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# An Existence Theorem for a Class of Infinite Horizon Optimal Control Problems

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**Abstract** In this paper, we deal with infinite horizon optimal control problems involving affine-linear dynamics and prove the existence of optimal solutions. The innovation of this paper lies in the special setting of the problem, precisely in the choice of weighted Sobolev and weighted Lebesgue spaces as the state and control spaces, respectively, which turns out to be meaningful for various problems. We apply the generalized Weierstraß theorem to prove the existence result. A lower semicontinuity theorem which is needed for that is shown under weakened assumptions.

**Keywords** Infinite horizon · Optimal control · Existence theorem · Weighted Sobolev and Lebesgue spaces · Weak lower semicontinuity

## 1 Introduction

The intensive theoretical investigation of infinite horizon optimal control problems began in the 1970s. Since then many results concerning necessary optimality conditions, e.g. [1–3], and sufficient optimality conditions, e.g. [4, 5], were established. The existence results were paid pretty much attention as well, so that various results came to light, see [6–10] and others. For various new applications of this class of problems we refer to [11, 12]. As was described in [5] and [13], the choice of Sobolev and Lebesgue spaces as state and control spaces, respectively, is unsatisfactory as some very simple examples showed. In [5] and [13] it was also shown that the weighted Sobolev spaces and weighted Lebesgue spaces are much more reasonable to work with, so that they are chosen as the functional spaces in the problem setting of

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the present paper. Another key point in the new problem formulation is the presence of a special pointwise state constraint which has an impact on the boundedness of the feasible set in the norms of underlying spaces and consequently on the existence of an optimal solution. Throughout the paper the integral in the improper integral objective is understood in Lebesgue sense. Thus we obtain a new class of infinite horizon optimal control problems for which it is important to find some existence results.

Main results of this paper include a lower semicontinuity theorem for integral functionals involving a Lebesgue integral with respect to weak topology and an existence theorem itself. The necessity of distinguishing between different interpretations of integral in the improper integral objective of infinite horizon control problems was addressed in [14]. The main difference from the existence theorems presented in [6, 7] is in the setting of the optimal control problem itself, namely in the choice of weighted Sobolev and weighted Lebesgue spaces as the state and control spaces, respectively. The corresponding choice of the weak topology of these spaces for the proof of lower semicontinuity of integral functionals makes the direct comparison of the existence result of this paper with those of [6, 7] rather difficult. Nevertheless, while using these spaces we followed the well known heuristics, cf. [15], which says that the vector space for the adjoint function should be a dual space of a space in which the feasible set of the optimal control problem has a nonempty interior. Motivated by this the considering of the reflexive Banach space, which the considered weighted Sobolev space is, is more comfortable than working with the space of locally absolutely continuous functions, which is considered in the named papers. In this case the desired relation between the state space and the space for the adjoint would be lost.

One of the classical tools for proving the existence of an optimal solution for a control problem on some compact interval via the generalized Weierstraß theorem is the compactness of the embedding of the weighted Sobolev space into the weighted Lebesgue space of the same index. However, in order to achieve the compactness of this embedding on an unbounded interval, it is necessary and sufficient to choose a weight function satisfying the so called "decay at infinity"-condition, cf. [16, 17], which was also applied for the proof of lower semicontinuity in [13]. In the present paper we make use of a pointwise state constraint instead of this restrictive condition, which requires some new technique of proof.

The present paper is structured as follows. Section 2 includes the problem formulation, definitions and main assumptions. Section 3 is devoted to the proof of a lower semicontinuity theorem for integral functionals in weak topology of weighted Sobolev and weighted Lebesgue spaces. In the next section we prove an existence theorem for the introduced class of problems. Section 5 presents an example illustrating the applicability of proved theorems. In Sect. 6 we finish with conclusions.

