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Game Control Problem for a Phase Field Equation

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Abstract A game control problem for a phase field equation is considered. This problem is investigated from the viewpoint of both the first player (the partner) and of the second player (the opponent). For both players, their own procedures for forming feedback controls are specified.

Keywords Game problem of control · Phase field equation · Feedback control · ε-saddle point

Mathematics Subject Classification 49J35 · 91A24

1 Introduction

A system modeling the solidification process and governed by so-called phase field equations is considered. The state variables are the order parameter (also called the phase function) and the temperature. In contrast to the classical Stefan problem, which models the solidification process with a sharp solid–liquid interface, the phase field equations are applicable to fuzzy domains. For the aforementioned system, we discuss a game control problem, which consists in the following. Some quality criterion, depending on the trajectory of the system, is given. At discrete time instants, the phase function is inaccurately measured. There are two antagonistic players. The problem undertaken by the first player (the partner) is to construct (using measurements of the

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phase functions) a law of forming a feedback control that minimizes some quality criterion. The goal of the second player (the opponent) is opposite.

One of the approaches to solving the problems of guaranteed control (they are also called positional differential games) for dynamical systems, described by ordinary differential equations, was developed in [1]. The fundamental theory of guaranteed control for some systems with distributed parameters within the framework of the formalization from [1] was presented in [2,3]. In all the works cited above, the cases when the full phase state of a system is inaccurately measured at frequent enough time instants were considered. In the present work, from the position of the approach [1–3], the problems of guaranteed control undertaken by the partner and opponent are investigated under measuring a "part" of system's phase state.

2 Problem Statement and Solution Method

Consider the system (introduced in [4])

$$\frac{\partial}{\partial t}\psi + l\frac{\partial}{\partial t}\varphi = k\Delta_L\psi + Bu - Cv \quad \text{in} \quad \Omega \times]t_0, \vartheta], \quad \vartheta = \text{const} < +\infty, \tag{1}$$

$$\tau\frac{\partial}{\partial t}\varphi = \xi^2\Delta_L\varphi + g(\varphi) + \psi \tag{2}$$

with the boundary condition $\frac{\partial}{\partial n}\psi = \frac{\partial}{\partial n}\varphi = 0$ on $\partial\Omega \times]t_0, \vartheta]$ and the initial condition $\psi(t_0) = \psi_0, \varphi(t_0) = \varphi_0$ in Ω. Here, ψ is the temperature, φ is the phase function, $\Omega \subset \mathbb{R}^n$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$, Δ_L is the Laplace operator, $\partial/\partial n$ is the outward normal derivative, $(U, |\cdot|_U)$ and $(V, |\cdot|_V)$ are Banach spaces, $B \in \mathscr{L}(U; H)$ and $C \in \mathscr{L}(V; H)$ are linear continuous operators, $H = L_2(\Omega)$, and the function $g(z)$ is the derivative of a so-called potential $G(z)$. Following [4], we assume that $g(z) = az + bz^2 - cz^3$.

System (1), (2) (we call it the system S) has been investigated by many authors. A rather detailed analysis of the previous results is presented in [5–7]. Among more recent works, we note [8]. Therefore, we do not dwell on this aspect. In what follows, for the sake of simplicity, we assume that $k = \xi = \tau = c = 1$. Furthermore, we assume that the following conditions are fulfilled: (A1) The domain $\Omega \subset \mathbb{R}^n, n = 2, 3$, has the boundary of C^2-class and (A2) the coefficients a and b are elements of $L_\infty([t_0, \vartheta]\times\Omega)$, and vrai sup $c(t, \eta) > 0$ for $(t, \eta) \in [t_0, \vartheta]\times\Omega$; (A3) $\{\psi_0, \varphi_o\} \in \mathscr{R}$, where $\mathscr{R} := \{\psi, \varphi \in W^2_\infty(\Omega)\colon \frac{\partial}{\partial n}\psi = \frac{\partial}{\partial n}\varphi = 0 \text{ on } \partial\Omega\}$.

Introduce the notation: $Q = \Omega \times]t_0, \vartheta[$;

$$W^{2,1}_p(Q) = \left\{u \mid u, \frac{\partial u}{\partial \eta_i}, \frac{\partial^2 u}{\partial\eta_i\partial\eta_j}, \frac{\partial u}{\partial t} \in L^p(Q)\right\} \quad \text{for} \quad p \in [1, \infty[$$

is the standard Sobolev space with the norm

$$\|u\|_{W^{2,1}_p(Q)} = \left(\int_\Omega |u|^p + \sum_{i=1}^n \left|\frac{\partial u}{\partial\eta_i}\right|^p + \sum_{i,j=1}^n \left|\frac{\partial^2 u}{\partial\eta_i\partial\eta_j}\right|^p + \left|\frac{\partial u}{\partial t}\right|^p \mathrm{d}\eta\mathrm{d}t\right)^{1/p};$$

$(\cdot,\cdot)_H$ and $|\cdot|_H$ are the scalar product and the norm in H. A solution of the system $Sx(\cdot;t_0,x_0,u(\cdot),v(\cdot)) = \{\psi(\cdot;t_0,\psi_0,u(\cdot),v(\cdot)),\varphi(\cdot;t_0,\varphi_0,u(\cdot),v(\cdot))\}$ is a unique function $x(\cdot) = x(\cdot;t_0,x_0,u(\cdot),v(\cdot)) \in V_T^{(1)} = V_1 \times V_1$, $V_1 = W_2^{2,1}(Q)$, satisfying relations (1) and (2). As is known (see [7, p. 25, Assertion 5]), under our conditions, there exists a unique solution of S for any $u(\cdot) \in L_\infty(T;U)$ and $v(\cdot) \in L_\infty(T;V)$.

Let the cost functional

$$I(x(\cdot;t_0,x_0,u_T(\cdot),v_T(\cdot))) = \int_{t_0}^{\vartheta}\int_{\Omega} f(t,\eta,x(t,\eta),\nabla x(t,\eta))\,\mathrm{d}\eta\,\mathrm{d}t$$

be given. Here, the symbol ∇x stands for the gradient of the function x; the function $f(t,\eta,x,y)$ satisfies the Carathéodory condition, i.e., $f(t,\eta,x,y)$ is measurable (in the Lebesgue sense) in $(t,\eta) \in T \times \Omega$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}^n$, and Lipschitz in $x \in \mathbb{R}$, $y \in \mathbb{R}^n$ for almost all $t,\eta \in T \times \Omega$. In addition, $|f(t,\eta,0,\ldots,0)| \le c_0(t,\eta)$ for almost all $t,\eta \in T \times \Omega$, and $c_0(t,\eta) \in L_\infty(T \times \Omega)$. At discrete time instants $\tau_i \in \Delta = \{\tau_i\}_{i=0}^m$, $\tau_0 = t_0$, $\tau_{i+1} = \tau_i + \delta$, $\tau_m = \vartheta$, the phase function φ is measured. The results of these measurements are functions $\xi_i^h \in H$ satisfying the inequalities

$$|\varphi(\tau_i) - \xi_i^h|_H \le h. \tag{3}$$

Here, $h \in]0,1[$ stands for the level of informational noise. There are two antagonistic players controlling the system S by means of various input actions. One of the players is called a partner, the other one is called an opponent. Let $P \subset U$ and $E \subset V$ be given convex, bounded, and closed sets. The problem undertaken by the partner is as follows. It is necessary to construct a law (a strategy) for forming the control u (with values from P) by the feedback principle (on the base of measuring the state $\varphi(\tau_i)$) in such a way that this control minimizes the quality criterion under any possible actions of opponent, whose goals are opposite. Thus, the partner solves the minimax control problem. The problem undertaken by the opponent is "inverse": it consists in the choice of a law (a strategy) for forming the control v (with values from E) also by the feedback principle [on the base of measuring the state $\varphi(\tau_i)$] in such a way that this control maximizes the quality criterion under any possible actions of the partner, whose goals, as mentioned above, are opposite. Consequently, the opponent solves the maximin control problem. This is the description of the problem considered in the paper. The minimax game control problem for systems with distributed parameters has been investigated by many authors (see, for example, [9–12]). In the present work, to solve this problem, we use the approach from [1–3,13–16].

