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On the Study of an Economic Equilibrium with Variational Inequality Arguments

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Abstract The main purpose of this paper is to investigate on the existence of a competitive equilibrium for a market with consumption and exchange. A variational representation is used to study this problem. More precisely, we consider an economy where utility functions are assumed concave and non-differentiable, and we characterize the equilibrium by means of a variational problem involving the subdifferential multimap. Thanks to this approach, by introducing suitable perturbed utility functions, we achieve an existence result when the operator is not coercive and the convex set might be unbounded.

Keywords Generalized quasi-variational inequalities · Competitive equilibrium · Perturbation procedure

Mathematics Subject Classification 49J53 · 91B50

1 Introduction

The aim of this paper is to deal with a competitive equilibrium model in a pure exchange economy. In this work we tackle the problem using the variational inequality approach: a widely used method (see, e.g., [1–5] and [6,7] for the state-of-the-art of this topic). The variational inequality theory was introduced by Fichera and Stampacchia, in the early 1960s, in connection with several equilibrium problems originating from mathematical physics. In 1973, to study impulse control problems, Bensoussan and

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Lions [8] introduced the idea of quasi-variational inequality which involves point-to-set maps. Later on, in order to study some problems from operations research, mathematical programming and optimization theory, Chan and Pang in [9] generalized the quasi-variational inequality considered in [8], by introducing the so-called generalized quasi-variational inequality. In [10–15] different equilibrium problems are studied by means of variational inequalities.

The founder of the theory of competitive economic equilibrium is Leon Walras [16], who built a system of simultaneous equations to describe an economy and then showed that the system could be solved to give the equilibrium prices and quantities of commodity. Yet, eventually, he did not achieve the solutions of this system. Only later on, with the work of Wald [17], the first rigorous result on the existence of equilibrium was established. Stimulated by the advances in linear programming, nonlinear analysis and game theory, many authors, like Arrow and Debreu [18], Mc Kenzie [19], Gale [20], Nikaido [21], obtained numerous existence results and developed algorithms for the calculus of the equilibrium by using fixed-point theory.

Starting from 1985, thanks to the development of the variational inequality theory, several authors provided a more sophisticated analysis, which was characterized by a considerably deeper approach to the competitive equilibrium problem. Indeed, the variational inequality method represents an innovative and powerful methodology to provide not only existence results, but also qualitative results such as sensitivity and stability analysis and to develop efficient computational processes for the calculation of the solution.

Our approach is based on the fundamental results presented in the work [4]. Here, the authors reformulated a competitive equilibrium model of consumption and production with trading in the market by means of a suitable variational inequality. Thus, they studied a variational problem which incorporates Lagrange multipliers, and they achieved the existence of the equilibrium. Following the idea of [4], in the papers [22–26] the authors propose the study of competitive equilibrium using a totally different variational approach. More precisely, they characterize the equilibrium as solution of a suitable quasi-variational inequality without appealing Lagrange multipliers and propose a new procedure in order to obtain its existence.

The aim of this paper is to improve the results obtained in the latter works, by relaxing the differentiability assumptions on utility functions. Thus, our equilibrium model is related to a suitable generalized quasi-variational inequality. By using variational arguments we are able to give the existence of equilibrium solutions. Such approach is useful not only from a mathematical point of view but also from an economic point of view, since it allows weakening some classical assumptions. Indeed, in the economic literature, in order to guarantees the existence of an equilibrium, the *strong survivability assumption* is requested: each consumer must be endowed with all goods to survive in the market. Here, as introduced in [11], we consider the minima prices which can be not equal to zero, and we assume a *weaker survivability assumption*. This condition ensures that, for any current price, the consumer has always the opportunity to earn on the sale of its endowment; hence, he can survive in the market.

We would like to conclude by underlining that usually the concept of equilibrium is connected with the optimization of a functional. Such conclusion can be applied to Nash, traffic, oligopolistic …equilibrium problems. The competitive equilibrium model represents a more complicate version of the equilibrium, that is, in addition to maximization to the actions of agents, the determination of market prices is crucial. Hence, the interest of our study is addressed to the competitive equilibrium since it represents a general framework into which we can set a wide class of equilibrium problems, including the aforementioned equilibria.

The paper is organized as follows. In Sect. 2 we recall basic definitions and results. In Sect. 3 we present the competitive economic equilibrium model. In Sect. 4 we reformulate the equilibrium as a solution to a suitable variational inequality involving point-to-set maps. Finally, in Sect. 5 we investigate on the existence of the equilibrium by using the variational inequality theory.

2 Preliminaries

For the reader's convenience, in this Section we recall some notations that will be useful in the sequel.

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multivalued map (multimap). The domain and the graph of $F(\cdot)$ are defined, respectively, as follows:

$$\text{Dom}\, F := \{x \in \mathbb{R}^n : \ \ F(x) \neq \emptyset\} \subseteq \mathbb{R}^n,$$

$$\text{Gph}\, F := \{(x, t) : \ \ t \in F(x), \ x \in \text{Dom} F\} \subseteq \mathbb{R}^n \times \mathbb{R}^m.$$

Definition 2.1 (see [27]) A multimap $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, with Dom $F \neq \emptyset$ is

(a) upper semicontinuous (usc) at $x \in \mathbb{R}^n$ iff for each open set $V \subset \mathbb{R}^m$, where $F(x) \subset V$, there exists a neighborhood $U \subset \mathbb{R}^n$ of x such that for all $x' \in U$: $F(x') \subset V$;
(b) lower semicontinuous (lsc) at $x \in \mathbb{R}^n$ iff for any sequence of elements $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$, $x_n \to x$, and for any $y \in F(x)$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$, with $y_n \in F(x_n)$ $\forall n$ and $y_n \to y$;
(c) closed iff for any sequences $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$, $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$, if $x_n \to x$ and $y_n \in F(x_n)$, $y_n \to y$ then $y \in F(x)$.

