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Local Boundedness of Minimizers with Limit Growth Conditions

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Abstract The energy integral of the calculus of variations, which we consider in this paper, has a limit behavior when the maximum exponent q, in the growth estimate of the integrand, reaches a threshold. In fact, if q is larger than this threshold, counterexamples to the local boundedness and regularity of minimizers are known. In this paper, we prove the local boundedness of minimizers (and also of quasi-minimizers) under this stated limit condition. Some other general and limit growth conditions are also considered.

Keywords Quasi-minimizer · Anisotropic growth conditions · Local boundedness · Non-coercive functional

Mathematics Subject Classification 49N60 · 35J60 · 35J25

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1 Introduction

We are interested in the local boundedness of minimizers of some integrals of the calculus of variations. The energy density is assumed to satisfy suitable growth conditions, precisely the p, q-growth and the *anisotropic* growth conditions. The regularity of local minimizers, under nonstandard growth, has been extensively studied in the last years, starting by Marcellini [1,2]. In the anisotropic case, if the exponents that appear in the growth estimate of the integrand from above are the same than those appearing in the estimate from below, the boundedness has been studied in Boccardo et al. [3], Stroffolini [4] and Fusco and Sbordone [5,6]; if the exponents from above may be different than those from below, the local boundedness of minimizers of functionals has been studied by the authors in [7]; see also [8–10]. We also point out a particular case of p, q-growth condition considered in the recent interesting papers by Colombo and Mingione [11,12]; see also Esposito et al. [13]: The functional, there considered, has an integrand that changes drastically its growth, sharply moving from a p-growth to a q-growth. Other related boundedness results are in Dall'Aglio et al. [14], Mascolo and Papi [15] and Moscariello and Nania [16].

The common feature, in these cited results, is that the local boundedness of the minimizers holds if the exponents are not too spread, otherwise the boundedness may fail; see Giaquinta [17], Marcellini [18], Hong [19] for counterexamples. Precisely, if the maximum exponent in the growth estimate from above of the integrand is greater than the Sobolev exponent of the harmonic average of the exponents appearing in the growth estimate from below of the integrand, then the minimizers may be locally unbounded.

In the present paper, we prove that, below this threshold, we get locally bounded minimizers. The equality case, more delicate, is also treated. We adopt a different strategy than in [7], where the Euler equation and the Moser iteration scheme were used. Here, we derive the local boundedness by the De Giorgi method of super(sub)-level sets; see [20]. This allows to improve the previous results in different directions: we consider a Carathéodory integrand f, thus not necessarily smooth; we admit the dependence of f not only on x and Du, but on u too; we obtain the boundedness of *quasi-minimizers* and not only of local minimizers. As noted above, we prove that if q is less then or equal to the Sobolev exponent of the harmonic average of the exponents $\{p_i\}$ appearing in the growth estimate from below of the integrand f, then the quasi-minimizers are locally bounded. We stress that we are able to include the equality case that the procedure of the Moser iteration argument was unable to include. The embedding results for anisotropic Sobolev spaces, see Troisi [21] and Acerbi and Fusco [22], play a crucial role. A delicate case is when the maximum of the summability exponents $\{p_i\}$ is equal to the Sobolev exponent of their harmonic average $\overline{p}$. In this case, the Sobolev space is no more embedded in the $\overline{p}^*$-Lebesgue space; see Kruzhkov and Kolodii [23] and Haskovec and Schmeiser [24] for counterexamples; see also Remark 3.1 for further details. In the known literature, this fact is sometimes not considered and the condition that the minimizers have to be assumed a priori in the $\overline{p}^*$-Lebesgue space is omitted.

Moreover, we study a class of variational integrals with linear growth from below. Under this assumption, the lack of coercivity is overcome using the relaxed functional, acting on a suitable subclass of the BV-functions, and a generalized definition of minimizers. We prove that there exists a locally bounded minimizer for this new functional. We refer to Beck and Schmidt [25] for related results.

The contents of the paper is described next briefly. In Sect. 2, we give the precise hypotheses and statements of the regularity results. We also state regularity results for minimizers of functionals in suitable Dirichlet classes, dealing with both the coercive case and the non-coercive case; see Theorems 2.4 and 2.5. Sections 5 and 6 contain the proofs of Theorems 2.1, 2.2 and 2.5. The proofs rely on embedding results for anisotropic Sobolev spaces and on a suitable Caccioppoli inequality; these results can be found in Sects. 3 and 4, respectively.

2 Assumptions and Statement of the Main Results

Define the integral functional

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x,u,Du(x))\,\mathrm{d}x, \tag{1}$$

where Ω is an open and bounded subset of $\mathbb{R}^n$, $n \geq 2$, and $u \in W^{1,1}(\Omega,\mathbb{R})$.

Assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, such that

(H1) either f is convex in the pair (s,ξ)
or
f is separately convex in s and ξ and $\lim_{|s|\to+\infty} f(x,s,\xi) = +\infty$ uniformly w.r.t. x and ξ.

(H2) there exist $c_1, c_2 > 0$ and $1 \leq p_i \leq q$, $i = 1,\dots,n$, such that

$$c_1 \sum_{i=1}^{n} [g(|\xi_i|)]^{p_i} \leq f(x,s,\xi) \leq c_2 \left\{1 + [g(|s|)]^q + [g(|\xi|)]^q\right\} \tag{2}$$

for a.e. x and every $\xi \in \mathbb{R}^n$.

Here, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is of class C^1, convex, non-decreasing, $g(0) = 0$, $g \not\equiv 0$, satisfying, for some $\mu \geq 1$ and some $t_0 \geq 0$,

$$g(\lambda t) \leq \lambda^\mu g(t) \quad \text{for every} \quad \lambda > 1 \quad \text{and every} \quad t \geq t_0. \tag{3}$$

Without any loss of generality, we assume t_0 large, so that $g(t) \geq 1$ for all $t \geq t_0$. Moreover, note that, if the second alternative in (H1) holds, then:

$$\begin{aligned}&\exists\, M \geq 0 \text{ such that } f(x,\cdot,\xi) \text{ is decreasing in }]-\infty,-M]\\ &\text{and increasing in } [M,+\infty[.\end{aligned} \tag{4}$$

In this case, we can also assume $t_0 \geq 2M$.

Now, we introduce some notation. Given a function φ, then $\operatorname{supp}\varphi$ is the support of φ. The set $B_R(x_0)$ is the ball in $\mathbb{R}^n$ of center x_0 and radius R. Moreover, given two sets $A, B \subseteq \mathbb{R}^n$, we write $A \Subset B$ whenever the closure of A is a subset of B.

Let us now give the definition of quasi-minimizers of (1).

Definition 2.1 A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a *quasi-minimizer* of (1) iff there exists $Q \geq 1$ such that $\mathcal{F}(u; \operatorname{supp}\varphi) < +\infty$ and

$$\mathcal{F}(u; \operatorname{supp}\varphi) \leq Q\mathcal{F}(u+\varphi; \operatorname{supp}\varphi),$$

for all $\varphi \in W^{1,1}(\Omega)$ with $\operatorname{supp}\varphi \Subset \Omega$. If $Q = 1$, then u is a *local minimizer* of (1).

It is well known that restrictions on the exponents $\{p_i\}$ and q are necessary to have the local boundedness of quasi-minimizers of (1). We denote by $\overline{p}$ the harmonic average of $\{p_i\}$; i.e., $\frac{1}{\overline{p}} := \frac{1}{n}\sum_{i=1}^n \frac{1}{p_i}$; finally, $\overline{p}^*$ is the Sobolev exponent of $\overline{p}$:

$$\overline{p}^* := \begin{cases} \frac{n\overline{p}}{n-\overline{p}}, & \text{if } \overline{p} < n, \\ \text{any } s > \overline{p}, & \text{if } \overline{p} \geq n. \end{cases} \tag{5}$$

Our first result deals with the case $q < \overline{p}^*$.

