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INVITED PAPER

A Critical View on Invexity

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Abstract The aim of this note is to show that a number of publications on invexity in prestigious journals contain unclear definitions, ambiguous statements and sometimes wrong proofs. By analyzing certain facts of some relevant works, we wish to call readers' attention to the literature on invexity when they use it in their research.

Keywords Invex function · Generalized invex function · Condition C

1 Introduction

The concept of invexity was first introduced by Hanson [1], Craven and Glover [2] and some others in the 1980s. The main focus of invexity is to find a class of functions for which any point where the derivative of a function vanishes, called a critical point, is also a global minimum of the function. The concept is then extended to non-differentiable functions and set-valued functions, too. The class of invex functions has nice properties and allows one to produce zero-gap duality for some optimization problems. However, since the 1990s a lot of generalizations have come to light with all sorts of names like quasi-invex functions, pre-invex functions, pseudo-invex functions, and so on. The number of publications on invexity is overwhelming, disproportional with the importance of this concept both from mathematical and practical points of view. However, what is worse is the fact that many papers in this domain contain unclear definitions, erroneous statements and false proofs which affect the community of researchers in applied mathematics.

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The aim of this short note is not to criticize the authors of certain papers (*errare humanum est*), but to analyze what is wrong in their publications in order to help the interested researchers avoid such mistakes and pay attention when using references on invexity.

2 About Statements and Proofs

Let us consider the following text quoted from [3] (Ref. [3] is cited 89 (resp., 106) times in Google Scholar and 10 (resp., 19) times in MathSciNet in April, 2012 (resp., in August, 2013)):

Definition 1.1 See Refs. 1–2. A set $K \subseteq \mathbb{R}^n$ is said to be invex if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$x, y \in K,\ \lambda \in [0, 1] \quad \Rightarrow \quad y + \lambda\eta(x, y) \in K.$$

Remark 1.1 A convex set is an invex set; i.e., take $\eta(x, y) = x - y$. But the converse does not hold.[1]

Of course, the converse does not hold because, by [3, Definition 1.1] (quoted above), any set is invex: just take $\eta(x, y) := 0$. The honest definition is: Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a function. The set $K \subset \mathbb{R}^n$ is said to be η-invex iff ...

Or there is another formulation: One says that f is (θ, α)-d invex iff there exist θ ... such that f satisfies a certain condition involving θ ... Of course, it is correct to first introduce θ, α, d and after that to say that f is (θ, α)-d invex iff ...

Let us quote from [4]:

Theorem 8 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is B-$(0, r)$-invex (B-$(0, r)$-incave) with respect to η and b on $\mathbb{R}^n$ if and only if its every stationary points is a global minimum (maximum) in $\mathbb{R}^n$.*

At least two remarks are in order with respect to this statement: first, if the statement is true, f is B-$(0, r)$-invex with respect to η and b (in the sense of Definition 1 in [4]) if and only if f is invex (because, as seen above, the invexity of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is equivalent to the fact that every stationary point is a global minimum); so why introduce B-$(0, r)$-invexity? Second, the statement gives the impression that the functions η and b, as well as $r \in \mathbb{R}$, are given. Consider $n = 1$, $r = 0$, $b(x, u) := 1$ and $\eta(x, u) := u^{-1}(x^2 - u^2) + \operatorname{sgn} u$ for $u \neq 0$, $\eta(x, u) := 0$ for $u = 0$. Taking $f(x) = \frac{1}{2}x^2$ for $x \in \mathbb{R}$, we see that every stationary point of f (that is, $u = 0$) is a global minimum, but f is not B-$(0, r)$-invex with respect to η and b on $\mathbb{R}$.

Let us quote also from Definition 8 in [5]:

Let $S \subset R^n$ be a nonempty invex set with respect to η. A function $f : S \to R$ is said to be pre-invex with respect to η if there exists a vector-valued function $\eta : S \times S \to R^n$ such that the relation ...

Definition 9 in [5] is obtained by changing pre-invex by invex. So, first, η is given, and one line below one asks for the existence of such an η.

