

Lagrange Duality for Evenly Convex Optimization Problems

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Abstract An evenly convex function on a locally convex space is an extended real-valued function, whose epigraph is the intersection of a family of open halfspaces. In this paper, we consider an infinite-dimensional optimization problem, for which both objective function and constraints are evenly convex, and we recover the classical Lagrange dual problem for it, via perturbational approach. The aim of the paper was to establish regularity conditions for strong duality between both problems, formulated in terms of even convexity.

Keywords Evenly convex function · Generalized convex conjugation · Lagrange dual problem

Mathematics Subject Classification 52A20 · 26B25 · 90C25

1 Introduction

A subset of a locally convex space is called *evenly convex* (*e-convex*, in brief) iff it is the intersection of an arbitrary family (possibly empty) of open halfspaces. This class of sets was introduced in the finite-dimensional case by Fenchel [1] in order to extend the polarity theory to non-closed and convex sets. Some characterizations of this kind

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of convex sets were given in [2,3]. E-convex sets have been applied in the geometric analysis of linear inequality systems containing strict inequalities [3,4], while their basic properties, related with sections and projections, have been studied in [5].

In a usual way, e-convex sets allow the definition of *e-convex functions*, which were introduced in [6] as those extended real-valued functions, whose epigraphs are e-convex. In [7], it was defined an appropriate conjugation scheme (the *c-conjugation*) for such a kind of functions, in the sense of getting the equality of a proper e-convex function and its double conjugate. The inspiration came from the survey of Martínez-Legaz [8], where generalized convex duality theory, based on Fenchel–Moreau conjugation, is applied to quasiconvex programming.

As any closed and convex set is e-convex, the class of lower semicontinuous and convex functions (whose epigraphs are closed and convex sets) is a subclass of the e-convex functions. In [9], some well-known results for lower semicontinuous and convex functions were extended to the more general framework of e-convex functions, and moreover, thanks to the c-conjugation pattern, it was built a new kind of Fenchel-type dual problem for optimization problems, in which both feasible set and objective function are e-convex. Optimization problems for which both objective function and feasible sets are e-convex are called *e-convex* and have potential applications in mathematical economics in the same way that evenly quasiconvex optimization is used in [10].

In the recent paper [11], via perturbational approach, an alternative dual problem for a general infinite-dimensional optimization primal one was built by means of the new conjugation pattern, introduced in [7], and two sufficient regularity conditions for strong duality were achieved. In particular, an interior point regularity condition was obtained by exploiting the existing relationship between e-convexity and other closedness-type notions. Fenchel duality was derived as a particular case, and the regularity conditions were compared with another one obtained in [9].

In this paper, we consider an infinite-dimensional e-convex optimization problem, and using c-conjugation, we obtain the Lagrange dual problem for it. Moreover, we establish three regularity conditions for strong duality between both problems.

In [12], Goberna et al. characterized strong Lagrange duality in terms of the Farkas–Minkowski property for convex optimization problems, which have infinitely many convex inequalities as constraints. That property turned out to be a version, in that particular case, of the *closed cone constraint qualification (CCCQ)* introduced in [13] and formulated later in an alternative way in [14]. In this paper, we give a regularity condition, the so-called *e-convex cone constraint qualification (ECCQ)*, for strong Lagrange duality, which can be presented as the version of the (CCCQ) in our setting, since it is formulated in terms of the epigraphs of the c-conjugates of the indicator function of the feasible set, and the involved functions in the constraints. On the other hand, we present two more regularity conditions, obtained by particularizing the general approach in [11], being one of them weaker than the (ECCQ) condition.

The organization is as follows. In Sect. 2, we summarize the basic properties for e-convex sets and functions, as well as all the necessary tools and results, in order to make the paper self-contained. In particular, the conjugation scheme for e-convex functions will be reminded, as well as its most important properties. In Sect. 3, we consider an e-convex optimization problem and obtain its Lagrange dual problem. The

(ECCQ) regularity condition for strong duality will be derived in Sect. 4. In Sect. 5, the two general regularity conditions obtained in [11] will be reformulated and compared with the one introduced in Sect. 4.

2 Preliminaries

Let X be a separated locally convex space and X^* its topological dual space endowed with the weak* topology induced by X . For a set $D \subseteq X$ ($D \subseteq X^*$, respectively), the closure of D (the weak* closure of D , respectively) is denoted by $\text{cl } D$, and the notation δ_D will stand for the indicator function of D . By $\langle x, x^* \rangle$, we denote $x^*(x)$ for all $(x, x^*) \in X \times X^*$. According to [2], a set $C \subseteq X$ is e-convex if and only if, for every $x_0 \notin C$, there exists $x^* \in X^*$ such that $\langle x - x_0, x^* \rangle < 0$, for all $x \in C$. An application of Hahn–Banach theorem leads to claim that every open or closed and convex set is e-convex. Given $K \subseteq X$, the *e-convex hull* of K , denoted by $\text{econv } K$, is the smallest e-convex set that contains K . This operator is well defined because X is e-convex, and the class of e-convex sets is closed under intersection. Moreover, if K is convex, then $K \subseteq \text{econv } K \subseteq \text{cl } K$. Another property, which appears in [3] for finite-dimensional spaces and can also be shown easily in the infinite-dimensional case, is that the cartesian product of a finite number of e-convex sets is also an e-convex set in the product space. For a function $f : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we denote by $\text{dom } f := \{x \in X : f(x) < +\infty\}$ and $\text{epi } f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ the effective domain and the epigraph of f , respectively. We say that f is *proper* iff $\text{dom } f \neq \emptyset$ and $f(x) > -\infty$, for all $x \in X$. The *lower semicontinuous (lsc, in short) hull* of f , $\text{cl } f : X \rightarrow \overline{\mathbb{R}}$, is defined such that $\text{epi } (\text{cl } f) := \text{cl } (\text{epi } f)$, and f is said to be *lsc at* $x \in X$ iff $f(x) = (\text{cl } f)(x)$. On the other hand, we define the *e-convex hull* of f , $\text{econv } f : X \rightarrow \overline{\mathbb{R}}$, as the largest e-convex minorant of f , that is,

$$\text{econv } f := \sup \{g : g \text{ is e-convex and } g \leq f\}.$$

This function is e-convex since the class of e-convex functions is closed under pointwise supremum. According to [6, Prop. 3.1], if $f : X \rightarrow \overline{\mathbb{R}}$ is an e-convex function and $\alpha > 0$, then αf is an e-convex function, and by [6, Prop. 3.3], if $f, g : X \rightarrow \overline{\mathbb{R}}$ are two proper e-convex functions with $\text{dom } f \cap \text{dom } g \neq \emptyset$, then $f + g$ is also an e-convex function.

Definition 2.1 A function $a : X \rightarrow \overline{\mathbb{R}}$ is said to be *e-affine* iff there exists $x^*, y^* \in X^*$ and $\alpha, \beta \in \mathbb{R}$ such that

$$a(x) = \begin{cases} \langle x, x^* \rangle - \beta, & \text{if } \langle x, y^* \rangle < \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

For any $f : X \rightarrow \overline{\mathbb{R}}$, \mathcal{E}_f denotes the set of all the e-affine functions minorizing f , that is, $\mathcal{E}_f := \{a : X \rightarrow \overline{\mathbb{R}} : a \text{ is e-affine and } a \leq f\}$.

From [7, Prop. 5], we have that any e-affine function is an e-convex function. In the same way that, by the biconjugation theorem, any proper, lsc and convex function is

the pointwise supremum of a family of continuous and affine functions, any proper e-convex function is the pointwise supremum of a family of e-affine functions.

