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Degree Theory and Solution Existence of Set-Valued Vector Variational Inequalities in Reflexive Banach Spaces

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Abstract In this paper, a degree theory for set-valued vector variational inequalities is built in reflexive Banach spaces. By using the method of degree theory, some existence results of solutions for set-valued vector variational inequalities are established under suitable conditions. Furthermore, some equivalent characterizations for the nonemptiness and boundedness of solution sets to single-valued vector variational inequalities are obtained under pseudomonotonicity assumption. To the best of our knowledge, there are still no papers dealing with the degree theory for vector variational inequalities.

Keywords Set-valued vector variational inequality · Topological degree · Existence of solutions · Nonempty and boundedness · C-pseudomonotone mapping

Mathematics Subject Classification 49J40 · 90C31

1 Introduction

Degree theory is a classical mathematical tool that has diverse applications. Particularly, it is useful for the study of the existence of a solution to an equation. Many authors have used degree theory as a tool to study the existence of solutions for various kinds of variational inequalities.

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In [1], Facchinei and Pang have used degree theory to study existence theorems for finite-dimensional variational inequalities (see Proposition 2.2.3 and Theorem 2.3.4). In [2], Kien et al. extended the degree theory of Facchinei and Pang to the case of generalized set-valued variational inequalities, where the generalized natural map has no convex values and so its degree is undefined generally. By using the degree theory developed in Kien et al. [2], He [3] proves some existence results of solutions to set-valued variational inequalities under a weak coercivity condition. Recently, Wang and Huang [4] built a degree theory for generalized set-valued variational inequalities in Banach spaces, which further generalized the results of Facchinei and Pang, Kien, Wong and Yao from finite-dimensional spaces to infinite-dimensional spaces. For more related works with respect to applications of degree theory to variational inequalities, we refer the readers to Robinson [5], Gowda [6] and the references therein.

On the other hand, Giannessi [7] first introduced the concept of vector variational inequality (VVI) in finite-dimensional spaces in 1980, which is the vector-valued version of the classical (scalar) variational inequality. Since then, extensive effort has been devoted to the study of various kinds of vector variational inequalities (VVIs) and their generalizations (see, e.g., [8–15] and the references therein). Nowadays, vector variational inequalities and their generalizations have become an effective and powerful tool in the study of vector optimization, applied sciences, mechanics, structural analysis and so on. Meanwhile, some important methods have been proposed to study the existence of solutions for vector variational inequalities, such as KKM theory and the scalarization method. Then, an interesting question is that, although the degree method is very effective and there have been a large amount of papers on the studies of degree theory for scalar variational inequalities, there is still no paper to deal with the degree theory for vector variational inequalities.

Motivated and inspired by the work mentioned above, we continue to study the degree theory for vector variational inequalities and investigate the solvability of vector variational inequalities by applying the method of degree theory directly. The main purpose of this paper is to build a degree theory for set-valued vector variational inequalities and to give some results on existence of solutions of this problem. To do this, we first establish the equivalence between the solvability of vector variational inequalities and scalar variational inequalities. Then, basing on this equivalence and using the degree theory due to [16,17], we construct the degree theory for vector variational inequalities in reflexive spaces. Finally, the degree theory is employed to prove some existence theorems of solutions for set-valued vector variational inequalities. Some equivalent characterizations of nonemptiness and boundedness of the solution set for C-pseudomonotone single-valued vector variational inequalities are also obtained. Different from most of previous existence results established in the literature via KKM theory or the scalarization method, we establish such results using directly the tool of degree method, which may provide a new perspective for dealing with the solvability for vector variational inequalities.

The paper is organized as follows. In Sect. 2, we introduce some basic notations and preliminary results and build a degree theory for set-valued vector variational inequalities. In Sect. 3, we prove some results on the solvability of vector variational inequalities, by applying the degree theory established in Sect. 2.

2 Preliminary

Throughout this paper, unless otherwise stated, let X always be a reflexive Banach space with X^* be the topological dual space and Y be a finite-dimensional space, respectively. By the result due to Lindenstrauss, Asplund and Trojanski, we know that X can be renormed so that X and X^* are both locally uniformly convex (see [18], Theorem 2.11). Let Ω be a bounded and open set in X. The boundary of Ω is denoted by $\partial\Omega$. The symbols "$\rightarrow$" and "$\rightharpoonup$" are used to denote strong and weak convergence, respectively. For a nonempty subset A of X, we denote the closure, interior and convex hull of A by cl A, int A and conv A, respectively.

Let C be a closed, convex and pointed cone in Y with $\operatorname{int} C \neq \emptyset$. The cone C introduces a partial ordering in Y, which is defined by $z_1 \leq_C z_2$ if and only if $z_2 - z_1 \in C$. Let

$$C^* := \left\{ y^* \in Y^* : \langle y^*, y \rangle \geq 0, \quad \forall y \in C \right\}$$

be the dual cone of C. Clearly,

$$y \in C \Leftrightarrow \langle y^*, y \rangle \geq 0, \quad \forall y^* \in C^*,$$
$$y \in \operatorname{int} C \Leftrightarrow \langle y^*, y \rangle > 0, \quad \forall y^* \in C^* \backslash \{0\}.$$

Let $e \in \operatorname{int} C$ be fixed and

$$C^{*0} := \left\{ x^* \in C^* : \langle x^*, e \rangle = 1 \right\}.$$

The dual cone C^* is said to admit a compact base iff there exists a compact set $S_1 \subset C^*$ such that $0 \notin S_1$ and $C^* \subset \bigcup_{t \geq 0} t S_1$. Since Y is finite dimensional, Lemma 3.4 of [19] shows that C^{*0} is a compact base of C^*.

Let $F : K \rightrightarrows L(X, Y)$ be a set-valued mapping with nonempty values. Let $\xi \in C^{*0}$ and $u \in L(X, Y)$. We define

$$\langle \xi u, x \rangle := \langle \xi, \langle u, x \rangle \rangle = \langle u^* \xi, x \rangle, \quad \forall x \in K$$

and

$$\xi F(x) := \bigcup_{u \in F(x)} \xi u, \quad \forall x \in K,$$

where $u^* \in L(Y^*, X^*)$ is the conjugate operator of $u \in L(X, Y)$.

Let X, Y be as before and $K \subset X$ be a nonempty, closed and convex set. Let $F : K \rightrightarrows L(X, Y)$ be a set-valued mapping with nonempty values. In this paper, we consider the following vector variational inequality (in short, VVI(K, F)), which consists in finding $x \in K$ and $u \in F(x)$ such that

$$\langle u, y - x \rangle \notin -\operatorname{int} C, \quad \forall y \in K. \tag{1}$$

If $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then vector variational inequality (1) reduces to the following scalar variational inequality (in short, VI(K, F)), which consists in finding $x \in K$ and $u \in F(x)$ such that

$$\langle u, y - x \rangle \geq 0, \quad \forall y \in K.$$

Now we introduce an important mapping $g : L(X, Y) \rightrightarrows X^*$, defined by

$$g(u) := \bigcup_{\xi \in C^{*0}} \xi u = \bigcup_{\xi \in C^{*0}} u^* \xi = u^*(C^{*0}), \quad \forall u \in L(X, Y), \tag{2}$$

where $u^* \in L(Y^*, X^*)$ is the conjugate operator of $u \in L(X, Y)$. We consider the problem (in short, VI$(K, g \circ T)$) of finding $x \in K$, $\xi \in C^{*0}$ and $u \in F(x)$ such that

$$\langle \xi u, y - x \rangle \geq 0, \quad \forall y \in K. \tag{3}$$

The solution sets of VVI(K, F) and VI$(K, g \circ F)$) are denoted by SVVI(K, F) and SVI$(K, g \circ F)$, respectively. The relationship between SVVI(K, F) and SVI$(K, g \circ F)$ will be discussed in Lemma 2.4.

