E}_{\text{int}\, P}(A_n, F_n)$.

(b) The proof follows on similar lines of (a). □

Theorem 4.2 *Suppose that all conditions of Theorem 3.5 hold. Then we have*

$$\text{E}_P^H(A_n, F_n) \xrightarrow{\text{P.K.}} \text{E}_P^H(A, F).$$

Proof As the proof is similar to the one of Theorem 4.1, we omit it. □

5 Conclusions

This paper considered the stability of a P-NQF set-valued optimization problem based on the concept of P.K. convergence of the feasible sets and objective set-valued mappings. The results improved and extended the corresponding results of the recent papers. The generalization is threefold: The objective mappings are extended from vector-valued mappings to set-valued ones; the convexity of objective mappings is weakened to P-NQF; the convergence of the sequences of objective mappings is weakened to P.K. convergence.

Acknowledgments This research was partially supported by the National Natural Science Foundation of China (11201509 and 11271389), the Basic and Advanced Research Project of Chongqing (cstc2014jcyjA00046), the Education Committee Project Research Foundation of Chongqing (KJ1400304) and the Program for Core Young Teacher of the Municipal Higher Education of Chongqing ([2014]47). The authors would like to express their deep gratitude to the anonymous referees and the associate editor for their valuable comments and suggestions which helped to improve the paper.

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