Lemma 2.1 ([7, Th. 16]) *Let $f : X \rightarrow \overline{\mathbb{R}}$, f not identically $+\infty$ or $-\infty$. Then, f is a proper e-convex function if and only if $f = \sup \{a : a \in \mathcal{E}_f\}$.*

The Fenchel conjugate of $f, f^* : X^* \rightarrow \overline{\mathbb{R}}$, is defined by

$$f^*(x^*) := \sup_{x \in X} \{ \langle x, x^* \rangle - f(x) \}. \tag{1}$$

From (1), the so-called Fenchel–Young inequality can be obtained

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle, \quad \forall x \in X \text{ and } \forall x^* \in X^*.$$

The classical Fenchel biconjugation theorem establishes the equivalence between a function f to be lsc convex, and the equality $f = f^{**}$. This theorem is not true for e-convex functions because if we take any e-convex non-lsc function f , then its biconjugate f^{**} is lsc convex and $f \neq f^{**}$. Based on the generalized convex conjugation theory, introduced by Moreau [15], a suitable conjugation scheme is provided in [7] for e-convex functions. Consider the space $W := X^* \times X^* \times \mathbb{R}$ with the coupling functions $c : X \times W \rightarrow \overline{\mathbb{R}}$ and $c' : W \times X \rightarrow \overline{\mathbb{R}}$ given by

$$c(x, (x^*, y^*, \alpha)) = c'((x^*, y^*, \alpha), x) := \begin{cases} \langle x, x^* \rangle, & \text{if } \langle x, y^* \rangle < \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a function $f : X \rightarrow \overline{\mathbb{R}}$, its c -conjugate $f^c : W \rightarrow \overline{\mathbb{R}}$ is defined by

$$f^c((x^*, y^*, \alpha)) := \sup_{x \in X} \{ c(x, (x^*, y^*, \alpha)) - f(x) \}.$$

Similarly, the c' -conjugate of a function $g : W \rightarrow \overline{\mathbb{R}}$ is the function $g^{c'} : X \rightarrow \overline{\mathbb{R}}$, given by

$$g^{c'}(x) := \sup_{(x^*, y^*, \alpha) \in W} \{ c'((x^*, y^*, \alpha), x) - g(x^*, y^*, \alpha) \}.$$

Throughout the paper, we adopt the conventions

$$(+\infty) + (-\infty) = (-\infty) + (+\infty) = (+\infty) - (+\infty) = (-\infty) - (-\infty) := -\infty.$$

Functions of the form $x \in X \mapsto c(x, (x^*, y^*, \alpha)) - \beta \in \overline{\mathbb{R}}$, with $(x^*, y^*, \alpha) \in W$ and $\beta \in \mathbb{R}$ are called c -elementary; in the same way, c' -elementary functions are those of the form

$$(x^*, y^*, \alpha) \in W \mapsto c(x, (x^*, y^*, \alpha)) - \beta \in \overline{\mathbb{R}},$$

with $x \in X$ and $\beta \in \mathbb{R}$. Note that c -elementary functions are actually e-affine functions.

In [7], it is shown that the family of the proper e -convex functions from X to $\overline{\mathbb{R}}$ along with the function identically equal to $-\infty$ is actually the family of pointwise suprema of sets of c -elementary functions. Using an analogous terminology, a function $g : W \rightarrow \overline{\mathbb{R}}$ is said e' -convex iff it is the pointwise supremum of sets of c' -elementary functions. Moreover, the e' -convex hull of any function $k : W \rightarrow \overline{\mathbb{R}}$ is the largest e' -convex minorant of k , and it is denoted by e' -conv k .

Lemma 2.2 ([16, Prop. 6.1 and 6.2, and Cor. 6.1]) *Let $f : X \rightarrow \overline{\mathbb{R}}$ and $g : W \rightarrow \overline{\mathbb{R}}$. Then,*

- (i) f^c is e' -convex; $g^{c'}$ is e -convex.
- (ii) If f has a proper e -convex minorant, then $e\text{conv } f = f^{cc'}$; e' -conv $g = g^{c'c}$.
- (iii) If f does not take on the value $-\infty$, then f is e -convex if and only if $f = f^{cc'}$; g is e' -convex if and only if $g = g^{c'c}$.
- (iv) $f^{cc'} \leq f$; $g^{c'c} \leq g$.

Definition 2.2 A set $D \subseteq W \times \mathbb{R}$ is e' -convex iff there exists an e' -convex function $k : W \rightarrow \overline{\mathbb{R}}$ such that $D = \text{epi } k$. The e' -convex hull of an arbitrary set $D \subseteq W \times \mathbb{R}$ is defined as the smallest e' -convex set containing D , and it is denoted by e' -conv D .

Definition 2.3 Consider two functions $f, g : X \rightarrow \overline{\mathbb{R}}$. A function $a : X \rightarrow \overline{\mathbb{R}}$ belongs to the set $\tilde{\mathcal{E}}_{f,g}$ iff there exists $a_1 \in \mathcal{E}_f, a_2 \in \mathcal{E}_g$ such that, if

$$a_1(\cdot) = \begin{cases} \langle \cdot, x_1^* \rangle - \beta_1, & \text{if } \langle \cdot, y_1^* \rangle < \alpha_1, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } a_2(\cdot) = \begin{cases} \langle \cdot, x_2^* \rangle - \beta_2, & \text{if } \langle \cdot, y_2^* \rangle < \alpha_2, \\ +\infty, & \text{otherwise,} \end{cases}$$

then

$$a(\cdot) = \begin{cases} \langle \cdot, x_1^* + x_2^* \rangle - (\beta_1 + \beta_2), & \text{if } \langle \cdot, y_1^* + y_2^* \rangle < \alpha_1 + \alpha_2, \\ +\infty, & \text{otherwise.} \end{cases}$$

We also associate with $\tilde{\mathcal{E}}_{f,g}$ the function $h_{f,g} : X \rightarrow \overline{\mathbb{R}}$, given by

$$h_{f,g} := \sup \{ a : a \in \tilde{\mathcal{E}}_{f,g} \}. \tag{2}$$

Lemma 2.3 ([9, Cor. 5]) *Let $f, g : X \rightarrow \overline{\mathbb{R}}$ be proper e -convex functions such that $\text{dom } f \cap \text{dom } g \neq \emptyset$, and let $h_{f,g}$ be the function defined in (2). Then, e' -conv $(\text{epi } f^c + \text{epi } g^c) = \text{epi } (f + g)^c$ if and only if $f + g = h_{f,g}$.*

One of the central objectives in optimization theory is the formulation of conditions, which guarantee strong duality between the primal problem

$$\inf F(x) \text{ s.t. } x \in X, \tag{GP}$$

where $F : X \rightarrow \overline{\mathbb{R}}$ is a proper function, and a dual one, whose definition can be given by means of the perturbational approach described first by Ekeland and Teman [17]. The key is to consider a *perturbation function* $\Phi : X \times \Theta \rightarrow \overline{\mathbb{R}}$, where Θ is also a separated locally convex space, named the *perturbation variable space*, such that

$\Phi(x, 0) = F(x)$, for all $x \in X$. In what we shall call a *classical framework*, a dual problem for (GP) can be built as follows:

$$\sup -\Phi^*(0, z^*) \text{ s.t. } z^* \in \Theta^*, \tag{GD}$$

where $\Phi^* : X^* \times \Theta^* \rightarrow \overline{\mathbb{R}}$ is the Fenchel conjugate of Φ , and Θ^* is the dual space of Θ .

A direct consequence of the Fenchel–Young inequality is that *weak duality* always holds, which means that the optimal value of (GP) , denoted by $v(GP)$, is greater than or equal to the optimal value of (GD) , denoted by $v(GD)$. Then, the challenge is to give conditions for the fulfillment of *strong duality*, the situation where both optimal values are equal and the dual problem is solvable. These conditions are called *regularity conditions*.

In the literature, there exist two main classes of this kind of conditions, named generalized interior point and closedness-type conditions. In [18], it provides an overview on some classical interior point regularity conditions from [17, 19, 20], as well as several new ones. We also mention [13] and [21], where we can find closedness-type regularity conditions for particular cases of (GP) and (GD) (see also [22] for a complete overview in this field). In most of these conditions, the lower semicontinuity and convexity of the proper function Φ are required, allowing that the Fenchel–Moreau theorem (or biconjugation theorem) can be applied.