Let $K \subset X$ be a nonempty, closed and convex set. The indicator of K, denoted by δ_K, is defined by

$$\delta_K(x) := \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

The normal cone of K at x is defined by

$$N_K(x) := \begin{cases} \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \quad \forall\, y \in K\}, & \text{if } x \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The barrier cone of K is defined by

$$\text{barr}(K) := \{x^* \in X^* : \sup_{x \in K} \langle x^*, x \rangle < \infty\}.$$

The recession cone of K is defined by

$$K_\infty := \left\{ d \in X : \exists t_k \to +\infty, x_k \in K \text{ such that } \frac{x_k}{t_k} \rightharpoonup d \right\}.$$

The negative polar cone of K is defined by

$$K^- := \{x^* \in X^* : \langle x^*, x \rangle \leq 0, \quad \forall x \in K\}.$$

The normalized duality mapping $J : X \rightrightarrows X^*$ is defined by

$$J(x) := \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X.$$

In locally uniformly convex spaces, the normalized duality mapping $J : X \rightrightarrows X^*$ is of class $(S)_+$, strictly monotone and a homeomorphism (see [20], Proposition 8 and [21], Corollary 32.24).

In the following, we will outline below some basic notations and results from [17] with respect to the degree theory for mappings of the form $f + T + G$ due to [16,17].

Definition 2.1 Let $\tilde{G} \subset X$ be a nonempty set and $f : \tilde{G} \to X^*$ be a mapping. Then,

(a) f is said to be of class $(S)_+$ iff for any sequence $\{x_n\}$ in $\tilde{G}$ which converges weakly to x and $\limsup_{n\to\infty} \langle f(x_n), x_n - x\rangle \leq 0$, one has $x_n \to x$;
(b) f is said to be norm to weak continuous iff for any sequence $\{x_n\}$ in $\tilde{G}$ which converges to x, one has $f(x_n) \rightharpoonup f(x)$;
(c) f is said to be compact iff f is continuous and $f(A)$ is relatively compact for each bounded subset A of $\tilde{G}$.

Let $T : X \rightrightarrows X^*$ be a set-valued mapping. We call the sets $D(T) = \{x \in X : T(x) \neq \emptyset\}$ and $R(T) = \{y \in T(x) : x \in D(A)\}$ the effective domain and the range of T, respectively. We denote the set $\mathrm{Gr}(T) = \{(x, y) : y \in T(x)\}$ the graph of T. Throughout this article, we always assume that $D(T)$ is nonempty and $K \subset D(T)$.

Definition 2.2 Let $T : X \rightrightarrows X^* \setminus \{\emptyset\}$ be a set-valued mapping. Then,

(a) T is said to be pseudomonotone iff for any $(x_1, y_1), (x_2, y_2) \in \mathrm{Gr}(T)$, one has

$$\langle y_1, x_2 - x_1\rangle \geq 0 \Rightarrow \langle y_2, x_2 - x_1\rangle \geq 0;$$

(b) T is said to be monotone iff for any $(x_1, y_1), (x_2, y_2) \in \mathrm{Gr}(T)$, one has

$$\langle y_2 - y_1, x_2 - x_1\rangle \geq 0;$$

(c) T is said to be maximal monotone iff T is monotone and it follows from

$$\langle v - y, u - x\rangle \geq 0, \quad \forall \langle x, y\rangle \in \mathrm{Gr}(T),$$

that $(u, v) \in \mathrm{Gr}(T)$.

Clearly, maximal monotonicity implies monotonicity and monotonicity implies pseudomonotonicity. The reverse implication is not true, in general.

The inverse $T^{-1} : R(T) \rightrightarrows X$ is defined by $T^{-1}(u) := \{x \in D(T) : u \in Tx\}$. Clearly, $T : X \rightrightarrows X^*$ is maximal monotone if and only if T^{-1} is maximal monotone. For each $\epsilon > 0$, we consider the generalized Yosida transformation T_ϵ corresponding to T, defined by the formula

$$T_\epsilon := \left\langle T^{-1} + \epsilon J^{-1}\right\rangle^{-1},$$

which is a single-valued mapping (see [18], Proposition 3.10).

Definition 2.3 ([20], *Definition 3*) Let $\Omega \subset X$ be a bounded and open set. Let $\{f_t, t \in [0, 1]\}$ be a family of mappings from Ω into X^*. Then, $\{f_t\}$ is said to be a homotopy of class $(S)_+$ iff for any sequence $\{x_n\}$ in Ω converging weakly to x and for any sequence $\{t_n\}$ in $[0, 1]$ converging to t for which $\limsup_{n\to\infty}\langle f_{t_n}(x_n), x_n - x\rangle \le 0$, one has $x_n \to x$ and $f_{t_n}(x_n) \rightharpoonup f_t(x)$.

Remark 2.1 Each affine homotopy between two norm to weak continuous mappings f and f_1 of class$(S)_+$ is a homotopy of class$(S)_+$ (see [20], Proposition 12).

Definition 2.4 [22] Let $\{T_t, t \in [0, 1]\}$ be a family of maximal monotone mappings from X into 2^{X^*} such that their effective domains are nonempty. Then, $\{T_t\}$ is said to be a pseudo-monotone homotopy iff for any $(x, y) \in \mathrm{Gr}(T_t)$ and a sequence $t_n \to t$ in $[0, 1]$, there exists a sequence $(x_n, y_n) \in \mathrm{Gr}(T_{t_n})$ such that $x_n \to x$ and $y_n \to y$.

Definition 2.5 Let X, Y be two topological spaces and $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values. Then,

(a) F is said to be upper semicontinuous (written u.s.c.) at $x \in X$ iff for any open set $V \subset Y$ with $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V$ for all $y \in U$. Iff F is upper semicontinuous at every $x \in X$, then we say that F is upper semicontinuous on X;
(b) F is said to be compact iff $F(A)$ is relatively compact for each bounded subset A of X.

Definition 2.6 ([16], *Definition 3*) Let $B \subset X$ be a nonempty subset. A mapping $G : B \rightrightarrows X^*\backslash\{\emptyset\}$ is said to belong to class (P) iff G satisfies the following conditions:

(i) G maps bounded sets to relatively compact sets;
(ii) for every $x \in B$, $G(x)$ is a closed and convex subset of X^*;
(iii) $G(\cdot)$ is upper semicontinuous on B.

Definition 2.7 ([18], *Definition 9*) Let $\Omega \subset X$ be a bounded and open set. A one-parameter family of set-valued mappings $G_t : \mathrm{cl}\,\Omega \rightrightarrows X^*\backslash\{\emptyset\}$, $t \in [0, 1]$, is said to be a homotopy class (P) iff G_t satisfies the following conditions:

(i) the mapping $(t, x) \to G_t(x)$ is upper semicontinuous on $[0, 1] \times \mathrm{cl}\,\Omega$;
(ii) for every $(t, x) \in [0, 1] \times \mathrm{cl}\,\Omega$, $G_t(x)$ is a closed and convex subset of X^*;
(iii) the set $\{\bigcup G_t(x) : t \in [0, 1], x \in \mathrm{cl}\,\Omega\}$ is compact in X^*.

Lemma 2.1 [23] *Let X and Y be two Banach spaces, $B \subset X$ be a nonempty set, and $G : B \rightrightarrows Y$ be an u.s.c. set-valued mapping with closed and convex values. Then, given $\epsilon > 0$, there exists a continuous mapping $g_\epsilon : B \to Y$ such that*

$$g_\epsilon(x) \in G((x + B_\epsilon) \cap B) + \widehat{B}_\epsilon$$

for all $x \in B$ and $g_\epsilon(B) \subset clconv\,(G(B))$, with $B_\epsilon = \{x \in X : \|x\| < \epsilon\}$ and $\widehat{B}_\epsilon = \{y \in Y : \|y\| < \epsilon\}$.

Obviously, if G is compact, then so is the approximate selection g_ϵ.