Definition 2.2 (see [28]) A multimap $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is

(a) monotone iff it has the property that

$$\langle h_1 - h_2, x_1 - x_2\rangle \geq 0 \quad \text{whenever} \quad h_1 \in F(x_1),\ h_2 \in F(x_2);$$

(b) strongly monotone iff there exists $\nu > 0$ such that

$$\langle h_1 - h_2, x_1 - x_2\rangle \geq \nu\|x_1 - x_2\|^2 \quad \text{whenever} \quad h_1 \in F(x_1),\ h_2 \in F(x_2);$$

(c) a monotone mapping F is maximal monotone iff for every pair $(\widehat{x}, \widehat{v}) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \text{Gph}\, F$ there exists $(\widetilde{x}, \widetilde{v}) \in \text{Gph}\, F$ with $\langle \widehat{v} - \widetilde{v}, \widehat{x} - \widetilde{x}\rangle < 0$,

where $\langle \cdot, \cdot\rangle$ is the usual inner product of $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function. The domain of f is denoted by $\mathrm{dom} f := \{x \in \mathbb{R}^n : f(x) < \infty\}$. A function f is called proper iff $f(x) < \infty$ for at least one $x \in \mathbb{R}^n$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

Definition 2.3 (see [28]) A proper function f is

(a) convex iff for every $x_0, x_1 \in \mathrm{dom} f$ one has

$$f((1-\tau)x_0 + \tau x_1) \leq (1-\tau) f(x_0) + \tau f(x_1) \quad \forall \tau \in [0, 1];$$

(b) strongly convex iff there is a constant $\sigma > 0$ such that

$$f\big((1-\tau)x_0 + \tau x_1\big) \leq (1-\tau) f(x_0) + \tau f(x_1) - \frac{1}{2}\sigma\tau(1-\tau)\|x_0 - x_1\|^2$$

for all $x_0,\ x_1 \in \mathrm{dom} f$ when $\tau \in (0, 1)$;

(c) concave iff $-f$ is convex.

Definition 2.4 (see [29]) A vector x^* is said to be a subgradient of a convex function f at a point $x \in \mathrm{dom} f$ iff

$$f(y) \geq f(x) + \langle x^*, y - x\rangle \quad \forall\ y \in \mathbb{R}^n.$$

The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$.

The map $x \to \partial f(x)$ is a multimap whose values are subsets of $\mathbb{R}^n$. In general, $\partial f(x)$ may be empty; when $\partial f(x)$ is not empty, f is subdifferentiable at x. We recall some properties of the subdifferential of convex functions. By $f : \mathbb{R}^n \to \mathbb{R}$ we mean a function defined on the whole of $\mathbb{R}^n$.

Proposition 2.1 (see [30,31]) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Then*

(a) for $x \in \mathbb{R}^n$, $\partial f(x)$ is a nonempty, convex, and compact set;
(b) $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a usc multimap.

Proposition 2.2 (see [28,29]) *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a proper and convex function. Then*

(a) for $x \in \mathrm{dom}\ f$, the set $\partial f(x)$ is convex and closed;
(b) the multimap $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is monotone;
(c) if f is lsc, its subdifferential ∂f is a maximal monotone operator.

Proposition 2.3 (see [28]) *For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and a value $\sigma > 0$, the following properties are equivalent:*

(a) ∂f is strongly monotone with constant σ;
(b) f is strongly convex with constant σ;
(c) $f - \frac{1}{2}\sigma\|\cdot\|^2$ is convex.

Proposition 2.4 (see [28]) *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *be a function such that* $f = g + f_0$ *with* g *finite at* $\overline{x}$*,* f_0 *smooth on a neighborhood of* $\overline{x}$*, and* f_0*,* g *convex functions. Then* $\partial f(\overline{x}) = \partial g(\overline{x}) + \nabla f_0(\overline{x})$

We recall a definition and some results about variational problems.

Definition 2.5 Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set and let $S : C \rightrightarrows \mathbb{R}^n$ and $\Phi : C \rightrightarrows \mathbb{R}^n$ be multimaps. A Generalized Quasi-Variational Inequality associated with C, S, Φ, denoted by GQVI, is the following problem: Find $\overline{u} \in S(\overline{u})$ and $\varphi \in \Phi(\overline{u})$ such that

$$\langle \varphi, u - \overline{u} \rangle \geq 0 \qquad \forall u \in S(\overline{u})\,. \tag{1}$$

In particular, when $S(u) = C$ for all $u \in C$, (1) is a Generalized Variational Inequality (GVI); when Φ is single-valued, (1) reduces to the Quasi-Variational Inequality (QVI). When both $\Phi(u)$ is singleton and $S(u) = C$, for all $u \in C$, we have the classical Variational Inequality (VI).