As announced, this paper deals with Lagrange duality for e-convex optimization problems, continuing the work of weakening the requirement for the perturbation function to be lower semicontinuous and convex in the regularity conditions, which started in [9, 11].

3 Lagrange Dual Problem

Let us consider the optimization problem

$$(P) \quad \inf f(x) \text{ s.t. } g_t(x) \leq 0, \ t \in T, \tag{3}$$

where $f, g_t : X \rightarrow \overline{\mathbb{R}}, t \in T$, are proper e-convex functions defined on X , T is an arbitrary index set, and the feasible set, denoted by

$$A := \{x \in X : g_t(x) \leq 0, t \in T\},$$

is non-empty. Since the sublevel sets of an e-convex function are e-convex and the class of e-convex sets is closed under intersection, the feasible set of (P) is an e-convex set.

We consider the perturbation function $\Phi : X \times \mathbb{R}^T \rightarrow \overline{\mathbb{R}}$, defined by

$$\Phi(x, b) := \begin{cases} f(x), & \text{if } g_t(x) \leq b_t \ \forall t \in T, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $b \in \mathbb{R}^T$ is the perturbation variable. Letting $\sigma := \{g_t(x) \leq 0, t \in T\}$, we can reformulate σ as $\{g(x) \in -\mathbb{R}_+^T\}$, where $g : X \rightarrow \overline{\mathbb{R}}^T$ is defined as $g(x)(t) := g_t(x)$ for all $t \in T$, and $x \in X$. Then, the perturbation function Φ can be rewritten as

$$\Phi(x, b) = \begin{cases} f(x), & \text{if } g(x) - b \in -\mathbb{R}_+^T, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

C-conjugating Φ makes possible to associate with (P) a dual problem verifying weak duality. Let us observe that the c-conjugation pattern can be applied in a more general framework than the Fenchel one, in the sense that it is based on a chosen coupling function which is defined on $X \times Y$, where Y is an arbitrary set. In our case, $Y = \mathbb{R}^T$, and we take the so-called *space of generalized finite sequences*, denoted by $\mathbb{R}^{(T)}$, as the dual space of \mathbb{R}^T . Recall that $\lambda = (\lambda_t)_{t \in T}$ belongs to $\mathbb{R}^{(T)}$ iff λ has finite support, which means that only finitely many λ_t are different from zero. We consider the following dual product for $\lambda \in \mathbb{R}^{(T)}$ and $b \in \mathbb{R}^T$, $\lambda b := \sum_{t \in T} \lambda_t b_t$. Hence, letting $Z := X \times \mathbb{R}^T$, the appropriate coupling function for building the c-conjugate of Φ will be $c_1 : Z \times Z^* \times Z^* \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ such that

$$c_1((x, b), ((x^*, \lambda), (y^*, \beta), \alpha)) := \begin{cases} \langle x, x^* \rangle + \lambda b, & \text{if } \langle x, y^* \rangle + \beta b < \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

We have $\Phi^c : Z^* \times Z^* \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ and

$$\Phi^c((x^*, \lambda), (y^*, \beta), \alpha) = \sup_{(x,b) \in Z} \{c_1((x, b), ((x^*, \lambda), (y^*, \beta), \alpha)) - \Phi(x, b)\}.$$

It is easy to see that, for all $x \in X, \lambda, \beta \in \mathbb{R}^{(T)}$ and $\alpha > 0$, it holds

$$\Phi(x, 0) \geq -\Phi^c((0, \lambda), (0, \beta), \alpha),$$

leading us to formulate the dual problem

$$\sup -\Phi^c((0, \lambda), (0, \beta), \alpha) \text{ s.t. } \lambda, \beta \in \mathbb{R}^{(T)}, \alpha > 0, \tag{D_L}$$

which verifies $v(D_L) \leq v(P)$. Defining the *infimum value function*, $p : \mathbb{R}^T \rightarrow \overline{\mathbb{R}}$,

$$p(b) := \inf_{x \in X} \Phi(x, b),$$

it holds $p(0) = v(P)$, whereas $\Phi^c((0, \lambda), (0, \beta), \alpha) = p^c(\lambda, \beta, \alpha)$, for all $\alpha > 0$ and $\lambda, \beta \in \mathbb{R}^{(T)}$, where $p^c : \mathbb{R}^{(T)} \times \mathbb{R}^{(T)} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is built using the coupling function $c_2 : \mathbb{R}^T \times \mathbb{R}^{(T)} \times \mathbb{R}^{(T)} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ such that

$$c_2(b, (\lambda, \beta, \alpha)) := \begin{cases} \lambda b, & \text{if } \beta b < \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, (D_L) can be rewritten as

$$(D_L) \quad \sup -p^c(\lambda, \beta, \alpha) \text{ s.t. } \lambda, \beta \in \mathbb{R}^{(T)}, \alpha > 0, \tag{5}$$

Now, for all $\lambda, \beta \in \mathbb{R}^{(T)}$ and $\alpha > 0$, we have

$$\begin{aligned} p^c(\lambda, \beta, \alpha) &= \sup_{b \in \mathbb{R}^T} \{c_2(b, (\lambda, \beta, \alpha)) - p(b)\} \\ &= \sup_{\substack{x \in X, b \in \mathbb{R}^T, \\ g(x) - b \in -\mathbb{R}_+^T}} \{c_2(b, (\lambda, \beta, \alpha)) - f(x)\}. \end{aligned}$$

Denoting by $s := b - g(x) \in \mathbb{R}_+^T$, we get

$$p^c(\lambda, \beta, \alpha) = \sup_{x \in X, s \in \mathbb{R}_+^T} \{c_2(s + g(x), (\lambda, \beta, \alpha)) - f(x)\}.$$

Since

$$c_2(s + g(x), (\lambda, \beta, \alpha)) = \begin{cases} \lambda s + \lambda g(x), & \text{if } \beta s + \beta g(x) < \alpha, \\ +\infty, & \text{otherwise,} \end{cases}$$

we consider the following three cases:

Case 1: If $\text{dom } f \subseteq \{x \in X : \beta g(x) < \alpha\}$ and $\beta \in -\mathbb{R}_+^{(T)}$ then, for all $\lambda \in \mathbb{R}^{(T)}$,

$$p^c(\lambda, \beta, \alpha) = \sup_{x \in X, s \in \mathbb{R}_+^T} \{\lambda s + \lambda g(x) - f(x)\}.$$

Case 2: If $\text{dom } f \not\subseteq \{x \in X : \beta g(x) < \alpha\}$, then there exists $x_0 \in \text{dom } f$ such that $\beta g(x_0) \geq \alpha$, and denoting the null vector of \mathbb{R}^T as 0_T , we have $\beta 0_T + \beta g(x_0) \geq \alpha$ so that $c_2(s + g(x_0), (\lambda, \beta, \alpha)) = +\infty$, while $f(x_0) \in \mathbb{R}$ and, for all $\lambda \in \mathbb{R}^{(T)}$, $p^c(\lambda, \beta, \alpha) = +\infty$.

Case 3: If $\text{dom } f \subseteq \{x \in X : \beta g(x) < \alpha\}$ but $\beta \notin -\mathbb{R}_+^{(T)}$, then there exists $t_0 \in T$ such that $\beta_{t_0} > 0$. Taking any point $x_0 \in \text{dom } f$ and $s_0 \in \mathbb{R}_+^T$, verifying that βs_0 is large enough to get $\beta s_0 + \beta g(x_0) \geq \alpha$, we have, for all $\lambda \in \mathbb{R}^{(T)}$, $p^c(\lambda, \beta, \alpha) = +\infty$.