We denote the function $u(t)$, $t \in [a,b]$, by $u_{a,b}(\cdot)$. The sets of all controls of the partner and the opponent are denoted by the symbols $P_T(\cdot)$ and $E_T(\cdot)$: $P_T(\cdot) := \{u(\cdot) \in L_2(T;U)\colon u(t) \in P \text{ a.e. } t \in T\}$, $E_T(\cdot) := \{v(\cdot) \in L_2(T;V)\colon v(t) \in E \text{ a.e. } t \in T\}$. Any function (perhaps, multifunction) $\mathscr{U} : T \times \mathscr{H} \to P$, $\mathscr{H} := H \times H \times H \times H$, is said to be a positional strategy of the partner. A positional strategy of the opponent is defined by analogy: $\mathscr{V} : T \times \mathscr{H} \to E$. Positional strategies adjust controls at discrete time moments, given by some partition of the interval T. Any function $\mathscr{V}_1 : T \times H \times H \to H$ is said to be a reconstruction strategy. The strategy $\mathscr{V}_1$ is destined to reconstruct the unknown component $\psi(\cdot)$.

Let us present the exact statement of the problems under consideration. Let the partition of T be any finite family $\Delta = \{\tau_i\}_{i=0}^m$, where $\tau_0 = t_0$, $\tau_m = \vartheta$, $\tau_{i+1} = \tau_i + \delta$; $\delta = \delta(\Delta)$ is the diameter of Δ. Auxiliary systems M_1 and M_2 (models) are introduced. The system M_1 has an input $u^*(\cdot)$ and an output $w(\cdot)$; the system M_2 has an input $p^h(\cdot)$ and an output $w_1(\cdot)$, respectively. In the process, $p^h(\cdot)$ is formed in such a way that $p^h(\cdot)$ approximates the unknown coordinate $\psi(\cdot)$ of the system S. A solution $x(\cdot)$ of the system S, starting from an initial state (t_*, x_*) and corresponding to piecewise constant controls $u^h(\cdot)$ and $p^h(\cdot)$ (formed by the feedback principle) and to a control $v_{t_*,\vartheta}(\cdot) \in E_{t_*,\vartheta}(\cdot)$, is called an $(h, \Delta, w, \mathscr{U}, \mathscr{V}_1)$-motion $x^h_{\Delta,w}(\cdot) = x^h_{\Delta,w}(\cdot; t_*, x_*, \mathscr{U}, \mathscr{V}_1, v_{t_*,\vartheta}(\cdot))$, generated by the positional strategies $\mathscr{U}$ and $\mathscr{V}_1$ on the partition Δ. The process of forming the motions $x^h_{\Delta,w}(\cdot)$, $w(\cdot)$, and $w_1(\cdot)$ is realized simultaneously. These three trajectories are all formed by the feedback principle, i.e., it is assumed that $x^h_{\Delta,w}(t) = x(t; \tau_i, x^h_{\Delta,w}(\tau_i), u^h_{\tau_i,\tau_{i+1}}(\cdot), v_{\tau_i,\tau_{i+1}}(\cdot))$, $w(t) = w(t; \tau_i, w(\tau_i), u^*_{\tau_i,\tau_{i+1}}(\cdot))$, $w_1(t) = w_1(t; \tau_i, w_1(\tau_i), p^h_{\tau_i,\tau_{i+1}}(\cdot))$, $t \in [\tau_i, \tau_{i+1}[$, where

$$u^h(t) = u^h_i \in \mathscr{U}(\tau_i, \xi^h_i, p^h_i, w(\tau_i)), \quad p^h(t) = p^h_i \in \mathscr{V}_1(\tau_i, \xi^h_i, w_1(\tau_i))$$
$$\text{for } t \in [\tau_i, \tau_{i+1}[, \quad i \in [i(t_*) : m-1], \quad |\xi^h_i - \varphi(\tau_i)|_H \le h,$$
$$u^h(t) = u^h_* \in P, \quad p^h(t) = p^h_* \in H \text{ for } t \in [t_*, \tau_{i(t_*)}[, \quad i(t_*) = \min\{i : \tau_i > t_*\}.$$

The set of all $(h, \Delta, w, \mathscr{U}, \mathscr{V}_1)$-motions is denoted by $X_h(t_*, x_*, \mathscr{U}, \mathscr{V}_1, \Delta, w)$.

Problem 1 (Problem of the partner) It is necessary to find models M_1 and M_2, a control $u^*(\cdot)$ for the model M_1, as well as a positional strategy of the partner $\mathscr{U} : T \times \mathscr{H} \to P$ and a positional reconstruction strategy $\mathscr{V}_1 : T \times H \to H$, and a number c_1 with the following properties: whatever the value $\varepsilon > 0$ may be, one can specify (explicitly) numbers $h_* > 0$ and $\delta_* > 0$ such that the inequality $I(x^h_{\Delta,w}(\cdot)) \le c_1 + \varepsilon$, $\forall x^h_{\Delta,w}(\cdot) \in X_h(t_0, x_0, \mathscr{U}, \mathscr{V}_1, \Delta, w)$ is fulfilled uniformly with respect to all measurements ξ^h_i with properties (3) if $h \le h_*$ and $\delta = \delta(\Delta) \le \delta_*$.

By analogy with the motion $x^h_{\Delta,w}(\cdot) = x^h_{\Delta,w}(\cdot; t_*, x_*, \mathscr{U}, \mathscr{V}_1, v_{t_*,\vartheta}(\cdot))$, we define the motion $x^h_{\Delta,z}(\cdot) := x^h_{\Delta,z}(\cdot; t_*, x_*, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2, u_{t_*,\vartheta}(\cdot))$ corresponding to piecewise constant controls $v^h(\cdot)$, $v^*(\cdot)$, and $p^h(\cdot)$ (formed by the feedback principle) and to a control $u_{t_*,\vartheta}(\cdot) \in P_{t_*,\vartheta}(\cdot)$. This motion is called an $(h, \Delta, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2)$-motion, generated by the positional strategies $\mathscr{V}$, $\mathscr{V}_1$, and $\mathscr{V}_2$ on the partition Δ. The set of all $(h, \Delta, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2)$-motions is denoted by $X_h(t_*, x_*, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2, \Delta)$. Note that the trajectory $x^h_{\Delta,z}(\cdot; t_*, x_*, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2, u_{t_*,\vartheta}(\cdot))$ is formed simultaneously with the other two trajectories, $z(\cdot)$ and $w_1(\cdot)$. Here, $z(\cdot)$ is the trajectory of some auxiliary system M_3 (a model), whereas $w_1(\cdot)$ is the trajectory of the system M_2. These three trajectories are all formed by the feedback principle, i.e., it is assumed that $x^h_{\Delta,z}(t) = x^h_{\Delta,z}(t; \tau_i, x^h_{\Delta,z}(\tau_i), u_{\tau_i,\tau_{i+1}}(\cdot), v^h_{\tau_i,\tau_{i+1}}(\cdot))$, $z(t) = z(t; \tau_i, z(\tau_i), v^*_{\tau_i,\tau_{i+1}}(\cdot))$, $w_1(t) = w_1(t; \tau_i, w_1(\tau_i), p^h_{\tau_i,\tau_{i+1}}(\cdot))$, $t \in [\tau_i, \tau_{i+1}[$, where

$$v^h(t) = v_i^h \in \mathscr{V}(\tau_i, \xi_i^h, p_i^h, z(\tau_i)), \quad v^*(t) = v_i^* \in \mathscr{V}_2(\tau_i, \xi_i^h, p_i^h, z(\tau_i)),$$
$$p^h(t) = p_i^h \in \mathscr{V}_1(\tau_i, \xi_i^h, w_1(\tau_i)) \text{ for } t \in [\tau_i, \tau_{i+1}[, \quad i \in [i(t_*) : m-1],$$
$$|\xi_i^h - \varphi(\tau_i)|_H \le h, \quad i(t_*) = \min\{i : \tau_i > t_*\},$$
$$p^h(t) = p_*^h \in H, \quad v^*(t) = v^* \in E, \quad v^h(t) = v_*^h \in E \text{ for } t \in [t_*, \tau_{i(t_*)}[.$$