Theorem 2.1 (see [32]) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a convex function and let* $C \subseteq \mathbb{R}^n$ *be a nonempty, convex, and closed set. Then,* $\overline{x} \in C$ *is a solution to problem* $\min_{x \in C} f(x)$ *if and only if there exists* $\varphi \in \partial f(\overline{x})$ *such that*

$$\langle \varphi, x - \overline{x} \rangle \geq 0 \quad \forall x \in C.$$

Theorem 2.2 (see [33]) *If* C *is a compact, convex and nonempty set and the function* $\Phi : C \to \mathbb{R}^n$ *is continuous, then the VI admits a solution.*

Theorem 2.3 (see [32]) *If* Φ *is usc, strongly monotone on* C *and has nonempty convex and compact values, then GVI has a unique solution.*

3 Equilibrium Model

Let us consider an economy of pure exchange with n consumers indexed by $a \in A = \{1, \ldots, n\}$ and l different goods indexed by $j \in J = \{1, \ldots, l\}$. To each consumer a is associated a consumption set $X_a \subset \mathbb{R}^l$. As usual in the classical theory, the set X_a is nonempty, convex, and closed. Furthermore, X_a is bounded from below: there exists a vector $\widehat{x}_a$ such that $x_a \geq \widehat{x}_a$ for all $x_a \in X_a$. However, it is not restrictive assuming $X_a \subseteq \mathbb{R}^l_+$; indeed, since X_a is bounded from below, with a change of variables from x_a to $x'_a = x_a - \widehat{x}_a$, it is possible to translate X_a into a set $X'_a \subseteq \mathbb{R}^l_+$. Thus, we can assume $X_a \subseteq \mathbb{R}^l_+$. In this market, the aggregate amount of each commodity is given exogenously, and the economic problem is that of allocating these among the consumers. In the study of the consumer behavior it is important to know how the income is generated. The standard way is to consider that the consumer has an initial endowment $e_a = (e^1_a, \ldots, e^l_a) \in X_a$ which represents the amount of various goods that he can consume or trade with other individuals. Moreover, each consumer chooses a consumption plan $x_a = (x^1_a, \ldots, x^l_a) \in X_a$. The matrix $x = (x_1, ..., x_n)^T$ is the total consumption of the market.

For each $a \in A$ we denote by I_a the set of indexes corresponding to positive initial endowments, namely $I_a := \{j \in J : e_a^j > 0\}$, and we assume that each consumer is endowed with at least one commodity, then $I_a \neq \emptyset$. To each commodity j is associated a non-negative price p^j such that $p^j \geq q^j$, for all $j \in J$, where q^j represents a fixed minimum price associated to each commodity j and such that $0 \leq q^j < \frac{1}{l}$. The vector $p = (p^1, \ldots, p^l) \in \mathbb{R}_+^l$ represents the market prices which all consumers take as given. In this market, the consumer is characterized by preferences over commodities so that he can compare and rank various goods available in the economy. These consumers' preferences can be represented by utility functions $u_a : \mathbb{R}^l \to \mathbb{R}$, with $a \in A$. Each consumer is operating in the market to maximize his utility subject to a natural budget constraint: the value of the consumption plan of the consumer a at the current price p, $\langle p, x_a \rangle$, cannot exceed his initial income $\langle p, e_a \rangle$. This leads to the following maximization problem: for all $a \in A$

$$\text{find } \bar{x}_a \in M_a(p) \text{ such that } u_a(\bar{x}_a) = \max\nolimits_{x_a \in M_a(p)} u_a(x_a),$$

where the set $M_a(p) := \{x_a \in X_a : \langle p, x_a \rangle \leq \langle p, e_a \rangle\}$ represents the budget constraint. From homogeneity of M_a, we can assume that p belongs to the set $P := \Big\{p \in \mathbb{R}_+^l : \sum_{j \in J} p^j = 1,\ p^j \geq q^j \, \forall j \in J\Big\}$. For all $a \in A$ and $p \in P$, the set $M_a(p)$ is convex and closed, and it is unbounded when $p^j = 0$ for some j.

Definition 3.1 Let $\overline{p} \in P$ and $\overline{x} \in M(\overline{p}) := \prod_{a \in A} M_a(\overline{p})$, we say that $(\overline{p}, \overline{x})$ is a competitive equilibrium for a pure exchange economy iff

$$\text{for all } a \in A \qquad u_a(\overline{x}_a) = \max_{x_a \in M_a(\overline{p})} u_a(x_a), \tag{2}$$

$$\text{for all } j \in J \qquad \sum_{a \in A} (\overline{x}_a^j - e_a^j) \leq 0. \tag{3}$$

From an economic point of view, condition (3) means that the total consumption cannot exceed the available commodity in the market. From now on, we will use following assumptions about the consumers' utility and endowments.

Assumptions

(A1) for all consumers $a \in A$ there exists $j \in J$ such that $q^j \neq 0$ and $e_a^j \neq 0$;
(A2) for all consumers $a \in A$, the utility function u_a is continuous and concave.

Assumption (A1) means that the consumer a is endowed with at least one commodity e_a^j with minimum price greater than zero, $q^j \neq 0$. More precisely, he can be active in the market even if he is not endowed with some goods. In fact the consumer a has always the opportunity to earn on the sale of its income $\langle p, e_a \rangle > 0$. We would like to stress that our assumption (A1) is not usual in the economic literature. That is, a classical equilibrium exists under the *strong survivability assumption*: in order to survive in the market, the consumer a must be endowed with a positive quantity of every commodity j, namely $I_a = J$. Thanks to the introduction of the minima prices and to our variational approach, we are able to obtain that: the existence of the equilibrium can be ensured without resorting to the strong survivability.