Now, for all (λ, β, α) verifying Case 1, we get

$$\begin{aligned} p^c(\lambda, \beta, \alpha) &= \sup_{s \in \mathbb{R}_+^T} \{\lambda s\} + \sup_{x \in X} \{\lambda g(x) - f(x)\} \\ &= \begin{cases} \sup_{x \in X} \{\lambda g(x) - f(x)\}, & \text{if } \lambda \in -\mathbb{R}_+^{(T)}, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned}$$

and therefore,

$$p^c(\lambda, \beta, \alpha) = \begin{cases} \sup_{x \in X} \{\lambda g(x) - f(x)\}, & \text{if } \beta g(x) < \alpha \forall x \in \text{dom } f \\ & \text{and } \lambda, \beta \in -\mathbb{R}_+^{(T)}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} \sup_{\substack{\lambda, \beta \in \mathbb{R}^{(T)}, \\ \alpha > 0}} -p^c(\lambda, \beta, \alpha) &= \sup_{\lambda \in -\mathbb{R}_+^{(T)}} \{-\sup_{x \in X} \{\lambda g(x) - f(x)\}\} \\ &= \sup_{\lambda \in \mathbb{R}_+^{(T)}} \{\inf_{x \in X} \{f(x) + \lambda g(x)\}\}, \end{aligned}$$

obtaining, finally, from (5) the Lagrange dual problem

$$\sup \{\inf_{x \in X} \{f(x) + \lambda g(x)\} \text{ s.t. } \lambda \in \mathbb{R}_+^{(T)}\}. \tag{D_L}$$

4 A Strong Duality Theorem for Evenly Convex Optimization

In the classical framework, Jeyakumar et al. in [13] introduced the so-called *closed cone constraint qualification (CCCQ)* in order to obtain strong duality between a convex optimization problem and its Lagrange dual problem. Later, in [14], (CCCQ) was reformulated in terms of the epigraphs of the Fenchel conjugates of the indicator function of the feasible set and the involved functions in the constraints, and it was proved that (CCCQ) implies strong Lagrange duality under weaker conditions than those considered in [13]. In this section, we introduce the so-called *e-convex cone constraint qualification (ECCQ)*, which can be viewed as a version of the (CCCQ) in our setting, and we prove that, under conditions related with e-convexity, (ECCQ) is a regularity condition for the dual pair $(P) - (D_L)$. From now on, W denotes the space $X^* \times X^* \times \mathbb{R}$.

Proposition 4.1 $\text{epi } \delta_A^c$ is the e' -convex hull of $\bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c$.

Proof First of all, it is easy to see that

$$\text{epi } \delta_A^c = \bigcap_{x \in A} \text{epi } c(x, \cdot). \tag{6}$$

Let $K := \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c$. Observe that, if $x \in A$, then $\lambda g(x) \leq 0$, for all $\lambda \in \mathbb{R}_+^{(T)}$. Therefore, $\lambda g \leq \delta_A$ and $\text{epi } (\lambda g)^c \subseteq \text{epi } \delta_A^c$, for all $\lambda \in \mathbb{R}_+^{(T)}$, which means that $K \subseteq \text{epi } \delta_A^c$. We shall show that $\text{epi } \delta_A^c \subseteq e'\text{-conv } K$, and since $\text{epi } \delta_A^c$ is an e' -convex set, we shall get $\text{epi } \delta_A^c = e'\text{-conv } K$.

We have that $e'\text{-conv } K = \text{epi } H$, where H is an e' -convex function, which, by definition, is the pointwise supremum of a certain set of c' -elementary functions, i.e., $H = \sup_{(x, \gamma) \in X_1 \times B} \{c(x, \cdot) - \gamma\}$, being $X_1 \times B$ a non-empty subset of $X \times \mathbb{R}$. Hence,

$$\text{epi } H = \bigcap_{(x, \gamma) \in X_1 \times B} \text{epi } \{c(x, \cdot) - \gamma\}. \tag{7}$$

Since $K \subseteq \text{epi } H$, taking $(x^*, y^*, \alpha, \beta) \in K$, we have $c(x, (x^*, y^*, \alpha)) - \gamma \leq \beta$, for all $(x, \gamma) \in X_1 \times B$, which means that $\langle x, x^* \rangle - \gamma \leq \beta$, for all $(x, \gamma) \in X_1 \times B$.

Observing that, for all $\delta > 0$, $\delta(x^*, y^*, \alpha, \beta) \in K$, we have

$$\delta \langle x, x^* \rangle - \gamma \leq \delta\beta, \tag{8}$$

for all $(x, \gamma) \in X_1 \times B$. Letting $\delta \rightarrow 0^+$, we get $\gamma \geq 0$, for all $\gamma \in B$, and $\text{epi } c(x, \cdot) \subseteq \text{epi } \{c(x, \cdot) - \gamma\}$, for all $x \in X_1$, implying that

$$\bigcap_{x \in X_1} \text{epi } c(x, \cdot) \subseteq \text{epi } H. \tag{9}$$

On the other hand, dividing in (8) by δ , we get $\langle x, x^* \rangle - \frac{\gamma}{\delta} \leq \beta$, for all $(x, \gamma) \in X_1 \times B$, or, equivalently, $c(x, (x^*, y^*, \alpha)) - \frac{\gamma}{\delta} \leq \beta$. Letting $\delta \rightarrow +\infty$, we get $c(x, (x^*, y^*, \alpha)) \leq \beta$, for all $x \in X_1$. Then, $K \subseteq \bigcap_{x \in X_1} \text{epi } c(x, \cdot)$, and, since

$\bigcap_{x \in X_1} \text{epi } c(x, \cdot)$ is an e' -convex set, we obtain

$$\text{epi } H = e' \text{-conv } K \subseteq \bigcap_{x \in X_1} \text{epi } c(x, \cdot). \tag{10}$$

Combining (10) with (9), we get

$$\text{epi } H = \bigcap_{x \in X_1} \text{epi } c(x, \cdot). \tag{11}$$

So, if we show that $X_1 \subseteq A$, from (6), then we shall arrive to the announced result $\text{epi } \delta_A^c = \bigcap_{x \in A} \text{epi } c(x, \cdot) \subseteq \bigcap_{x \in X_1} \text{epi } c(x, \cdot) = \text{epi } H$.

Take any point $x \in X_1$, and consider also a generalized sequence $\lambda \in \mathbb{R}_+^{(T)}$. Since $K \subseteq \text{epi } H$, from (11), we get $\text{epi } (\lambda g)^c \subseteq \text{epi } c(x, \cdot)$, and

$$c(x, (x^*, y^*, \alpha)) - (\lambda g)^c(x^*, y^*, \alpha) \leq 0,$$

for all $(x^*, y^*, \alpha) \in \text{dom } (\lambda g)^c$, getting to $(\lambda g)^{cc'}(x) \leq 0$, and since λg is an e -convex function, according to Proposition 2.2, $(\lambda g)(x) \leq 0$. The choice of the generalized sequence λ is indifferent, meaning that $x \in A$. □

Definition 4.1 σ verifies (ECCQ) iff $\bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c$ is an e' -convex set.

Remark 4.1 According to Proposition 4.1, we can reformulate

$$\bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c = \text{epi } \delta_A^c. \tag{ECCQ}$$

Proposition 4.2 σ verifies (ECCQ) if and only if, for all $(x^*, y^*, \alpha) \in W$ such that $A \subseteq \{x \in X : \langle x, y^* \rangle < \alpha\}$, it holds

$$\inf_{x \in A} c(x, (x^*, y^*, \alpha)) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \left\{ \inf_{x \in X} \{c(x, (x^*, y^*, \alpha)) + \lambda g(x)\} \right\}, \tag{12}$$

and there exists $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that

$$\inf_{x \in A} c(x, (x^*, y^*, \alpha)) = \inf_{x \in \text{dom } \bar{\lambda}g} \{-c(x, (-x^*, y^*, \alpha)) + \bar{\lambda}g(x)\}. \tag{13}$$

Proof Let us suppose that σ verifies (ECCQ) and take a point $(x^*, y^*, \alpha) \in W$ such that $A \subseteq \{x \in X : \langle x, y^* \rangle < \alpha\}$. Then, we consider the primal problem

$$\inf c(x, (x^*, y^*, \alpha)) \text{ s.t. } x \in A, \tag{P_1}$$

and its Lagrange dual problem

$$\sup \{ \inf_{x \in X} \{c(x, (x^*, y^*, \alpha)) + \lambda g(x)\} \} \text{ s.t. } \lambda \in \mathbb{R}_+^{(T)}. \tag{D_{1L}}$$

Since $v(P_1) \geq v(D_{1L})$, if $v(P_1) = -\infty$, then (12) holds, because $v(D_{1L}) = -\infty$ and $\inf_{x \in X} \{c(x, (x^*, y^*, \alpha)) + \lambda g(x)\} = -\infty$ for all $\lambda \in \mathbb{R}_+^{(T)}$. Taking into account that $-c(x, (-x^*, y^*, \alpha)) \leq c(x, (x^*, y^*, \alpha))$, for all $x \in \text{dom } \lambda g$, we obtain

$$\inf_{x \in \text{dom } \lambda g} \{-c(x, (-x^*, y^*, \alpha)) + \lambda g(x)\} \leq \inf_{x \in \text{dom } \lambda g} \{c(x, (x^*, y^*, \alpha)) + \lambda g(x)\},$$

with $\inf_{x \in \text{dom } \lambda g} \{c(x, (x^*, y^*, \alpha)) + \lambda g(x)\} = -\infty$. Hence, (13) also holds.