## 2 Preliminaries

### 2.1 Definitions and Notations

Let us introduce $\mathbb{B}$ as a measurable set in s-dimensional Euclidean space. We denote by $\mathcal{M}^n(\mathbb{B})$, $L_p^n(\mathbb{B})$ and $C^{0,n}(\mathbb{R}^+)$ the spaces of all vector functions $x : \mathbb{B} \to \mathbb{R}^n$ with

Lebesgue measurable, in the $p$th power Lebesgue integrable or continuous components, respectively ([18], p. 146 and pp. 285 ff.; [19], pp. 228 ff.). For $n = 1$, we suppress the superscript in the labels of the spaces. We write $[0, \infty[= \mathbb{R}^+$.

**Definition 2.1**

(a) A continuous function $\nu : \mathbb{R}^+ \to \mathbb{R}^+ \backslash \{0\}$ is called a *weight function*.
(b) A weight function $\nu$ is called a *density function* iff it satisfies

$$L\text{-}\int_0^\infty \nu(t)\, dt < \infty. \tag{1}$$

*Remark 2.1* The notation $L\text{-}\int$ stands for the Lebesgue interpretation of the integral. The Lebesgue measure will be denoted by $\mu$, while the measure induced by a density function $\nu$ will be denoted by $\mu_\nu$. We also remark that because of the positivity and continuity of the function $\nu$ the sets of measure zero are the same for both measures, so that the meanings of $\mu$-a.e. and $\mu_\nu$-a.e. are in fact identical. Therefore, throughout the paper we just use the abbreviation "a.e." without concretizing the corresponding measure.

**Definition 2.2**

(a) By means of a weight function $\nu$, we define for any $1 \le p < \infty$ the *weighted Lebesgue space*

$$L_p^n(\mathbb{B}, \nu) := \left\{ x \in \mathcal{M}^n(\mathbb{B}) \,\middle|\, \left( L\text{-}\int_{\mathbb{B}} \left|x(t)\right|^p \nu(t)\, dt \right)^{1/p} < \infty \right\} \tag{2}$$

as well as

$$L_\infty^n(\mathbb{B}, \nu) := \left\{ x \in \mathcal{M}^n(\mathbb{B}) \,\middle|\, \operatorname*{ess\,sup}_{t \in \mathbb{B}} \left|x(t)\nu(t)\right| < \infty \right\} \tag{3}$$

and
(b) the *weighted Sobolev space*

$$W_p^{1,n}\big(\mathbb{R}^+, \nu\big) := \big\{ x \in \mathbb{M}^n\big(\mathbb{R}^+\big) \mid x \in L_p^n\big(\mathbb{R}^+, \nu\big), \dot{x} \in L_p^n\big(\mathbb{R}^+, \nu\big) \big\}, \tag{4}$$

where $\dot{x}$ denotes the distributional derivative; see [20], p. 11 f. Equipped with the norm

$$\|x\|_{W_p^{1,n}(\mathbb{R}^+,\nu)} = \|x\|_{L_p^n(\mathbb{R}^+,\nu)} + \|\dot{x}\|_{L_p^n(\mathbb{R}^+,\nu)}, \tag{5}$$

$W_p^{1,n}(\mathbb{R}^+, \nu)$ becomes a Banach space (this can be confirmed analogously to [20], p. 19, Theorem 3.6).

**Lemma 2.1** *Let $\nu$ be a density function. Then any linear, continuous functional $\varphi \colon L_p(\mathbb{R}^+, \nu) \to \mathbb{R}$ can be represented by a function $y \in L_q(\mathbb{R}^+, \nu)$ with* $\frac{1}{p} + \frac{1}{q} = 1$

*if* $1 < p < \infty$ *and* $q = \infty$ *if* $p = 1$:

$$\langle \varphi, x \rangle = L\text{-}\int_0^\infty y(t)x(t)\nu(t)\,dt, \quad \forall x \in L_p(\mathbb{R}^+, \nu). \tag{6}$$

*We can apply* [19], *p.* 287, *Theorem* 3.2, *since the measure generated by the density function* $\nu$ *is* $\sigma$*-finite on* $\mathbb{R}^+$.