[1] To see the precise coordinates of the references referred in the quoted texts, one might consult the reviews on MathSciNet of the corresponding articles listed at the end of this note.

Reading the next text quoted from [6], one could ask if invexity is a topic in mathematics (Ref. [6] is cited 77 (resp., 88) times in Google Scholar and 15 (resp., 21) times in MathSciNet in April, 2012 (resp., in August, 2013)):

Remark 2.3 We will show that Assumption C holds if $\eta(x,y)=x-y+o(\|x-y\|)$. In fact, the following two equalities hold:[2]

$$\begin{aligned}
\text{(i)}\quad & \eta\bigl(y, y+\lambda\eta(x,y)\bigr) \\
&= \eta(y, y+\lambda\bigl(x-y+o(\|x-y\|)\bigr) \\
&= -\lambda\bigl(x-y+o(\|x-y\|)\bigr)+o\bigl(\lambda(\|x-y+o(\|x-y\|))\|\bigr) \\
&= -\lambda\bigl[x-y+o(\|x-y\|)+o(\|x-y+o(\|x-y\|))\|\bigr] \\
&= -\lambda\bigl[x-y+o(\|x-y\|)\bigr] \\
&= -\lambda\eta(x,y); \ldots
\end{aligned}$$

Which is this Condition C (or Assumption C)?

I quote again from p. 610 of [6] (see also p. 116 of [7]):

Assumption C *See Ref.* 6. *Let* $\eta: X\times X\to\mathbb{R}^n$. *Then, for any* $x,y\in\mathbb{R}^n$ *and for any* $\lambda\in[0,1]$,
$\eta(y, y+\lambda\eta(x,y)) = -\lambda\eta(x,y)$,
$\eta(x, y+\lambda\eta(x,y)) = (1-\lambda)\eta(x,y)$.

Somewhere it is written that $X\subset\mathbb{R}^n$; probably for the authors it is not very important to speak about $\eta(x,y)$ when x or y is not in X. Let us consider $X=\mathbb{R}^n$. (Moreover, note that in Definition 2.4 of [6] $\eta: X\times X\to\mathbb{R}$, that is, η takes its values in $\mathbb{R}$ instead of $\mathbb{R}^n$.)

To see that the assumption $\eta(x,y)=x-y+o(\|x-y\|)$ does not imply Condition C, let us consider $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by $\eta(x,y):=x-y+(x-y)^2$. Of course, η satisfies the hypothesis of the statement in Remark 2.3 of [6]. For $\lambda=1$, the first relation of Condition C is equivalent to each of the following: $-\eta(x,y)+(\eta(x,y))^2=-\eta(x,y)$, $(\eta(x,y))^2=0$, $\eta(x,y)=0$, $(x-y)(1+x-y)=0$, $x-y\in\{0,-1\}$. So, taking $x=0$, $y=2$ and $\lambda=1$, one sees that η does not verify Condition C.

In [7], one finds:

Example 2.1 Let

$$\eta(x,y)=\begin{cases} x-y & \text{if } x\ge 0, y\ge 0;\\ x-y & \text{if } x\le 0, y\le 0;\\ -2-y & \text{if } x>0, y\le 0;\\ 2-y & \text{if } x\le 0, y>0.\end{cases}$$

Then, it is easy to verify that η satisfies Condition C.

[2] Related to this piece of non-mathematics, let us quote from what the authors say in the first footnote on the first page of [6]: "The authors are thankful to ..., and *three anonymous referees* for their many valuable comments on an early version of this paper. The authors are also grateful to Professor B.D. Craven for some discussion on this paper".

A similar example is the following quoted from [6] (which is very close to that quoted above from [7], as well as to Example 1.1 in [3]):

Example 2.2 Let
$f = -|x|, \forall x \in K = [-2, 2]$, and let

$$\eta(x, y) = \begin{cases} x - y, & \text{if } x \geq 0, y \geq 0; \\ x - y, & \text{if } x < 0, y < 0; \\ -2 - y, & \text{if } x > 0, y \leq 0; \\ 2 - y, & \text{if } x \leq 0, y > 0. \end{cases}$$

Then, it is easy to verify that f is invex with respect to η on K and that f and η satisfy Assumptions A and C. However, f is not convex.

