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An Existence Result for the Generalized Vector Equilibrium Problem on Hadamard Manifolds

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Abstract We present a sufficient condition for the existence of a solution to the generalized vector equilibrium problem on Hadamard manifolds using a version of the Knaster–Kuratowski–Mazurkiewicz Lemma. In particular, the existence of solutions to optimization, vector optimization, Nash equilibria, complementarity, and variational inequality problems is a special case of the existence result for the generalized vector equilibrium problem.

Keywords Vector equilibrium problem · Vector optimization · Hadamard manifold

AMS subject classification 90C33 · 65K05 · 47J25 · 91E10

1 Introduction

The generalized vector equilibrium problem (GVEP) has been widely studied and continues to be an active topic for research. One of the primary reasons for this is that multiple problems can be formulated as generalized vector equilibrium problems, such as optimization, vector optimization, Nash equilibria, complementarity, fixed point,

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and variational inequality problems. Extensive developments of these problems can be found in Fu [1], Fu and Wan [2], Konnov and Yao[3], Ansari et al. [4], Farajzadeh et al. [5], and the references therein. An important question concerns the conditions under which a solution to the GVEP exists. In a linear setting, multiple authors have provided results that answer this question, such as Ansari and Yao [7], Fu [1], Fu and Wan [2], Konnov and Yao[3], Ansari et al. [4], Farajzadeh et al. [5], and the authors referenced in their work. Moreover, it should be noted that Ky Fan studied inequalities in [6], which prompted present equilibrium theory.

Colao et al. [8] and Zhou and Huang [9] were the first authors to examine the existence of solutions for equilibrium problems in the Riemannian context by generalizing the Knaster–Kuratowski–Mazurkiewicz (KKM) Lemma to a Hadamard manifold. Applying the KKM Lemma in a Riemannian setting allowed Zhou and Huang [10] to confirm solution existence for vector optimization problems and vector variational inequalities in this context. Similarly, Li and Huang [11] presented results concerning solution existence for a special class of the GVEP. In this paper, we apply the KKM Lemma in a Riemannian setting in order to prove solution existence for the GVEP. To the best of our knowledge, our contribution is unprecedented. However, it should be noted that the results of this paper include the results presented in [8,10] and are not included in [11].

This paper is organized as follows. In Sect. 2, we present the notations and basic results used in the paper. Our main results are stated and proved in Sect. 3, and conclusions are discussed in Sect. 4.

2 Notations and Basics Definitions

In this paper, every manifold M is assumed to be Hadamard and finite dimensional. The notations, results, and concepts used throughout this paper can be found in Bento et al. [12].

A set $\Omega \subseteq M$ is said to be *convex* iff any geodesic segment with end points in Ω is contained in Ω, that is, iff $\gamma : [a, b] \to M$ is a geodesic such that $x = \gamma(a) \in \Omega$ and $y = \gamma(b) \in \Omega$; then $\gamma((1-t)a + tb) \in \Omega$ for all $t \in [0, 1]$. Given an arbitrary set, $\mathscr{B} \subset M$, the minimal convex subset that contains $\mathscr{B}$ is called the *convex hull* of $\mathscr{B}$ and is denoted by $\operatorname{conv}(\mathscr{B})$; see [8].

Suppose $\Omega \subseteq M$ is a convex set. Then for any set $\mathscr{A}$, we let $2^{\mathscr{A}}$ represent the set of all subsets of $\mathscr{A}$. Let $\Omega \subseteq M$ be a nonempty set and $\mathbb{Y}$ a topological vector space. Given a set-valued mapping $T : \Omega \rightrightarrows \mathbb{Y}$, the domain and range are the sets, respectively, defined by the following:

$$\operatorname{dom} T := \{x \in \Omega \ : \ T(x) \neq \varnothing\}, \quad \operatorname{rge} T := \{y \in \mathbb{Y} \ : \ y \in T(x) \text{ for some } x \in \Omega\}. \tag{1}$$

Moreover, the *inverse* of T is the set-valued mapping $T^{-1} : \mathbb{Y} \rightrightarrows \Omega$ defined by

$$T^{-1}(y) := \{x \in \Omega \ : \ y \in T(x)\}. \tag{2}$$

A set-valued mapping $T : \Omega \rightrightarrows \mathbb{Y}$ is said to be *upper semicontinuous* on Ω iff, for each $x_0 \in \Omega$ and any open set V in $\mathbb{Y}$ containing $T(x_0)$, there exists an open neighborhood U of x_0 in Ω such that $T(x) \subset V$ for all $x \in U$.