Let us assume that $v(P_1) \in \mathbb{R}$. Since $A \subseteq \{x \in X : \langle x, y^* \rangle < \alpha\}$, we have $(-x^*, y^*, \alpha, -v(P_1)) \in \text{epi } \delta_A^c$.

According to (ECCQ), there exists $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that

$$\sup_{x \in \text{dom } \bar{\lambda}g} \{c(x, (-x^*, y^*, \alpha)) - \bar{\lambda}g(x)\} \leq -v(P_1),$$

and, in addition, $\text{dom } \bar{\lambda}g \subseteq \{x \in X : \langle x, y^* \rangle < \alpha\}$, which allow us to write

$$\begin{aligned} v(P_1) &\leq \inf_{x \in \text{dom } \bar{\lambda}g} \{-c(x, (-x^*, y^*, \alpha)) + \bar{\lambda}g(x)\} \\ &= \inf_{x \in \text{dom } \bar{\lambda}g} \{c(x, (x^*, y^*, \alpha)) + \bar{\lambda}g(x)\} \leq v(D_{1L}) \leq v(P_1). \end{aligned}$$

Then, $v(P_1) = v(D_{1L})$ and $v(D_{1L}) = \inf_{x \in X} \{c(x, (x^*, y^*, \alpha)) + \bar{\lambda}g(x)\}$, so that (12) fulfills. Moreover, $v(P_1) = \inf_{x \in \text{dom } \bar{\lambda}g} \{-c(x, (-x^*, y^*, \alpha)) + \bar{\lambda}g(x)\}$, in such a way that (13) is also true.

For the converse statement, we shall prove that $\text{epi } \delta_A^c \subseteq \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c$.

Take any point $(x^*, y^*, \alpha, \beta) \in \text{epi } \delta_A^c$. Then, $\delta_A^c(x^*, y^*, \alpha) \leq \beta$, and moreover, $\delta_A^c(x^*, y^*, \alpha) < +\infty$. We can write

$$-\beta \leq -\sup_{x \in A} c(x, (x^*, y^*, \alpha)) = \inf_{x \in A} c(x, (-x^*, y^*, \alpha)). \tag{14}$$

Clearly, $A \subseteq \{x \in X : \langle x, y^* \rangle < \alpha\}$, whence

$$\inf_{x \in A} \{-c(x, (x^*, y^*, \alpha))\} = \inf_{x \in A} c(x, (-x^*, y^*, \alpha)).$$

By hypothesis, there exists $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that, from (12) and (13),

$$\begin{aligned} \inf_{x \in A} c(x, (-x^*, y^*, \alpha)) &= \inf_{x \in X} \{c(x, (-x^*, y^*, \alpha)) + \bar{\lambda}g(x)\} \\ &= \inf_{x \in \text{dom } \bar{\lambda}g} \{-c(x, (x^*, y^*, \alpha)) + \bar{\lambda}g(x)\}. \end{aligned} \tag{15}$$

Since

$$\begin{aligned} \inf_{x \in \text{dom } \bar{\lambda}g} \{-c(x, (x^*, y^*, \alpha)) + \bar{\lambda}g(x)\} &= -\sup_{x \in \text{dom } \bar{\lambda}g} \{c(x, (x^*, y^*, \alpha)) - \bar{\lambda}g(x)\} \\ &= -(\bar{\lambda}g)^c(x^*, y^*, \alpha), \end{aligned} \tag{16}$$

combining (14), (15), and (16), we get $-\beta \leq -(\bar{\lambda}g)^c(x^*, y^*, \alpha)$ and, consequently, $(x^*, y^*, \alpha, \beta) \in \text{epi } (\bar{\lambda}g)^c$. \square

Next theorem represents the main result of this section.

Theorem 4.1 *If σ verifies (ECCQ), $f + \delta_A = h_{f, \delta_A}$ and $\text{epi } f^c + \text{epi } \delta_A^c$ is e' -convex, then*

$$\inf_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \left\{ \inf_{x \in X} \{f(x) + \lambda g(x)\} \right\}.$$

Proof If $v(P) = -\infty$, the result holds trivially. Let $v(P) \in \mathbb{R}$. We write

$$v(P) = \inf_{x \in A} f(x) = \inf_{x \in X} \{f(x) + \delta_A(x)\}. \tag{17}$$

According to [9, Cor. 15], we obtain

$$\inf_{x \in X} \{f(x) + \delta_A(x)\} = -f^c(x^*, y^*, \alpha_1) - \delta_A^c(-x^*, -y^*, \alpha_2), \tag{18}$$

for certain $x^*, y^* \in X^*, \alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 > 0$.

As $v(P) \in \mathbb{R}$, by (17) and (18), $f^c(x^*, y^*, \alpha_1)$ and $\delta_A^c(-x^*, -y^*, \alpha_2)$ are also finite, whence

$$\delta_A^c(-x^*, -y^*, \alpha_2) = \sup_{x \in A} c(x, (-x^*, -y^*, \alpha_2)) < +\infty, \tag{19}$$

and

$$A \subseteq \{x \in X : \langle x, -y^* \rangle < \alpha_2\}, \tag{20}$$

implying that $c(x, (-x^*, -y^*, \alpha_2)) = -c(x, (x^*, -y^*, \alpha_2))$, for all $x \in A$. Thus,

$$\delta_A^c(-x^*, -y^*, \alpha_2) = \sup_{x \in A} \{-\langle x, x^* \rangle\} = -\inf_{x \in A} c(x, (x^*, -y^*, \alpha_2)). \tag{21}$$

Now, since $(ECCQ)$ and (20) hold, we apply Proposition 4.2 to obtain

$$\inf_{x \in A} c(x, (x^*, -y^*, \alpha_2)) = \inf_{x \in \text{dom } \bar{\lambda}g} \{-c(x, (-x^*, -y^*, \alpha_2)) + \bar{\lambda}g(x)\},$$

for a certain $\bar{\lambda} \in \mathbb{R}_+^{(T)}$, and since this infimum is finite, we have that $\text{dom } \bar{\lambda}g$ is contained in $\{x \in X : \langle x, -y^* \rangle < \alpha_2\}$ and

$$\inf_{x \in \text{dom } \bar{\lambda}g} \{-c(x, (-x^*, -y^*, \alpha_2)) + \bar{\lambda}g(x)\} = \inf_{x \in X} \{\langle x, x^* \rangle + \bar{\lambda}g(x)\}. \tag{22}$$

Recalling $f^c(x^*, y^*, \alpha_1)$ is finite, $\text{dom } f \subseteq \{x \in X : \langle x, y^* \rangle < \alpha_1\}$ and

$$\begin{aligned} f^c(x^*, y^*, \alpha_1) &= \sup_{x \in \text{dom } f} \{c(x, (x^*, y^*, \alpha_1)) - f(x)\} \\ &= -\inf_{x \in \text{dom } f} \{f(x) - \langle x, x^* \rangle\} = -\inf_{x \in X} \{f(x) - \langle x, x^* \rangle\}. \end{aligned} \tag{23}$$

From (17), (18), (21), (22), and (23), we get

$$\begin{aligned} v(P) &= \inf_{x \in X} \{f(x) - \langle x, x^* \rangle\} + \inf_{x \in X} \{\langle x, x^* \rangle + \bar{\lambda}g(x)\} \\ &\leq \inf_{x \in X} \{f(x) + \bar{\lambda}g(x)\} \leq v(D_L). \end{aligned}$$

□

5 Comparing Regularity Conditions

Let us denote the obtained regularity condition

$(C1_L)\sigma$ verifies $(ECCQ)$, $f + \delta_A = \sup \{a \mid a \in \tilde{\mathcal{E}}_{f, \delta_A}\}$ and $\text{epi } f^c + \text{epi } \delta_A^c$ is e' -convex.