*Remark 2.2* The continuity of the weight function $\nu$ in Definition 2.1 is essential, since in the proofs of the main theorems we are going to apply the Rellich–Kondrachov embedding theorem for non-weighted Sobolev spaces, which remains valid also for the weighted Sobolev spaces if the weight function $\nu$ is continuous.

### 2.2 Infinite Horizon Optimal Control Problem

The infinite horizon control problem consists in minimizing the integral objective

$$J_\infty(x, u) := L\text{-}\int_0^\infty r\big(t, x(t), u(t)\big)\widetilde{\nu}(t)\,dt \tag{7}$$

with respect to all pairs

$$(x, u) \in W_p^{1,n}\big(\mathbb{R}^+, \nu\big) \times L_p^m\big(\mathbb{R}^+, \nu\big), \quad 1 < p < \infty, \tag{8}$$

governed by the differential equation

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) \quad \text{a.e. on } \mathbb{R}^+, \tag{9}$$

$$x(0) = x_0, \tag{10}$$

and satisfying the constraints

$$\big|x(t)\big| \le \beta(t), \quad \text{with } \beta(\cdot) \in L_p^1\big(\mathbb{R}^+, \nu\big), \tag{11}$$

$$u(t) \in U \subset \mathbb{R}^m \quad \text{a.e. on } \mathbb{R}^+. \tag{12}$$

Hereby $U$ denotes a compact convex subset of $\mathbb{R}^m$, $\nu$ is a density function and $\widetilde{\nu}$ is a weight function as in Definition 2.1. The functions $x$ and $u$ are called the state and the control function, respectively. The integral in (7) is understood in Lebesgue sense. We refer to problem (7)–(12) as to the problem $(P_\infty)$. The choice of such sophisticated spaces as in (8) is motivated by the fact that very often the solution trajectory $x$, as well as the control function $u$, do not belong to any non-weighted Sobolev space over the unbounded interval $[0, \infty[$; cf. [13].

*Remark 2.3*

(a) State and control functions are weighted by the same density function $\nu$ which seems to be natural and becomes clear if one considers the simplest state equation $\dot{x}(t) = u(t)$, where the functions on both hand sides are from the same weighted Lebesgue space.

(b) Since the weight function $\nu$ is a density, the inclusion $L_\infty(\mathbb{R}^+) \subseteq L_p(\mathbb{R}^+, \nu)$ holds for a fixed $1 \leq p < \infty$ in contrast to the $L_\infty(\mathbb{R}^+) \nsubseteq L_p(\mathbb{R}^+)$ for non-weighted spaces. This allows to consider a larger space for the control function.

We now introduce the following assumptions:

**Assumption 2.1** *The function $r : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies the following conditions: $r(\cdot, \xi, v)$ is continuous for all $(\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m$, $\nabla_\xi r(t, \cdot, v)$ is continuous for all $(t, v) \in \mathbb{R}^+ \times \mathbb{R}^m$, $\nabla_v r(t, \xi, \cdot)$ is continuous for all $(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n$, $r(t, \xi, \cdot)$ is convex for all $(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n$.*

**Assumption 2.2** *$r(t, \xi, v)$ satisfies the growth condition*

$$\left|r(t, \xi_1, \ldots, \xi_n, v_1, \ldots, v_m)\right| \leq \frac{A_1(t)}{\widetilde{\nu}(t)} + B_1 \cdot \sum_{i=1}^{n} \frac{|\xi_i|^p}{\widetilde{\nu}(t)} \cdot \nu(t) + B_1 \cdot \sum_{k=1}^{m} \frac{|v_k|^p}{\widetilde{\nu}(t)} \cdot \nu(t),$$
$$\forall (t, \xi, v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \tag{13}$$