Remark 3.1 If $(\overline{p}, \overline{x})$ is a competitive equilibrium, then

$$\left\langle \overline{p} - q, \sum_{a \in A} (\overline{x}_a - e_a) \right\rangle \leq 0. \tag{4}$$

Indeed, from condition (3) and being $\overline{p}^j - q^j \geq 0$, it follows (4). It is interesting to observe that when $q^j = 0$ for all $j \in J$, the inequality (4) represents the well-known Walras' law in the general sense:

$$\left\langle \overline{p}, \sum_{a \in A} (\overline{x}_a - e_a) \right\rangle \leq 0. \tag{5}$$

When equality is verified in (5), the corresponding identity is referred to the Walras' law in the narrow sense. Moreover, for each element $\overline{x} \in M(\overline{p})$ one has $\langle \overline{p}, \overline{x}_a - e_a \rangle \leq 0$ for all $a \in A$; then the Walras' law (5) holds.

4 Variational Approach

Now, our aim is to relate the competitive equilibria with the solutions of a suitable variational inequality problem. We consider the following problem:

Find $\overline{p} \in P$, $\overline{x} \in M(\overline{p}) = \prod_{a \in A} M_a(\overline{p})$ and $h = \{h_a\}_{a \in A}$ with $h_a \in \partial(-u_a(\overline{x}_a))$:

$$\sum_{a \in A} \langle h_a, x_a - \overline{x}_a \rangle + \left\langle \sum_{a \in A} (e_a - \overline{x}_a), p - \overline{p} \right\rangle \geq 0 \qquad \forall\, (p, x) \in P \times M(\overline{p}). \tag{6}$$

We observe that, according to Definition 2.5, with

$$\Phi(p, x) := \left(\partial(-u_1(x_1)), \ldots, \partial(-u_n(x_n)), \sum_{a \in A} (e_a - x_a) \right), \quad \forall (p, x) \in P \times \mathbb{R}^{n \times l},$$

$$x = (x_1, \ldots, x_n), \qquad S(p, x) := P \times M(p), \qquad \forall (p, x) \in P \times \mathbb{R}^{n \times l},$$

the problem (6) represents a generalized quasi-variational inequality. It will be useful the following:

Remark 4.1 The pair $(\overline{p}, \overline{x})$ is a solution to GQVI (6) if and only if for all $a \in A$, $\overline{x}_a$ and $h_a \in \partial(-u_a(\overline{x}_a))$ are solutions to

$$\langle h_a, x_a - \overline{x}_a \rangle \geq 0 \qquad \forall x_a \in M_a(\overline{p}), \tag{7}$$

and $\overline{p}$ is a solution to

$$\left\langle \sum_{a \in A} (e_a - \overline{x}_a),\, p - \overline{p} \right\rangle \geq 0 \qquad \forall p \in P. \tag{8}$$

This is easily seen by testing equation (6), respectively, with $(\overline{p}, x)$ for $x \in M(\overline{p})$ such that

$$x_s = \begin{cases} \overline{x}_s, & if\ s \neq a, \\ x_a, & \text{for all } \ x_a \in M_a(\overline{p}), \end{cases}$$

and with $(p, \overline{x})$ for all $p \in P$.

Theorem 4.1 *Under assumptions (A1)–(A2), let $(\overline{p}, \overline{x})$ be a solution to GQVI (6). Then, if $(\overline{p}, \overline{x})$ verifies inequality (4), it is a competitive equilibrium.*

Proof From Remark 4.1, for all $a \in A$, $\overline{x}_a$ is a solution to (7) and $\overline{p}$ is a solution to (8). Hence, from Theorem 2.1, $\overline{x}_a$ is a maximum point of u_a in $M_a(\overline{p})$. It remains to prove condition (3). We pose

$$J^+ := \left\{ j \in J : \sum_{a \in A} (\overline{x}_a^j - e_a^j) > 0 \right\},$$

$$J_0^- := \left\{ j \in J : \sum_{a \in A} (\overline{x}_a^j - e_a^j) \leq 0,\ \overline{p}^j = q^j \right\},$$

and $$J^- := \left\{ j \in J : \sum_{a \in A} (\overline{x}_a^j - e_a^j) \leq 0,\ \overline{p}^j > q^j \right\}.$$

We suppose by contradiction that $J^+ \neq \emptyset$. Firstly, we observe that $J^- \neq \emptyset$. Indeed, if $J^- = \emptyset$, since $\sum_{j \in J} q^j < 1$, we have $\overline{p} \neq q$, then there exists at least one index $j \in J^+$ such that $\overline{p}^j > q^j$. Hence

$$\begin{aligned} \left\langle \sum_{a \in A} (e_a - \overline{x}_a), \overline{p} - q \right\rangle &= \sum_{j \in J^+} \left(\sum_{a \in A} (e_a^j - \overline{x}_a^j) \right) (\overline{p}^j - q^j) \\ &\quad + \sum_{j \in J_0^-} \left(\sum_{a \in A} (e_a^j - \overline{x}_a^j) \right) (\overline{p}^j - q^j) \\ &= \sum_{j \in J^+} \left(\sum_{a \in A} (e_a^j - \overline{x}_a^j) \right) (\overline{p}^j - q^j) < 0, \end{aligned}$$

but this contradicts (4), then $J^- \neq \emptyset$. We consider

$$\widetilde{p}^j = \begin{cases} \overline{p}^j + \varepsilon, & \forall j \in J^+, \\ \overline{p}^j, & \forall j \in J_0^-, \\ \overline{p}^j - \varepsilon \frac{|J^+|}{|J^-|}, & \forall j \in J^-, \end{cases}$$