Let Ω be a bounded and open set in X. From Lemma 2.1, we know that if $G : \Omega \rightrightarrows X^*\backslash\{\emptyset\}$ is a set-valued mapping of class (P) and $\epsilon > 0$, then we can find a continuous mapping $g_\epsilon : \Omega \to X^*$ such that $g_\epsilon(\text{cl}\,\Omega) \subset clconv\,(G(\text{cl}\,\Omega))$ and $g_\epsilon(x) \in G((x + B_\epsilon) \cap \text{cl}\,\Omega) + \widehat{B}_\epsilon$ for all $x \in \Omega$, where $B_\epsilon = \{x \in X : \|x\| < \epsilon\}$ and $\widehat{B}_\epsilon = \{y \in Y : \|y\| < \epsilon\}$. In what follows, $g_\epsilon(\cdot)$ will always denote this approximate selection of $G(\cdot)$.

Lemma 2.2 ([17], Theorem 2.11) *Let Ω be a bounded and open set in X. Let $T : X \rightrightarrows X^*$ be a maximal monotone mapping, $f : cl\,\Omega \to X^*$ be bounded, norm to weak continuous and of class $(S)_+$ and $G : cl\,\Omega \rightrightarrows X^*\backslash\{\emptyset\}$ be a set-valued mapping of class (P). Let $\bar{y} \in X^*$ such that $\bar{y} \notin (T + f + G)(\partial\Omega)$. Then, there exists some $\bar{\epsilon} > 0$ such that the following assertions hold:*

- *(i) $\bar{y} \notin (T_\epsilon + f + g_\epsilon)(\partial\Omega)$ for all $\epsilon \in]0, \bar{\epsilon}]$;*
- *(ii) $T_\epsilon + f + g_\epsilon$ is a mapping of class $(S)_+$ and so the Browder degree $d(T_\epsilon + f + g_\epsilon, \Omega, \bar{y})$ is defined for all $\epsilon \in]0, \bar{\epsilon}]$;*
- *(iii) $d(T_\epsilon + f + g_\epsilon, \Omega, \bar{y}) = d(T_{\epsilon'} + f + g_{\epsilon'}, \Omega, \bar{y})$ for all $\epsilon, \epsilon' \in]0, \bar{\epsilon}]$.*

Definition 2.8 ([17], *Definition 2.12*) Let Ω be a bounded and open set in X, $T : X \rightrightarrows X^*$ be a maximal monotone mapping, $f : \text{cl}\,\Omega \to X^*$ be bounded, norm to weak continuous and of class $(S)_+$ and $G : \text{cl}\,\Omega \rightrightarrows X^*\backslash\{\emptyset\}$ be a set-valued mapping of class (P). Let $\bar{y} \in X^*$ such that $\bar{y} \notin (T+f+G)(\partial\Omega)$. The degree $d_1(T+f+G, \Omega, \bar{y})$ is assigned to be the common value of $d(T_\epsilon + f + g_\epsilon, \Omega, \bar{y})$ for $\epsilon > 0$ sufficiently small.

Some properties of the degree defined in Definition 2.8 are listed as follows.

Lemma 2.3 ([17], Theorem 2.14) *Let Ω be a bounded and open set in X. Then, degree function defined by Definition 2.8 has the following properties:*

- *(i) Normalization: $d_1(J - \bar{y}, \Omega, 0) = d_1(J, \Omega, \bar{y}) = 1$ for all $\bar{y} \in J(\Omega)$;*
- *(ii) Existence: If $d_1(T + f + G, \Omega, \bar{y}) \neq 0$, then there exists an $x \in \Omega$ such that*

$$\bar{y} \in f(x) + T(x) + G(x);$$

- *(iii) Additivity: If Ω_1, Ω_2 are disjoint open subsets of Ω and $\bar{y} \notin (T + f + G)(cl\,\Omega\backslash(\Omega_1 \cup \Omega_2))$, then*

$$d_1(T + f + G, \Omega, \bar{y}) = d_1(T + f + G, \Omega_1, \bar{y}) + d_1(T + f + G, \Omega_2, \bar{y});$$

- *(iv) Homotopy invariance: Let $\{f_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+$ of mappings from $cl\,\Omega$ into a bounded subset of X^*, $\{T_t\}_{t\in[0,1]}$ be a pseudomonotone homotopy of maximal monotone mappings from X into 2^{X^*} and $\{G_t\}_{t\in[0,1]}$ is a homotopy of class (P) of set-valued mappings from $cl\,\Omega$ into the nonempty, closed and convex subsets of X^*. Let $\{y_t : t \in [0, 1]\}$ be a continuous path in X^* such that $y_t \notin (T_t + f + G_t)(\partial\Omega)$ for all $t \in [0, 1]$. Then, $d_1(f_t + T_t + G_t, \Omega, y_t)(\partial\Omega)$ is independent of $t \in [0, 1]$.*

The following lemma establishes the equivalence between the solvability of vector variational inequalities and scalar variational inequalities.

Lemma 2.4 *Let X be a reflexive Banach space, and Y be a finite-dimensional space. Let K be an nonempty, closed and convex subset of X. Let $F : K \rightrightarrows L(X, Y)$ be a set-valued mapping with nonempty values. Let $g : L(X, Y) \rightrightarrows X^*$ be defined as in* (2). Then, the following conclusions are equivalent:

(i) $x_0 \in K$ is a solution of VVI(K, F);
(ii) $x_0 \in K$ is a solution of VI$(K, g \circ F)$, i.e., $0 \in g \circ F(x_0) + N_K(x_0)$.

Proof $(i) \Rightarrow (ii)$. Suppose that $x_0 \in K$ solves VVI(K, F). Then, there exists some $u_0 \in F(x_0)$ such that

$$\langle u_0, y - x_0 \rangle \notin -\text{int}\, C, \quad \forall y \in K.$$

By using a similar discussion as in Theorem 2.1 of [13], there exists some $\xi_0 \in C^* \backslash \{0\}$ such that

$$\langle \xi_0 u_0, y - x_0 \rangle \geq 0, \quad \forall y \in K.$$

Note that C^{*0} is a base of C^*, we have $C^* \backslash \{0\} = \bigcup_{t>0} tC^{*0}$. Without any loss of generality, we can further assume that $\xi_0 \in C^{*0}$. That is, there exists some $\xi_0 \in C^{*0}$ and $u_0 \in F(x_0)$ such that

$$\langle \xi_0 u_0, y - x_0 \rangle \geq 0, \quad \forall y \in K,$$

which implies that $x_0 \in K$ is a solution of VI$(K, g \circ F)$.

$(ii) \Rightarrow (i)$. Suppose that $x_0 \in K$ solves VI$(K, g \circ F)$. Then, there exists some $\xi_0 \in C^{*0}$ and $u_0 \in F(x_0)$ such that

$$\langle \xi_0 u_0, y - x_0 \rangle \geq 0, \quad \forall y \in K. \tag{4}$$

We claim that

$$\langle u_0, y - x_0 \rangle \notin -\text{int}\, C, \quad \forall y \in K,$$

which implies that $x_0 \in K$ is a solution of VVI(K, F). Otherwise, there exists some $y_0 \in K$ such that

$$\langle u_0, y_0 - x_0 \rangle \in -\text{int}\, C,$$

and so

$$\langle \xi_0 u_0, y_0 - x_0 \rangle < 0.$$

A contradiction with (4). Thus, the implication (ii) $\Rightarrow$ (i) holds. This completes the proof. □

Remark 2.2 (i) From Lemma 2.4, it is known that the relationship between SVI(K, $g\circ F$) and SVVI(K, F) can be concluded as follows:

$$\mathrm{SVVI}(K, F) = \mathrm{SVI}(K, g \circ F) = \bigcup_{\xi\in C^{*0}} \mathrm{SVI}(K, \xi F).$$

(ii) As for scalar variational inequalities, many powerful methods and abundant research achievements have been obtained by the authors. Thus, Lemma 2.4 provides an effective means to study the existence of solutions for VVI(K, F), by changing vector variational inequality VVI(K, F) to scalar variational inequality VI(K, $g \circ F$).