Problem 2 (Problem of the opponent) It is necessary to find models M_3 and M_2, as well as a positional strategy of the opponent $\mathscr{V} : T \times \mathscr{H} \to E$, a positional strategy $\mathscr{V}_2 : T \times \mathscr{H} \to E$, a positional reconstruction strategy $\mathscr{V}_1 : T \times H \to H$ with the following properties: whatever the value $\varepsilon > 0$ may be, one can specify (explicitly) numbers $h_* > 0$ and $\delta_* > 0$ such that the inequality $I(x_{\Delta,z}^h(\cdot)) \ge c_1 - \varepsilon, \forall x_{\Delta,z}^h(\cdot) \in X_h(t_0, x_0, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2, \Delta)$, is fulfilled uniformly with respect to all measurements ξ_i^h with properties (3) if $h \le h_*$ and $\delta = \delta(\Delta) \le \delta_*$. Here, the number c_1 is the same as in Problem 1.

Remark 2.1 The fact that the value c_1 in Problem 2 is the same as in Problem 1 means that c_1 is the value of the game, in which the partner aims to minimize a maximally possible value of the quality criterion. At the same time, the goal of the opponent is opposite: He aims to maximize a minimally possible value of the criterion. In this case, a strategy of the partner $\mathscr{U}$ solving Problem 1 is called an ε-optimal minimax strategy, whereas a strategy of the opponent $\mathscr{V}$ solving Problem 2 is called an ε-optimal maximin strategy. Moreover, the pair $(\mathscr{U}, \mathscr{V})$ constitutes an ε-saddle point.

3 Algorithm for Solving Problem 1

Let the following condition be fulfilled.

Condition 3.1 *There exists a convex and closed set $D \subset H$ such that $BP = CE + D$.*

Here, we use the notation $BP := \{Bu : u \in P\}$, $CE := \{Cv : v \in E\}$, $CE + D := \{u : u = u_1 + u_2, u_1 \in CE, u_2 \in D\}$. Let $u^*(\cdot)$ be an optimal control solving

Problem 3 It is necessary to minimize $I(w(\cdot; t_0, x_0, u(\cdot)))$ over the set $D_T(\cdot) = \{u(\cdot) \in L_2(T; H) : u(t) \in D \text{ for a. a. } t \in T\}$. Here, the symbol $w(\cdot) = \{w^{(1)}(\cdot), w^{(2)}(\cdot)\} = w(\cdot; t_0, x_0, u(\cdot))$, $u(\cdot) \in D_T(\cdot)$, denotes the solution of the system

$$\begin{aligned} &\frac{\partial}{\partial t} w^{(1)} + l \frac{\partial}{\partial t} w^{(2)} = \Delta_L w^{(1)} + u \quad \text{in} \quad \Omega \times]t_0, \vartheta], \\ &\frac{\partial}{\partial t} w^{(2)} = \Delta_L w^{(2)} + g(w^{(2)}) + w^{(1)} \end{aligned} \tag{4}$$

with the boundary condition $\frac{\partial}{\partial n} w^{(1)} = \frac{\partial}{\partial n} w^{(2)} = 0$ on $\partial\Omega \times]t_0, \vartheta]$ and the initial condition $w^{(1)}(t_0) = \psi_0$, $w^{(2)}(t_0) = \varphi_0$ in Ω.

Note that Problem 3 was investigated in a number of papers (see, for example, [6,7,17], where some optimality conditions were stated). Let $w(\cdot) = w(\cdot; t_0, x_0, u^*(\cdot))$

be an optimal trajectory in Problem 3 and $C_{\text{opt}} = \inf\{I(w(\cdot; t_0, x_0, u(\cdot))) : u(\cdot) \in D_T(\cdot)\}$ be the optimal value of the quality criterion. As the model M_1, we take system (4) with the control $u(\cdot) = u^*(\cdot)$; as the model M_2, we take the equation

$$\frac{\partial w_1(t,\eta)}{\partial t} = \Delta_L w_1(t,\eta) + p^h(t,\eta) + g(w_1(t,\eta)) \quad \text{in} \quad \Omega \times]t_0, \vartheta] \tag{5}$$

with the boundary condition $\frac{\partial w_1}{\partial n} = 0$ on $\partial\Omega \times]t_0, \vartheta]$ and the initial condition $w_1(t_0) = \varphi_0$ in Ω. The strategies $\mathscr{U}$ and $\mathscr{V}_1$ are defined in such a way that:

$$\mathscr{U}(t, \xi, p, w) = \arg\max\{L(u, y) : u \in P\}, \tag{6}$$

$$\mathscr{V}_1(t, \xi, w_1) = \arg\min\{l(t, \alpha, u, s) : u \in U_d\}, \tag{7}$$

where $w = \{w^{(1)}, w^{(2)}\}, L(u, y) = (y, Bu)_H, y = w^{(1)} - p + l(w^{(2)} - \xi), l(t, \alpha, u, s) = \exp(-2bt)(s, u)_H + \alpha|u|_H^2, s = w_1 - \xi, U_d := \{u \in H : |u|_H \le d\}$.

Let us describe the algorithm for solving Problem 1. Before the algorithm starts, we fix a value $h \in]0, 1[$, a partition

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,i} = \tau_{h,i-1} + \delta, \quad \delta = \delta(h), \quad \tau_{h,0} = t_0, \quad \tau_{h,m_h} = \vartheta,$$

with the diameter $\delta(h) = \tau_{h,i+1} - \tau_{h,i}$, and a function $\alpha = \alpha(h)$: $]0, 1[\to \mathbb{R}^+$,

$$\alpha(h) \to h, \quad (h + \delta(h))\alpha^{-1}(h) \to 0 \quad \text{as } h \to 0. \tag{8}$$

The work of the algorithm is decomposed into $m - 1, m = m_h$, identical steps. We assume that $u^h(t) = u_0^h \in \mathscr{U}(t_0, \xi_0^h, p_0^h, w(t_0)), p^h(t) = p_0^h \in \mathscr{V}_1(t_0, \xi_0^h, \varphi_0), |\xi_0^h - \varphi_0|_H \le h$ on the interval $[t_0, \tau_1[$. Under the action of these piecewise constant controls, as well as of an unknown disturbance $v_{t_0,\tau_1}(\cdot)$, the $(h, \Delta, w, \mathscr{U}, \mathscr{V}_1)$-motion $\{x_{\Delta,w}^h(\cdot)\}_{t_0,\tau_1} = \{x_{\Delta,w}^h(\cdot; t_0, x_0, u_{t_0,\tau_1}^h(\cdot), v_{t_0,\tau_1}(\cdot))\}_{t_0,\tau_1}$ of the system S, the trajectory $\{w_1(\cdot)\}_{t_0,\tau_1} = \{w_1(\cdot; t_0, w_1(t_0), p_{t_0,\tau_1}^h(\cdot))\}_{t_0,\tau_1}$ of the model M_2, and the trajectory $\{w(\cdot)\}_{t_0,\tau_1} = \{w(\cdot; t_0, x_0, u_{t_0,\tau_1}^*(\cdot))\}_{t_0,\tau_1}$ of the model M_1 are realized. At the moment $t = \tau_1$, we determine u_1^h and p_1^h from the condition