The following result is a version of the KKM lemma in Riemannian context due to [8], which is an extension of KKM theorem that can be found, for example, in [13].

Lemma 2.1 *Let $\Omega \subseteq M$ be a nonempty, closed, and convex set, and $G : \Omega \rightrightarrows \Omega$ a set-valued mapping, such that, for each $y \in \Omega$, $G(y)$ is closed. Suppose that there exists $y_0 \in \Omega$ such that $G(y_0)$ is compact, and for all $y_1, \ldots, y_m \in \Omega$, we have* $\operatorname{conv}(\{y_1, \ldots, y_m\}) \subset \bigcup_{i=1}^{m} G(y_i)$. *Then $\bigcap_{y \in \Omega} G(y) \neq \varnothing$.*

Proof See [8, Lemma 3.1]. □

3 Generalized Vectorial Equilibrium Problem

In this section, we present a sufficient condition for the existence of a solution to the generalized vector equilibrium problem on Hadamard manifolds. We should note that this material is motivated by the results found in [7]. Henceforth, we let $\Omega \subseteq M$ denote a nonempty, closed and convex set, and $\mathbb{Y}$ denote a metric vector space. Assume $C : \Omega \rightrightarrows \mathbb{Y}$ is a set-valued mapping such that

$$C(x) \text{ is a closed and convex cone}, \quad \operatorname{int} C(x) \neq \varnothing, \quad \forall\, x \in \Omega. \tag{3}$$

Also suppose $x \in \Omega$. A set-valued mapping $F : \Omega \times \Omega \rightrightarrows \mathbb{Y}$ is called $C(x)$ *-quasiconvex-like* iff for any geodesic segment $\gamma : [0, 1] \to \Omega$, either $F(x, \gamma(t)) \subseteq F(x, \gamma(0)) - C(x)$ or $F(x, \gamma(t)) \subseteq F(x, \gamma(1)) - C(x)$, for all $t \in [0, 1]$.

Example 3.1 Let $\mathbb{H}^2 := \left\{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > 0, \right\}$ be the two-dimensional hyperbolic space endowed with the Riemannian metric $g_{ij}(x_1, x_2) := \delta_{ij}/x_2^2$, for $i, j = 1, 2$. The curvature of $\mathbb{H}^2$ is $K = -1$, and the geodesics in $\mathbb{H}^2$ are semicircles centered on x_1-axis and vertical lines. Udriste discusses more details of this in [15]. In addition, assume that $F : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$ is the bifunction given by

$$F\left((x_1, x_2), (y_1, y_2)\right) = \left| y_1^2 + y_2^2 - x_1^2 - x_2^2 \right|.$$

Since, for every $c \in \mathbb{R}$, the sub-level set

$$L_{\psi,\Omega}(c) = \left\{(y_1, y_2) \in \mathbb{R}^2 \;:\; -c + x_1^2 + x_2^2 \leq y_1^2 + y_2^2 \leq c + x_1^2 + x_2^2,\; y_2 > 0\right\},$$

is convex in $\mathbb{H}^2$, where $\psi(y_1, y_2) = F((x_1, x_2), (y_1, y_2))$ and $(x_1, x_2) \in \Omega$ is a fixed point, we can conclude that F is $C(x)$-quasiconvex-like. It should be noted that F is not $C(x)$-quasiconvex-like in the Euclidean setting.

Given a set-valued mapping $F : \Omega \times \Omega \rightrightarrows \mathbb{Y}$, the *generalized vector equilibrium problem* (GVEP) in the Riemannian context consists in

$$\text{Find } x^* \in \Omega : \quad F(x^*, y) \nsubseteq -\operatorname{int} C(x^*), \quad \forall\, y \in \Omega. \tag{4}$$

Remark 3.1 Let $M = \mathbb{R}^n$, $\mathbb{Y} = \mathbb{R}^m$ and $\operatorname{int} C(x) = K$ for all $x \in \mathbb{R}^n$, where $K \subset \mathbb{R}^m$ is a closed pointed and convex cone such that $\operatorname{int} K \neq \varnothing$. Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, if $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is defined by $F(x, y) = f(y) - f(x)$, then we can transform the GVEP in (4) into the classic vector optimization problem $\min_K f(x)$; see [14].