Lemma 2.5 *Let X be a reflexive Banach space and Y be a finite-dimensional space. Let K be an nonempty, closed and convex subset of X. Let $g : L(X, Y) \rightrightarrows X^*$ be a set-valued mapping defined as in (2) and $F : K \rightrightarrows L(X, Y)$ be an upper semicontinuous set-valued mapping with nonempty, compact and convex values. Consider the following two conclusions:*

(i) $J - g \circ F : K \rightrightarrows X^*$ is compact;
(ii) for any $\xi \in C^{*0}$, $J - \xi F : K \rightrightarrows X^*$ is compact.

Then, (i) $\Rightarrow$ (ii). Moreover, (i) $\Leftrightarrow$ (ii) if X is finite dimensional.

Proof (i) $\Rightarrow$ (ii). Since $J - g\circ F : K \rightrightarrows X^*$ is compact, for any bounded set $B \subset K$, we have $(J - g \circ F)(B)$ is a relative compact set. Then, for any $\xi \in C^{*0}$, it follows from the definition of $g(\cdot)$ that

$$(J - \xi F)(B) \subset (J - g \circ F)(B),$$

which implies that $(J-\xi F)(B)$ is also a relative compact set and so $J-\xi F : K \rightrightarrows X^*$ is compact.

If X is finite dimensional, then we can further claim that (ii) $\Rightarrow$ (i) and so (i) $\Leftrightarrow$ (ii). To claim that (ii) $\Rightarrow$ (i), we only need to show that $(J - g \circ F)(x_i)$ has a convergent subsequence for any bounded sequence $\{x_i\} \subset K$. Let $y_i = J(x_i) - \xi_i u_i \in J(x_i) - \xi_i F(x_i)$ with $\xi_i \in C^{*0}$ and $u_i \in F(x_i)$. Note that C^{*0} is a compact base of Y^* and $\xi_i \in C^{*0}$; without any loss of generality, we can assume that $\xi_i \to \xi_0 \in C^{*0}$. Since X is finite dimensional and $\{x_i\}$ is bounded, there exists a compact set D such that $\{x_i\} \subset D$. Then, by the upper semicontinuity of F, we have $\{u_i\} \subset F(D)$, where $F(D)$ is compact. This implies that $\{u_i\}$ and $\{u_i^*\}$ are bounded sequences.

Moreover, since $J - \xi F$ is compact for any $\xi \in C^{*0}$, the sequence $\{y_i'\}$ with $y_i' = J(x_i) - \xi_0 u_i$ belonging to $J(x_i) - \xi_0 F(x_i)$ has a convergent sequence. Without any loss of generality, still denote it by $\{y_i'\}$. Then, we have

$$\|y_i - y_i'\| = \langle \xi_i - \xi_0, u_i\rangle = \langle u_i^*, \xi_i - \xi_0\rangle \le \|u_i^*\| \cdot \|\xi_i - \xi_0\| \to 0,$$

which implies that y_i has a convergent sequence and so $J - g \circ F : K \rightrightarrows X^*$ is compact. This completes the proof. □

Lemma 2.6 *Let X be a reflexive Banach space, Y be a finite-dimensional space, and $g : L(X, Y) \rightrightarrows X^*$ be a set-valued mapping defined as in (2). Then, g is upper semicontinuous with nonempty, compact and convex values.*

Proof By the definition of g, we have

$$g(u) = u^*(C^{*0}), \quad \forall u \in L(X, Y),$$

where $u^* \in L(Y^*, X^*)$ is the conjugate operator of $u \in L(X, Y)$. Since C^{*0} ia a compact and convex base of $C^* \subset Y^*$, the compactness and convexity of the values of g is obvious.

Now we claim that g is upper semicontinuous on $L(X, Y)$. Let $u_0 \in L(X, Y)$ be any given point and V be any open set containing $g(u_0)$. Since $g(u_0)$ is a compact set, there exists some $\varepsilon_0 > 0$ such that $g(u_0) + \varepsilon_0 \operatorname{cl}B_{X^*} \subset V$, where $\operatorname{cl}B_{X^*}$ denotes the closed unit ball in X^*. Moreover, the compactness of C^{*0} implies that there exists some constant $k_0 > 0$ such that $k_0 = \max_{\xi \in C^{*0}} \|\xi\|$. For any $u \in L(X, Y)$ with $\|u - u_0\| \leq \frac{\varepsilon_0}{k_0}$ and $v \in g(u)$, there exists some $\xi_0 \in C^{*0}$ such that $v = \langle \xi_0, u\rangle = u^*(\xi_0)$. Set $v_0 = \langle \xi_0, u_0\rangle = u_0^*(\xi_0)$. Clearly, $\langle \xi_0, u_0\rangle = u_0^*(\xi_0) \in g(u_0)$. Moreover, from the isometric isomorphism property of the mapping $u \to u^*$, we have

$$\begin{aligned} \|v - v_0\| &= \|u^*(\xi_0) - u_0^*(\xi_0)\| \\ &\leq \|u^* - u_0^*\| \cdot \|\xi_0\| \\ &= \|u - u_0\| \cdot \|\xi_0\| \\ &\leq \frac{\varepsilon_0}{k_0} \cdot k_0 \\ &\leq \varepsilon_0, \end{aligned}$$

which implies that

$$g(u) \subset g(u_0) + \varepsilon_0 \operatorname{cl}B_{X^*} \subset U, \quad \forall u \in L(X, Y) \text{ with } \|u - u_0\| \leq \frac{\varepsilon_0}{k_0}.$$

From the above discussion, we conclude that g is an upper semicontinuous mapping with nonempty, compact and convex values on $L(X, Y)$. This completes the proof. □

Let $g : L(X, Y) \rightrightarrows X^*$ be defined as in (2) and $F : K \rightrightarrows L(X, Y)$ be an upper semicontinuous mapping with nonempty, compact and convex values. Suppose that $J - g \circ F : K \rightrightarrows X^*$ is a compact and upper semicontinuous mapping. From Lemma 2.4, we know that VVI(K, F) is equivalent to the following inclusion problem: find $x \in K$ satisfies

$$0 \in G(x) + f(x) + T(x), \tag{5}$$

where $G = g \circ F - J$ is a compact and upper semicontinuous mapping with nonempty and compact values, $f = J$ is a mapping of bounded, norm to weak continuous and of class $(S)_+$, and $T = N_K$ is a maximal monotone mapping. We would like to mention that although the values of $G = g \circ F - J$ may not be convex, there still exists an approximate selection g_ϵ of $G = g \circ F - J$, where g_ϵ can be taken as an approximate selection of $\xi F - J$ with $\xi \in C^{*0}$ being a given point. Indeed, for each $x \in K$, we have $(\xi F - J)(x) \subset G(x)$. Moreover, since the mapping $\xi F - J$ is upper semicontinuous with nonempty, closed and convex values, by Lemma 2.1, there exists an approximate selection g_ϵ for the mapping $\xi F - J$. Clearly, g_ϵ is also an approximate selection of $G = g \circ F - J$. Consequently, the degree $d_1(T + f + G, \Omega, \bar{y}) := d(T_\epsilon + f + g_\epsilon, \Omega, \bar{y})$ in Definition 2.8 is still well defined for $\epsilon > 0$ sufficiently small.

Now we introduce the degree of vector variational inequality VVI(K, F) as follows.