$$u_1^h \in \mathscr{U}(\tau_1, \xi_1^h, p_1^h, w(\tau_1)), \quad |\xi_1^h - \varphi_{\Delta,w}^h(\tau_1)|_H \le h, \quad p_1^h \in \mathscr{V}_1(\tau_1, \xi_1^h, w_1(\tau_1));$$

i.e., we assume that $u^h(t) = u_1^h$ and $p^h(t) = p_1^h$ for $t \in [\tau_1, \tau_2[$. Then, we calculate the $(h, \Delta, w, \mathscr{U}, \mathscr{V}_1)$-motion $\{x_{\Delta,w}^h(\cdot)\}_{\tau_1,\tau_2} = \{x_{\Delta,w}^h(\cdot; \tau_1, x_{\Delta,w}^h(\tau_1), u_{\tau_1,\tau_2}^h(\cdot), v_{\tau_1,\tau_2}(\cdot))\}_{\tau_1,\tau_2}$, the trajectory $\{w_1(\cdot)\}_{\tau_1,\tau_2} = \{w_1(\cdot; \tau_1, w_1(\tau_1), p_{\tau_1,\tau_2}^h(\cdot))\}_{\tau_1,\tau_2}$ of the model M_2, and the trajectory $\{w(\cdot)\}_{\tau_1,\tau_2} = \{w(\cdot; \tau_1, w(\tau_1), u_{\tau_1,\tau_2}^*(\cdot)\}_{\tau_1,\tau_2}$ of the model M_1. Let the $(h, \Delta, w, \mathscr{U}, \mathscr{V}_1)$-motion $x_{\Delta,w}^h(\cdot)$, the trajectory $w_1(\cdot)$ of the model M_2, and the trajectory $w(\cdot)$ of the model M_1 be defined on the interval $[t_0, \tau_i]$. At the moment $t = \tau_i$, we assume that

$$u_i^h \in \mathscr{U}(\tau_i, \xi_i^h, p_i^h, w(\tau_i)), \quad |\xi_i^h - \varphi_{\Delta,w}^h(\tau_i)|_H \le h, \quad p_i^h \in \mathscr{V}_1(\tau_i, \xi_i^h, w_1(\tau_i)); \tag{9}$$

i.e., we set $u^h(t) = u_i^h$ and $p^h(t) = p_i^h$ for $t \in [\tau_i, \tau_{i+1}[$. As the result of the action of these controls and of an unknown disturbance $v_{\tau_i,\tau_{i+1}}(\cdot)$, the $(h, \Delta, w, \mathscr{U}, \mathscr{V}_1)$-motion $\{x^h_{\Delta,w}(\cdot)\}_{\tau_i,\tau_{i+1}} = \{x^h_{\Delta,w}(\cdot; \tau_i, x^h_{\Delta,w}(\tau_i), u^h_{\tau_i,\tau_{i+1}}(\cdot), v_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}}$, the trajectory $\{w_1(\cdot)\}_{\tau_i,\tau_{i+1}} = \{w_1(\cdot; \tau_i, w_1(\tau_i), p^h_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}}$ of the model M_2, and the trajectory $\{w(\cdot)\}_{\tau_i,\tau_{i+1}} = \{w(\cdot; \tau_i, w(\tau_i), u^*_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}}$ of the model M_1 are realized on the interval $[\tau_i, \tau_{i+1}]$. The procedure of forming the $(h, \Delta, w, \mathscr{U}, \mathscr{V}_1)$-motion and the trajectories of models M_2 and M_1 stops at the moment ϑ.

Theorem 3.1 *Let $c_1 = C_{\text{opt}}$ and let the models M_1 and M_2 be specified by relations* (4) *and* (5)*. Then, the strategies $\mathscr{U}$ and $\mathscr{V}_1$ of form* (6), (7) *solve Problem* 1.

Proof To prove the theorem, we estimate the variation in the functional

$$\Lambda(t, x^h_{\Delta,w}(\cdot), w(\cdot)) = \Lambda^0(t, x^h_{\Delta,w}(\cdot), w(\cdot)) + 0.5\int_0^t \Big\{ \int_\Omega |\nabla\pi^h(\rho,\eta)|^2\,\mathrm{d}\eta + l^2 \int_\Omega |\nabla\mu^h(\rho,\eta)|^2\,\mathrm{d}\eta \Big\}\mathrm{d}\rho,$$

where $\Lambda^0(t, x^h_{\Delta,w}(\cdot), w(\cdot)) = 0.5|g^h(t)|^2_H + 0.5l^2|\mu^h(t)|^2_H$, $\pi^h(t) = w^{(1)}(t) - \psi^h_{\Delta,w}(t)$, $\mu^h(t) = w^{(2)}(t) - \varphi^h_{\Delta,w}(t)$, $g^h(t) = \pi^h(t) + l\mu^h(t)$. It is easily seen that the functions $\pi^h(\cdot)$ and $\mu^h(\cdot)$ are solutions of the system

$$\begin{aligned} &\frac{\partial\pi^h(t,\eta)}{\partial t} + l\frac{\partial\mu^h(t,\eta)}{\partial t} = \Delta_L\pi^h(t,\eta) + u^*(t,\eta) - (Bu^h)(t,\eta) + (Cv)(t,\eta) \\ &\text{in } \Omega \times]t_0, \vartheta], \quad \frac{\partial\mu^h(t,\eta)}{\partial t} = \Delta_L\mu^h(t,\eta) + R^h(t,\eta)\mu^h(t,\eta) + \pi^h(t,\eta) \end{aligned} \tag{10}$$

with the initial condition $\pi^h(t_0) = \mu^h(t_0) = 0$ in Ω and with the boundary condition $\dfrac{\partial\pi^h}{\partial n} = \dfrac{\partial\mu^h}{\partial n} = 0$ on $\partial\Omega \times]t_0, \vartheta]$. Here, $R^h(t,\eta) = a(t,\eta) + b(t,\eta)(w^{(1)}(t,\eta) + \varphi^h_{\Delta,w}(t,\eta)) - ((w^{(1)}(t,\eta))^2 + w^{(1)}(t,\eta)\varphi^h_{\Delta,w}(t,\eta) + (\varphi^h_{\Delta,w})^2(t,\eta))$. Multiplying scalarly the first equation of (10) by $g^h(t)$, and the second one by $\mu^h(t)$, we obtain

$$\begin{aligned} &(g^h(t), g^h_t(t))_H + \int_\Omega \{|\nabla\pi^h(t,\eta)|^2 + l\nabla\pi^h(t,\eta)\nabla\mu^h(t,\eta)\}\,\mathrm{d}\eta \\ &\quad = (g^h(t), u^*(t) - Bu^h(t) + Cv(t))_H, \\ &(\mu^h(t), \mu^h_t(t))_H + \int_\Omega |\nabla\mu^h(t,\eta)|^2\,\mathrm{d}\eta \le (\pi^h(t), \mu^h(t))_H + b|\mu^h(t)|^2_H \quad \text{for a.a. } t \in T. \end{aligned} \tag{11}$$

Here, we use the inequality $\operatorname*{vrai\,max}_{(t,\eta)\in T\times\Omega} \{a(t,\eta) + b(t,\eta)(v_1+v_2) - (v_1^2 + v_1v_2 + v_2^2)\} \le b$, which is valid for any $v_1, v_2 \in \mathbb{R}$. It is evident that the inequality

$$\int_\Omega l(\nabla\pi^h(t,\eta), \nabla\mu^h(t,\eta))\,\mathrm{d}\eta \ge -0.5\int_\Omega \{|\nabla\pi^h(t,\eta)|^2 + l^2|\nabla\mu^h(t,\eta)|^2\}\,\mathrm{d}\eta, \tag{12}$$

for a.a. $t \in T$, is fulfilled. Let us multiply the first inequality of (11) by l^2 and add to the second one. Taking into account (12), we have for a.a. $t \in T$

$$(g^h(t), g^h_t(t))_H + l^2(\mu^h(t), \mu^h_t(t))_H + 0.5 \int\limits_{\Omega} \{|\nabla \pi^h(t,\eta)|^2 + l^2 |\nabla \mu^h(t,\eta)|^2\}\, \mathrm{d}\eta$$
$$\le (g^h(t), u^*(t) - Bu^h(t) + Cv(t))_H + l^2(\pi^h(t), \mu^h(t))_H + bl^2|\mu^h(t)|^2_H. \quad (13)$$