Remark 3.2 Although variational inequality theory provides us with a toll for formulating multiple equilibrium problems, Iusem and Sosa [16, Proposition 2.6] demonstrated that the generalization given by equilibrium problem (EP) formulation with respect to variational inequality (VI) is genuine, meaning there are EP formulations that do not fit the format of a VI. When compared with VIs, EP formulations may also guarantee genuineness by considering the important class of quasiconvex optimization problems, which appear, for instance, in many microeconomical models that are devoted to maximizing utility. Indeed, the absence of convexity allows us to obtain situations in which this important class of problems cannot be considered to be a VI because its possible representation given this format produces a problem whose solution set contains points that do not necessarily belong to the solution set of the original optimization problem. For example, let $\Omega \subseteq M$ be a nonempty, closed and convex set, and $f : M \to \mathbb{R}$ be a differentiable and $(\mathbb{R}_+)$-quasiconvex-like function. Then consider the following optimization problem:

$$\text{Find } x^* \in \Omega : \quad f(y) - f(x^*) \notin -\operatorname{int} \mathbb{R}_+, \quad \forall\, y \in \Omega. \tag{5}$$

Note that, if $F : \Omega \times \Omega \to \mathbb{R}$ is the bifunction given by $F(x, y) = f(y) - f(x^*)$, then the optimization problem in (6) is equivalent to the following equilibrium problem:

$$\text{Find } x^* \in \Omega : \quad F(x^*, y) \notin -\operatorname{int} \mathbb{R}_+, \quad \forall\, y \in \Omega. \tag{6}$$

On the other hand, in the absence of convexity, the optimization problem in (6) is not equivalent to the associated variational inequality,

$$\text{Find } x^* \in \Omega : \quad \left\langle \nabla f(x^*), y - x^* \right\rangle \notin -\operatorname{int} \mathbb{R}_+, \quad \forall\, y \in \Omega,$$

because, for instance, point $x^* \in \Omega$, in which $\nabla f(x^*) = 0$, is a solution to this variational inequality, but it cannot be a solution to the equilibrium problem in (6).

The following result is closed related to [7, Theorem 2.1] and establishes an existence result of solution for GVEP as an application of Lemma 2.1.

Theorem 3.1 *Let $F : \Omega \times \Omega \rightrightarrows \mathbb{Y}$ be a set-valued mapping such that, for each $x, y \in \Omega$, we have:*

h1. *$F(x, x) \not\subset -\operatorname{int} C(x)$;*
h2. *$F(\cdot, y)$ is upper semicontinuous;*
h3. *F is $C(x)$-quasiconvex-like;*
h4. *there exist $D \subset \Omega$ compact and $y_0 \in \Omega$ such that $x \in \Omega \backslash D \Rightarrow F(x, y_0) \subset -\operatorname{int} C(x)$.*

Then, the solution set, S^*, of the GVEP defined in (4) is a nonempty compact set.

Remark 3.3 In particular, when $M = \mathbb{R}^n$, our problem (4) retrieves a particular instance of the generalized vector equilibrium problem studied in [7]. In the case where $C(x) = \mathbb{R}_+$, for each $x \in \Omega$ fixed, $\mathbb{Y} = \mathbb{R}$ and F is single-valued map from $\Omega \times \Omega$ to $\mathbb{R}$, and then the problem in (4) reduces to the equilibrium problem on Hadamard manifold that was studied in [8]. Let us consider the following vector optimization problem on Hadamard manifolds:

$$\min_{\mathbb{R}^m_+} f(x), \quad \text{such that} \quad x \in \Omega, \tag{7}$$

in which $f : M \to \mathbb{R}^m$ is a vector function and $\min_{\mathbb{R}^m_+}$ represents the weak minimum. In the main result of [10], namely Theorem 3.2, the existence of a solution to the problem in (7) was achieved by demonstrating the equivalence of this and the variational inequality on Hadamard manifolds (studied by Németh in [17]):

$$\text{Find } x^* \in \Omega : \quad \langle A(x^*), \exp_{x^*}^{-1} y \rangle \notin -\mathbb{R}^m_{++}, \quad \forall\, y \in \Omega, \tag{8}$$

in the particular case where f is a differentiable and convex vector function, and A is the Riemannian Jacobian of f. When we consider that $x^* \in \Omega$ is a weak minimum of (7), i.e., $f(x) - f(x^*) \notin -\mathbb{R}^m_{++}$, for all $x \in \Omega$, then Theorem 3.1 increases the applicability of [10, Theorem 3.2] to genuine Hadamard manifolds and quasiconvex nondifferentiable vector functions.