The authors seem not to realize that η, defined in these two examples, is not a function because $\eta(2, 0)$ gives 2 using the first expression and -2 using the third expression. A possible modification for η, defined in Example 2.1 of [7], could be:

$$\eta(x, y) = \begin{cases} x - y & \text{if } xy \geq 0, \\ 2 - y & \text{if } xy < 0. \end{cases}$$

Take $x, y \in \mathbb{R}$ with $x > 0 > y$ and $\lambda \in [0, 1]$ and let us look at the second relation in Condition C. We have that $\eta(x, y) = 2 - y$ and $y' := y + \lambda\eta(x, y) = y + \lambda(2 - y)$. Assuming that $y' \geq 0$, then $\eta(x, y') = x - y' = x - 2 + (1 - \lambda)\eta(x, y)$. Hence, in such a situation ($y < 0$ and $y + \lambda(2 - y) \geq 0$), one has $\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y)$ if and only if $x = 2$. Is it possible to have $y < 0$, $y + \lambda(2 - y) \geq 0$ and $\lambda \in [0, 1]$? The answer is YES! Just take $\lambda = 1$. Hence, for $x = 1$, $y = -1$ and $\lambda = 1$ the second relation in Condition C is not verified because $\eta(1, -1) = 3$, $\eta(1, -1 + 3) = \eta(1, 2) = -1 \neq 0$.

In fact, an adequate modification of the function η in Example 2.1 from [7] (or Example 2.2 in [6]) is

$$\eta(x, y) = \begin{cases} x - y & \text{if } xy \geq 0, \\ 2 - y & \text{if } x < 0, y > 0, \\ -2 - y & \text{if } x > 0, y < 0. \end{cases}$$

The function η defined in this way indeed satisfies Condition C.

Somewhere (say [S]) it was said that

$$\eta\bigl(y + \lambda_2\eta(x, y), y + \lambda_1\eta(x, y)\bigr) = (\lambda_2 - \lambda_1)\eta(x, y) \\ \forall x, y \in \mathbb{R}^n, \ \forall\lambda_1, \lambda_2 \in [0, 1], \quad (1)$$

whenever η verifies Condition C; and for this, the proof of Theorem 3.1 in [6] was cited. In fact, I was determined by [S] to look at [6] and [7]. Of course, relation (1) is nice and good to have; moreover, for $\lambda_2 = 0$ one recovers the first relation in Condition C.

Looking at the proof of Theorem 3.1 in [6] (but one can look also at the proof of Theorem 2.1 in [6] for the same text), one observes that one takes $0 < \lambda_2 < \lambda_1 < 1$ and one obtains relation (14) of [6] I am quoting below:

$$\begin{aligned}
&\eta\bigl(y + \lambda_1\eta(x, y), y + \lambda_2\eta(x, y)\bigr) \\
&\quad = \eta\bigl(y + \lambda_1\eta(x, y), y + \lambda_1\eta(x, y) - (\lambda_1 - \lambda_2)\eta(x, y)\bigr) \\
&\quad = \eta\bigl(y + \lambda_1\eta(x, y), y + \lambda_1\eta(x, y) + \eta\bigl(y, y + (\lambda_1 - \lambda_2)\eta(x, y)\bigr)\bigr) \\
&\quad = -\eta(y, y + \bigl(\lambda_1 - \lambda_2\eta(x, y)\bigr) \\
&\quad = (\lambda_1 - \lambda_2)\eta(x, y).
\end{aligned}$$

The first equality is obvious, the second as well as the fourth follow from the first relation in Condition C [however, it is $= -\eta(y, y + (\lambda_1 - \lambda_2)\eta(x, y))$ instead of $= -\eta(y, y + (\lambda_1 - \lambda_2\eta(x, y)))$]. What is used to obtain the third equality? Setting $y' := y + \lambda_1\eta(x, y)$, the expression on the third line becomes $\eta(y', y' + \eta(y, y + (\lambda_1 - \lambda_2)\eta(x, y)))$. In order to get the expression on the fourth line (using Condition C directly), we should have $\eta(y'', y'' + \eta(y, y''))$ with $y'' := y + (\lambda_1 - \lambda_2)\eta(x, y)$. Is $y' = y''$? In fact, $y' = y''$ if and only if $\eta(x, y) = 0$ or $\lambda_2 = 0$.