Note that $\pi^h(t) = g^h(t) - l\mu^h(t)$. In this case, for a.a. $t \in T$

$$\begin{aligned}(\pi^h(t), \mu^h(t))_H + b|\mu^h(t)|^2_H &= (g^h(t) - l\mu^h(t), \mu^h(t))_H + b|\mu^h(t)|^2_H \\ &= (g^h(t), \mu^h(t))_H + (b-l)|\mu^h(t)|^2_H \\ &\le 0.5(|g^h(t)|^2_H + (0.5 + |b-l|)|\mu^h(t)|^2_H. \end{aligned} \quad (14)$$

Combining (13) and (14), we obtain for a.a. $t \in T$

$$\frac{\mathrm{d}}{\mathrm{d}t} \Lambda^0(t, x^h_{\Delta,w}(\cdot), w(\cdot)) + 0.5 \int\limits_{\Omega} \{|\nabla \pi^h(t,\eta)|^2 + l^2|\nabla \mu^h(t,\eta)|^2\}\, \mathrm{d}\eta$$
$$\le 2l^2\lambda^2 \Lambda^0(t, x^h_{\Delta,w}(\cdot), w(\cdot)) + (g^h(t), u^*(t) - Bu^h(t) + Cv(t))_H. \quad (15)$$

Estimate the last term in the right-hand side of inequality (15). For $t \in [\tau_i, \tau_{i+1}[$,

$$|g^h(t) - y^h_i|_H = |\pi^h(t) + l\mu^h(t) - y^h_i|_H \le \lambda_{1,i}(t) + \lambda_{2,i}(t), \quad (16)$$

where $y^h_i = w^{(1)}(\tau_i) - p^h_i - l(w^{(2)}(\tau_i) - \xi^h_i)$, $\lambda_{1,i}(t) = |w^{(1)}(t) - \psi^h_{\Delta,w}(t) - w^{(1)}(\tau_i) + p^h_i|_H$, $\lambda_{2,i}(t) = l|w^{(2)}(t) - \varphi^h_{\Delta,w}(t) - w^{(2)}(\tau_i) + \xi^h_i|_H$. By virtue of (3), we have

$$\begin{aligned}\lambda_{1,i}(t) &\le |p^h_i - \psi^h_{\Delta,w}(t)|_H + \int\limits_{\tau_i}^{t} |\dot w^{(1)}(\tau)|_H\, \mathrm{d}\tau, \\ \lambda_{2,i}(t) &\le lh + l\int\limits_{\tau_i}^{t} \{|\dot\varphi^h_{\Delta,w}(\tau)|_H + |\dot w^{(2)}(\tau)|_H\}\, \mathrm{d}\tau, \quad t \in \delta_i = [\tau_i, \tau_{i+1}[. \end{aligned} \quad (17)$$

From (16) and (17), for $t \in \delta_i$, it follows that

$$|g^h(t) - y^h_i|_H \le lh + \int\limits_{\tau_i}^{t} \{l|\dot\varphi^h_{\Delta,w}(\tau)|_H + |\dot w^{(1)}(\tau)|_H + |\dot w^{(2)}(\tau)|_H\}\, \mathrm{d}\tau + |p^h_i - \psi^h_{\Delta,w}(t)|_H. \quad (18)$$

It follows from [16] that $|p^h(\cdot) - \psi(\cdot)|^2_{L_2(T;H)} \le K\mu(h)$, $K = \mathrm{const} > 0$, $\mu(h) = (h + \delta(h) + \alpha(h))^{1/2} + (h + \delta(h))\alpha^{-1}(h))$, i.e., the strategy $\mathscr{V}_1$ of form (7) is a reconstruction strategy. By virtue of (3), taking into account the latter inequality, from estimate (18), we derive

$$\sum_{i=0}^{m-1}\int_{\tau_i}^{\tau_{i+1}} M(t;\tau_i)\,\mathrm{d}t \le k_1(h+\delta)+k_2\int_{t_0}^{\vartheta}|p^h(\tau)-\psi^h_{\Delta,w}(\tau)|_H\,\mathrm{d}\tau \le k_3\mu^{1/2}(h), \tag{19}$$

where $M(t;\tau_i) = |g^h(t)-y_i^h|_H\{|Bu_i^h|_H+|Cv(t)|_H+|u^*(t)|_H\}$ for a.a. $t\in\delta_i$. Then, it follows from (15) that for a.a. $t\in\delta_i$,

$$\begin{aligned}(g^h(t), g_t^h(t))_H + l^2(\mu^h(t),\mu_t^h(t))_H + 0.5\int_{\Omega}\{|\nabla\pi^h(t,\eta)|^2 + l^2|\nabla\mu^h(t,\eta)|^2\}\,\mathrm{d}\eta \\ \le 2l^2\lambda^2\Lambda^0(t, x^h_{\Delta,w}(\cdot), w(\cdot)) + (y_i^h, u^*(t) - Bu_i^h + Cv(t))_H + M(t;\tau_i).\end{aligned} \tag{20}$$

Here (see (9)), $u_i^h \in \mathscr{U}(\tau_i, \xi_i^h, p_i^h, w^{(1)}(\tau_i), w^{(2)}(\tau_i))$; ξ_i^h is an inaccurate measurement of the phase state $\varphi^h_{\Delta,w}(\tau_i)$; $v_{\tau_i,\tau_{i+1}}(\cdot)$ is an unknown realization of the control of the opponent; the strategy $\mathscr{U}$ is determined from formula (6). By virtue of Condition 3.1, there exists a control $u^{(1)}_{\tau_i,\tau_{i+1}}(\cdot)\in P_{\tau_i,\tau_{i+1}}(\cdot)$ such that

$$Bu^{(1)}(t) = Cv(t) + u^*(t) \quad \text{for a. a.} \quad t\in[\tau_i,\tau_{i+1}]. \tag{21}$$

From (21), we have that for a. a. $t\in[\tau_i,\tau_{i+1}]$

$$(y_i^h, Cv(t) + u^*(t) - Bu_i^h)_H = (B(u^{(1)}(t) - u_i^h), y_i^h)_H \le 0. \tag{22}$$

We deduce from (20) and (22) that

$$\begin{aligned}\frac{\mathrm{d}\Lambda(t, x^h_{\Delta,w}(\cdot), w(\cdot))}{\mathrm{d}t} &= (g^h(t), g_t^h(t))_H + l^2(\mu^h(t),\mu_t^h(t))_H + 0.5\int_{\Omega}\{|\nabla\pi^h(t,\eta)|^2 \\ &\quad + l^2|\nabla\mu^h(t,\eta)|^2\}\mathrm{d}\eta \le l^2\lambda^2\Lambda^0(t, x^h_{\Delta,w}(\cdot), w(\cdot)) \\ &\quad + M(t;\tau_i) \quad \text{for a.a. } t\in\delta_i.\end{aligned} \tag{23}$$

Using (23) and (19), by virtue of the Gronwall Lemma, we obtain

$$\Lambda^0(t, x^h_{\Delta,w}(\cdot), w(\cdot)) \le k_4\sum_{i=0}^{m-1}\int_{\tau_i}^{\tau_{i+1}} M(t;\tau_i)\,\mathrm{d}t \le k_5\mu^{1/2}(h), \quad \forall t\in T.$$

Hence and from (23), we derive $\Lambda(t, x^h_{\Delta,w}(\cdot), w(\cdot)) \le k_6\mu^{1/2}(h)$, $\forall t\in T$. The statement of the theorem follows from the last inequality. The theorem is proved. □