where $0 < \varepsilon < \min_{j \in J^-} \left\{ (\overline{p}^j - q^j) \frac{|J^-|}{|J^+|} \right\}$, with $0 < |J^-|,\ |J^+| < l$ denoting, respectively, the cardinality of sets J^- and J^+. From choosing of ε, we have $\widetilde{p}^j \geq q^j$ for all $j \in J$ and

$$\begin{aligned} \sum_{j \in J} \widetilde{p}^j &= \sum_{j \in J^+} (\overline{p}^j + \varepsilon) + \sum_{j \in J_0^-} \overline{p}^j + \sum_{j \in J^-} \left(\overline{p}^j - \varepsilon \frac{|J^+|}{|J^-|} \right) \\ &= 1 + \varepsilon |J^+| - |J^-| \varepsilon \frac{|J^+|}{|J^-|} = 1, \end{aligned}$$

then $\widetilde{p} \in P$. Replacing $\widetilde{p}$ in (8); it results

$$\left\langle \sum_{a \in A} (e_a - \overline{x}_a),\ \widetilde{p} - \overline{p} \right\rangle = \varepsilon \sum_{j \in J^+} \left(\sum_{a \in A} (e_a^j - \overline{x}_a^j) \right) - \varepsilon \frac{|J^+|}{|J^-|} \sum_{j \in J^-} \left(\sum_{a \in A} (e_a^j - \overline{x}_a^j) \right) < 0.$$

But this is false because $\overline{p}$ is a solution to VI (8). Then $J^+ = \emptyset$, hence, condition (3) holds. Then, we conclude that $(\overline{p}, \overline{x})$ is a competitive equilibrium. □

Remark 4.2 Under the classical survivability assumption one has that, for all $a \in A$, $e_a^j > 0$ for all $j \in J$. Then, it is possible to assume that the minimum price is equal to zero: $q^j = 0$ for all $j \in J$. Since $\overline{x} \in M(\overline{p})$, the Walras' law in the general sense holds; hence, Theorem 4.1 can be reformulated as follows:

Theorem 4.2 *Let assumption (A2) be satisfied and* $e_a^j > 0\ \forall a \in A\ \ \forall j \in J$. *Then a solution* $(\overline{p}, \overline{x})$ *to GQVI (6) is a competitive equilibrium.*

5 Existence Result

In this Section, we investigate on the existence of the competitive equilibrium. Our aim is to give an existence result without assuming strong concavity and differentiability conditions on utility functions. Firstly, we observe that, the variational problem (6) fits in the following general formulation:

Find $(\overline{u}, \overline{v}) \in S_1(\overline{v}) \times S_2$ *and* $(\varphi_1, \varphi_2) \in \Phi_1(\overline{u}, \overline{v}) \times \Phi_2(\overline{u}, \overline{v})$ such that

$$\langle \varphi_1, u - \overline{u} \rangle + \langle \varphi_2, v - \overline{v} \rangle \geq 0 \qquad \forall (u, v) \in S_1(\overline{v}) \times S_2\,, \tag{9}$$

where $S_2 \subseteq \mathbb{R}^m$ and $\Phi \equiv (\Phi_1, \Phi_2)$ and S_1 are multimaps defined, respectively, from $C_1 \times C_2$ and C_2, with $C_1 \subseteq \mathbb{R}^n$ and $C_2 \subseteq \mathbb{R}^m$ nonempty, closed, and convex sets. In order to give an existence result, we use the following two-level procedure (see, e.g., [15]): fixed $v \in S_2$, we consider the parametric GVI

$$\text{find} \;\; \overline{u} \in S_1(v) \;\; \text{and} \;\; \varphi_1 \in \Phi_1(\overline{u}, v) \;\; \text{such that} \;\; \langle \varphi_1, u - \overline{u} \rangle \geq 0 \qquad \forall u \in S_1(v)\,. \quad (10)$$

For this problem we obtain existence and regularity results of the solution $\overline{u}(v)$. Then we consider the VI

$$\text{find} \;\; \overline{v} \in S_2 \;\; \text{and} \;\; \varphi_2 \in \Phi_2(\overline{u}(\overline{v}), \overline{v}) \;\; \text{such that}$$

$$\langle \varphi_2, v - \overline{v} \rangle \geq 0 \qquad \forall v \in S_2\,. \quad (11)$$

For this problem we also obtain the existence of the solution. Finally, $(\overline{u}(\overline{v}), \overline{v})$ represents the solution to (9). It will be later useful the following:

Proposition 5.1 *Let assumption (A1) holds. For all $a \in A$, the multimap $M_a : P \rightrightarrows X_a$ defined as, for all $p \in P$, $\;M_a(p) := \{x_a \in X_a : \langle p, x_a - e_a \rangle \leq 0\}$ is closed and lsc and with closed and convex values.*

Proof The multimap M_a is closed and with closed and convex values. We prove that M_a is lsc. Let $\{p_n\}_{n \in \mathbb{N}} \subseteq P$ such that $p_n \to p$ and we fix $x_a \in M_a(p)$.