Maybe (1) is true whenever Condition C holds, but some additional arguments must be provided.

I have no purpose to mention all doubtful sentences or statements in articles about invexity, but the majority I had occasion to browse are like that.

3 About the Triviality of Results and Generalizations

Another problem with invexity is given by the triviality of some results or generalizations. Let us mention some of them found in recent articles published in prestigious journals.

It is well known that for a Gâteaux differentiable function $f : D \to \mathbb{R}$ with D an open subset of a normed vector space X (but X could be a topological vector space), for any (distinct) points $a, b \in D$ with $[a, b] := \{\lambda a + (1 - \lambda)b \mid \lambda \in [0, 1]\} \subset D$, there exists $c \in\,]a, b[\, := \{\lambda a + (1 - \lambda)b \mid \lambda \in (0, 1)\}$ such that $f(b) - f(a) = \nabla f(c)(b - a)$ (the proof being immediate using the real-valued function φ defined by $\varphi(t) := f(ta + (1 - t)b)$ for those $t \in \mathbb{R}$ with $ta + (1 - t)b \in D$). What are the main results obtained in [5]? I don't speak about Theorems 11 and 12 which are just rewritten definitions of convexity and pre-invexity (in Theorem 11 of [5] no need to have the differentiability of f). Let us quote Theorem 14 in [5]:

Theorem 14 *Let $S \subset R^n$ be a nonempty invex set with respect to $\eta : S \times S \to R^n$, and P_{ab} be an arbitrary η-path contained in* int S. *Moreover, we assume that $f : S \to R$ is defined on S and differentiable on* int S. *Then, for any $a, b \in S$, there exists $c \in P^0_{ab}$ such that the following relation*

$$f\bigl(a + \eta(b, a)\bigr) - f(a) = \bigl[\eta(b, a)\bigr]^T \nabla f(c) \tag{8}$$

holds.

Because (c.f. Definition 5 in [5]) $P_{ab} := [a, a+\eta(a,b)]$ and $P^0_{ab} :=]a, a+\eta(a,b)[$, we see that Theorem 14 in [5] is an immediate consequence of the usual mean-value theorem mentioned above. (Note that it is not said what kind of differentiability is asked for f—Gâteaux or Fréchet.) Probably the next paper will deal with such a result in infinite dimensional spaces, then with α–η invex functions (mentioned below). Theorem 17 in [5] deals with a Taylor's expansion (of order 2) for f. Other "important" results (Theorems 21, 22) are immediate consequences of known results for differentiable functions of one real variable. They could constitute easy exercises for students following a first course in analysis.

Another example in this sense is provided by [8]. As seen in the title of [8], there is some G there. What is it? It is a function defined on a certain set $A \subset \mathbb{R}$ with values in $\mathbb{R}$ which is increasing ($s, t \in A$, $s < t \Rightarrow G(s) < G(t)$), and moreover, G is differentiable. In fact, G is defined on the image of a real-valued function f defined in its turn on an η-invex set $X \subset \mathbb{R}^n$. (By the way, if $A = I_f(X)$ is $\{0, 1\}$, what does differentiability of G mean?) One defines G-invex and G-pre-invex functions. Let us quote Definition 3 in [8]:

> **Definition 3** Let X be a nonempty invex (with respect to η) subset of R^n and $f : X \to R$ be a differentiable function defined on X. Further, we assume that there exists a differentiable real-valued increasing function $G : I_f(X) \to R$. Then f is said to be (strictly) G-invex at $u \in X$ on X with respect to η if there exists a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X$ $(x \neq u)$,
>
> $$G\big(f(x)\big) - G\big(f(u)\big) \geq G'\big(f(u)\big)\nabla f(u)\eta(x,u) \qquad (>). \tag{2}$$
>
> If (2) is satisfied for any $u \in X$ then f is G-invex on X with respect to η.