4 Algorithm for Solving Problem 2

We design an algorithm for solving Problem 2. Assume that, as everywhere above, Condition 3.1 is fulfilled. As the model M_3, we take the system

$$\begin{aligned}&\frac{\partial}{\partial t}z^{(1)}+l\frac{\partial}{\partial t}z^{(2)}=\Delta_L z^{(1)}+v^* \quad \text{in} \quad \Omega\times\,]t_0,\vartheta],\\&\frac{\partial}{\partial t}z^{(2)}=\Delta_L z^{(2)}+g(z^{(2)})+z^{(1)}\end{aligned} \tag{24}$$

with the boundary condition $\frac{\partial}{\partial n}z^{(1)}=\frac{\partial}{\partial n}z^{(2)}=0$ on $\partial\Omega\times\,]t_0,\vartheta]$ and the initial condition $z^{(1)}(t_0)=\psi_0$, $z^{(2)}(t_0)=\varphi_0$ in Ω. Its solution is denoted by the symbol $z(\cdot)=\{z^{(1)}(\cdot),z^{(2)}(\cdot)\}=z(\cdot;t_0,z_0,v^*(\cdot))$, where $z_0=\{\psi_0,\varphi_0\}$. The model M_2 is described by relations (5). The strategies $\mathscr{V}$, $\mathscr{V}_1$, and $\mathscr{V}_2$ are defined as follows:

$$\mathscr{V}(t,\xi,p,z):=\arg\max\{L_1(v,\chi)\colon v\in E\}, \tag{25}$$

$$\mathscr{V}_1(t,\xi,w_1):=\arg\min\{l(t,\alpha,u,s)\colon u\in U_d\}, \tag{26}$$

$$\mathscr{V}_2(t,\xi,p,z):=B\tilde{u}-C\tilde{v}, \tag{27}$$

where $\tilde{u}\in\arg\min\{L(u,\chi):u\in P\}$, $L_1(v,\chi)=(\chi,Cv)_H$, $\chi=z^{(1)}-p+l(z^{(2)}-\xi)$, $z=\{z^{(1)},z^{(2)}\}$, $L(u,\chi)=(\chi,Bu)_H$, $\tilde{v}=\tilde{v}(\tilde{u})$ is an arbitrary element from the set E such that $B\tilde{u}-C\tilde{v}\in D$.

Let us pass to the description of the algorithm for solving Problem 2. Before the algorithm starts, we fix a value $h\in\,]0,1[$, a function $\alpha=\alpha(h)\colon\,]0,1[\,\to\mathbb{R}^+$ with properties (8), and a partition $\Delta_h=\{\tau_{h,i}\}_{i=0}^{m_h}$ with the diameter $\delta(h)$. The work of the algorithm is decomposed into $m-1$, $m=m_h$, identical steps. We assume that

$$\begin{aligned}v^h(t)=v_0^h\in\mathscr{V}(t_0,\xi_0^h,p_0^h,z(t_0)),\quad |\xi_0^h-\varphi_0|_H\le h,\\ p^h(t)=p_0^h\in\mathscr{V}_1(t_0,\xi_0^h,w_1(t_0)),\quad v^*(t)=v_0^*\in\mathscr{V}_2(t_0,\xi_0^h,p_0^h,z(t_0))\end{aligned} \tag{28}$$

on the interval $[t_0,\tau_1[$. Under the action of these piecewise constant controls, as well as of an unknown control $u_{t_0,\tau_1}(\cdot)$, the $(h,\Delta,\mathscr{V},\mathscr{V}_1,\mathscr{V}_2)$-motion $\{x^h_{\Delta,z}(\cdot)\}_{t_0,\tau_1}=\{x^h_{\Delta,z}(\cdot;t_0,x_0,u_{t_0,\tau_1}(\cdot),v^h_{t_0,\tau_1}(\cdot))\}_{t_0,\tau_1}$ of the system S, the trajectory $\{w_1(\cdot)\}_{t_0,\tau_1}=\{w_1(\cdot;t_0,w_1(t_0),p^h_{t_0,\tau_2}(\cdot))\}_{t_0,\tau_1}$ of the model M_2, and the trajectory $\{z(\cdot)\}_{t_0,\tau_1}=\{z(\cdot;t_0,z_0,v^*_{t_0,\tau_1}(\cdot))\}_{t_0,\tau_1}$ of the model M_3 are realized. At the moment $t=\tau_1$, we determine v_1^h, p_1^h, and v_1^* from the condition

$$v_1^h\in\mathscr{V}(\tau_1,\xi_1^h,p_1^h,z(\tau_1)),\quad p_1^h\in\mathscr{V}_1(\tau_1,\xi_1^h,w_1(\tau_1)),\quad v_1^*\in\mathscr{V}_2(\tau_1,\xi_1^h,p_1^h,z(\tau_1)), \tag{29}$$

$|\xi_1^h-\varphi^h_{\Delta,z}(\tau_1)|_H\le h$, where $v_1^*=B\tilde{u}_1^h-C\tilde{v}_1^h$, $\tilde{u}_1^h=\arg\min\{L(u,\chi_1^h)\colon u\in P\}$, $\chi_1^h=z^{(1)}(\tau_1)-p_1^h+l(z^{(2)}(\tau_1)-\xi_1^h)$, $\tilde{v}_1^h=\tilde{v}_1^h(\tilde{u}_1^h)$ is an arbitrary element from the set E such that $B\tilde{u}_1^h-C\tilde{v}_1^h\in D$. We assume that

$$v^h(t)=v_1^h,\quad p^h(t)=p_1^h,\quad v^*(t)=v_1^*\quad \text{for}\quad t\in[\tau_1,\tau_2[.$$

Then, we calculate the realization of the $(h, \Delta, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2)$-motion $\{x^h_{\Delta,z}(\cdot)\}_{\tau_1,\tau_2} = \{x^h_{\Delta,z}(\cdot;\tau_1, x^h_{\Delta,z}(\tau_1), u_{\tau_1,\tau_2}(\cdot), v^h_{\tau_1,\tau_2}(\cdot))\}_{\tau_1,\tau_2}$, the trajectory $\{w_1(\cdot)\}_{\tau_1,\tau_2} = \{w_1(\cdot;\tau_1, w_1(\tau_1), p^h_{\tau_1,\tau_2}(\cdot)))\}_{\tau_1,\tau_2}$ of the model M_2, and the trajectory $z_{\tau_1,\tau_2}(\cdot) = \{z(\cdot;\tau_1, z(\tau_1), v^*_{\tau_1,\tau_2}(\cdot)\}_{\tau_1,\tau_2}$ of the model M_3. Let the $(h, \Delta, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2)$-motion $x^h_{\Delta,z}(\cdot)$, the trajectory $w_1(\cdot)$ of the model M_2, and the trajectory $z(\cdot)$ of the model M_3 be defined on the interval $[t_0, \tau_i]$. At the moment $t = \tau_i$, we assume that

$$v^h_i \in \mathscr{V}(\tau_i, \xi^h_i, p^h_i, z(\tau_i)), \quad p^h_i \in \mathscr{V}_1(\tau_i, \xi^h_i, w_1(\tau_i)), \quad v^*_i \in \mathscr{V}_2(\tau_i, \xi^h_i, p^h_i, z(\tau_i)), \tag{30}$$

$|\xi^h_i - \varphi^h_{\Delta,z}(\tau_i)|_H \le h$, where $v^*_i = B\tilde{u}^h_i - C\tilde{v}^h_i$, $\tilde{u}^h_i = \arg\min\{L(u, \chi^h_i) : u \in P\}$, $\chi^h_i = z^{(1)}(\tau_i) - p^h_i + l(z^{(2)}(\tau_i) - \xi^h_i)$, $\tilde{v}^h_i = \tilde{v}^h_i(\tilde{u}^h_i)$ is an arbitrary element from the set E such that $B\tilde{u}^h_i - C\tilde{v}^h_i \in D$. We assume that

$$v^h(t) = v^h_i, \quad p^h(t) = p^h_i, \quad v^*(t) = v^*_i \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}[.$$