*with a function $A_1 \in L_1(\mathbb{R}^+)$ and a constant $B_1 > 0$.*

**Assumption 2.3** *The gradient $\nabla_v r(t, \xi, v)$ satisfies the growth condition*

$$\left|\nabla_v r(t, \xi_1, \ldots, \xi_n, v_1, \ldots, v_m) \cdot \frac{\widetilde{\nu}(t)}{\nu(t)}\right|$$
$$\leq A_2(t)\nu(t)^{-1/q} + B_2 \cdot \sum_{i=1}^{n} |\xi_i|^{p/q} + B_2 \cdot \sum_{k=1}^{m} |v_k|^{p/q},$$
$$\forall (t, \xi, v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \tag{14}$$

*with $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$ with a function $A_2 \in L_q(\mathbb{R}^+)$ and a constant $B_2 > 0$.*

**Assumption 2.4** *Let the function $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ from (9) be defined as follows:*

$$f\big(t, x(t), u(t)\big) := A\big(t, x(t)\big) + B\big(t, x(t)\big)u(t), \tag{15}$$

*where the elements of the matrix $B(\cdot, x(\cdot))$ satisfy the growth conditions*

$$\left|B_{ij}(t, \xi_1, \ldots, \xi_n)\right| \leq A_{3ij}(t)\big(\nu(t)\big)^{-1/q} + B_{3ij} \sum_{k=1}^{n} |\xi_k|^{p/q},$$
$$\forall (t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n \tag{16}$$

*for all $(i, j) : i \in \{1, \ldots, n\}; j \in \{1, \ldots, m\}$. Hereby $\frac{1}{p} + \frac{1}{q} = 1$, $A_{3ij} \in L_q(\mathbb{R}^+)$, $B_{3ij} > 0$. Besides, it is assumed that there exist nonnegative constants $C_1$, $C_2$, $C_3$, $C_4$ such that*

$$\left|A(t, \xi) + B(t, \xi)v\right| \leq C_1 + C_2|\xi| + C_3|\xi| \cdot |v| + C_4|v|,$$
$$\forall (t, \xi, v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m. \tag{17}$$

*Remark 2.4*

(a) The condition (16) as well as the growth conditions from Assumption 2.3 can be written in a slightly different way, namely

$$\left|B_{ij}(t,\xi_1,\ldots,\xi_n)\right| \leq A_{3ij}(t) + B_{3ij}\sum_{k=1}^{n}|\xi_k|^{p/q}, \quad \forall(t,\xi)\in\mathbb{R}^+\times\mathbb{R}^n \quad (18)$$

for all $(i,j): i\in\{1,\ldots,n\};\ j\in\{1,\ldots,m\}$. Hereby $A_{3ij}\in L_q(\mathbb{R}^+,\nu)$, $B_{3ij}>0$, $\frac{1}{p}+\frac{1}{q}=1$ hold as before.

(b) The growth conditions in Assumptions 2.2 and 2.3 are in such a manner that if they are satisfied for the integrand $r$ and some weight function $\widetilde{\nu}$, then they are also satisfied for the integrand $r_1 := r\cdot\widetilde{\nu}$ and the weight function $\widetilde{\nu}_1(t)\equiv 1$ and vice versa. Consequently, it is not necessary to separate some weight function $\widetilde{\nu}$ in the integral functional $J_\infty$ and one could just set $\widetilde{\nu}(t)\equiv 1$. However, in view of numerous economical and biological applications of infinite horizon optimal control problems which contain some special weight function, such as discount rate $e^{-\rho t}$, in the integrand of the objective, we prefer here to explicitly introduce the weight function $\widetilde{\nu}$ in the integral functional $J_\infty$.

**Definition 2.3**

(a) A pair $(x,u)$ is called *admissible* for the problem $(P_\infty)$, if it satisfies the conditions (8)–(12) and the Lebesgue integral in (7) exists and has a finite value.