Taking into account that, for f (Fréchet) differentiable, one has $\nabla h(u) = G'(f(u))\nabla f(u)$, where $h := G \circ f$, the inequality above says that $h(x) - h(u) \geq \nabla h(u)\eta(x,u)$. Having this inequality for all $x, u \in X$ means that h is invex. So, one can simply say that f is G-invex (at u) iff $G \circ f$ is invex (at u). This simple remark is not made in [8], but one has (quoted from [8]):

> We remark that the G-invexity assumption generalizes a hypothesis of Avriel et al. [6], Avriel [7], Hanson [11] and Antczak [3] for differentiable functions. Thus, the following remarks are true:
>
> *Remark 5* In the case when $\eta(x,u) = x - u$, we obtain a definition of a differentiable G-convex function introduced Avriel et al. [6].
>
> *Remark 6* Every invex function with respect to η introduced by Hanson [11] is G-invex with respect to the same function η, where $G : I_f(X) \to R$ is defined by $G(a) \equiv a$. The converse result is, in general, not true (see also Remark 13 and Example 14).
>
> *Remark 7* Every r-invex function with respect to η introduced by Antczak [1,3] is G-invex with respect to the same function η, where $G : I_f(X) \to R$ is defined by $G(a) = e^{ra}$, where r is any finite real number.

(However, note that, for $r \leq 0$, the function G defined by $G(a) = e^{ra}$ is not increasing.)

It is suggestive to quote also the definition a G-pre-invex function (but probably the reader already guesses it):

Definition 9 Let X be a nonempty invex (with respect to η) subset of R^n. A function $f : X \to R$ is said to be (strictly) G-pre-invex at u on X with respect to η if there exist a continuous real-valued increasing function $G : I_f(X) \to R$ and a vector-valued function $\eta : X \times X \to R^n$ such that for all $x \in X$ $(x \neq u)$,

$$f\big(u + \lambda\eta(x, u)\big) \leq G^{-1}\big(\lambda G\big(f(x)\big) + (1 - \lambda)G\big(f(u)\big)\big) \qquad (<). \tag{3}$$

If (2) is satisfied for any $u \in X$ then f is G-pre-invex on X with respect to η.

Of course, the author does not (want to) observe that this means that $h := G \circ f$ is pre-invex at u. What does one obtain in Theorem 10 of [8]? One obtains that f is G-invex provided f and G are differentiable and f is G-pre-invex. I quote from p. 646 in [3]:

Recently, Pini (Ref. 6) showed that, if f is defined on an invex set $K \subseteq R^n$ and if it is preinvex and differentiable, then f is also invex with respect to η, i.e., $f(y) - f(x) \geq \eta(y, x)^T \nabla f(x)$.

Of course, in [8] one gives a detailed proof. On p. 646 of [3], one continues with:

But the converse is not true, in general. A counterexample was given in Ref. 6. However, Mohan and Neogy (Ref. 9) proved that a differentiable invex function is also preinvex under the following condition. *Condition C* ...

On p. 1620 of [8], one says:

The converse result is not true in general, that is, there exist G-invex functions with respect to η which are not G-pre-invex with respect to the same function η. To prove the converse theorem, the function η should satisfy the following Condition C (see [16]). *Condition C* ...

Of course, one states Theorem 11 and gives a detailed proof. As a conclusion for paper [8]: The function f is G-"..." iff $G \circ f$ is "...". If an existing result holds for "..." then in [8] one has a result for G-"..." with detailed proof. And this is published in a prestigious journal.