– If $\langle p, x_a - e_a \rangle < 0$, from the strict inequality and the continuity of the scalar product, for sufficiently large n, one has $\langle p_n, x_a - e_a \rangle < 0$. So $x_a \in M_a(p_n)$.
– If $\langle p, x_a - e_a \rangle = 0$. We pose $I_+ := \{j \in J : x_a^j > 0\}$; from (A1), since $\langle p, x_a \rangle = \langle p, e_a \rangle > 0$, there exists $s \in I_+$ such that $p^s > 0$. We pose

$$x_{a,n}^j = \begin{cases} 0, & \text{if} \;\; j \notin I_+, \\ x_a^j - \dfrac{p_n^j}{\sum_{j \in I_+} (p_n^j)^2} \langle p_n, x_a - e_a \rangle, & if \;\; j \in I_+. \end{cases}$$

It results: $x_{a,n} \to x_a$, $x_{a,n}^j \geq 0$ for all $j \in J$ and

$$\langle p_n, x_{a,n} - e_a \rangle = \langle p_n, x_a - e_a \rangle - \sum_{j \in I_+} p_n^j \frac{p_n^j}{\sum_{j \in I_+} (p_n^j)^2} \langle p_n, x_a - e_a \rangle = 0;$$

hence $x_{a,n} \in M_a(p_n)$.

Then, for all $x_a \in M_a(p)$ there exists $\{x_{a,n}\}_{n \in \mathbb{N}}$ such that $x_{a,n} \in M_a(p_n)$ and $x_{a,n} \to x_a$; namely, M_a is a lsc multimap. □

Firstly, we give a preliminary existence result for the variational problem (6), when utility functions are strongly concave. □

Theorem 5.1 *Let assumptions (A1)–(A2) be satisfied and u_a be strongly concave. Then there exists $(\overline{p}, \overline{x})$ solution to GQVI (6).*

Proof We fix $p \in P$, $a \in A$, and we consider the following parametric GVI: Find $\overline{x}_a \in M_a(p)$ and $h_a \in \partial(-u_a(\overline{x}_a))$ such that

$$\langle h_a, x_a - \overline{x}_a \rangle \geq 0 \qquad \forall x_a \in M_a(p)\,. \tag{12}$$

In accordance with Proposition 2.1, $\partial(-u_a)$ is usc with compact and convex values. Moreover, from Propositions 2.2 and 2.3, $\partial(-u_a)$ is a maximal monotone and strongly monotone map. Hence, by Theorem 2.3, there exists a unique solution to GVI (12). So, we can define the function:

$$\overline{x}_a : P \to \mathbb{R}^l \text{ such that } \forall p \in P,\ \overline{x}_a\,(p) \text{ is the solution to (12).}$$

We prove that $\overline{x}_a(\cdot)$ is continuous on P. Let $\{p_n\}_{n\in\mathbb{N}} \subseteq P$ be a sequence converging to p for all $n \in \mathbb{N}$ there exist $\overline{x}_{a,n}$ and $h_{a,n} \in \partial(-u_a(\overline{x}_{a,n}))$:

$$\langle h_{a,n}, x_a - \overline{x}_{a,n} \rangle \geq 0 \qquad \forall x_a \in M_a(p_n)\,. \tag{13}$$

We consider the sequence $\{\overline{x}_{a,n}\}_{n\in\mathbb{N}}$, it results:

(i) *there is at least one subsequence of* $\{\overline{x}_{a,n}\}_{n\in\mathbb{N}}$ *converging to some* $t \in \mathbb{R}^l$. By Proposition 2.1, $\partial(-u_a(e_a)) \neq \emptyset$; thus, we consider $h_0 \in \partial(-u_a(e_a))$. Thanks to the strong monotonicity of $\partial(-u_a)$, there exists $\nu > 0$ such that

$$\langle h_1 - h_2, x_1 - x_2 \rangle \geq \nu \|x_1 - x_2\|^2 \quad \forall x_1\ x_2 \in \mathbb{R}^l\,, \tag{14}$$

whenever $h_1 \in \partial(-u_a(x_1))$, $h_2 \in \partial(-u_a(x_2))$. In particular for $x_1 = e_a$, $x_2 = \overline{x}_{a,n}$ and $h_1 = h_0 \in \partial(-u_a(e_a))$, $h_2 = h_{a,n} \in \partial(-u_a(\overline{x}_{a,n}))$ and being $\overline{x}_{a,n}$ the solution to n-GVI (13) and $e_a \in M_a(p_n)$ for all n, one has

$$\nu\|\overline{x}_{a,n} - e_a\|^2 \leq \langle h_0 - h_{a,n}, e_a - \overline{x}_{a,n} \rangle \leq \|h_0\| \cdot \|\overline{x}_{a,n} - e_a\|.$$

Namely, $\quad \|\overline{x}_{a,n}\| \leq \|e_a\| + \|\overline{x}_{a,n} - e_a\| \leq \|e_a\| + \frac{\|h_0\|}{\nu} = M\,.$
So, there exists at least one subsequence $\{\overline{x}_{a,n_k}\}$ such that $\overline{x}_{a,n_k} \to t \in \mathbb{R}^l$.

(ii) *t is the solution to* (12): $t = \overline{x}_a(p)$.
Since M_a is a closed multimap, $t \in M_a(p)$. Being $\overline{x}_{a,n}$ the solution to (13):

$$u_a(\overline{x}_{a,n}) = \max_{x_{a,n}\in M_a(p_n)} u_a(x_{a,n}) \quad \Leftrightarrow \quad u_a(x_{a,n}) \leq u_a(\overline{x}_{a,n})\ \forall x_{a,n} \in M_a(p_n)\,. \tag{15}$$

For all $x_a \in M_a(p)$, being M_a lsc, one has $\exists\ \{x'_{a,n}\}_{n\in\mathbb{N}}$: $x'_{a,n} \in M_a(p_n)$, $x'_{a,n} \to x_a$. If we replace $x'_{a,n} = x_{a,n}$ in (15) and we pass to the limit, then one has $u_a(x_a) \leq u_a(t)$. That is t is the unique solution to GVI (12): $t = \overline{x}_a(p)$.