Definition 2.9 Let X be a reflexive Banach space with the dual space X^*, K be a nonempty, closed and convex subset of X. Let $g : L(X, Y) \rightrightarrows X^*$ be defined as in (2) and $F : K \rightrightarrows L(X, Y)$ be an upper semicontinuous mapping with nonempty, compact and convex values. Suppose that $J - g \circ F : K \rightrightarrows X^*$ is a compact and upper semicontinuous mapping. The degree of vector variational inequality VVI(K, F), denoted by $d(g \circ F + N_K, \Omega, 0)$, is the degree $d_1(G + f + T, \Omega, 0)$ defined in Definition 2.8, where $G = g \circ F - J$ is a compact and upper semicontinuous mapping with nonempty and compact values, $f = J$ is a mapping of bounded, norm to weak continuous and of class $(S)_+$, and $T = N_K$ is a maximal monotone mapping.

Remark 2.3 If $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then from Lemma 2.5, the assumption "$J - g \circ F : K \rightrightarrows X^*$ is a compact and upper semicontinuous mapping" reduces to "$J - F$ is a compact and upper semicontinuous mapping," which was applied extensively to ensure the existence of solutions for scalar complementarity problems and variational inequalities (see, e.g., [4,24,25] and the references therein).

From Lemma 2.3, we know that the degree $d(g \circ F + N_K, \Omega, 0)$ of vector variational inequality VVI(K, F) has the following properties.

Lemma 2.7 *Let Ω be a bounded and open set in X. Then, degree function defined by Definition 2.9 has the following properties:*

(i) Normalization: $d(J - \bar{y}, \Omega, 0) = d(J, \Omega, \bar{y}) = 1$ for all $\bar{y} \in J(\Omega)$;
(ii) Existence: If $d(g \circ F + N_K, \Omega, 0) \neq 0$, then there exists an $x \in \Omega$ such that

$$0 \in g \circ F(x) + N_K(x), \text{ i.e., } x \text{ is a solution of VVI}(K, F);$$

(iii) Additivity: If Ω_1, Ω_2 are disjoint open subsets of Ω and $\bar{y} \notin (g \circ F + N_K)(cl\,\Omega \backslash (\Omega_1 \cup \Omega_2))$, then

$$d(g \circ F + N_K, \Omega, \bar{y}) = d(g \circ F + N_K, \Omega_1, \bar{y}) + d(g \circ F + N_K, \Omega_2, \bar{y});$$

(iv) Homotopy invariance: Suppose that $g \circ F_1 - J$ and $g \circ F_2 - J$ are compact and upper semicontinuous mappings with nonempty and compact values. If $0 \notin (tg \circ F_1(x) + (1 - t)g \circ F_2(x) + N_K(x))(\partial\Omega)$ for all $t \in [0, 1]$, then $d(g \circ F_1(x) + N_K(x), \Omega, 0) = d(g \circ F_2(x) + N_K(x), \Omega, 0)$.

3 Solution Existence of Vector Variational Inequalities

In this section, we will apply directly the degree theory developed in Sect. 2 to study the existence of solution for VVI(K, F) under some suitable conditions. In the sequel, unless otherwise stated, we always assume that $F : K \rightrightarrows L(X, Y)$ is an upper semicontinuous mapping with nonempty, compact and convex values. First, we obtain the following existence result for VVI(K, F), when $J - g \circ F : K \rightrightarrows X^*$ is compact and upper semicontinuous.

Theorem 3.1 *Let X be a reflexive Banach space, Y be a finite-dimensional space, and $K \subset X$ be a nonempty, closed and convex subset. Let $g : L(X, Y) \rightrightarrows X^*$ be defined as in (2). Suppose that $J - g \circ F : K \rightrightarrows X^*$ is a compact and upper semicontinuous mapping. If there exists a vector $\widehat{x} \in K$ such that the set*

$$L_<(\widehat{x}) := \{x \in K : \exists u \in F(x) \text{ such that } \langle u, x - \widehat{x}\rangle \notin C\} \tag{6}$$

is bounded (possibly empty), then VVI(K, F) has a solution.

Proof Let $\Omega' \subset K$ be a bounded and open set containing $L_<(\widehat{x}) \cup \{\widehat{x}\}$. Then, we have

$$L_<(\widehat{x}) \cap \partial\Omega' = \emptyset,$$

and so

$$\langle u, x - \widehat{x}\rangle \in C, \quad \forall x \in K \cap \partial\Omega', u \in F(x). \tag{7}$$

If $0 \in (g \circ F + N_K)(\partial\Omega') = (g \circ F - J + J + N_K)(\partial\Omega')$, then VVI$(K, F)$ has a solution. Otherwise, the degree $d(g \circ F + N_K, \Omega', 0)$ is well defined. Define a homotopy by

$$H(t, x) := t(g \circ F(x) + N_K(x)) + (1 - t)(J(x) - J(\widehat{x})), \quad \forall (t, x) \in [0, 1] \times \operatorname{cl}\Omega'.$$

Then, we have $H(0, x) = J(x) - J(\widehat{x})$ and $H(1, x) = g \circ F(x) + N_K(x)$.

We claim that $0 \notin H(t, \partial\Omega')$ for all $t \in [0, 1]$. In fact, if $0 \in H(0, \partial\Omega')$, then there exists some $x_0 \in K \cap \partial\Omega'$ such that $0 = J(x_0) - J(\widehat{x})$, which implies that $\widehat{x} = x_0$ by the strictly monotonicity of $J(\cdot)$. This contradicts $x_0 \in \partial\Omega'$ and $\widehat{x} \in \Omega'$. Thus, $0 \notin H(0, \partial\Omega')$. If $0 \in H(1, \partial\Omega')$, then $0 \in (g \circ F + N_K)(\partial\Omega')$. This contradicts the assumption of $0 \notin (g \circ F + N_K)(\partial\Omega')$.

If there exists some $t_0 \in]0, 1[$ and $x_0' \in K \cap \partial\Omega'$ such that

$$0 \in t_0(g \circ F(x_0') + N_K(x_0')) + (1 - t_0)(J(x_0') - J(\widehat{x})), \tag{8}$$

then (8) implies that there exists $u_0' \in F(x_0')$ and $\xi_0 \in C^{*0}$ such that

$$-\xi_0 u_0' - \frac{1 - t_0}{t_0}(J(x_0') - J(\widehat{x})) \in N_K(x_0').$$

By the definition of $N_K(\cdot)$, we have

$$\langle \xi_0 u_0' + \frac{1-t_0}{t_0}(J(x_0') - J(\widehat{x})), y - x_0' \rangle \geq 0, \quad \forall y \in K,$$

and so

$$\langle \xi_0 u_0', y - x_0' \rangle \geq -\frac{1-t_0}{t_0} \langle J(x_0') - J(\widehat{x}), y - x_0' \rangle. \tag{9}$$

Taking $y = \widehat{x}$ in (9), by the strictly monotonicity of $J(\cdot)$, $t_0 \in]0, 1[$ and $\widehat{x} \neq x_0'$, we obtain

$$\langle \xi_0 u_0', \widehat{x} - x_0' \rangle \geq -\frac{1-t_0}{t_0} \langle (J(x_0') - J(\widehat{x})), \widehat{x} - x_0' \rangle > 0,$$

and so

$$\langle u_0', x_0' - \widehat{x} \rangle \notin C.$$

This contradicts with (7) and so the claim holds true. According to Lemma 2.7 (iv), we obtain that $d(g \circ F + N_K, \Omega', 0) = d(J - J(\widehat{x}), \Omega', 0) = d(J, \Omega', J(\widehat{x})) = 1$. Moreover, by Lemma 2.7 (ii), there exists $x_0 \in K \cap \Omega'$ such that $0 \in G \circ F(x_0) + N_K(x_0)$, which implies that x_0 is a solution of VVI(K, F). This completes the proof. □

Remark 3.1 Theorem 3.1 establishes a new existence result for vector variational inequalities with the mapping $J - g \circ F$ being compact and upper semicontinuous, by using the degree method. Meanwhile, Theorem 3.1 generalizes the corresponding results of Proposition 2.2.3 of [1], Theorem 2.3 of [26], and Theorem 3.1 of [3] from scalar variational inequalities to vector variational inequalities. Moreover, the spaces involved are extended from finite-dimensional spaces to infinite-dimensional spaces.