Theorem 2.1 *Assume* (H1) *and* (H2). *If $q < \overline{p}^*$, then any quasi-minimizer u of* (1) *is locally bounded. Moreover, fixed $B_R(x_0) \Subset \Omega$, there exists a constant c, depending on q, p_i, μ, Q, t_0, c_1, c_2, such that*

$$\|g(|u|)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq c\left\{1 + \frac{1}{R^{\mu\frac{q\overline{p}^*}{p(\overline{p}^*-q)}}}\left(\int_{B_R(x_0)} g^q(|u|)\,dx\right)^{\frac{1+\theta}{q}}\right\}, \tag{6}$$

where $\theta := \frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}$, with $p := \min\{p_i\}$.

As far as the borderline case $q = \overline{p}^*$ is concerned, we have the following result.

Theorem 2.2 *Assume* (H1) *and* (H2). *If $q = \overline{p}^*$ and*

$$\textit{either} \quad \max\{p_i\} < \overline{p}^* \quad \textit{or} \quad g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega),$$

then any quasi-minimizer u of (1) *is locally bounded.*

Example 2.1 Let us consider the functional

$$\mathcal{F}(u) := \int_\Omega \left(\sum_{i=1}^n |u_{x_i}|^{p_i} + a(x)|u_{x_n}|^q\right) \mathrm{d}x,$$

with $1 \leq p_1 \leq \cdots \leq p_n$. Assume that $a \not\equiv 0$, with $a(x) = 0$ on a set of positive measure: If $p_n < q = \overline{p}^*$, then the quasi-minimizers of $\mathcal{F}$ are locally bounded.

Assume now $a(x) \equiv 1$: If $p_n = q = \overline{p}^*$, then we can conclude that any quasi-minimizer $u \in L^{\overline{p}^*}_{\text{loc}}(\Omega)$ of $\mathcal{F}$ is locally bounded.

Note that, if the p_i's are equal, then a straightforward consequence of the above results is the following.

Theorem 2.3 *Assume* (H1) *and that there exists* $c_1, c_2 > 0$*, such that*

$$c_1|\xi|^p \leq f(x,s,\xi) \leq c_2\left\{1+|s|^q+|\xi|^q\right\},$$

for a.e. x and every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^n$.

If $1 \leq p < q \leq p^*$, then the quasi-minimizers of $\mathcal{F}$ are locally bounded.

Now, we deal with the minimization problem in a Dirichlet class. To do this, we consider $g(t) := t$; i.e.,

(H3) there exist $c_1, c_2 > 0$ and $1 \leq p_i \leq q$, $i = 1, \dots, n$, such that

$$c_1 \sum_{i=1}^{n} |\xi_i|^{p_i} \leq f(x,s,\xi) \leq c_2\left\{1+|s|^q+|\xi|^q\right\}$$

for a.e. x and every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^n$.

A first result, with $\min\{p_i\} > 1$, is the following.

Theorem 2.4 *Assume* (H1) *and* (H3)*, with $1 < p_i \leq q \leq \overline{p}^*$, $i = 1, \dots, n$. Let $u_0 \in W^{1,1}(\Omega) \cap L^{\overline{p}^*}_{\text{loc}}(\Omega)$ be such that $\mathcal{F}(u_0;\Omega) < +\infty$. If u is a minimizer of $\mathcal{F}(\cdot;\Omega)$ in $u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$, then u is locally bounded.*

Now, let us consider the analogue of Theorem 2.4, under the assumption $\min\{p_i\} = 1$.

Fix $u_0 \in W^{1,1}(\Omega)$, such that $\mathcal{F}(u_0;\Omega) < +\infty$. Since $\min\{p_i\} = 1$, then $W^{1,(p_1,\dots,p_n)}(\Omega)$ is a non-reflexive space and the direct method generally fails. So, minimizers of $\mathcal{F}$ in $u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$ may not exist. We claim that minimizers in BV of the relaxed functional in $BV(\Omega)$ of $\mathcal{F}$, i.e.,

$$\overline{\mathcal{F}}(u) := \inf\left\{\liminf_{k\to+\infty} \mathcal{F}(u_k) \,:\, u_k \to u \text{ in } L^1(\Omega),\ u_k \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)\right\},$$

exist and are locally bounded.

Theorem 2.5 *Assume* (H1) *and* (H3)*, with $1 \leq p_i \leq q < \overline{p}^*$, $\min\{p_i\} = 1$. Fixed $u_0 \in W^{1,1}(\Omega)$, such that $\mathcal{F}(u_0;\Omega) < +\infty$, there exists a minimizer $\bar{u} \in BV(\Omega)$ of $\overline{\mathcal{F}}$, such that $\bar{u} \in L^\infty_{\text{loc}}(\Omega)$ and, for all $B_R(x_0) \Subset \Omega$,*

$$\|\bar{u}\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq c\left\{1+\frac{1}{R^{\mu\frac{q\overline{p}^*}{\overline{p}^*-q}}}\left(\overline{\mathcal{F}}(\bar{u})+1+\|u_0\|_{W^{1,(p_1,\dots,p_n)}(\Omega)}\right)^{1+\theta}\right\},$$

where $\theta := \frac{\overline{p}^*(q-1)}{\overline{p}^*-q}$.

3 Anisotropic Sobolev Spaces

To prove our results, we use a suitable anisotropic Sobolev space. Precisely, we consider

$$W^{1,(p_1,\dots,p_n)}(\Omega) := \left\{ u \in W^{1,1}(\Omega) \,:\, u_{x_i} \in L^{p_i}(\Omega), \text{ for all } i = 1,\dots,n \right\},$$

endowed with the norm

$$\|u\|_{W^{1,(p_1,\dots,p_n)}(\Omega)} := \|u\|_{L^1(\Omega)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

We write $W_0^{1,(p_1,\dots,p_n)}(\Omega)$ in place of $W_0^{1,1}(\Omega) \cap W^{1,(p_1,\dots,p_n)}(\Omega)$. Note that, in this last space, an equivalent norm of u is given by $\sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}$.

We recall the following embedding results for anisotropic Sobolev spaces. We refer to [21,22].

Theorem 3.1 *Let $p_i \geq 1$, $i = 1,\dots,n$, and $\overline{p}^*$ be as in (5).*
If $u \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$, with Ω open and bounded set in $\mathbb{R}^n$, then there exists c, depending on n, p_i and, only in the case $\overline{p} \geq n$, also on $\overline{p}^*$ and the measure of the support of u, such that

$$\|u\|_{L^{\overline{p}^*}(\Omega)} \leq c \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

Theorem 3.2 *Let $Q \subset \mathbb{R}^n$ be a cube with edges parallel to the coordinate axes and consider $u \in W^{1,(p_1,\dots,p_n)}(Q)$, $p_i \geq 1$ for all $i = 1,\dots,n$. If $\overline{p} < n$, assume also that* $\max\{p_i\} < \overline{p}^*$.
Then, $u \in L^{\overline{p}^*}(Q)$. Moreover, there exists c, depending on n, p_i and, if $\overline{p} \geq n$, also on $\overline{p}^*$ and the measure of the support of u, such that

$$\|u\|_{L^{\overline{p}^*}(Q)} \leq c \left\{ \|u\|_{L^1(Q)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(Q)} \right\}. \tag{7}$$

We also need the following result; see Proposition 1 in [7].

Proposition 3.1 *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and let g satisfy the assumptions described in* Sect. 2. *Suppose that $g(|u_{x_i}|) \in L^{p_i}_{\mathrm{loc}}(\Omega)$, with $1 \leq p_i < \overline{p}^*$ for every $i = 1,\dots,n$. Then, $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$.*

Remark 3.1 Let $n \geq 2$. In general, the inclusion $W^{1,(p_1,\dots,p_n)}(\Omega) \subset L^{\overline{p}^*}(\Omega)$ does not hold, even if Ω is a rectangular domain. Let assume $\overline{p} < n$, that is $\sum_{i=1}^{n} \frac{1}{p_i} > 1$, and, without loss of generality, assume $p_1 \leq p_2 \leq \dots \leq p_n$. Define, for $k = 1,\dots,n$,

$$q^k := \begin{cases} \dfrac{k}{\sum_{i=1}^{k} \frac{1}{p_i} - 1}, & \text{if } \sum_{i=1}^{k} \frac{1}{p_i} > 1, \\ +\infty, & \text{else.} \end{cases}$$

If $p_n = \overline{p}^*$, we have $q^{n-1} = q^n = \overline{p}^*$. Thus, by Lemma 1 and Theorem 6 in [24], $W^{1,(p_1,\ldots,p_n)}(\Omega)$ is continuously embedded into every $L^q(\Omega)$ with $q < \overline{p}^*$. In [24], it is also proved that, if $q^{n-1} > q^n$, then $W^{1,(p_1,\ldots,p_n)}(\Omega)$ is continuously embedded into $L^{\overline{p}^*}(\Omega)$.