The trajectory $\{w_1(\cdot)\}_{\tau_i,\tau_{i+1}} = \{w_1(\cdot;\tau_i, w_1(\tau_i), p^h_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}}$ of the model M_2, the trajectory $\{z(\cdot)\}_{\tau_i,\tau_{i+1}} = \{z(\cdot;\tau_i, z(\tau_i), v^*_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}}$ of the model M_3, and the $(h, \Delta, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2)$-motion

$$\{x^h_{\Delta,z}(\cdot)\}_{\tau_i,\tau_{i+1}} = \{x^h_{\Delta,z}(\cdot;\tau_i, x^h_{\Delta,z}(\tau_i), u_{\tau_i,\tau_{i+1}}(\cdot), v^h_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}}$$

are realized on the interval $[\tau_i, \tau_{i+1}]$ as the result of the action of these controls and an unknown control $u_{\tau_i,\tau_{i+1}}(\cdot)$. The above procedure of forming the $(h, \Delta, \mathscr{V}, \mathscr{V}_1, \mathscr{V}_2)$-motion and the trajectories of the models M_2 and M_3 stops at the moment ϑ.

Theorem 4.1 *Let $c_1 = C_{\text{opt}}$ and let the models M_3 and M_2 be specified by relations (24) and (5). Then, the strategies $\mathscr{V}$, $\mathscr{V}_1$, and $\mathscr{V}_2$ of form (25)–(27) solve Problem 2.*

Proof Note that $I(w(\cdot)) \ge c_1, \forall w(\cdot) \in W_T(\cdot)$. Here and below, the symbol $W_T(\cdot) = W_T(\cdot; t_0, x_0)$ stands for the bundle of solutions of system (4), i.e., $W_T(\cdot; t_0, x_0) = \{w(\cdot; t_0, x_0, u(\cdot)),\ u(\cdot) \in D_T(\cdot)\}$, whereas the symbol $W_T(t)$ denotes the section of this bundle at the moment t. Therefore, by virtue of the Lipschitz property of the function f, for any $\varepsilon > 0$, it is sufficient to find $h_1 > 0$ and $\delta_1 > 0$ such that the inequality

$$\lambda(\vartheta, x^h_{\Delta,z}(\cdot), W_T(\cdot)) = \inf\{\Lambda(\vartheta, x^h_{\Delta,z}(\cdot), z(\cdot)) : z(\cdot) \in W_T(\cdot)\} \le \varepsilon$$

is fulfilled for $h \in]0, h_1[$ and $\delta \in]0, \delta_1[$. Here,

$$\Lambda(t, x^h_{\Delta,z}(\cdot), z(\cdot)) = \Lambda^0(t, x^h_{\Delta,z}(\cdot), z(\cdot)) + 0.5 \int_0^t \Big\{ \int_\Omega |\nabla \tilde{\pi}^h(\rho,\eta)|^2 \,\mathrm{d}\eta$$

$$+ l^2 \int_\Omega |\nabla \tilde{\mu}^h(\rho,\eta)|^2 \,\mathrm{d}\eta \Big\} \mathrm{d}\rho, \quad \Lambda^0(t, x^h_{\Delta,z}(\cdot), z(\cdot)) = 0.5|\tilde{g}^h(t)|^2_H + 0.5 l^2 |\tilde{\mu}^h(t)|^2_H,$$

$$\tilde{\pi}^h(t) = z^{(1)}(t) - \psi^h_{\Delta,z}(t), \quad \tilde{\mu}^h(t) = z^{(2)}(t) - \varphi^h_{\Delta,z}(t), \quad \tilde{g}^h(t) = \tilde{\pi}^h(t) + l\tilde{\mu}^h(t).$$

It is easily seen that the inequality

$$\lambda(\tau_{i+1}, x^h_{\Delta,z}(\cdot), W_T(\cdot)) \le \Lambda(\tau_{i+1}, x^h_{\Delta,z}(\cdot), z(\cdot))$$

is valid, since $z(\tau_{i+1}; \tau_i, z(\tau_i), v^*_{\tau_i,\tau_{i+1}}(\cdot)) \in W_T(\tau_{i+1})$ by the choice of the control $v^*(t)$, $t \in [\tau_i, \tau_{i+1}[$. Thus, to prove the theorem, it is sufficient to estimate the variation in the value $\Lambda(t, x^h_{\Delta,z}(\cdot), z(\cdot))$. Note that the functions $\tilde{\pi}^h(\cdot)$ and $\tilde{\mu}^h(\cdot)$ are solutions of the system

$$\begin{aligned} &\frac{\partial \tilde{\pi}^h(t,\eta)}{\partial t} + l\frac{\partial \tilde{\mu}^h(t,\eta)}{\partial t} = \Delta_L \tilde{\pi}^h(t,\eta) + v^*(t,\eta) - (Bu)(t,\eta) + (Cv^h)(t,\eta) \\ &\text{in } \Omega \times]t_0, \vartheta], \quad \frac{\partial \tilde{\mu}^h(t,\eta)}{\partial t} = \Delta_L \tilde{\mu}^h(t,\eta) + \tilde{R}^h(t,\eta)\tilde{\mu}^h(t,\eta) + \tilde{\pi}^h(t,\eta) \end{aligned} \tag{31}$$

with the initial condition $\tilde{\pi}^h(t_0) = \tilde{\mu}^h(t_0) = 0$ in Ω and the boundary condition $\frac{\partial \tilde{\pi}^h}{\partial n} = \frac{\partial \tilde{\mu}^h}{\partial n} = 0$ on $\partial\Omega \times]t_0, \vartheta]$. Here, $\tilde{R}^h(t,\eta) = a(t,\eta) + b(t,\eta)(z^{(1)}(t,\eta) + \varphi^h_{\Delta,z}(t,\eta)) - ((z^{(1)}(t,\eta))^2 + z^{(1)}(t,\eta)\varphi^h_{\Delta,z}(t,\eta) + (\varphi^h_{\Delta,z})^2(t,\eta))$. Multiplying scalarly the first equation of (31) by $\tilde{g}^h(t)$, and the second one by $\tilde{\mu}^h(t)$, we have

$$\begin{aligned} &(\tilde{g}^h(t), \tilde{g}^h_t(t))_H + \int_\Omega \{|\nabla \tilde{\pi}^h(t,\eta)|^2 + l\nabla \tilde{\pi}^h(t,\eta)\nabla \tilde{\mu}^h(t,\eta)\}\, \mathrm{d}\eta \\ &\quad = (\tilde{g}^h(t), v^*(t) - Bu(t) + Cv^h(t))_H, \\ &(\tilde{\mu}^h(t), \tilde{\mu}^h_t(t))_H + \int_\Omega |\nabla \tilde{\mu}^h(t,\eta)|^2\, \mathrm{d}\eta \le (\tilde{\pi}^h(t), \tilde{\mu}^h(t))_H + b|\tilde{\mu}^h(t)|^2_H \quad \text{for a.a. } t \in T. \end{aligned} \tag{32}$$

Now, multiply the second inequality of (32) by l^2 and add to the first one. We obtain for a.a. $t \in T$

$$\begin{aligned} &(\tilde{g}^h(t), \tilde{g}^h_t(t))_H + l^2(\tilde{\mu}^h(t), \tilde{\mu}^h_t(t))_H + 0.5\int_\Omega \{|\nabla \tilde{\pi}^h(t,\eta)|^2 + l^2|\nabla \tilde{\mu}^h(t,\eta)|^2\}\, \mathrm{d}\eta \\ &\le (\tilde{g}^h(t), v^*(t) - Bu(t) + Cv^h(t))_H + l^2(\tilde{\pi}^h(t), \tilde{\mu}^h(t))_H + bl^2|\tilde{\mu}^h(t)|^2_H. \end{aligned} \tag{33}$$