(b) An admissible pair $(x^*,u^*)$ is called *global optimal solution* of the problem $(P_\infty)$, if for any admissible pair $(x,u)$ the inequality $J_\infty(x^*,u^*)\leq J_\infty(x,u)$ holds.

## 3 Lower Semicontinuity Result

**Theorem 3.1** *Let $1<p<\infty$ be given. Furthermore, let the integrand $r:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, a density function $\nu$, a weight function $\widetilde{\nu}$ satisfy Assumptions 2.1–2.3. With a function $\beta\in L_p(\mathbb{R}^+,\nu)$ and a compact nonempty set $U\subset\mathbb{R}^m$ we define the sets*

$$\mathbb{X}_p := \left\{x\in W_p^{1,n}\left(\mathbb{R}^+,\nu\right)\mid \left|x(t)\right|\leq\beta(t),\forall t\in\mathbb{R}^+\right\}, \quad (19)$$

$$\mathbb{U}_p := \left\{u\in L_p^m\left(\mathbb{R}^+,\nu\right)\mid u(t)\in U \text{ a.e. on } \mathbb{R}^+\right\}. \quad (20)$$

*Then, the integral functional $J_\infty$ from (7) is lower semicontinuous on the set $\mathbb{X}_p\times\mathbb{U}_p$ with respect to the weak topology of spaces $W_p^{1,n}(\mathbb{R}^+,\nu)$ and $L_p^m(\mathbb{R}^+,\nu)$, i.e. the inequality*

$$\begin{aligned} J_\infty(x_0,u_0) &= L\text{-}\int_0^\infty r\big(t,x_0(t),u_0(t)\big)\tilde{\nu}(t)\,dt \\ &\leq \liminf_{N\to\infty} L\text{-}\int_0^\infty r\big(t,x_N(t),u_N(t)\big)\tilde{\nu}(t)\,dt = \liminf_{N\to\infty} J_\infty(x_N,u_N) \quad (21)\end{aligned}$$

*holds for any weak convergent sequences*

$$\{x_N\} \rightharpoonup x_0\big(in\ W_p^{1,n}\big(\mathbb{R}^+,\nu\big)\big), \qquad \{u_N\} \rightharpoonup u_0\big(in\ L_p^m\big(\mathbb{R}^+,\nu\big)\big), \quad N \to \infty. \tag{22}$$

*Remark 3.1* We prove the lower semicontinuity of the integral functional on the set $\mathbb{X}_p \times \mathbb{U}_p$, since we need it merely on the admissible set of optimal control problem $(P_\infty)$. The existence of a majorant function $\beta \in L_p(\mathbb{R}^+,\nu)$ and of a nonempty compact set $U \subset \mathbb{R}^m$ results from the statement of the optimal control problem.

*Proof* Consider an arbitrary sequence $\{(x_N,u_N)\}_{N=1}^\infty \subset (\mathbb{X}_p \times \mathbb{U}_p)$ such that $\{x_N\} \rightharpoonup x_0$ (in $W_p^{1,n}(\mathbb{R}^+,\nu)$) and $\{u_N\} \rightharpoonup u_0$ (in $L_p^m(\mathbb{R}^+,\nu)$). We show the validity of the inequality (21).

The growth condition (13), the definition of the set $\mathbb{X}_p$ and the compactness of the set $U$ imply the existence of a function $\alpha \in L_1(\mathbb{R}^+,\widetilde{\nu})$ such that all pairs of functions $(x,u) \in \mathbb{X}_p \times \mathbb{U}_p$ satisfy the inequality $r(t,x(t),u(t)) \geq \alpha(t)$ for all $t > 0$. Therefore, without any loss of generality, we suppose that the function $r$ has non-negative values. Otherwise we could consider the function $\widetilde{r}$ defined by

$$\widetilde{r}\big(t,x(t),u(t)\big) := r\big(t,x(t),u(t)\big) - \alpha(t) \geq 0, \quad \forall t > 0.$$