4 Caccioppoli Inequality

First of all, we recall some properties of the Δ_2-functions; see [7] for the proof.

Lemma 4.1 *Consider $g : \mathbb{R}_+ \to \mathbb{R}_+$ of class C^1, convex, non-decreasing and satisfying* (3). *Then,*

$$g(\lambda t) \leq \lambda^\mu (g(t) + g(t_0)) \quad \text{and} \quad g'(t)t \leq \mu(g(t) + g(t_0)),$$

for all $t \geq 0$ and all $\lambda > 1$.
Moreover, for every $(t_1, \ldots, t_k) \in \mathbb{R}_+^k$, we have:

$$k^{-1} \sum_{i=1}^{k} g(t_i) \leq g\left(\sum_{i=1}^{k} t_i\right) \leq k^\mu \left\{ g(t_0) + \sum_{i=1}^{k} g(t_i) \right\}.$$

Now, we state a lemma related to the convexity assumptions on f.

Lemma 4.2 *If the second alternative in* (H1) *holds, then, for all $\xi_1, \xi_2 \in \mathbb{R}^n$, we have*

$$\begin{aligned} f(x, ts_1 + (1-t)s_2, t\xi_1 + (1-t)\xi_2) \leq & t^2 f(x, s_1, \xi_1) + (1-t) f(x, s_2, \xi_2) \\ & + t(1-t) f(x, s_2, \xi_1), \end{aligned}$$

whenever $0 \leq t \leq 1$ and $M \leq s_1 \leq s_2$ or $s_2 \leq s_1 \leq -M$. Here, M is as in (4).

Proof Using the convexity of f in the second and in the third variable, we have

$$\begin{aligned} f(x, ts_1 + (1-t)s_2, t\xi_1 + (1-t)\xi_2) \leq & t^2 f(x, s_1, \xi_1) \\ & + t(1-t)\{f(x, s_1, \xi_2) + f(x, s_2, \xi_1)\} + (1-t)^2 f(x, s_2, \xi_2). \end{aligned}$$

Since $f(x, s_1, \xi_2) \leq f(x, s_2, \xi_2)$, then the thesis follows. □

The following is a well-known classical result; see, e.g., [26].

Lemma 4.3 *Let $\phi(t)$ be a nonnegative and bounded function, defined in $[\tau_0, \tau_1]$. Suppose that, for all s, t, such that $\tau_0 \leq s < t \leq \tau_1$, ϕ satisfies*

$$\phi(s) \leq \theta\phi(t) + \frac{A}{(t-s)^\alpha} + B,$$

where A, B, α are nonnegative constants and $0 < \theta < 1$.
Then, for all ρ and R, such that $\tau_0 \leq \rho \leq R \leq \tau_1$, we have

$$\phi(\rho) \leq C \left\{ \frac{A}{(R-\rho)^\alpha} + B \right\}.$$

If $u \in W^{1,1}(\Omega)$ and $B_R(x_0) \subseteq \Omega$, we define the super-level sets:

$$A_{k,R} := \{x \in B_R(x_0) \,:\, u(x) > k\}, \qquad k \in \mathbb{R}.$$

The following Caccioppoli inequality holds.

Theorem 4.1 *Assume* (H1), (H2) *and let $u \in W^{1,1}_{\text{loc}}(\Omega)$ be a quasi-minimizer of $\mathcal{F}$, such that $g(|u|) \in L^q_{\text{loc}}(\Omega)$. Then, there exists a constant $c > 0$, such that, for any $B_R(x_0) \Subset \Omega$, $0 < \rho < R \leq \rho + 1$, and for any k and d, such that $\frac{t_0}{2} \leq k \leq d$,*

$$\int_{A_{k,\rho}} f(x, u, Du)\, dx \leq \frac{c}{(R-\rho)^{\mu q}} \int_{A_{k,R}} \left\{ g^q(u-k) + g^q(d) \right\} dx. \tag{8}$$

Proof Let $B_R(x_0) \Subset \Omega$. Let ρ, s, t be such that $\rho \leq s < t \leq R \leq \rho + 1$. Let $\eta \in C_0^\infty(B_t)$ be a cutoff function, satisfying the following assumptions:

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \leq \frac{2}{t-s}. \tag{9}$$

Fixed $k \in \mathbb{R}_+$, define

$$w := \max(u-k, 0) \quad \text{and} \quad \varphi := -\eta^{\mu q} w.$$

Consider a number d, such that $d \geq k$. By the quasi-minimality of u, we get

$$\begin{aligned}
&\int_{A_{k,s}} f(x, u, Du)\, \mathrm{d}x \leq Q \int_{A_{k,t}} f(x, u+\varphi, Du + D\varphi)\, \mathrm{d}x \\
&= Q \int_{A_{k,t}} f\left(x, \left(1-\eta^{\mu q}\right) u + \eta^{\mu q} k, \left(1-\eta^{\mu q}\right) Du + \mu q \eta^{\mu q - 1}(k-u) D\eta \right) \mathrm{d}x.
\end{aligned}$$

Case 1 Let us assume that the first alternative in (H1) holds.
If f is convex in (s, ξ), by (H2) we have that, for a.e. $x \in \{\eta \neq 0\}$,

$$\begin{aligned}
& f\left(x, \left(1-\eta^{\mu q}\right) u + \eta^{\mu q} k, \left(1-\eta^{\mu q}\right) Du + \eta^{\mu q} \left(\mu q \frac{k-u}{\eta} D\eta \right)\right) \\
&\leq \left(1-\eta^{\mu q}\right) f(x, u, Du) + \eta^{\mu q} f\left(x, k, \mu q \frac{k-u}{\eta} D\eta\right) \\
&\leq (1-\eta^{\mu q}) f(x, u, Du) + c_2 \eta^{\mu q} \left\{ 1 + g^q\left(\mu q \left|\frac{u-k}{\eta} D\eta\right|\right) + g^q(d) \right\}.
\end{aligned}$$

Lemma 4.1 and (9) imply

$$g(|\mu q \frac{u-k}{\eta} D\eta|) \leq \frac{(2\mu q)^\mu}{(t-s)^\mu \eta^\mu} \left\{g(|u-k|) + g(t_0)\right\}. \tag{10}$$

Taking into account that $\operatorname{supp}(1-\eta^{\mu q}) \subset A(k,t) \setminus A(k,s)$ and $t \leq R$, we obtain

$$\begin{aligned}\int_{A_{k,s}} f(x,u,Du)\,\mathrm{d}x &\leq Q \int_{A_{k,t}} (1-\eta^{\mu q}) f(x,u,Du)\,\mathrm{d}x \\ &\quad + Q \frac{c}{(t-s)^{\mu q}} \int_{A_{k,t}} \left\{g^q(u-k) + g^q(t_0) + g^q(d) + 1\right\} \mathrm{d}x \\ &\leq Q \int_{A_{k,t}\setminus A_{k,s}} f(x,u,Du)\,\mathrm{d}x + \frac{c_3}{(t-s)^{\mu q}} \int_{A_{k,R}} \left(g^q(u-k) + g^q(d)\right) \mathrm{d}x, \end{aligned} \tag{11}$$

with $c_3 = c_3(n, \mu, q, Q, c_2)$.