The following example is used to illustrate Theorems 3.1.

Example 3.1 Let $X = Y = \mathbb{R}^2$, $K = C = \mathbb{R}^2_+$ and $e = (1, 1) \in \text{int}\, C$. Let $F : K \rightrightarrows L(X, Y)$ be defined by

$$F(x) := \left\{ \begin{pmatrix} x_1 & 0 \\ a & x_2 \end{pmatrix} : 1 \leq a \leq 10 \right\}, \quad \forall x = (x_1, x_2)^T \in K,$$

where (x_1, x_2) denotes a row vector in $\mathbb{R}^2$ and $(x_1, x_2)^T$ denotes its transposition.

Then, $C^{*0} = \{(\xi_1, \xi_2) \in \mathbb{R}^2_+ : \xi_1 + \xi_2 = 1\}$ and F is upper semicontinuous on K with nonempty, compact and convex values, which implies that $J - g \circ F$ is upper semicontinuous and compact on K. Moreover, there exists a vector $\widehat{x} = (0, 0)^T \in K$ such that the set

$$L_<(\widehat{x}) := \{x \in K : \exists u \in F(x) \text{ such that } \langle u, x - \widehat{x} \rangle \notin C\} \tag{10}$$

is empty. Thus, all the assumptions of Theorem 3.1 are satisfied. By a simple computation, we have

$$\mathrm{SVVI}(K, F) = \{(0, x_2)^T : x_2 \geq 0\} \neq \emptyset,$$

where $(0, x_2)^T$ denotes the transposition of a row vector $(0, x_2) \in \mathbb{R}^2$. Thus, the conclusion of Theorem 3.1 holds true.

Definition 3.1 Let $F : K \rightrightarrows L(X, Y)$ be a set-valued mapping with nonempty values. Then, F is said to be C-pseudomonotone on K iff for any $(x_1, u_1), (x_2, u_2) \in \mathrm{Gr}(F)$, one has

$$\langle u_1, x_2 - x_1\rangle \notin -\mathrm{int}\, C \Rightarrow \langle u_2, x_2 - x_1\rangle \in C.$$

Remark 3.2 The concepts of various pseudomonotonicity were introduced and discussed in [8,10,14,15] and the references therein. Indeed, the C-pseudomonotonicity in Definition 3.1 is a set-valued version of Definition 2.2 (4) in [15]. The following lemma establishes the equivalence between C-pseudomonotonicity of F and pseudomonotonicity of $g \circ F$ in the sense of Definition 2.2 (a), where $g : L(X, Y) \rightrightarrows X^*$ is defined as in (2).

Lemma 3.1 *Let X be a reflexive Banach space, Y be a finite-dimensional space, and $K \subset X$ be a nonempty, closed and convex subset. Let $F : K \rightrightarrows L(X, Y)$ be a set-valued mapping with nonempty values and $g : L(X, Y) \rightrightarrows X^*$ be defined as in (2). Then, F is C-pseudomonotone on K if and only if $T = g \circ F : K \rightrightarrows X^*$ is pseudomonotone on K.*

Proof Suppose that F is C-pseudomonotone on K. Then, for any $x_1, x_2 \in K$ and $u_1 \in F(x_1), u_2 \in F(x_2)$, we have

$$\langle u_1, x_2 - x_1\rangle \notin -\mathrm{int}\, C \Rightarrow \langle u_2, x_2 - x_1\rangle \in C. \tag{11}$$

For any $x_1, x_2 \in K$, $u_1 \in F(x_1)$ and $\xi_1 \in C^{*0}$, if

$$\langle \xi_1(u_1), x_2 - x_1\rangle \geq 0,$$

then

$$\langle u_1, x_2 - x_1\rangle \notin -\mathrm{int}\, C.$$

From (11), we have

$$\langle u_2, x_2 - x_1\rangle \in C,$$

and so

$$\langle \xi_2(u_2), x_2 - x_1\rangle \geq 0, \quad \forall \xi_2 \in C^{*0},$$

which implies that $T = g \circ F$ is pseudomonotone on K.

Conversely, suppose that $T = g \circ F$ is pseudomonotone on K. Then, for any $x_1, x_2 \in K$, $u_1 \in F(x_1)$, $u_2 \in F(x_2)$ and $\xi_1, \xi_2 \in C^{*0}$, we have

$$\langle \xi_1(u_1), x_2 - x_1 \rangle \geq 0 \Rightarrow \langle \xi_2(u_2), x_2 - x_1 \rangle \geq 0. \tag{12}$$

For any $x_1, x_2 \in K$ and $u_1 \in F(x_1)$, if

$$\langle u_1, x_2 - x_1 \rangle \notin -\operatorname{int} C,$$

then there exists some $\xi_1 \in C^{*0}$ such that

$$\langle \xi_1(u_1), x_2 - x_1 \rangle \geq 0.$$

From (12), it follows that

$$\langle \xi_2(u_2), x_2 - x_1 \rangle \geq 0, \quad \forall \xi_2 \in C^{*0},$$

and so

$$\langle u_2, x_2 - x_1 \rangle \in C.$$

This implies that F is C-pseudomonotone on K. This completes the proof. □

In the case when F further satisfies the C-pseudomonotonicity assumption on K, we can obtain the following equivalent conditions for VVI(K, F) to have a solution.

Theorem 3.2 *Let X be a reflexive Banach space, Y be a finite-dimensional space, and $K \subset X$ be a nonempty, closed and convex subset. Let $g : L(X, Y) \rightrightarrows X^*$ be defined as in (2). Suppose that $J - g \circ F : K \rightrightarrows X^*$ is a compact and upper semicontinuous mapping. Suppose in addition that F is C-pseudomonotone on K. Then, the following statements are equivalent:*

(i) There exists a vector $\widehat{x} \in K$ such that the set

$$L_<(\widehat{x}) := \{x \in K : \exists u \in F(x) \text{ such that } \langle u, x - \widehat{x} \rangle \notin C\}$$

is bounded (possibly empty);

(ii) There exists an open ball Ω of X and $\widehat{x} \in K \cap \Omega$ such that

$$\langle u, x - \widehat{x} \rangle \in C, \quad \forall x \in K \cap \partial\Omega, u \in F(x);$$

(iii) VVI(K, F) has a solution.

Proof (i)⇒(ii). If (i) holds, then there exists a vector $\widehat{x} \in K$ such that the set $L_<(\widehat{x})$ is bounded (possibly empty). In this case, there exists an open ball Ω of X such that the set $L_<(\widehat{x}) \cup \{\widehat{x}\} \subset \Omega$. This yields that for any $x \in C \cap \partial\Omega$, one has $x \notin L_<(\widehat{x})$ and so

$$\langle u, x - \widehat{x} \rangle \in C, \quad \forall x \in K \cap \partial\Omega, u \in F(x).$$

(ii)⇒(iii). The implication follows directly from the proof of Theorem 3.1.

(iii)⇒(i). Let $\widehat{x}$ be a solution of VVI(K, F). Then, there exists some $\widehat{u} \in F(\widehat{x})$ such that

$$\langle \widehat{u}, x - \widehat{x} \rangle \notin -\text{int}\, C, \quad \forall x \in K.$$

This implies that there exists some $\xi \in C^{*0}$ such that

$$\langle \xi\widehat{u}, x - \widehat{x} \rangle \geq 0, \quad \forall x \in K.$$

Since $\xi\widehat{u} \in g \circ F(\widehat{x})$, by the pseudomonotonicity of $g \circ F$, we have

$$\langle x^*, x - \widehat{x} \rangle \geq 0, \quad \forall x^* \in g \circ F(x),$$

and so

$$\langle u, x - \widehat{x} \rangle \in C, \quad \forall u \in F(x).$$

Consequently,

$$L_<(\widehat{x}) := \{x \in K : \exists u \in F(x) \text{ such that } \langle u, x - \widehat{x} \rangle \notin C\} = \emptyset$$

and so (i) holds. This completes the proof. □

Remark 3.3 Theorem 3.2 generalizes the corresponding results of Theorem 3.2 of [3] from scalar variational inequalities to vector variational inequalities.