So, for all $a \in A$, each function $\overline{x}_a$ is continuous on P. Finally, we consider

$$\left\langle \sum_{a\in A} (e_a - \overline{x}_a(\overline{p})),\, p - \overline{p} \right\rangle \geq 0 \qquad \forall\ p \in P\,. \tag{16}$$

Since the operator $g(p) = \sum_{a \in A}(\overline{x}_a(p) - e_a)$ is continuous and P is compact, from Theorem 2.2 there exists at least one solution $\overline{p}$ to VI (16). Hence $(\overline{x}(\overline{p}), \overline{p})$ is a solution to GQVI (6). □

Now, by using perturbation arguments, we obtain another existence result when utility functions are concave and non-differentiable.

Proposition 5.2 *Let assumption (A2) be satisfied. For each* $\varepsilon > 0$*, we define the perturbed utility function:* $u_{a,\varepsilon}(x) := u_a(x) - \varepsilon\|x\|^2 \ \ \forall\, x \in \mathbb{R}^l$*. It results*

(i) $u_{a,\varepsilon}$ *verifies assumption (A2);*
(ii) for all $x \in \mathbb{R}^l$ *one has* $\partial(-u_{a,\varepsilon})(x) = \partial(-u_a(x)) + 2\varepsilon x$*;*
(iii) $u_{a,\varepsilon}$ *is strongly concave.*

Proof Clearly (i) holds; the equality (ii) follows from Proposition 2.4. We prove (iii) with standard arguments. From (ii), for all $x_1,\ x_2 \in \mathbb{R}^l$, $h_1 \in \partial(-u_{a,\varepsilon}(x_1))$ and $h_2 \in \partial(-u_{a,\varepsilon})(x_2)$, one has $h_1 = v_1 + 2\varepsilon x_1$ and $h_2 = v_2 + 2\varepsilon x_2$, for some $v_1 \in \partial(-u_a(x_1))$ and $v_2 \in \partial(-u_a(x_2))$. Then, being $\partial(-u_a)$ a monotone map (see Proposition 2.2), one has:

$$\langle h_1 - h_2, x_1 - x_2\rangle = \langle v_1 + 2\varepsilon x_1 - v_2 - 2\varepsilon x_2, x_1 - x_2\rangle = \\ = \langle v_1 - v_2, x_1 - x_2\rangle + 2\varepsilon\langle x_1 - x_2, x_1 - x_2\rangle \geq 2\varepsilon\|x_1 - x_2\|^2\,;$$

that is $\partial(-u_{a,\varepsilon})$ is strongly monotone; so $u_{a,\varepsilon}$ is strongly concave. □

Theorem 5.2 *Let assumptions (A1)-(A2) be satisfied, then there exists at least one solution to GQVI (6).*

Proof Fixed $n \in \mathbb{N}$ and $a \in A$, we consider the perturbed function $u_{a,\varepsilon}$ with $\varepsilon = \frac{1}{n}$: $u_{a,n}(x) = u_a(x) - \frac{1}{n}\|x\|^2 \ \ \forall x \in \mathbb{R}^l$. By Proposition 5.2, $u_{a,n}$ satisfies assumption (A2), and it is strongly concave. Then, from Theorem 5.1 there exist $(\overline{p}_n, \overline{x}_n) \in P \times M(\overline{p}_n)$ and $h_n = \{h_{a,n}\}_{a \in A}$ with $h_{a,n} \in \partial(-u_{a,n}(\overline{x}_{a,n}))$:

$$\sum_{a \in A}\langle h_{a,n}, x_{a,n} - \overline{x}_{a,n}\rangle + \left\langle \sum_{a \in A}(e_a - \overline{x}_{a,n}), p - \overline{p}_n \right\rangle \geq 0 \qquad \forall\ (p, x_n) \in P \times M(\overline{p}_n)\,. \tag{17}$$

The sequence $\{\overline{x}_{a,n}\}_{n \in \mathbb{N}}$ is bounded. Indeed, since $\partial(-u_{a,n}(e_a)) \neq \emptyset$, there exists $h_{0,n} \in \partial(-u_{a,n}(e_a))$ and $h_{0,n} = v_0 + \frac{2}{n}e_a$ with $v_0 \in \partial(-u_a(e_a))$. From the strong monotonicity of $\partial(-u_{a,n})$, there exists $\nu > 0$ such that

$$\langle h_{1,n} - h_{2,n}, x_1 - x_2\rangle \geq \nu\|x_1 - x_2\|^2 \qquad \forall x_1\ x_2 \in \mathbb{R}^l\,, \tag{18}$$

whenever $h_{1,n} \in \partial(-u_{a,n}(x_1))$, $h_{2,n} \in \partial(-u_{a,n}(x_2))$. In particular for $x_1 = e_a$, $x_2 = \overline{x}_{a,n}$ and $h_{1,n} = h_{0,n} \in \partial(-u_{a,n}(e_a))$, $h_{2,n} = h_{a,n} \in \partial(-u_{a,n}(\overline{x}_{a,n}))$, being $\overline{x}_{a,n}$ the solution to n-GVI:

$$\langle h_{a,n}, x_{a,n} - \overline{x}_{a,n}\rangle \geq 0 \qquad \forall x_{a,n} \in M_a(\overline{p}_n) \tag{19}$$