Case 2 Let us assume that the second alternative in (H1) holds.

By Lemma 4.2, with $t := \eta^{\mu q}(x)$, $s_1 := k$, $s_2 := u(x)$, $\xi_1 := \mu q \frac{k-u}{\eta} D\eta$, $\xi_2 := Du(x)$, and, using $k \geq M$, we get that, for a.e. $x \in \{u \geq k\} \cap \{\eta \neq 0\}$,

$$\begin{aligned} & f\left(x, (1-\eta^{\mu q})u + \eta^{\mu q}k, (1-\eta^{\mu q})Du + \eta^{\mu q}\mu q \frac{k-u}{\eta} D\eta\right) \\ &\leq (1-\eta^{\mu q})^2 f(x,u,Du) + \eta^{2\mu q} f(x,k,\mu q \frac{k-u}{\eta} D\eta) \\ &\quad + \eta^{\mu q} f(x,u,\mu q \frac{k-u}{\eta} D\eta). \end{aligned}$$

Now, using (H2), $k \leq d$, and (10),

$$\begin{aligned} f(x,k,\mu q \frac{k-u}{\eta} D\eta) &\leq c_2 \left\{1 + g^q(d) + g^q(\mu q \frac{|u-k|}{\eta} |D\eta|)\right\} \\ &\leq \frac{c}{(t-s)^{\mu q} \eta^{\mu q}} \{g^q(|u-k|) + g^q(d)\}. \end{aligned}$$

Analogously, taking into account that, in $A_{k,R}$,

$$\begin{aligned} g^q(|u|) = g^q(|u-k|+k) &\leq \frac{1}{2} g^q(2|u-k|) + \frac{1}{2} g^q(2k) \\ &\leq 2^{\mu q - 1} \left\{g^q(|u-k|) + g^q(d)\right\}, \end{aligned}$$

we obtain

$$f(x,u,\mu q \frac{k-u}{\eta} D\eta) \leq \frac{c}{(t-s)^{\mu q} \eta^{\mu q}} \{g^q(|u-k|) + g^q(d)\}.$$

So, we get

$$f\left(x, (1-\eta^{\mu q})u + \eta^{\mu q}k, (1-\eta^{\mu q})Du + \eta^{\mu q}\mu q\frac{k-u}{\eta}D\eta\right)$$
$$\leq (1-\eta^{\mu q})^2 f(x,u,Du) + \frac{c}{(t-s)^{\mu q}}\left\{g^q(|u-k|) + g^q(d)\right\}.$$

Therefore, estimate (11) follows.

Conclusion

By (11), adding to both sides Q times the left-hand side, we get:

$$\int_{A_{k,s}} f(x,u,Du)\,\mathrm{d}x \leq \frac{Q}{Q+1}\int_{A_{k,t}} f(x,u,Du)\,\mathrm{d}x$$
$$+ \frac{c_3}{(t-s)^{\mu q}}\int_{A_{k,R}}\left\{g^q(u-k) + g^q(d)\right\}\mathrm{d}x.$$

Thus, by Lemma 4.3, with $\tau_0 := \rho$, $\tau_1 := R$, and

$$\phi(t) := \int_{A_{k,t}} f(x,u,Du)\,\mathrm{d}x, \qquad A := \int_{A_{k,R}}\left\{g^q(u-k) + g^q(d)\right\}\mathrm{d}x,$$

we get (8). □

5 Proof of Theorems 2.1, 2.2 and 2.4

We will use the following classical result; see, e.g., [26].

Lemma 5.1 *Let $\alpha > 0$ and (J_h) a sequence of real positive numbers, such that*

$$J_{h+1} \leq A\,\lambda^h J_h^{1+\alpha},$$

with $A > 0$ and $\lambda > 1$.
If $J_0 \leq A^{-\frac{1}{\alpha}}\lambda^{-\frac{1}{\alpha 2}}$, then $J_h \leq \lambda^{-\frac{h}{\alpha}} J_0$ and $\lim_{h\to\infty} J_h = 0$.

We now need to introduce some notation.
Fixed $B_{R_0}(x_0) \Subset \Omega$, with $R \leq R_0$, define the decreasing sequences

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2}\left(1 + \frac{1}{2^h}\right), \qquad \bar{\rho}_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2}\left(1 + \frac{3}{4\cdot 2^h}\right).$$

Fixed a positive constant $d \geq t_0$, to be chosen later, define the increasing sequence of positive real numbers

$$k_h := d\left(1 - \frac{1}{2^{h+1}}\right), \quad h \in \mathbb{N}.$$

Moreover, whenever $g(|u|) \in L^q_{\mathrm{loc}}(\Omega)$, define the sequence (J_h),

$$J_h := \int_{A_{k_h,\rho_h}} g^q(u-k_h)\,\mathrm{d}x.$$

We begin proving an inequality that will be the common root to prove Theorems 2.1 and 2.2.

Lemma 5.2 *Assume* (H1) *and* (H2). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a quasi-minimizer of $\mathcal{F}$. Assume that $q < \overline{p}^*$ or, if $q = \overline{p}^*$, that $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$.*
If $2^h J_h \geq 1$ for all h, then there exists a constant $C > 0$, such that, for all $h \in \mathbb{N} \cup \{0\}$,

$$J_{h+1} \leq \frac{C}{(g(d))^{q-\frac{q^2}{\overline{p}^*}}} \left(\frac{1}{R}\right)^{\mu\frac{q^2}{p}} \lambda^h J_h^{1+\alpha},$$

where $\lambda = 4^{\mu\frac{q^2}{p}}$ and $\alpha = \frac{q}{p} - \frac{q}{\overline{p}^*}$.

Proof Since u is quasi-minimizer of $\mathcal{F}$ and (H2) holds, then $g(|u_{x_i}|) \in L^{p_i}_{\mathrm{loc}}(\Omega)$.

If $q < \overline{p}^*$, then $\max\{p_i\} < \overline{p}^*$ and, by Proposition 3.1, $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$.

If $q = \overline{p}^*$, we have, by assumption, that $g(|u|) \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$. In particular, $g(|u|) \in L^{p_i}_{\mathrm{loc}}(\Omega)$, $i = 1, \ldots, n$, and J_h is finite. Moreover, $J_{h+1} \leq J_h$, since the following chain of inequalities holds:

$$J_{h+1} \leq \int_{A_{k_{h+1},\rho_h}} g^q(u-k_{h+1})\,\mathrm{d}x \leq \int_{A_{k_{h+1},\rho_h}} g^q(u-k_h)\,\mathrm{d}x \leq J_h. \tag{12}$$

Let, now, define a sequence (ζ_h) of cutoff functions, satisfying the following properties: $\zeta_h \in C^\infty_c(B_{\bar{\rho}_h}(x_0))$, $\zeta_h \equiv 1$ in $B_{\rho_{h+1}}$, $|D\zeta_h| \leq \frac{2^{h+4}}{R}$.