Example 3.2 Let $(\mathbb{H}^2, g_{ij})$ be the two-dimensional hyperbolic space, as defined in Example 3.1. The bifunction $F : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$, which is given by $F((x_1, x_2), (y_1, y_2)) = \ln^2\left(y_1^2 + y_2^2\right) - \ln^2\left(x_1^2 + x_2^2\right)$, satisfies all the assumptions in Theorem 3.1 if $\Omega = \left\{x = (x_1, x_2) \in \mathbb{H}^2 : x_2 \geq 1/2\right\}$, $C(x) \equiv \mathbb{R}_+$, $y_0 = (0, 1)$, and

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1,\ x_2 \geq 1/2\right\}.$$

Indeed, it is clear that $F((x_1, x_2), (x_1, x_2)) = 0$ for all $(x_1, x_2) \in \Omega$, which implies that F satisfies **h1**. In addition, for fixed $(y_1, y_2) \in \Omega$, we know that $\varphi(x_1, x_2) = F((x_1, x_2), (y_1, y_2))$ is continuous, and F consequently satisfies **h2**. Moreover, for all $c \in \mathbb{R}$, the sub-level set,

$$L_{\psi,\Omega}(c) = \left\{(y_1, y_2) \in \mathbb{R}^2 : e^{-\sqrt{d}} \leq y_1^2 + y_2^2 \leq e^{\sqrt{d}},\ y_2 > 0\right\},$$
$$d = c + \ln^2\left(x_1^2 + x_2^2\right),$$

is convex in $\mathbb{H}^2$, where $\psi(y_1, y_2) = F((x_1, x_2), (y_1, y_2))$, and $(x_1, x_2) \in \Omega$ is a fixed point. Hence, F satisfies **h3**. Finally, because we have $F((x_1, x_2), (0, 1)) > 0$ for all $x \in \Omega \backslash D$, then we know that F satisfies **h4**. Moreover, according to Theorem 3.1, we can conclude that $S^* = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1,\ x_2 \geq 1/2\}$, and the set is compact.

Remark 3.4 One reason for the successful extension, to the Riemannian setting, is the possibility to transform nonconvex or quasiconvex problems in linear context into convex or quasiconvex problems by introducing a suitable metric; see Rapcsák [18]. For instance, in Example 3.2, for a fixed point $(x_1, x_2) \in \Omega$, the function $\psi(y_1, y_2) = \ln^2\left(y_1^2 + y_2^2\right) - \ln^2\left(x_1^2 + x_2^2\right)$ is not usual quasiconvex in $\{(y_1, y_2) \in \mathbb{R}^2 : y_2 > 0\}$, because its sub-level $L_{\psi,\Omega}(0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1,\ y_2 > 0\}$ is not convex. Therefore, [7, Theorem 2.1] cannot be applied to the GVEP. However, we can apply Theorem 3.1.

Henceforth, we assume that every assumption made in Theorem 3.1 holds. In order to prove this theorem, we must establish some preliminary concepts. First, we define the set-valued mapping, $P : \Omega \rightrightarrows \Omega$, by

$$P(x) := \{y \in \Omega \ : \ F(x, y) \subset -\mathrm{int}\, C(x)\}. \tag{9}$$

Lemma 3.1 *If $S^* = \varnothing$, then for each $x, y \in \Omega$, the set-valued mapping P satisfies the following conditions:*

(i) set $P(x)$ is nonempty and convex;
(ii) $P^{-1}(y)$ is an open set, and $\bigcup_{y\in\Omega} P^{-1}(y) = \Omega$;
(iii) there exists $y_0 \in \Omega$ such that $P^{-1}(y_0)^c$ is compact.

Proof Because solution set $S^* = \varnothing$, the definition in (9) lets us to conclude that $P(x) \neq \varnothing$, for all $x \in \Omega$, which proves the first statement, (i). Assume $x \in \Omega$. To prove $P(x)$ is convex, we consider $y_1, y_2 \in P(x)$ and a geodesic $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = y_1$ and $\gamma(1) = y_2$. Applying assumption **h3** gives us

$$F(x, \gamma(t)) \subseteq F(x, y_1) - C(x) \quad \text{or} \quad F(x, \gamma(t)) \subseteq F(x, y_2) - C(x). \tag{10}$$

As $y_1, y_2 \in P(x)$, the definition of $P(x)$ in (9) implies that $F(x, y_1) \subset -\mathrm{int}\, C(x)$ and $F(x, y_2) \subset -\mathrm{int}\, C(x)$. Therefore, given $-\mathrm{int}\, C(x) - C(x) \subset -\mathrm{int}\, C(x)$, which is obtained using Proposition 1.3 and Proposition 1.4 of [19], it follows from (10) that $F(x, \gamma(t)) \subset -\mathrm{int}\, C(x)$, and this concludes the proof of (i).