In [11], two regularity conditions for strong duality for a general problem

$$\inf \Phi(x, 0) \text{ s.t. } x \in X, \tag{GP}$$

and its dual one obtained through c-conjugation

$$\sup -\Phi^c((0, u^*), (0, v^*), \alpha) \text{ s.t. } u^*, v^* \in \Theta^*, \alpha > 0, \tag{GD_c}$$

were found. One of them is a closedness-type condition, expressed in terms of the projection operator on $W \times \mathbb{R}$, $\text{Pr}_{W \times \mathbb{R}} : X^* \times \Theta^* \times X^* \times \Theta^* \times \mathbb{R} \rightarrow W \times \mathbb{R}$, and the second one, an interior point condition, is related to an extension of the classical interior of a set. We recall that, if $A \subseteq X$ is a non-empty and convex subset in X , then the *relative algebraic interior* of A is the set of all the points $a \in A$ verifying that cone $(A - a)$ is a linear subspace of X . It is denoted by iA . Moreover, a point $a \in {}^{ic}A$ iff cone $(A - a)$ is a closed and linear subspace of X (i.e., ${}^{ic}A = {}^iA$, if the affine hull of A is closed, and ${}^{ic}A = \emptyset$, otherwise).

The conditions are the following, keeping the numeration used in [11]:

(C4) X, Θ are Fréchet spaces, Φ is proper e-convex and $0 \in {}^{ic}(\text{Pr}_\Theta(\text{dom } \Phi))$.

(C5) Φ is a proper e-convex function, and $\text{Pr}_{W \times \mathbb{R}}(\text{epi } \Phi^c)$ is e' -convex.

Our aim now was to reformulate (C4) and (C5) for the dual pair $(P) - (D_L)$ analyzed in this paper. In that setting, $\Theta = \mathbb{R}^T$. This space is metrizable, and moreover, Fréchet with the product topology if and only if T is at most countable (see [23, 24] for more details). In order to reformulate condition (C4) and (C5), we give the following result.

Proposition 5.1 *Let $\Phi : X \times \mathbb{R}^T \rightarrow \overline{\mathbb{R}}$ be the perturbation function, defined as in (4). Then, Φ is proper and e-convex, and*

$$\text{Pr}_{\mathbb{R}^T}(\text{dom } \Phi) = g(\text{dom } f) + \mathbb{R}_+^T := \left\{ \bigcup_{x \in \text{dom } f, g(x) \in \mathbb{R}^T} g(x) \right\} + \mathbb{R}_+^T.$$

Proof Since $\text{dom } f \cap A \neq \emptyset$, there exists $x \in A$ such that $f(x) < +\infty$, and then $\Phi(x, 0) < +\infty$. Due to f is proper, Φ cannot take the value $-\infty$. Hence, it is proper.

We shall see that $\text{epi } \Phi$ is an e-convex set by expressing it in a convenient way. We have $\text{epi } \Phi = \{(x, b, \beta) \in X \times \mathbb{R}^T \times \mathbb{R} : f(x) \leq \beta, g(x) - b \in -\mathbb{R}_+^T\}$.

If we define the sets $C := \{(x, b, \beta) \in X \times \mathbb{R}^T \times \mathbb{R} : (x, \beta) \in \text{epi } f\}$ and $D := \{(x, b) \in X \times \mathbb{R}^T : g_t(x) \leq b_t, t \in T\}$, then $\text{epi } \Phi = C \cap (D \times \mathbb{R})$. Since $\text{epi } f$ and \mathbb{R}^T are e-convex, it is easy to see that C is e-convex.

On the other hand, if we take any point $(x_0, b_0) \notin D$, then there exists $\bar{t} \in T$ such that $(x_0, b_{0\bar{t}}) \notin \text{epi } g_{\bar{t}}$, which is an e-convex set in $X \times \mathbb{R}$. Hence, there exists $(x^*, \alpha) \in X^* \times \mathbb{R}$ such that $\langle x - x_0, x^* \rangle + (b_{\bar{t}} - b_{0\bar{t}})\alpha < 0$, for all $(x, b_{\bar{t}}) \in \text{epi } g_{\bar{t}}$. If we take $\lambda \in \mathbb{R}^{(T)}$ as

$$\lambda_t = \begin{cases} \alpha, & \text{if } t = \bar{t}, \\ 0, & \text{otherwise,} \end{cases}$$

then it holds that $\langle x - x_0, x^* \rangle + (b - b_0)\lambda < 0$, for all $(x, b) \in D$. Thus, D is e-convex, $D \times \mathbb{R}$ is e-convex, and $\text{epi } \Phi = C \cap (D \times \mathbb{R})$ is e-convex. Finally,

$$\begin{aligned} \text{Pr}_{\mathbb{R}^T}(\text{dom } \Phi) &= \left\{ b \in \mathbb{R}^T : \exists x \in \text{dom } f \text{ such that } b \in g(x) + \mathbb{R}_+^T \right\} \\ &= g(\text{dom } f) + \mathbb{R}_+^T. \end{aligned}$$

□

The above proposition and the product topology on \mathbb{R}^T , when T is at most countable, allow us to particularize the regularity conditions (C4) and (C5) for the dual pair $(P) - (D_L)$ in the following way:

- (C4_L) T is at most countable, X is a Fréchet space, and $0 \in {}^{ic} (g (\text{dom } f) + \mathbb{R}_+^T)$.
- (C5_L) $\text{Pr}_{W \times \mathbb{R}} (\text{epi } \Phi^c)$ is e' -convex.

Remark 5.1 According to Proposition 5.1 in [11], condition (C5) can alternatively be formulated as $\Phi (\cdot, 0)^c = \min_{\lambda, \beta \in \mathbb{R}^T} \Phi^c (\cdot, (\lambda, \beta))$.

Our objective in this section was to compare the regularity conditions (C4_L), (C5_L) and (C1_L). As we shall see, the unique relationship between them is that (C1_L) implies (C5_L).

Proposition 5.2 *Regularity condition (C1_L) implies (C5_L).*

Proof Following the same steps than in the proof of [11, Prop. 6.2], it holds $\text{Pr}_{W \times \mathbb{R}} (\text{epi } \Phi^c) \subseteq \text{epi } f^c + \text{epi } \delta_A^c$. Hence, proving the opposite inclusion, we conclude that $\text{Pr}_{W \times \mathbb{R}} (\text{epi } \Phi^c)$ is e' -convex, since, by hypothesis, $\text{epi } f^c + \text{epi } \delta_A^c$ is e' -convex.