By the Hölder inequality, denoting $(u-k_{h+1})_+ := \max\{u-k_{h+1}, 0\}$, we get

$$\begin{aligned} J_{h+1} &\leq |A_{k_{h+1},\bar{\rho}_h}|^{1-\frac{q}{\overline{p}^*}} \left(\int_{A_{k_{h+1},\bar{\rho}_h}} (g(u-k_{h+1})\zeta_h)^{\overline{p}^*}\,\mathrm{d}x\right)^{\frac{q}{\overline{p}^*}} \\ &= |A_{k_{h+1},\bar{\rho}_h}|^{1-\frac{q}{\overline{p}^*}} \left(\int_{B_{\bar{\rho}_h}} (\zeta_h g((u-k_{h+1})_+))^{\overline{p}^*}\,\mathrm{d}x\right)^{\frac{q}{\overline{p}^*}}. \end{aligned}$$

To apply the Sobolev embedding Theorem 3.1 to the function $g((u-k_{h+1})_+)\zeta_h$, we need to prove that $g((u-k_{h+1})_+)\zeta_h \in W^{1,(p_1,\ldots,p_n)}_0(B_{\bar{\rho}_h}(x_0))$. Precisely, it remains only to prove that $(\zeta_h g((u-k_{h+1})_+))_{x_i} \in L^{p_i}(B_{\bar{\rho}_h}(x_0))$. By Lemma 4.1 and using $(g((u(x)-k_{h+1})_+)))_{x_i} = g'(u(x)-k_{h+1})u_{x_i}(x)\chi_{A_{k_{h+1},\bar{\rho}_h}}(x)$, for a.e. $x \in B_{\bar{\rho}_h}(x_0)$

(here $\chi_{A_{k_{h+1},\bar\rho_h}}$ is, as usual, the characteristic function of the set $A_{k_{h+1},\bar\rho_h}$) we get that, for a.e. $x \in B_{\bar\rho_h}(x_0)$,

$$|(\zeta_h g((u-k_{h+1})_+))_{x_i}| \\ \le c(\mu)\frac{2^h}{R}\{g(u-k_{h+1})+g(t_0)\}\,\chi_{A_{k_{h+1},\bar\rho_h}} + \mu g(|u_{x_i}|)\chi_{A_{k_{h+1},\bar\rho_h}}. \tag{13}$$

Indeed,

$$\begin{aligned} &|(\zeta_h g((u-k_{h+1})_+))_{x_i}| \\ &\quad\le\ g((u-k_{h+1})_+)|(\zeta_h)_{x_i}| + \zeta_h g'(u-k_{h+1})|u_{x_i}|\chi_{A_{k_{h+1},\bar\rho_h}} \\ &\quad\le\ g((u-k_{h+1})_+)|D\zeta_h| \\ &\qquad +\ \zeta_h\left\{g'(u-k_{h+1})(u-k_{h+1}) + g'(|u_{x_i}|)|u_{x_i}|\right\}\chi_{A_{k_{h+1},\bar\rho_h}} \\ &\quad\le\ g(u-k_{h+1})|D\zeta_h|\chi_{A_{k_{h+1},\bar\rho_h}} \\ &\qquad +\ \zeta_h\mu\left\{g(u-k_{h+1}) + g(|u_{x_i}|) + 2g(t_0)\right\}\chi_{A_{k_{h+1},\bar\rho_h}} \end{aligned}$$

and the claim follows. Since both $g(|u|)$ and $g(|u_{x_i}|)$ are in $L^{p_i}_{\mathrm{loc}}(\Omega)$, we have proved that $(\zeta_h g((u-k_{h+1})_+))_{x_i} \in L^{p_i}(B_{\bar\rho_h}(x_0))$.
Thus, by the Sobolev embedding Theorem 3.1,

$$J_{h+1} \le c\,|A_{k_{h+1},\bar\rho_h}|^{1-\frac{q}{\bar p^*}}\left\{\sum_{i=1}^{n}\left(\int_{B_{\bar\rho_h}}|(\zeta_h g((u-k_{h+1})_+)_{x_i}|^{p_i}\,\mathrm{d}x\right)^{\frac{1}{p_i}}\right\}^{q}. \tag{14}$$

By (13), since $(a+b)^{\frac{1}{p_i}} \le a^{\frac{1}{p_i}} + b^{\frac{1}{p_i}}$, $g(d) \ge g(t_0) \ge 1$, and $\bar\rho_h \le \rho_h$, we get

$$\left(\int_{B_{\bar\rho_h}}|(\zeta_h g((u-k_{h+1})_+)_{x_i}|^{p_i}\,\mathrm{d}x\right)^{\frac{1}{p_i}} \le \mu\left(\int_{A_{k_{h+1},\bar\rho_h}}[g(|u_{x_i}|)]^{p_i}\,\mathrm{d}x\right)^{\frac{1}{p_i}} \\ + \frac{c2^h}{R}\left(\int_{A_{k_{h+1},\rho_h}}\left\{g^q(u-k_{h+1}) + g^q(d)\right\}\,\mathrm{d}x\right)^{\frac{1}{p_i}}.$$

By (2) and the Caccioppoli inequality (8), we obtain

$$\begin{aligned} c_1\int_{A_{k_{h+1},\bar\rho_h}} g^{p_i}(|u_{x_i}|)\,\mathrm{d}x &\le \int_{A_{k_{h+1},\bar\rho_h}} f(x,u,Du)\,\mathrm{d}x \\ &\le c\left(\frac{2^h}{R}\right)^{\mu q}\int_{A_{k_{h+1},\rho_h}}\{g^q(u-k_{h+1}) + g^q(d)\}\,\mathrm{d}x, \end{aligned}$$

with c possibly depending on diam Ω. Collecting the above inequalities, we have

$$\left(\int_{A_{k_{h+1},\bar{\rho}_h}} |(\zeta_h g(u-k_{h+1}))_{x_i}|^{p_i}\,\mathrm{d}x\right)^{\frac{1}{p_i}}$$
$$\leq c\left(\frac{2^h}{R}\right)^{\mu\frac{q}{p_i}}\left(\int_{A_{k_{h+1},\rho_h}}\left\{g^q(u-k_{h+1})+g^q(d)\right\}\,\mathrm{d}x\right)^{\frac{1}{p_i}}.$$

By the above inequality, (14) and (12), it follows that

$$J_{h+1}\leq c|A_{k_{h+1},\bar{\rho}_h}|^{1-\frac{q}{\overline{p}^*}}\left\{\sum_{i=1}^{n}\left(\frac{2^h}{R}\right)^{\mu\frac{q}{p_i}}\left(J_h\;+\;g^q(d)|A_{k_{h+1},\rho_h}|\right)^{\frac{1}{p_i}}\right\}^q. \tag{15}$$

Note that

$$J_h\geq\int_{A_{k_{h+1},\rho_h}} g^q(u-k_h)\,\mathrm{d}x\geq g^q(k_{h+1}-k_h)|A_{k_{h+1},\rho_h}|$$
$$=g^q(\frac{d}{2^{h+2}})|A_{k_{h+1},\rho_h}|\geq\frac{g^q(d)}{2^{(h+2)\mu q}}|A_{k_{h+1},\rho_h}|,$$

therefore

$$|A_{k_{h+1},\bar{\rho}_h}|\leq|A_{k_{h+1},\rho_h}|\leq\frac{2^{(h+2)\mu q}}{g^q(d)}J_h. \tag{16}$$

Since $2^h J_h\geq 1$ for all h, by (12), (15), (16), denoting $p:=\min\{p_i\}$, we obtain

$$J_{h+1}\leq c\left(\frac{2^{h\mu q}}{g^q(d)}J_h\right)^{1-\frac{q}{\overline{p}^*}}\left\{\sum_{i=1}^{n}\left(\frac{2^h}{R}\right)^{\mu\frac{q}{p_i}}\left(2^{h\mu q}J_h\right)^{\frac{1}{p_i}}\right\}^q$$
$$\leq c\left(\frac{2^{h\mu q}}{g^q(d)}J_h\right)^{1-\frac{q}{\overline{p}^*}}\left(\frac{2^h}{R}\right)^{\mu\frac{q^2}{p}}\left(2^{h\mu q}J_h\right)^{\frac{q}{p}}$$
$$\leq\frac{C}{R^{\mu\frac{q^2}{p}}(g^q(d))^{\frac{\overline{p}^*-q}{\overline{p}^*}}}\left(4^{\mu\frac{q^2}{p}}\right)^h J_h^{1+\frac{q}{p}-\frac{q}{\overline{p}^*}}$$

and the conclusion follows. □

We are now ready to prove the first of our main results.

Proof of Theorem 2.1 Let us assume that $2^h J_h\geq 1$ for all h, and let d be a positive constant, $d\geq t_0$, to be chosen later.
By Lemma 5.2, we have that, for all h,

$$J_{h+1}\leq\frac{C}{(g(d))^{q-\frac{q^2}{\overline{p}^*}}}\left(\frac{1}{R}\right)^{\mu\frac{q^2}{p}}\lambda^h J_h^{1+\alpha},$$

with $\lambda := 4^{\mu \frac{q^2}{p}}$ and $\alpha := \frac{q}{p} - \frac{q}{\overline{p}^*} > 0$.