When $X = \mathbb{R}^n$, $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the following corollary establishes the existence of solutions for pseudomonotone scalar variational inequality VI(K, F).

Corollary 3.1 *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex subset and $F : K \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty, compact and convex values. Suppose that $F : K \rightrightarrows \mathbb{R}^n$ is pseudomonotone on K. Then, the following statements are equivalent:*

(i) There exists a vector $\widehat{x} \in K$ such that the set

$$L_<(\widehat{x}) := \{x \in K : \inf_{u \in F(x)} \langle u, x - \widehat{x} \rangle < 0\}$$

is bounded (possibly empty);

(ii) There exists an open ball Ω *of* X *and* $\widehat{x} \in K \cap \Omega$ *such that*

$$\inf_{u \in F(x)} \langle u, x - \widehat{x} \rangle \geq 0, \quad \forall x \in K \cap \partial\Omega;$$

(iii) VI(K, F) *has a solution.*

Proof Since $F : K \rightrightarrows \mathbb{R}^n$ is an upper semicontinuous mapping with nonempty, compact and convex values, from Lemma 2.6, it is easy to see that $g \circ F : K \rightrightarrows \mathbb{R}^n$ is a set-valued mapping of class (P) and so is $I - g \circ F$. Then, Corollary 3.1 follows directly from Theorem 3.2. This completes the proof. □

In the following, we further discuss the nonemptiness and boundedness property of solution sets for VVI(K, F) in finite-dimensional spaces $\mathbb{R}^n$ by using the degree method.

Theorem 3.3 *Let* $K \subset \mathbb{R}^n$ *be a nonempty, closed and convex subset,* $F : K \to L(\mathbb{R}^n, \mathbb{R}^n)$ *be a single-valued continuous mapping, and* $g : L(\mathbb{R}^n, \mathbb{R}^n) \rightrightarrows \mathbb{R}^n$ *be defined as in* (2)*. Suppose that* F *is* C*-pseudomonotone on* K*. Then, the following statements are equivalent:*

(i) $SVVI(K, F)$ *is nonempty and bounded;*
(ii) $K_\infty \cap [(g \circ F)(K)]^- = \{0\}$*;*
(iii) VVI(K, F) *is strictly feasible; i.e.,* $(g \circ F)(K) \bigcap int(-barr(K)) \neq \emptyset$.

Proof From Lemmas 2.6 and 3.1, we know that $g \circ F : K \rightrightarrows \mathbb{R}^n$ is upper semicontinuous and pseudomonotone on K with nonempty, compact and convex values. Then, the implication (i)⇔(ii) follows directly from Theorem 2 of [27]. We only need to claim that (ii)⇔(iii).

(ii)⇒(iii). Suppose that (ii) holds, that is

$$K_\infty \cap [(g \circ F)(K)]^- = \{0\}. \tag{13}$$

Let $\widehat{x} \in K$ be any given vector. Define a homotopy by

$$H(t, x) := t(g \circ F(x) + N_K(x)) + (1 - t)(J(x) - J(\hat{x})), \quad \forall (t, x) \in [0, 1] \times K. \tag{14}$$

Then, we have $H(0, x) = J(x) - J(\hat{x})$ and $H(1, x) = g \circ F(x) + N_K(x)$. Define a set

$$C := \{x \in K : 0 \in H(t, x) \text{ for some } t \in [0, 1]\}.$$

We claim that C is bounded. Otherwise, there exist $\{t_k\} \subset [0, 1]$ and $\{x_k\} \subset K$ such that

$$\lim_{k \to \infty} \|x_k\| = \infty \text{ and } 0 \in H(t_k, x_k), \quad \forall k \in \mathbb{N}.$$

That is,

$$0 \in t_k(g \circ F(x_k) + N_K(x_k)) + (1 - t_k)(J(x_k) - J(\hat{x})).$$

Then, there exists $\xi_k \in C^{*0}$ such that

$$0 \in t_k \xi_k F(x_k) + (1 - t_k)(J(x_k) - J(\hat{x})) + N_K(x_k).$$

Thus,

$$\left\langle t_k \xi_k F(x_k) + (1 - t_k)(J(x_k) - J(\hat{x})), y - x_k \right\rangle \geq 0, \quad \forall y \in K. \tag{15}$$

If $t_k = 0$, then (15) implies that

$$\left\langle J(x_k) - J(\hat{x}), y - x_k \right\rangle \geq 0, \quad \forall y \in K.$$

Taking $y = \hat{x}$, it follows that

$$0 \geq \left\langle J(x_k) - J(\hat{x}), x_k - \hat{x} \right\rangle,$$

which implies that $x_k = \hat{x}$ for all $k = 1, 2, \ldots$. However, $\lim_{k \to \infty} \|x_k\| = \infty$. Thus, it is impossible that $t_k = 0$. Now assuming $t_k > 0$ for k large enough, by (15), we have

$$\langle \xi_k F(x_k), y - x_k \rangle \geq -\frac{1 - t_k}{t_k} \left\langle J(x_k) - J(\hat{x}), y - x_k \right\rangle, \quad \forall y \in K. \tag{16}$$

Since $\langle J(x_k), x_k \rangle = \|x_k\|^2$ and $\|x_k\| \to +\infty$, for each fixed but arbitrary $y \in K$, the right-hand side of (16) is nonnegative for all k sufficiently large. Thus, by the pseudomonotonicity of $g \circ F$ on K, we get that

$$\langle y^*, y - x_k \rangle \geq 0, \quad \forall y^* \in g \circ F(y) \tag{17}$$

for all k sufficiently large. Since $\{x_k\}$ is an unbounded sequence in the closed and convex set K, passing to a subsequence, we obtain that

$$\lim_{k \to \infty} \frac{x_k}{\|x_k\|} = d,$$

where $d \neq 0$ and $d \in K_\infty$. It follows from (17) that

$$\left\langle y^*, \frac{y - x_k}{\|x_k\|} \right\rangle \geq 0.$$

Taking the limit in the above inequality, we obtain

$$0 \geq \langle y^*, d \rangle, \quad \forall y^* \in g \circ F(K),$$

and so

$$d \in K_\infty \backslash \{0\} \text{ and } \langle y^*, d\rangle \leq 0, \quad \forall y^* \in g \circ F(y),$$

which contradicts the expression (13). Therefore, C is a bounded set. Let Ω be an open and bounded set in X containing C. If there exists $x \in \mathrm{cl}\,\Omega$ such that $0 \in H(t, x)$ for some $t \in [0, 1]$, then $x \in C$, and so $x \notin \partial\Omega$ as Ω is an open set. Thus, $0 \notin H([0, 1] \times \partial\Omega)$ and

$$0 \notin H(1, \partial\Omega) = (g \circ F + N_K)(\partial\Omega) \text{ and } 0 \notin H(0, \partial\Omega) = (J - J(\hat{x}))(\partial\Omega). \quad (18)$$

It follows that $d(g \circ F + N_K, \Omega, 0)$ and $d(J - J(\hat{x}), \Omega, 0)$ are well defined. By Lemma 2.7, we have

$$d(g \circ F + N_K, \Omega, 0) = d(J - J(\hat{x}), \Omega, 0) = 1. \quad (19)$$

Let q be an arbitrary vector in $\mathrm{int}(-\mathrm{barr}(K))$. For any $\varepsilon > 0$, define

$$F_\varepsilon(x) := g \circ F(x) - \varepsilon q, \quad \forall\, x \in K$$

and

$$\widetilde{H}(t, x) := t(g \circ F(x) + N_K(x)) + (1 - t)F_\varepsilon(x), \quad \forall (t, x) \in [0, 1] \times \mathrm{cl}\,\Omega.$$