Take any two points $(x_1^*, y_1^*, \alpha_1, \beta_1) \in \text{epi } f^c$ and $(x_2^*, y_2^*, \alpha_2, \beta_2) \in \text{epi } \delta_A^c$. Since (ECCQ) holds, we have $\bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c = \text{epi } \delta_A^c$. Hence, there exists $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $(x_2^*, y_2^*, \alpha_2, \beta_2) \in \text{epi } (\bar{\lambda} g)^c$. We obtain, for all $x \in X$,

$$c (x, (x_1^*, y_1^*, \alpha_1)) + c (x, (x_2^*, y_2^*, \alpha_2)) - f (x) - \bar{\lambda} g (x) \leq \beta_1 + \beta_2.$$

Taking into account that, for all $x \in X$,

$$c (x, (x_1^* + x_2^*, y_1^* + y_2^*, \alpha_1 + \alpha_2)) \leq c (x, (x_1^*, y_1^*, \alpha_1)) + c (x, (x_2^*, y_2^*, \alpha_2)),$$

we obtain that, for all $x \in X$,

$$c (x, (x_1^* + x_2^*, y_1^* + y_2^*, \alpha_1 + \alpha_2)) - f (x) - \bar{\lambda} g (x) \leq \beta_1 + \beta_2. \tag{24}$$

Let $(\bar{x}, \bar{b}) \in \text{dom } \Phi$. We have to find $\gamma, \delta \in \mathbb{R}_+^{(T)}$ such that

$$c ((\bar{x}, \bar{b}), (x_1^* + x_2^*, \gamma), (y_1^* + y_2^*, \delta), \alpha_1 + \alpha_2) - \Phi (\bar{x}, \bar{b}) \leq \beta_1 + \beta_2, \tag{25}$$

which means that $\Phi^c ((x_1^* + x_2^*, \gamma), (y_1^* + y_2^*, \delta), \alpha_1 + \alpha_2) \leq \beta_1 + \beta_2$, and then $(x_1^* + x_2^*, y_1^* + y_2^*, \alpha_1 + \alpha_2, \beta_1 + \beta_2) \in \text{Pr}_{W \times \mathbb{R}} (\text{epi } \Phi^c)$.

Let us observe that if $(\bar{x}, \bar{b}) \notin \text{dom } \Phi$, then inequality (25) will hold for all $\gamma, \delta \in \mathbb{R}_+^{(T)}$.

Now, since $(\bar{x}, \bar{b}) \in \text{dom } \Phi$, we have $\bar{x} \in \text{dom } f$ and $g (\bar{x}) - \bar{b} \in -\mathbb{R}_+^T$. Then, $g_t (\bar{x}) \leq \bar{b}_t$, for all $t \in T$ and $\bar{\lambda} g (\bar{x}) \leq \bar{\lambda} \bar{b}$. Using this inequality in (24), we get

$$c (\bar{x}, (x_1^* + x_2^*, y_1^* + y_2^*, \alpha_1 + \alpha_2)) - f (\bar{x}) - \bar{\lambda} \bar{b} \leq \beta_1 + \beta_2.$$

Taking $\gamma = -\bar{\lambda}$ and $\delta = 0$, we have

$$c_1((\bar{x}, \bar{b}), (x_1^* + x_2^*, \gamma), (y_1^* + y_2^*, \delta), \alpha_1 + \alpha_2) - f(\bar{x}) \leq \beta_1 + \beta_2,$$

and $f(\bar{x}) = \Phi(\bar{x}, \bar{b})$, so (25) fulfills. □

The following example shows that $(C5_L)$ does not imply $(C1_L)$.

Example 5.1 Let us take $X = \mathbb{R}$, $f = \delta_{[0, +\infty[}$ and $\sigma = \{tx \leq 0, t \in T\}$, where $T = [0, +\infty[$. We have $A =]-\infty, 0]$. In [11, Ex. 6.1], it was shown the equalities $f + \delta_A = h_{f, \delta_A}$ and $\text{epi } f^c + \text{epi } \delta_A^c = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+$, being the last set e' -convex. We shall see that $(ECCQ)$ does not hold, i.e., $\bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c \not\subseteq \text{epi } \delta_A^c$.

Since $\text{epi } \delta_A^c = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$ (see again [11, Ex. 6.1]), a point $(\alpha, \beta, \gamma, \delta) \in \text{epi } \delta_A^c$ verifies $\alpha \geq 0, \beta \geq 0, \gamma > 0$ and $\delta \geq 0$. This point will be in $\text{epi } (\lambda g)^c$ for some $\lambda \in \mathbb{R}_+^{(T)}$, if $c(x, (\alpha, \beta, \gamma)) - \lambda g(x) \leq \delta$, for all $x \in \text{dom } (\lambda g) = \mathbb{R}$, which implies that $\beta x < \gamma$, for all $x \in \mathbb{R}$, and this is impossible if $\beta \neq 0$. Hence, $(C1_L)$ does not fulfill.

We now show that $(C5_L)$ holds. The set $\text{Pr}_{W \times \mathbb{R}}(\text{epi } \Phi^c)$ is e' -convex if and only if $\text{epi } \Phi(\cdot, 0)^c \subseteq \text{Pr}_{W \times \mathbb{R}}(\text{epi } \Phi^c)$, since $\text{epi } \Phi(\cdot, 0)^c$ is its e' -convex hull, according to [11, Lemma 5.3]. Applying Lemma 2.3, we have that $\text{epi } \Phi(\cdot, 0)^c = \text{epi } (f + \delta_A)^c = \text{epi } f^c + \text{epi } \delta_A^c$. Hence, we will see that

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+ = \text{epi } f^c + \text{epi } \delta_A^c \subseteq \text{Pr}_{W \times \mathbb{R}}(\text{epi } \Phi^c).$$

Let $(\alpha, \beta, \gamma, \delta) \in \text{epi } f^c + \text{epi } \delta_A^c$, i.e., $\alpha \in \mathbb{R}, \beta \in \mathbb{R}, \gamma > 0$ and $\delta \geq 0$. We will prove that $(\alpha, \beta, \gamma, \delta) \in \text{Pr}_{W \times \mathbb{R}}(\text{epi } \Phi^c)$, finding $\lambda_1, \lambda_2 \in \mathbb{R}^{(T)}$ such that $\Phi^c((\alpha, \lambda_1), (\beta, \lambda_2), \gamma) \leq \delta$. Therefore, we have to find $\lambda_1, \lambda_2 \in \mathbb{R}^{(T)}$ such that, for all $(x, b) \in \text{dom } \Phi$, $c_1((x, b), (\alpha, \lambda_1), (\beta, \lambda_2), \gamma) \leq \delta$, or equivalently,

$$\beta x + \lambda_2 b < \gamma \text{ and } \alpha x + \lambda_1 b \leq \delta. \tag{26}$$

Since $\text{dom } \Phi = \{(x, b) \in \mathbb{R} \times \mathbb{R}^T : x \geq 0, b_t \geq tx, \forall t \geq 0\}$, taking in particular $x = 0$ and $b \in \mathbb{R}_+^T$, from (26), we deduce that $\lambda_1, \lambda_2 \in -\mathbb{R}_+^{(T)}$. Now, for $x > 0$ and $b_t \geq tx$, for all $t \geq 0$,

$$\beta x + \lambda_2 b = \beta x + \sum_{t \geq 0} \lambda_{2,t} b_t \leq \beta x + \sum_{t \geq 0} \lambda_{2,t} tx = x \left(\beta + \sum_{t \geq 0} t \lambda_{2,t} \right).$$

Forcing $(\beta + \sum_{t \geq 0} t \lambda_{2,t}) \leq 0$, we will have $\beta x + \lambda_2 b < \gamma$, and the question is if there exists $\lambda_2 \in -\mathbb{R}_+^{(T)}$ verifying that. However,

$$\left(\beta + \sum_{t \geq 0} t \lambda_{2,t} \right) \leq 0 \text{ if and only if } \sum_{t \geq 0} t \lambda_{2,t} \leq -\beta,$$

so that the choice of λ_2 only depends on the chosen β , which is fixed, and clearly, $\lambda_2 \in -\mathbb{R}_+^{(T)}$ can be found. If we now take the second inequality in (26), we also deduce that the choice of λ_1 only depends on the chosen α , which is fixed, and $\lambda_1 \in -\mathbb{R}_+^{(T)}$ can be calculated. We conclude that $(\alpha, \beta, \gamma, \delta) \in \text{Pr}_{W \times \mathbb{R}}(\text{epi } \Phi^c)$.

We continue with an example showing that $(C1_L)$ does not imply $(C4_L)$. Note that, with this example and Proposition 5.2, we also have that $(C5_L)$ does not imply $(C4_L)$.