Using Lemma 5.1, with $A := \dfrac{C}{R^{\mu \frac{q^2}{p}} (g^q(d))^{\frac{\overline{p}^*-q}{\overline{p}^*}}}$, we have that, if

$$J_0 \leq K[g(d)]^{p \frac{\overline{p}^*-q}{\overline{p}^*-p}}, \quad \text{with} \quad K := \left\{ \frac{C}{R^{\mu \frac{q^2}{p}}} \right\}^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}}, \tag{17}$$

then $\lim_{h \to +\infty} J_h = 0$.
Since

$$J_0 := \int_{A_{\frac{d}{2},R}} g^q \left(u - \frac{d}{2} \right) \mathrm{d}x \leq \int_{B_R} g^q(|u|) \,\mathrm{d}x,$$

it is easy to check that (17) is satisfied, if we choose d such that

$$g(d) = g(t_0) + \left\{ \frac{1}{K} \int_{B_R} g^q(|u|) \,\mathrm{d}x \right\}^{\frac{\overline{p}^*-p}{p(\overline{p}^*-q)}}. \tag{18}$$

Hence, since $\lim\limits_{h \to +\infty} J_h = \int_{A_{d,\frac{R}{2}}} g^q(u-d) \,\mathrm{d}x$, we get $|A_{d,\frac{R}{2}}| = 0$. So, we conclude that $B_{\frac{R}{2}} \subseteq \{u \leq d\}$.

On the other hand, since $-u$ is a quasi-minimizer of the functional

$$\mathcal{I}(v) := \int \overline{f}(x, u, Du) \,\mathrm{d}x,$$

where $\overline{f}(x, u, \xi) := f(x, -u, -\xi)$, which satisfies the same assumptions of f, we obtain that $B_{\frac{R}{2}} \subseteq \{u \geq -d\}$.
Therefore, by (18) and the monotonicity of g,

$$g(|u|) \leq g(t_0) + \left\{ \left(\frac{C}{R^{\mu \frac{q^2}{p}}} \right)^{\frac{1}{\alpha}} \lambda^{\frac{1}{\alpha^2}} \int_{B_R} g^q(|u|) \,\mathrm{d}x \right\}^{\frac{\overline{p}^*-p}{p(\overline{p}^*-q)}} \quad \text{a.e. in} \quad B_{\frac{R}{2}},$$

that is

$$\|g(|u|)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq g(t_0) + \frac{c}{R^{\mu \frac{q \overline{p}^*}{p(\overline{p}^*-q)}}} \left(\int_{B_R} g^q(|u|) \,\mathrm{d}x \right)^{\frac{\overline{p}^*-p}{p(\overline{p}^*-q)}}.$$

The estimate (6) follows.

Now, let us assume that it fails that $2^h J_h \geq 1$ for all h. Then, for a suitable subsequence, $J_{h_m} \to 0$, hence $g^q(u-d) = 0$ a.e. in $A_{d,\frac{R}{2}}$, for any $d \geq t_0$. Choosing d as in (18), we get the same estimate than in the previous case. □

We now turn to the proof of our boundedness result, under the assumption $q = \overline{p}^*$.

Proof of Theorem 2.2 As in the proof of Theorem 2.1, we observe that, without any loss of generality, we may suppose that $2^h J_h \geq 1$ for all h.

If $\max\{p_i\} = \overline{p}^*$, we know, by assumption, that $g(|u|) \in L^{\overline{p}^*}_{\text{loc}}(\Omega)$. The same conclusion holds if $\max\{p_i\} < \overline{p}^*$. Indeed, (H2) implies $g(|u_{x_i}|) \in L^{p_i}_{\text{loc}}(\Omega)$; so, by Proposition 3.1, $g(|u|) \in L^{\overline{p}^*}_{\text{loc}}(\Omega)$.
By Lemma 5.2,

$$J_{h+1} \leq C \left(\frac{1}{R}\right)^{\mu \frac{(\overline{p}^*)^2}{p}} \lambda^h J_h^{1+\alpha},$$

with $\lambda := 4^{\mu \frac{(\overline{p}^*)^2}{p}}$ and $\alpha := \frac{\overline{p}^*}{p} - 1 > 0$. Therefore, by Lemma 5.1 we have that $\lim_{h\to+\infty} J_h = 0$, if

$$J_0 \leq \left(C \left(\frac{1}{R}\right)^{\mu \frac{(\overline{p}^*)^2}{p}} \right)^{-\frac{1}{\alpha}} \left(4^{\mu \frac{(\overline{p}^*)^2}{p}}\right)^{-\frac{1}{\alpha^2}}. \tag{19}$$

By definition, $J_0 = \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*}\left(u - \frac{d}{2}\right) \mathrm{d}x$. We choose $d > 0$ large, such that (19) holds; this is possible, because $g^{\overline{p}^*}(|u|) \in L^1(B_R)$ and

$$J_0 = \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*}\left(u - \frac{d}{2}\right) \mathrm{d}x \leq \int_{A_{\frac{d}{2},R}} g^{\overline{p}^*}(|u|)\, \mathrm{d}x \to_{d\to+\infty} 0.$$

With this choice of d, we get $J_h \longrightarrow 0$; i.e.,

$$\int_{A_{d,\frac{R}{2}}} g^{\overline{p}^*}(u-d)\, \mathrm{d}x = 0.$$

Therefore, $u \leq d$ a.e. in $B_{\frac{R}{2}}(x_0)$.
To get a bound from below, we proceed as in the proof of Theorem 2.1. □

We conclude the section with the proof of Theorem 2.4.

Proof of Theorem 2.4 If $q < \overline{p}^*$, then we get the thesis by Theorem 2.1. Assume $q = \overline{p}^*$. By $\mathcal{F}(u_0) < +\infty$ and (H3), we get $u_0 \in W^{1,(p_1,\ldots,p_n)}(\Omega)$. Theorem 3.1 implies $u - u_0 \in L^{\overline{p}^*}(\Omega)$. Thus, $u \in L^{\overline{p}^*}_{\text{loc}}(\Omega)$. The conclusion follows by Theorem 2.2. □

6 Proof of Theorem 2.5

In this section, we assume the growth condition (H3), with $\min\{p_i\} = 1$. For the reader's convenience, we now recall the main notations. The functional is

$$\mathcal{F}(u) := \int_\Omega f(x, u, Du)\, \mathrm{d}x.$$

We assume that there exist $c_1, c_2 > 0$ and $1 = \min\{p_i\} \le p_i \le q$, $i = 1, \ldots, n$, such that

$$c_1 \sum_{i=1}^n |\xi_i|^{p_i} \le f(x, s, \xi) \le c_2 \left\{1 + |\xi|^q + |s|^q\right\}, \tag{20}$$

for a.e. x, every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^n$.