Then, we have (see (19))

$$\sum_{i=0}^{m-1} \int_{\tau_i}^{\tau_{i+1}} M_*(t; \tau_i)\, \mathrm{d}t \le c_1 \mu^{1/2}(h), \tag{34}$$

where

$$M_*(t; \tau_i) = |\tilde{g}^h(t) - \chi^h_i|_H \{|Bu(t)|_H + |Cv^h_i|_H + |v^*(t)|_H\} \text{ for a.a. } t \in \delta_i.$$

By analogy with (20), from (33), we deduce that for a.a. $t \in \delta_i$,

$$\begin{aligned} &(\tilde{g}^h(t), \tilde{g}^h_t(t))_H + l^2(\tilde{\mu}^h(t), \tilde{\mu}^h_t(t))_H + 0.5\int_\Omega \{|\nabla \tilde{\pi}^h(t,\eta)|^2 + l^2|\nabla \tilde{\mu}^h(t,\eta)|^2\}\, \mathrm{d}\eta \\ &\le 2l^2\lambda^2 \Lambda^0(t, x^h_{\Delta,z}(\cdot), z(\cdot)) + (\chi^h_i, v^*(t) - Bu(t) + Cv^h_i)_H + M_*(t; \tau_i). \end{aligned} \tag{35}$$

For $t \in [\tau_i, \tau_{i+1}[$, by virtue of (28)–(30), we get

$$(\chi_i^h, v^*(t) - Bu(t) + Cv_i^h)_H = (\chi_i^h, B(\tilde{u}_i^h - u(t)))_H + (\chi_i^h, C(\tilde{v}_i^h - v_i^h))_H.$$

Taking into account the rules for forming the controls $\tilde{u}_i^h$ and v_i^h, we conclude that

$$(\chi_i^h, v^*(t) - Bu(t) + Cv_i^h)_H \le 0 \quad \text{for } t \in [\tau_i, \tau_{i+1}[. \tag{36}$$

We deduce from (35) and (36) that

$$\frac{\mathrm{d}\Lambda(t, x_{\Delta,z}^h(\cdot), z(\cdot))}{\mathrm{d}t} = (\tilde{g}^h(t), \tilde{g}_t^h(t))_H + l^2(\tilde{\mu}^h(t), \tilde{\mu}_t^h(t))_H + 0.5 \int_\Omega \{|\nabla \tilde{\pi}^h(t,\eta)|^2 + l^2|\nabla \tilde{\mu}^h(t,\eta)|^2\}\,\mathrm{d}\eta \le l^2\lambda^2 \Lambda^0(t, x_{\Delta,z}^h(\cdot), z(\cdot)) + M_*(t;\tau_i) \quad \text{for a.a. } t \in \delta_i. \tag{37}$$

Thus,

$$\frac{\mathrm{d}\Lambda^0(t, x_{\Delta,z}^h(\cdot), z(\cdot))}{\mathrm{d}t} \le 2l^2\lambda^2 \Lambda^0(t, x_{\Delta,z}^h(\cdot), z(\cdot)) + M_*(t;\tau_i) \quad \text{for a.a. } t \in \delta_i.$$

Using (34), by virtue of the Gronwall Lemma, we derive

$$\Lambda^0(t, x_{\Delta,z}^h(\cdot), z(\cdot)) \le c_2 \sum_{i=0}^{m-1} \int_{\tau_i}^{\tau_{i+1}} M_*(t;\tau_i)\,\mathrm{d}t \le c_3 \mu^{1/2}(h), \quad \forall t \in T.$$

Hence and from (37), we get $\Lambda(t, x_{\Delta,z}^h(\cdot), z(\cdot)) \le c_4 \mu^{1/2}(h)$, $\forall t \in T$. The theorem is proved. □

Remark 2.1 and Theorems 3.1 and 4.1 imply the main result of the paper.

Theorem 4.2 *The strategy $\mathscr{U}$, defined by (6), is an ε-optimal minimax strategy, and the strategy $\mathscr{V}$, defined by (25), is an ε-optimal maximin strategy. Thus, the pair $(\mathscr{U}, \mathscr{V})$ constitutes an ε-saddle point in the game, and $c_1 = C_{\text{opt}} = \inf\{I(w(\cdot; t_0, x_0, u(\cdot))) : u(\cdot) \in D_T(\cdot)\}$ (see Problem 3)* is the value of the game.

5 Conclusions

In this paper, we studied the game control problem for the nonlinear distributed system described by the phase field equations. The work was aimed at building stable algorithms solving the problem. The suggested algorithms are based on constructions from the dynamical reconstruction theory and on the method of extremal shift, which is known in the theory of positional differential games.

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References

1. Krasovskii, N.N., Subbotin, A.I.: Game-Theoretical Control Problems. Springer, Berlin (1988)
2. Osipov, YuS: On the theory of differential games in systems with distributed parameters. Sov. Math. Dokl. **16**(4), 1093–1097 (1975)
3. Maksimov, V.: Feedback Minimax Control for Parabolic Variational Inequality. C.R. Academy of Science, Paris, Serie IIb (2000)
4. Caginalp, G.: An analysis of a phase field model of a free boundary. Arch. Ration. Mech. Anal. **92**, 205–245 (1986)
5. Brokate, M., Sprekels, J.: Hysteresis and Phase Transitions. Springer, New York (1996)
6. Heinkenschloss, M., Tröltzsch, F.: Analysis of the lagrange-SQP-newton method for the control of a phase field equation. Control. Cybern. **28**(2), 177–211 (1999)
7. Hoffman, K.-H., Jiang, L.S.: Optimal control problem of a phase field model for solidification. Numer. Funct. Anal. Optim. **13**(1–2), 11–27 (1992)
8. Benincasa, T., Favini, A., Morosanu, C.: A product formula approach to a nonhomogeneous boundary optimal control problem governed by nonlinear phase-field transition systems. Part I: A phase-field model. J. Optim. Theory Appl. **148**(1), 14–20 (2011)
9. McMillan, C., Triggiani, R.: Min-max game theory and algebraic Riccati equations for boundary control problems with analytic semigroups: Part I: The stable case. Marcel Dekker Lecture Notes in Pure and Applied Mathematics. **152**, 757–780 (1994)
10. Lasiecka, I., Triggiani, R.: Control Theory for Partial Differential Equations: Continuous and Approximation Theories. Part I: Abstract Parabolic Systems. Cambridge University Press, Cambridge (2000)
11. Mordukhovich, B.S.: Optimal control and feedback design of state-constrained parabolic systems in uncertainty conditions. Appl. Anal. **90**, 1075–1109 (2011)
12. Mordukhovich, B.S.: Suboptimal minimax design of constrained parabolic systems with mixed boundary control. Appl. Math. Comp. **204**, 580–588 (2008)
13. Maksimov, V., Pandolfi, L.: Dynamical reconstruction of inputs for contraction semigroup systems: Boundary input case. J. Optim. Theory Appl. **103**(2), 401–420 (1999)
14. Favini, A., Maksimov, V., Pandolfi, L.: A deconvolution problem related to a singular system. J. Math. Anal. Appl. **292**(1), 60–72 (2004)
15. Maksimov, V., Tröltzsch, F.: One inverse problem for phase field equations. Dokl. Math. **69**(3), 377–380 (2004)
16. Maksimov, V., Tröltzsch, F.: Dynamical state and control reconstruction for a phase field model. Dynamics of continuous, discrete and impulsive systems. A: Mathematical analysis **13**(3–4), 419–444 (2006)
17. Neittaanmaki, P., Tiba, D.: Optimal Control of Nonlinear Parabolic Systems. Theory, Algorithms and Applications. Marcel Dekker, New York (1994)