Example 5.2 Take $X = \mathbb{R}$, $f = \delta_{[0, +\infty[}$ and $\sigma = \{tx + \delta_{]-\infty, t]}(x) \leq 0, t \in T\}$, being $T = \mathbb{N} \cup \{0\}$. We have $A =]-\infty, 0]$. Then, as in the previous example, $f + \delta_A = h_{f, \delta_A}$ and $\text{epi } f^c + \text{epi } \delta_A^c$ is e' -convex. For the fulfillment of $(C1_L)$, we only need to show that $(ECCQ)$ holds, i.e.,

$$\text{epi } \delta_A^c = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+ \subseteq \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c.$$

Take any point $(\alpha, \beta, \gamma, \delta) \in \text{epi } \delta_A^c$, with $\alpha \geq 0, \beta \geq 0, \gamma > 0$ and $\delta \geq 0$. Then, $(\alpha, \beta, \gamma, \delta) \in \text{epi } (\lambda g)^c$ for some $\lambda \in \mathbb{R}_+^{(T)}$, if $c(x, (\alpha, \beta, \gamma)) - \lambda g(x) \leq \delta$, for all $x \in \mathbb{R}$, i.e., $\text{dom } (\lambda g) \subseteq \{x \in \mathbb{R} : \beta x < \gamma\}$ and $\alpha x - \lambda g(x) \leq \delta$, for all $x \in \text{dom } (\lambda g)$. We distinguish two cases.

Case 1: If $\beta = 0$, it is enough to take $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$\lambda_t = \begin{cases} \alpha, & \text{if } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\text{dom } (\lambda g) = \text{dom } g_1 =]-\infty, 1] \subseteq \{x \in \mathbb{R} : \beta x < \gamma\} = \mathbb{R}$. Moreover, $\alpha x - \lambda g(x) = (\alpha - \alpha)x = 0 \leq \delta$, for all $x \leq 1$, since $\delta \geq 0$.

Case 2: If $\beta > 0$, then we will take $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$\lambda_t = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\text{dom } (\lambda g) = \text{dom } g_0 =]-\infty, 0] \subseteq \{x \in \mathbb{R} : \beta x < \gamma\}$. Moreover, it holds $\alpha x - \lambda g(x) = \alpha x \leq \delta$, for all $x \in \text{dom } (\lambda g)$, since $\alpha \geq 0$ and $\delta \geq 0$.

Then, we conclude that $\text{epi } \delta_A^c \subseteq \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi } (\lambda g)^c$. Now, we shall prove that $(C4_L)$ does not hold, i.e., $\text{cone}(g(\text{dom } f) + \mathbb{R}_+^T)$ is not a linear subspace of \mathbb{R}^T . Since $\text{dom } f = [0, +\infty[$, $x = 0$ is the only point verifying $g(0) \in \mathbb{R}^T$, then $g(\text{dom } f) + \mathbb{R}_+^T = \mathbb{R}_+^T$.

As $(C5_L)$ does not imply $(C4_L)$, from Proposition 5.2, we obtain that $(C4_L)$ does not imply $(C1_L)$. We finish with an example showing that $(C4_L)$ does not imply $(C5_L)$.

Example 5.3 Take $X = \mathbb{R}^2, \sigma = \{x_1 - tx_2 \leq 0, t \in T\}, T = \{\frac{1}{n}, n \in \mathbb{N}\}$. From the following function $\bar{f} : X \rightarrow \bar{\mathbb{R}}$

$$\bar{f}(x_1, x_2) = \begin{cases} \frac{x_2^2}{2x_1}, & \text{if } x_1 > 0, \\ 0, & \text{if } x_1 = x_2 = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

defined in [25, p. 51], where it is shown its properness and e-convexity, we build $f := 2\bar{f}$, a function which preserves both properties from \bar{f} . We have

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0, x_1 \leq x_2\}.$$

Clearly, (C4_L) holds, since $\text{dom } f = (\emptyset, +\infty[\times\mathbb{R}) \cup \{0_2\}$ and, taking into account that $g_t(x_1, x_2) = x_1 - tx_2$, for all $t \in T, g(\text{dom } f) + \mathbb{R}_+^T = \mathbb{R}^T$. Now, we shall use the equivalent condition to (C5_L), which appears in Remark 5.1, and we shall see that there exists at least a point $(x^*, y^*, \alpha) \in W$ such that $\Phi(\cdot, 0)^c(x^*, y^*, \alpha) < \min_{\lambda, \beta \in \mathbb{R}^{(T)}} \Phi^c((x^*, \lambda), (y^*, \beta), \alpha)$.

Let $y^* = (1, 0)$, any $x \in \mathbb{R}^2$ and $\alpha = 1$. Then,

$$\begin{aligned} \Phi(\cdot, 0)^c(x^*, y^*, \alpha) &= \sup_{x \in A \cap \text{dom } f} \{c(x, (x^*, y^*, \alpha)) - \Phi(x, 0)\} \\ &= \sup_{x=0_2} \{\langle x, x^* \rangle\} = 0. \end{aligned}$$

Now, take any $(\lambda, \beta) \in \mathbb{R}^{(T)} \times \mathbb{R}^{(T)}$. Then,

$$\Phi^c((x^*, \lambda), (y^*, \beta), \alpha) = \sup_{\substack{x \in \text{dom } f \\ g(x) - b \in -\mathbb{R}_+^T}} \{c_1((x, b), (x^*, \lambda), (y^*, \beta), \alpha) - f(x)\}.$$

Clearly, $c_1((x, b), (x^*, \lambda), (y^*, \beta), \alpha) < +\infty$ only if

$$\langle x, y^* \rangle + \sum_{t \in T} \beta_t b_t < \alpha, \tag{27}$$

for all (x, b) such that $x \in \text{dom } f$ and $g(x) - b \in -\mathbb{R}_+^T$. Since the point (x_0, b_0) , where $x_0 = (1, 0)$ and $b_0 = (b_{0,t})$, with $b_{0,t} = 1$ for all $t \in T$, belongs to the set $\{(x, b) \mid x \in \text{dom } f, g(x) - b \in -\mathbb{R}_+^T\}$, the fulfillment of (27) will force β not to be 0_T , and moreover, since b_t can be as large as we want, for all $t \in T$, it is also necessary for the fulfillment of (27) that $\beta \in -\mathbb{R}_+^{(T)}$.

Define $T_\beta := \{t \in T : \beta_t < 0\}$ and $\delta_\beta := \max\{\frac{1}{t} : t \in T_\beta\} > 0$. Let us observe that δ_β is well defined, since T_β is a non-empty finite set. Now, we take the point (x_β, b_β) , where $x_\beta = (1, \delta_\beta)$ and $b_\beta = (b_{\beta,t})$ is defined

$$b_{\beta,t} = \begin{cases} g_t(x_\beta), & \text{if } t \notin T_\beta, \\ 0, & \text{otherwise.} \end{cases}$$

It belongs to the set $\{(x, b) \mid x \in \text{dom } f, g(x) - b \in -\mathbb{R}_+^T\}$, and (27) will hold only if $1 + \sum_{t \in T} \beta_t b_{\beta,t} < 1$, which is impossible, since $\sum_{t \in T} \beta_t b_{\beta,t} = 0$. Hence, $c_1((x, b), (x^*, \lambda), (y^*, \beta), \alpha) = +\infty$, for all $(\lambda, \beta) \in \mathbb{R}^{(T)} \times \mathbb{R}^{(T)}$, and

$$\min_{\lambda, \beta \in \mathbb{R}^{(T)}} \Phi^c((x^*, \lambda), (y^*, \beta), \alpha) = +\infty.$$

6 Conclusions

In this paper, we have obtained the Lagrange dual problem for an e-convex optimization problem via perturbational approach by means of c-conjugation. We have established three regularity conditions for strong duality between both problems, formulated in terms of even convexity. The first condition we state can be viewed as the e-convex counterpart of the so-called closed cone constrained qualification related to a convex optimization problem in the classical context. Another two conditions are derived as particular cases of two already existing regularity conditions which were obtained in a previous work. We finally compare the three regularity conditions.

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