Fixed $u_0 \in W^{1,1}(\Omega)$ such that $\mathcal{F}(u_0) < +\infty$, we consider the relaxed functional in $BV(\Omega)$ of $\mathcal{F}$

$$\overline{\mathcal{F}}(u) := \inf \left\{ \liminf_{k\to+\infty} \mathcal{F}(u_k) \,:\, u_k \to u \text{ in } L^1(\Omega),\ u_k \in u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega) \right\}.$$

Proof of Theorem 2.5 By Rellich's Theorem in BV, every minimizing sequence for $\mathcal{F}$ in $u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)$ has a L^1-convergent subsequence. The lower semicontinuity of $\overline{\mathcal{F}}$ gives the existence of a minimizer $\bar{u}$ in BV, such that

$$\overline{\mathcal{F}}(\bar{u}) = \min_{u\in BV} \overline{\mathcal{F}}(u) = \inf_{u\in u_0+W_0^{1,(p_1,\ldots,p_n)}(\Omega)} \mathcal{F}(u). \tag{21}$$

We prove now that $\bar{u}$ is locally bounded. By the minimality of $\bar{u}$ and (21), there exists a sequence (u_k) in $u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)$, such that, for all k,

$$\mathcal{F}(u_k) \le \inf_{u_0+W_0^{1,(p_1,\ldots,p_n)}(\Omega)} \mathcal{F} + \frac{1}{k}, \qquad \text{and} \qquad u_k \to_{k\to+\infty} \bar{u} \text{ in } L^1(\Omega). \tag{22}$$

By the Ekeland's variational principle, see [27], for every k there exists a function $v_k \in u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)$, such that

$$\mathcal{F}(v_k) \le \mathcal{F}(u) + \frac{1}{\sqrt{k}} \sum_{i=1}^n \left(\int_\Omega |(v_k - u)_{x_i}|^{p_i}\, \mathrm{d}x \right)^{\frac{1}{p_i}} \ \forall u \in u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega), \tag{23}$$

and

$$\sum_{i=1}^n \left(\int_\Omega |(v_k - u_k)_{x_i}|^{p_i}\, \mathrm{d}x \right)^{\frac{1}{p_i}} \le \frac{1}{\sqrt{k}} \quad \forall k. \tag{24}$$

Since $u_k - v_k \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$, then (24) implies that $u_k - v_k$ converges to 0 in L^1. Thus, by the second item in (22), we get $v_k \to \bar{u}$ in L^1.
Note that there exists $\tilde{c} > 0$, depending on $|\Omega|$, such that

$$a^{1/p_i} \le a + \tilde{c}|\Omega| \qquad \forall\, i = 1, \dots, n, \text{ and } \forall\, a > 0.$$

Thus, using (23) and (H2), we get that, for all $u \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$,

$$\begin{aligned}\mathcal{F}(v_k) &\le \mathcal{F}(u) + \frac{1}{\sqrt{k}}\left\{\sum_{i=1}^{n}\left(\int_\Omega |(v_k)_{x_i}|^{p_i}\,\mathrm{d}x\right)^{1/p_i} + \sum_{i=1}^{n}\left(\int_\Omega |u_{x_i}|^{p_i}\,\mathrm{d}x\right)^{1/p_i}\right\}\\ &\le \left(1 + \frac{1}{c_1\sqrt{k}}\right)\mathcal{F}(u) + \frac{1}{c_1\sqrt{k}}\mathcal{F}(v_k) + \frac{2\tilde{c}|\Omega|}{\sqrt{k}},\end{aligned}$$

that implies

$$\left(1 - \frac{1}{c_1\sqrt{k}}\right)\mathcal{F}(v_k) \le \left(1 + \frac{1}{c_1\sqrt{k}}\right)\mathcal{F}(u) + \frac{2\tilde{c}|\Omega|}{\sqrt{k}}.$$

Therefore, the above inequality implies that v_k is a quasi-minimizer of the functional

$$\mathcal{I}(u) := \int_\Omega (f(x, u, Du) + 1)\,\mathrm{d}x,$$

with Q independent of k. Since $(x, s, \xi) \mapsto f(x, s, \xi) + 1$ satisfies properties analogous to (H1) and (20), we can apply Theorem 2.1. Thus, $v_k \in L^\infty_{\mathrm{loc}}(\Omega)$ and it satisfies an estimate analogous to (6) that we now write using cubes instead than balls. Precisely, fixed $x_0 \in \Omega$, consider $Q_R(x_0) \Subset \Omega$, cube centered at x_0, with edges, of length $2R$, parallel to the coordinate axes. Then, there exists a constant c, independent of k, such that

$$\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \le c\left\{1 + \frac{1}{R^{\mu\frac{q\overline{p}^*}{\overline{p}^*-q}}}\left(\int_{Q_R(x_0)} |v_k|^q\,\mathrm{d}x\right)^{\frac{1+\theta}{q}}\right\}, \tag{25}$$

where $\theta := \frac{\overline{p}^*(q-1)}{\overline{p}^*-q}$.
Since $\mathcal{F}(u_0) < +\infty$, then we have that $u_0 \in W^{1,(p_1,\dots,p_n)}(\Omega)$. By Theorem 3.2, $u_0 \in L^{\overline{p}^*}(Q_R)$ and it satisfies an estimate as in (7) on the cube Q_R. Moreover, since $v_k \in u_0 + W_0^{1,(p_1,\dots,p_n)}(\Omega)$, we can apply Theorem 3.1 to the function $v_k - u_0 \in W_0^{1,(p_1,\dots,p_n)}(\Omega)$. Thus,

$$\begin{aligned}
&\left\{\int_{Q_R(x_0)} |v_k|^q \, \mathrm{d}x\right\}^{\frac{1}{q}} \\
&\quad \leq |\Omega|^{1-\frac{q}{\overline{p}^*}} \left\{\left(\int_\Omega |v_k - u_0|^{\overline{p}^*} \, \mathrm{d}x\right)^{\frac{1}{\overline{p}^*}} + \left(\int_{Q_R(x_0)} |u_0|^{\overline{p}^*} \, \mathrm{d}x\right)^{\frac{1}{\overline{p}^*}}\right\} \\
&\quad \leq c(\Omega) \sum_{i=1}^n \left(\int_\Omega |(v_k - u_0)_{x_i}|^{p_i} \, \mathrm{d}x\right)^{\frac{1}{p_i}} + c(\Omega)\|u_0\|_{W^{1,(p_1,\ldots,p_n)}(\Omega)}.
\end{aligned}$$

Using (20), it is easy to prove that there exists $c > 0$, independent of k, such that

$$\begin{aligned}
\sum_{i=1}^n \left(\int_\Omega |(v_k - u_0)_{x_i}|^{p_i} \, \mathrm{d}x\right)^{\frac{1}{p_i}} &\leq c\,\{\mathcal{F}(v_k) + 1\} + c \sum_{i=1}^n \left\{\int_\Omega |(u_0)_{x_i}|^{p_i} \, \mathrm{d}x\right\}^{\frac{1}{p_i}} \\
&\leq c\,\{\mathcal{F}(v_k) + 1\} + c\|u_0\|_{W^{1,(p_1,\ldots,p_n)}(\Omega)}. \qquad (26)
\end{aligned}$$

Thus, collecting (25)–(26), we get

$$\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \leq c\left\{1 + \frac{1}{R^{\mu\frac{q\overline{p}^*}{\overline{p}^*-q}}}\left(\mathcal{F}(v_k) + 1 + \|u_0\|_{W^{1,(p_1,\ldots,p_n)}(\Omega)}\right)^{1+\theta}\right\}, \qquad (27)$$

for some positive c depending also on Ω, but independent of k and u_0.
By (23), applied with u_k in place of u, by (24) and the first property in (22), we have

$$\mathcal{F}(v_k) \leq \mathcal{F}(u_k) + \frac{1}{k} \leq \inf_{u_0+W_0^{1,(p_1,\ldots,p_n)}(\Omega)} \mathcal{F} + \frac{2}{k}.$$

Therefore, (27) implies

$$\begin{aligned}
&\|v_k\|_{L^\infty(Q_{\frac{R}{2}}(x_0))} \\
&\quad \leq c\left\{1 + \frac{1}{R^{\mu\frac{q\overline{p}^*}{\overline{p}^*-q}}}\left(\inf_{u_0+W_0^{1,(p_1,\ldots,p_n)}(\Omega)} \mathcal{F} + \frac{2}{k} + 1 + \|u_0\|_{W^{1,(p_1,\ldots,p_n)}(\Omega)}\right)^{1+\theta}\right\}.
\end{aligned}$$

So, up to subsequences, v_k converges to a function v in the $*$-weak topology of L^∞. Since v_k also converges to $\bar{u}$ in L^1, then $v = \bar{u}$. By the lower semicontinuity of the L^∞-norm and (21), we conclude. □

7 Perspectives

We studied above the local boundedness of minimizers of general integrals of the calculus of variations of the type

$$\mathcal{F}(u; \Omega) := \int_\Omega f(x, u, Du(x)) \, \mathrm{d}x, \qquad (28)$$

where Ω is an open and bounded subset of $\mathbb{R}^n$, $n \geq 2$, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, satisfying some convexity and growth conditions. As a particular case, we considered $f(x, s, \xi)$ convex in (s, ξ) and satisfying growth conditions such as

$$c_1 \sum_{i=1}^{n} |\xi_i|^{p_i} \leq f(x, s, \xi) \leq c_2 \left\{1 + |s|^q + |\xi|^q\right\}, \tag{29}$$

for some $1 \leq p_i \leq q$ and $c_1, c_2 > 0$. The boundedness of minimizers is not guaranteed if the exponents are too spread; see Giaquinta [17], Marcellini [18], Hong [19] for counterexamples. In this note, we proved that the reverse limit condition

$$q \leq \frac{n\overline{p}}{n - \overline{p}}, \quad \text{where} \quad \frac{1}{\overline{p}} := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i},$$

is sufficient for local boundedness of minimizers. In the paper, we also considered another limit case, when at least one of the exponents p_i in (29) is equal to 1, by extending the functional to BV.