The case of [5] and [8] is not singular. Let us have a look at [9] and its follower [10]. Let us first quote two interesting phrases from [9]:

In recent years, the concept of convexity has been generalized and extended in several directions using *novel and innovative* techniques ...

and

Motivated and inspired by the research going on in this *fascinating field*, we introduce a new class of generalized functions.

Let us quote again from p. 698 of [9]:

> Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm, respectively. Let $F : K \to H$ and $\eta(\cdot,\cdot) : K \times K \to R$ be continuous functions. Let $\alpha : K \times K \to R \setminus \{0\}$ be a bifunction. First of all, we recall the following well-known results and concepts.
>
> **Definition 2.1** Let $u \in K$. Then the set K is said to be α-invex at u with respect to $\eta(\cdot,\cdot)$ and $\alpha(\cdot,\cdot)$, if, for all $u, v \in K, t \in [0,1]$, $u + t\alpha(v,u)\eta(v,u) \in K$. K is said to be an α-invex set with respect to η and α, if K is α-invex at each $u \in K$. The α-invex set K is also called $\alpha\eta$-connected set. Note that the convex set with $\alpha(v,u) = 1$ and $\eta(v,u) = v - u$ is an invex set, but the converse is not true.

First note that $u + t\alpha(v,u)\eta(v,u)$ above does not make sense if $H \neq \mathbb{R}$ because $u \in H$ and $t\alpha(v,u)\eta(v,u) \in \mathbb{R}$; next, if $\eta(\cdot,\cdot) : K \times K \to H$ (as in [10]), then K is an α-invex set with respect to η if and only if K is η'-invex, where $\eta' := \alpha\eta$ (apparently not observed in [9, 10]). Of course, in Definition 2.2 of [9], one says:

> The function F on the α-invex set K is said to be α-preinvex with respect to α and η, if $F(u + t\alpha(v,u)\eta(v,u)) \le (1-t)F(u) + tF(v), \forall u, v \in K, t \in [0,1]$,

that is, I say, F is η'-preinvex (however, one must take $F : K \to \mathbb{R}$ as in [10] instead of $F : K \to H$). In a similar way, one obtains the corresponding definitions for "α-invex" replaced by "quasi α-preinvex" (see Definition 2.3 in [9]), "logarithmic α-preinvex" (see Definition 2.4 in [9]), "pseudo α-preinvex" (see Definition 2.5 in [9]) from the definitions without α. (Note the interesting inequality $\max\{F(u), F(v)\} < \max\{F(u), F(v)\}$ from the displayed relation after Definition 2.4 in [9].) Maybe the next one is an exception:

> **Definition 2.6** A differentiable function F on K is said to be an α-invex function with respect to α and η, if
>
> $$F(v) - F(u) \ge \langle \alpha(v,u)F'(u), \eta(v,u) \rangle, \quad \forall u, v \in K,$$
>
> where $F'(u)$ is the differential of F at $u \in K$. *The concepts of the α-invex and α-preinvex functions have played very important role in the development of convex programming*; *see [6,7]*. Note that for $\alpha(v,u) = 1$, Definition 2.6 is mainly due to Hanson [1].

Unfortunately, even in this case, F is α-invex with respect to α and η if and only if F is η'-invex. What is new and surprising for me is the emphasized text above.

Similar remarks are valid for the notions of "$\alpha\eta$-monotone", "strictly $\alpha\eta$-monotone", "$\alpha\eta$-pseudomonotone", "quasi $\alpha\eta$-monotone", "strictly $\alpha\eta$-pseudomonotone" referred to an operator $T : K \to H$ (defined in Definition 2.7 of [9]).