and since $e_a \in M_a(p_n)$ for all n, one has

$$\nu\|\overline{x}_{a,n} - e_a\|^2 \leq \langle h_{0,n} - h_{a,n}, e_a - \overline{x}_{a,n}\rangle = \langle h_{0,n}, e_a - \overline{x}_{a,n}\rangle - \langle h_{a,n}, e_a - \overline{x}_{a,n}\rangle$$

$$\leq \|h_{0,n}\| \cdot \|\overline{x}_{a,n} - e_a\| = \|v_0 + \frac{2}{n}\, e_a\| \cdot \|\overline{x}_{a,n} - e_a\|$$

$$\leq \left(\|v_0\| + \frac{2}{n}\,\|e_a\|\right) \|\overline{x}_{a,n} - e_a\|;$$

which yields $\quad \|\overline{x}_{a,n} - e_a\| \leq \frac{1}{\nu}\left(\|v_0\| + \frac{2}{n}\|e_a\|\right) \leq M \qquad \forall n \in \mathbb{N}.$

Namely $\quad \|\overline{x}_{a,n}\| \leq \|e_a\| + \|\overline{x}_{a,n} - e_a\| \leq \|e_a\| + M = K$, that is the sequence $\{\overline{x}_{a,n}\}_{n\in\mathbb{N}}$ is bounded. Then, since $\{\overline{p}_n\} \subseteq P$ with P compact, there exists a subsequence $\{(\overline{p}_{n_k}, \overline{x}_{n_k})\}_{k\in\mathbb{N}}$ converging to $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$. The limit point $(\overline{p}, \overline{x})$ is a solution to GQVI (6). Indeed, one has

$$\forall a \in A \qquad \langle h_{a,n}, x_{a,n} - \overline{x}_{a,n}\rangle \geq 0 \qquad \forall x_{a,n} \in M_a(\overline{p}_n), \tag{20}$$

$$\left\langle \sum_{a\in A}(e_a - \overline{x}_{a,n}),\, p - \overline{p}_n \right\rangle \geq 0 \qquad \forall p \in P. \tag{21}$$

For all $p \in P$, passing to the limit in (20) one has

$$\left\langle \sum_{a\in A}(e_a - \overline{x}_a),\, p - \overline{p} \right\rangle \geq 0\,. \tag{22}$$

Moreover, for all $x_a \in M_a(\overline{p})$, since M_a is lsc, there exists a sequence $\{x_{a,n}\}_{n\in\mathbb{N}}$ such that $x_{a,n} \in M_a(\overline{p}_n)$ and $x_{a,n} \to x_a$. From Theorem 2.1 one has

$$u_{a,n}(x_{a,n}) \leq u_{a,n}(\overline{x}_{a,n}) \qquad \Leftrightarrow \qquad u_a(x_{a,n}) - \frac{1}{n}\|x_{a,n}\|^2 \leq u_a(\overline{x}_{a,n}) - \frac{1}{n}\|\overline{x}_{a,n}\|^2\,.$$

Passing to the limit, from continuity of u_a, one has $u_a(x_a) \leq u_a(\overline{x}_a)$. Hence, $\overline{x}_a$ is a maximal point of u_a in $M_a(\overline{p})$. Then, from Theorem 2.1, there exists $h_a \in \partial(-u_a(\overline{x}_a))$ such that

$$\langle h_a, x_a - \overline{x}_a\rangle \geq 0 \qquad \forall x_a \in M_a(\overline{p})\,. \tag{23}$$

So, from (23) and (22), it follows that $(\overline{p}, \overline{x})$ is a solution to GQVI (6). □

We observe that, under assumptions (A1) and (A2), there exists $(\overline{p}, \overline{x})$ solution to GQVI (6) (Theorem 5.2). Moreover, if $(\overline{p}, \overline{x})$ verifies inequality (4), then it is a competitive equilibrium (Theorem 4.1).

Theorem 5.3 *Let assumption (A2) be satisfied and* $e_a^j > 0\ \forall a \in A,\ \ \forall j \in J$. *Then there exists the competitive equilibrium for a pure exchange economy.*

Proof It follows from Theorems 4.2 and 5.2. □

6 Conclusions

In this paper we investigated on the existence of a competitive equilibrium for a market with consumption and exchange using a variational inequality approach. The variational representation, which we set up for a competitive economic model, fits in the general formulation (9). In this paper we achieved a new and useful methodology to obtain both the existence of the solution and the computational procedures for the calculus by using a two-level procedure (see Sect. 5). These methods, which we applied to an economic equilibrium problem, can be used in a more general context. We point out that the above procedure is useful not only to provide existence results but also to provide efficient computational procedure for the calculus of solutions (see, e.g., [24], where an example is illustrated). Indeed, instead of solving the GQVI (9), it is certainly easier and more convenient to solve two variational inequalities where the convex set does not depend on the solution.

Furthermore, it deserves emphasis the fact that in the variational problem (9) the operator is not coercive and the convex set might be unbounded. This represents a considerable difficulty in the tractability of the problem. Indeed, in literature, existence results are mainly based either on coercivity conditions over the operator or on compactness conditions over the convex set. But these conditions are not satisfied directly by our problem. We overcome this difficulty by introducing suitable perturbed utility functions, in order to study perturbed variational problems with strongly monotone operator converging to the initial variational problem. Then, thanks to our approach, we are able to provide an existence theorem when the operator is not coercive and the set is unbounded.

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