In the classical theory of regularity and in some recent developments, the integrand f satisfies growth conditions depending on Du through its modulus $|Du|$, such as

$$c_1 |\xi|^p \leq f(x, s, \xi) \leq c_2 \left(1 + |\xi|^q\right). \tag{30}$$

If $p = q$, then (30) is called p-growth or standard or natural growth. If $p < q$, we are in the framework of p, q-growth.

In (29), we may have $\max\{p_i\} < q$, as for instance for the functional

$$\mathcal{F}_1(u) := \int_{\Omega} \left(\sum_{i=1}^{n} |u_{x_i}|^{p_i} + a(x)|Du|^q \right) \mathrm{d}x,$$

whenever $a(x) \geq 0$ is measurable and $a(x) = 0$ on a set of positive measure; see, in this context, the recent and interesting papers by Colombo and Mingione [11,12]; see also Esposito et al. [13]. There, the local boundedness of minimizers is a fundamental initial step for further regularity; the restriction on p_i, q is necessary only at this stage of local boundedness. The local boundedness of minimizers of functionals with anisotropic growth (29) has already been studied by the authors in [7]; related results are in [8,9], for the vector valued case, and [10], for the existence and regularity of solutions to elliptic equations.

In the present paper, we adopted a different strategy than in [7], where the Euler equation and the Moser iteration scheme were used. Here, we derived the local boundedness by the De Giorgi method of super(sub)-level sets. This allowed us to improve the previous results in different directions, as already mentioned in the Introduction. Moreover, it suggests to study, as a further step, the Hölder continuity of the solutions. Note that local minimizers of integrals, with Carathéodory integrands $g(x, s, \xi)$ possibly neither convex nor regular, and satisfying

$$m(f(x,s,\xi)-1) \leq g(x,s,\xi) \leq M(f(x,s,\xi)+1), \qquad \text{with}\, m, M > 0,$$

are quasi-minimizers of $\int_\Omega \{f(x,u,Du)+1\}\,\mathrm{d}x$. Thus, our results applied to local minimizers too.

A more general assumption on q has been considered. With the notations for the harmonic average $\overline{p}$ of $\{p_i\}$ and for the Sobolev exponent $\overline{p}^*$:

$$\frac{1}{\overline{p}} := \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}, \qquad \text{and} \qquad \overline{p}^* := \begin{cases} \frac{n\overline{p}}{n-\overline{p}}, & \text{if}\, \overline{p} < n, \\ \text{any}\, \mu > \overline{p}, & \text{if}\, \overline{p} \geq n, \end{cases}$$

we proved that if $q \leq \overline{p}^*$, then the quasi-minimizers of $\mathcal{F}$ are locally bounded.

We emphasize that we were able to include the limit case $q = \overline{p}^*$ and that the procedure of the Moser iteration argument was unable to include. Precisely, given a quasi-minimizer u, if $q = \overline{p}^*$ and if one of the following two assumptions holds:

$$\max\{p_i\} < \overline{p}^* \quad \text{or} \quad u \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega),$$

then u is locally bounded. We point out that, in the known literature, for the case $q = \overline{p}^*$, the condition that u a-priori belongs to $L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$ is sometimes omitted. On the contrary, this condition is needed for the embedding results of the anisotropic Sobolev spaces. In fact, the natural space, to consider minimizers of $\mathcal{F}$, is

$$W^{1,(p_1,\ldots,p_n)}(\Omega) := \left\{u \in W^{1,1}(\Omega) : u_{x_i} \in L^{p_i}(\Omega), \quad i = 1,\ldots,n\right\}.$$

For a rectangular domain Ω, with edges parallel to the coordinate axes, if $\max\{p_i\} < \overline{p}^*$, then $W^{1,(p_1,\ldots,p_n)}(\Omega) \subset L^{\overline{p}^*}(\Omega)$; see Troisi [21] and Acerbi and Fusco [22]. Otherwise, if $\max\{p_i\} = \overline{p}^*$, then the embedding is not guaranteed; see Kruzhkov and Kolodii [23] and Haskovec and Schmeiser [24] for counterexamples; see also Remark 3.1 for further details and insight. However, the embedding theory for anisotropic Sobolev spaces is now entirely completed.

Theorems 2.1 and 2.2 actually cover functionals more general than (28), (29). More precisely, the growth condition considered (see assumption (H2) in Sect. 2) takes into account a function g satisfying the $\triangle_2$ property (see (3)). Related boundedness results are in Dall'Aglio et al. [14], Mascolo and Papi [15] and Moscariello and Nania [16].

We also studied a class of variational integrals with linear growth from below; i.e., (29) with $\min\{p_i\} = 1$. Due to the lack of coerciveness, we considered the relaxed functional of $\mathcal{F}$ in $BV(\Omega)$; that is, for fixed $u_0 \in W^{1,1}(\Omega)$ with $\int_\Omega f(x,u_0,Du_0) < +\infty$, the relaxed functional, defined in $BV(\Omega)$, is

$$\overline{\mathcal{F}}(u) := \inf\left\{\liminf_{k\to+\infty}\mathcal{F}(u_k) : u_k \to u \text{ in } L^1(\Omega),\ u_k \in u_0 + W_0^{1,(p_1,\ldots,p_n)}(\Omega)\right\}.$$

We proved that there exists a locally bounded minimizer $\bar{u} \in BV$ of $\overline{\mathcal{F}}$ and an estimate of the L^∞-norm is given; see Theorem 2.5.

The use of the De Giorgi method for local boundedness suggests to go on: Under which conditions either the minimizers or the quasi-minimizers are Hölder continuous? This problem is at the same time appealing and difficult to be treated. Few recent tentative approaches for functionals and operators with special structures are available in the literature; see Liskevich and Skrypnik [28], Düzgün et al. [29,30], Colombo and Mingione [11,12].

8 Conclusions

By means of De Giorgi's method, which is still considered in the mathematical literature with plenty of interest and full of new applications to the regularity context, we are able to arrive to a limit threshold condition for the growth exponents.

The contents of the paper is described next briefly. In Sect. 2, we gave the precise hypotheses and the statements of the regularity results: Theorems 2.1 and 2.2 (cases $q < \overline{p}^*$ and $q = \overline{p}^*$, respectively) and Theorem 2.3 ($p_i = p$ for all i). We also stated regularity results for minimizers of functionals in suitable Dirichlet classes, dealing both with the coercive case ($\min\{p_i\} > 1$), Theorem 2.4, and with the non-coercive case ($\min\{p_i\} = 1$), Theorem 2.5. In this last result, a generalized definition of minimizers is used, e.g., minimizers in BV for the associated relaxed functional.

We state here one of the main results in this paper: minimizers of a class of integrals of the calculus of variations, satisfying p, q-growth conditions with

$$q \le p^* := \frac{np}{n-p}$$

(note the possibility of the equality sign), are locally bounded.

Sections 5 and 6 contain the proofs of Theorems 2.1, 2.2 and 2.5. They rely on some embedding results for anisotropic Sobolev spaces and on a suitable Caccioppoli inequality, which can be found in Sects. 3 and 4.

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