However, there are some notions which do not correspond to those for $\eta' := \alpha\eta$. These are those containing the word "strongly" in their definition: "strongly $\alpha\eta$-monotone" and "strongly $\alpha\eta$-pseudomonotone" operators (see Definition 2.7 in [9]) as well as "strongly α-preinvex" (see Definition 2.8 in [9]), "strongly α-invex" (see Definition 2.9 in [9]), "strongly pseudo $\alpha\eta$-invex" (see Definition 2.10 in [9]) and "strongly quasi α-invex" (see Definition 2.11 in [9]) functions. The results which refer to these notions are Theorems 3.1–3.5 in [9]. I do not aim at verifying the correctness of these results (however, see Example 6.1 in [10]), but some of them are probably not true having in mind that Theorems 6.1 and 6.4 in [10] give alternative formulations for the sufficiency parts of Theorems 3.2 and 3.5 in [9], respectively. What I want to point out are the following facts:

1. If $\eta(u,u)=0$ for some $u\in K$, then there do not exist pseudo α-preinvex, strictly α-invex and strictly pseudo α-invex functions with respect to α and η, as well as strictly $\alpha\eta$-monotone and strictly pseudo $\alpha\eta$-monotone operators. Note that, if α and η satisfy Condition C on p. 702 of [9] or condition (ii) in Theorem 6.1 of [10], then $\eta(u,u)=0$ for every $u\in K$.
2. In some proofs of the statements in [9] and [10], one uses the relation $g(1)-g(0)=\int_0^1 g'(t)\,dt$, where g is a real-valued differentiable function on a subset of $\mathbb{R}$ containing [0, 1]. In fact, $g(t):=F(u+t\alpha(v,u)\eta(v,u))$ for $t\in[0,1]$, where F is differentiable. It is a well-known fact that the formula $g(1)-g(0)=\int_0^1 g'(t)\,dt$ might not be true if g' is not Riemann integrable on [0, 1]. As an example, take $g(t):=t^2\sin(t^{-2})$ for $t\in\,]0,1]$, $g(0):=0$.
3. In Theorems 6.1–6.4 of [10], one uses the condition

$$\alpha\bigl(u,u+t\alpha(v,u)\eta(v,u)\bigr)=t\alpha(v,u),\quad \forall u,v\in K,\ t\in[0,1].$$

   Taking $t=0$, this implies that $\alpha(u,u)=0$ for every $u\in K$, contradicting the assumption made before Definition 1.1 in [10] that α takes its values in $\mathbb{R}\setminus\{0\}$. This shows that the domain of applicability of Theorems 6.1–6.4 in [10] is the empty set.

4 Conclusions

In this note, we pointed out that several papers published in prestigious journals contain important drawbacks in the formulation of the notions and in the statements of the results, as well as very serious mistakes in the proofs. Furthermore, there are many trivial generalizations of notions and results. In this sense, it is useful to mention that there are several reviews in Mathematical Reviews and Zentralblatt für Mathematik which are concordant with our opinions; let us cite the reviews *MR1989930 (2004e:90091)* (for [6], by S. Komlosi), in which it is mentioned explicitly that Remark 2.3 of [6] is false by giving a counterexample; *Zbl 1094.26008 Noor, Muhammad Aslam On generalized preinvex functions and monotonicities. (English) [J] JIPAM, J. Inequal. Pure Appl. Math. 5, No. 4, Paper No. 110, 9 p., electronic only (2004). ISSN 1443-5756* (by J. E. Martínez-Legaz), in which it is mentioned that all the results in the paper follow from a simple observation; *Zbl 1096.26006 Noor, Muhammad Aslam; Noor, Khalida Inayat On strongly generalized preinvex functions. (English) [J] JIPAM, J. Inequal. Pure Appl. Math. 6, No. 4, Paper No. 102, 8 p., electronic only (2005). ISSN 1443-5756* (by J. E. Martínez-Legaz), in which, besides other remarks, a definition is mentioned which does not make sense; *Zbl 1093.26006* (for [9], by N. Hadjisavvas), where it is mentioned that "Many other notions and properties introduced in this paper can be derived in the same way from the usual generalized invexity notions that can be found in other papers in the field. When this is not the case, mistakes occur frequently". In conclusion, we consider that there are too many papers related to invexity, much more than the domain deserves. We consider that the editors of mathematical journals have to pay much more attention when accepting to publish such papers, taking into account at least the lack of criticism in the Invexity Community.

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