In order to prove (ii), we must first note that the definition in (2) provides

$$P^{-1}(y) = \{x \in \Omega \ : \ y \in P(x)\} = \{x \in \Omega \ : \ F(x, y) \subset -\mathrm{int}\, C(x)\}, \tag{11}$$

where the second equality follows from the definition of the set, $P(x)$, in (9). Given $x_0 \in P^{-1}(y)$, the second equality in (11), and the fact that $-\mathrm{int}\, C(x)$ is an open set, if we apply **h2**, then we know there exists an open set, $V_{x_0} \subset \Omega$, such that $F(x, y) \subset -\mathrm{int}\, C(x)$, for all $x \in V_{x_0}$. Hence, $P^{-1}(y)$ is open, which proves the first statement in (ii). The definition in (11) implies that $P^{-1}(y) \subseteq \Omega$ for all $y \in \Omega$. In order to complete the proof of (ii), it is sufficient to prove that $\Omega \subseteq \bigcup_{y\in\Omega} P^{-1}(y)$. Therefore, suppose $x \in \Omega$. Item (i) ensures that $P(x) \neq \varnothing$, which implies that there exists $y \in P(x)$. Thus, $x \in P^{-1}(y)$ for some $y \in \Omega$, which concludes the proof of item (ii).

To prove (iii), we note that **h4** and (11) imply that $P^{-1}(y_0)^c = \{x \in \Omega \ : \ F(x, y_0) \not\subset -\mathrm{int}\, C(x)\} \subset D$, for some $y_0 \in \Omega$, and $D \subset \Omega$ is a compact set. Given item i, we know $P^{-1}(y_0)$ is an open set. Furthermore, because D is compact, we can conclude from the last inclusion that $P^{-1}(y_0)^c$ is a compact set, and this completes the proof our proposition. □

Now we are ready to prove our main result: Theorem 3.1.

Proof In order to create a contradiction, let us suppose that solution set $S^* = \varnothing$. Also, assume $G : \Omega \rightrightarrows \Omega$ is the set-valued mapping defined by

$$G(y) := P^{-1}(y)^c. \tag{12}$$

Further define set $D := \bigcap_{y\in\Omega} G(y)$. Therefore, we have two possibilities for set D: $D \neq \varnothing$ or $D = \varnothing$. If $D \neq \varnothing$, i.e., $\bigcap_{y\in\Omega} P^{-1}(y)^c \neq \varnothing$, then we have $\bigcup_{y\in\Omega} P^{-1}(y) \neq \Omega$, which contradicts (ii) in Lemma 3.1. Hence, we can conclude that $D = \varnothing$, i. e., $\bigcap_{y\in\Omega} G(y) = \varnothing$.

Thus, given our assumption that $S^* = \varnothing$, combining the definition in (12) and statements (ii) and (iii) in Lemma 3.1, we can conclude that for each $y \in \Omega$, set $G(y)$ is closed, and there exists $y_0 \in \Omega$ such that $G(y_0)$ is a compact set. Hence, because $\bigcap_{y\in\Omega} G(y) = \varnothing$, Lemma 2.1 implies that there exist $y_1, \ldots, y_m \in \Omega$ such that $\mathrm{conv}\{y_1, \ldots, y_m\} \not\subset \bigcup_{i=1}^m G(y_i)$. Therefore, there also exists $x \in \mathrm{conv}\{y_1, \ldots, y_m\}$ such that $x \notin G(y_i) = P^{-1}(y_i)^c$ for all $i = 1, \ldots m$. Equivalently, there exists $x \in \mathrm{conv}\{y_1, \ldots, y_m\}$ such that $x \in P^{-1}(y_i)$ for all $i = 1, \ldots m$. Hence, we can conclude that

$$\exists\, y_1, \ldots, y_m \in \Omega, \quad \exists\, x \in \mathrm{conv}\{y_1, \ldots, y_m\}; \quad y_i \in P(x), \quad \forall\, i = 1, \ldots m. \tag{13}$$

Considering $S^* = \varnothing$, items (i) in Lemma 3.1 implies that $P(x)$ is convex. When combined with the relations in (13), this implies that there exists $x \in \Omega$ such that $x \in P(x)$. These inclusions and the definition in (9) imply that there exists $x \in \Omega$ such that $F(x, x) \subset -\mathrm{int}\, C(x)$. This contradicts assumption **h1** in Theorem 3.1. Therefore, solution set $S^* \neq \varnothing$, and this concludes the proof of Theorem 3.1. □

4 Conclusions

In this paper, we examined the basic intrinsic properties of the generalized vector equilibrium problem on Hadamard manifolds, and we briefly discussed equilibrium problem theory in this context. Our results should provide the first step to a more general theory, which includes hyperbolic spaces and algorithms for solving problems on Hadamard manifolds.

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