Then,

$$\widetilde{H}(t, x) = g \circ F(x) - (1 - t)\varepsilon q + N_K(x), \quad \forall (t, x) \in [0, 1] \times \mathrm{cl}\,\Omega,$$

and so

$$\widetilde{H}(0, x) = F_\varepsilon(x) + N_K(x), \quad \widetilde{H}(1, x) = g \circ F(x) + N_K(x).$$

Now we claim that for some $\varepsilon > 0$, $0 \notin \widetilde{H}([0, 1] \times \partial\Omega)$. Otherwise, for any $n \in \mathbb{N}$, there exist $t_n \in [0, 1]$, $x_n \in K \cap \partial\Omega$ and $\xi_n \in C^{*0}$ such that

$$0 \in \xi_n F(x_n) - \frac{1 - t_n}{n} q + N_K(x_n),$$

and so

$$\langle \xi_n F(x_n) - \frac{1 - t_n}{n} q, y - x_n\rangle \geq 0, \quad \forall\, y \in K. \quad (20)$$

Since Ω is bounded, $\partial\Omega$ is a compact set. Without any loss of generality, we may assume that the sequence $\{x_n\}$ converges to some point $x_0 \in K \cap \partial\Omega$. Similarly, we assume that $\xi_n \to \xi_0 \in C^{*0}$. It follows from (20) that

$$\langle \xi_0 F(x_0), y - x_0\rangle \geq 0, \quad \forall\, y \in K,$$

which implies that x_0 is a solution of VI$(K, g \circ F)$ and so

$$0 \in g \circ F(x_0) + N_K(x_0).$$

This contradicts the first expression in (18), and so the claim is established.

By Lemma 2.7 and (19), we get

$$d(F_\varepsilon + N_K, \Omega, 0) = d(g \circ F + N_K, \Omega, 0) = 1.$$

So there is a vector $x_0 \in K$ such that $0 \in F(x_0) - \varepsilon q + N_K(x_0)$. Thus, there exists $u_0 \in F(x_0)$ and $\xi_0 \in C^{*0}$ such that

$$\langle \xi_0 u_0 - \varepsilon q, y - x_0 \rangle \geq 0, \quad \forall y \in K. \tag{21}$$

For any $d \in K_\infty \backslash \{0\}$, we have $x_0 + d \in K$. It follows from (21) that

$$\langle \xi_0 u_0 - \varepsilon q, x_0 + d - x_0 \rangle \geq 0,$$

and so

$$\langle \xi_0 u_0 - \varepsilon q, d \rangle \geq 0,$$

which implies that

$$\langle \xi_0 u_0, d \rangle \geq \langle \varepsilon q, d \rangle > 0, \quad \forall d \in K_\infty \backslash \{0\}.$$

Since $barr(K)^- = K_\infty$ (see Proposition 3.10 in [28]) and $d \in K_\infty \backslash \{0\}$ is arbitrary, we obtain that $\xi_0 u_0 \in \text{int}(-\text{barr}(K))$ and so

$$(g \circ F)(K) \bigcap \text{int}(-\text{barr}(K)) \neq \emptyset.$$

(iii)⇒(ii). If $(g \circ F)(K) \bigcap \text{int}(-\text{barr}(K)) \neq \emptyset$, then we have

$$0 \in \text{int}[\text{barr}(K) + g \circ F(K)],$$

and so

$$[\text{barr}(K) + g \circ F(K)]^- = \{0\}.$$

Since $\text{barr}(K)^- = K_\infty$, it follows that

$$\begin{aligned} \{0\} &= [\text{barr}(K) + g \circ F(K)]^- \supset \text{barr}(K)^- \bigcap (g \circ F(K))^- \\ &= K_\infty \bigcap (g \circ F(K))^- \\ &= \{0\}, \end{aligned}$$

which implies that

$$K_\infty \cap (g \circ F(K))^- = \{0\}.$$

Thus, one can apply again Theorem 2 of [27] to conclude that the solution set of $\mathrm{VI}(g \circ F, K)$ is nonempty and bounded, and so is the solution set of $\mathrm{VVI}(F, K)$. This completes the proof. □

Remark 3.4 Theorem 3.3 establishes some new equivalent characterizations for C-pseudomonotone $\mathrm{VVI}(K, F)$ to have nonempty and bounded solution set via the degree theory directly.

Example 3.2 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}_+$, $C = \mathbb{R}^2_+$ and $e = (1, 1) \in \operatorname{int} C$. Let $F : K \to L(X, Y)$ be defined by

$$F(x) := (x^2 + 1, \mathrm{e}^x)^T, \quad \forall x \in K,$$

where $(x^2 + 1, \mathrm{e}^x)^T$ denotes the transposition of a row vector $(x^2 + 1, \mathrm{e}^x)$ in $\mathbb{R}^2$. Clearly, F is a continuous mapping on K. Moreover, it is not hard to verify that F is $\mathbb{R}^2_+$-pseudomonotone on K. Thus, all the assumptions of Theorem 3.3 are satisfied. By a simple computation, we have

$$\mathrm{SVVI}(K, F) = \{0\}.$$

Furthermore, we get that

$$\begin{aligned} K_\infty &= K = \mathbb{R}_+, \quad \mathrm{barr}(K) = -\mathbb{R}_+, \quad \mathrm{int}(-\mathrm{barr}(K)) = \mathbb{R}_+ \backslash \{0\}, \\ C^{*0} &= \{(\xi_1, \xi_2) \in \mathbb{R}^2_+ : \xi_1 + \xi_2 = 1\} \end{aligned}$$

and

$$\begin{aligned} (g \circ F)(K) &= \{\xi_1 (x^2 + 1) + \xi_2 \mathrm{e}^x : \forall \xi = (\xi_1, \xi_2) \in C^{*0}, \forall x \in K\} \\ &= \{x \in \mathbb{R} : x \geq 1\}, \quad [(g \circ F)(K)]^- = -\mathbb{R}_+. \end{aligned}$$

Consequently,

$$K_\infty \cap [(g \circ F)(K)]^- = \{0\}$$

and

$$(g \circ F)(K) \bigcap \mathrm{int}(-\mathrm{barr}(K)) = \{x \in \mathbb{R} : x \geq 1\} \neq \emptyset.$$

From the above discussion, we know that the conclusion of Theorem 3.3 holds true.

When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, from Theorem 3.3, we obtain the following corollary, which gives some classical characterizations for the nonempty and boundedness of solution set for the scalar variation inequalities (in this case, F can be set-valued).

Corollary 3.2 *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex subset and $F : K \rightrightarrows \mathbb{R}^n$ be a pseudomonotone and upper semicontinuous mapping with nonempty, compact and convex values. Then, the following conclusions are equivalent:*

(i) *SVI(K, F) is nonempty and bounded;*
(ii) $K_\infty \cap [F(K)]^- = \{0\}$;
(iii) *VI(K, F) is strictly feasible, i.e., $F(K) \bigcap int(-barr(K)) \neq \emptyset$.*

4 Conclusions

This paper aims to construct a degree theory for set-valued vector variational inequalities in reflexive Banach spaces. As we know, the degree theory is very effective and has extensive applications in many fields such as differential equation, fixed point theory. In recent years, many authors further use degree theory as a tool to study the solution existence for various kinds of scalar variational inequalities and achieve abundant research results. However, there is still no paper to establish the degree theory for vector variational inequalities, and this motivates us to consider such a problem. By introducing a set-valued mapping $g(\cdot)$ from $L(X, Y)$ to X^* and using the degree theory for mappings of the form $f + T + G$, we establish a degree theory for vector variational inequalities. This enables us to apply degree method directly to obtain some new existence results for vector variational inequalities, not via some existing classical method such as KKM theory and the scalarization method. In some sense, the research of this paper may provide a new and valuable method to study the existence of solutions for vector variational inequalities and other related problems.

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