The convexity of $r(t,\xi,\cdot)$, due to Assumption 2.1, yields

$$\begin{aligned} &r\big(t,x_N(t),u_N(t)\big)\widetilde{\nu}(t) \\ &\quad \geq r\big(t,x_N(t),u_0(t)\big)\widetilde{\nu}(t) + \nabla_v^T r\big(t,x_N(t),u_0(t)\big)(u_N-u_0)(t)\widetilde{\nu}(t) \end{aligned} \tag{23}$$

pointwise on $\mathbb{R}^+$ for all $\{x_N,u_N\}$ satisfying $x_N \rightharpoonup x_0$ in $W_p^{1,n}(\mathbb{R}^+,\nu)$ and $u_N \rightharpoonup u_0$ in $L_p^m(\mathbb{R}^+,\nu)$. We rewrite this inequality as follows:

$$\begin{aligned} &r\big(t,x_N(t),u_N(t)\big)\widetilde{\nu}(t) \\ &\quad \geq r\big(t,x_N(t),u_0(t)\big)\widetilde{\nu}(t) \\ &\qquad + \nabla_v^T r\big(t,x_0(t),u_0(t)\big)\big(u_N(t)-u_0(t)\big)\widetilde{\nu}(t) \\ &\qquad + \big[\nabla_v^T r\big(t,x_N(t),u_0(t)\big) - \nabla_v^T r\big(t,x_0(t),u_0(t)\big)\big] \cdot \big(u_N(t)-u_0(t)\big)\widetilde{\nu}(t) \end{aligned} \tag{24}$$

and integrate it over the half-line. The inequality sign remains valid:

$$\begin{aligned} &\int_0^\infty r\big(t,x_N(t),u_N(t)\big)\widetilde{\nu}(t)\,dt \\ &\quad \geq \int_0^\infty r\big(t,x_N(t),u_0(t)\big)\widetilde{\nu}(t)\,dt \\ &\qquad + \int_0^\infty \nabla_v^T r\big(t,x_0(t),u_0(t)\big)\big(u_N(t)-u_0(t)\big)\widetilde{\nu}(t)\,dt \\ &\qquad + \int_0^\infty \big[\nabla_v^T r\big(t,x_N(t),u_0(t)\big) - \nabla_v^T r\big(t,x_0(t),u_0(t)\big)\big]\big(u_N(t)-u_0(t)\big)\widetilde{\nu}(t)\,dt. \end{aligned} \tag{25}$$

We now should ensure that all three integrals on the right hand side of the last inequality exist and have finite values. The first one exists due to Assumption 2.2, the second and the third summands are finite by Assumption 2.3. Taking the limit inferior of the both sides of the inequality (25) and having in mind that for arbitrary real number sequences $(a_N)$, $(b_N)$, $(c_N)$ the inequality

$$\liminf_{N\to\infty}(a_N+b_N+c_N) \geq \liminf_{N\to\infty}(a_N)+\liminf_{N\to\infty}(b_N)+\liminf_{N\to\infty}(c_N) \tag{26}$$

holds, we obtain

$$l := \liminf_{N\to\infty}\int_0^\infty r\big(t, x_N(t), u_N(t)\big)\widetilde{\nu}(t)\,dt \geq l_1+l_2+l_3 \tag{27}$$

with notations

$$l_1 := \liminf_{N\to\infty}\int_0^\infty r\big(t, x_N(t), u_0(t)\big)\widetilde{\nu}(t)\,dt, \tag{28}$$

$$l_2 := \liminf_{N\to\infty}\int_0^\infty \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\big(u_N(t)-u_0(t)\big)\widetilde{\nu}(t)\,dt, \tag{29}$$

$$l_3 := \liminf_{N\to\infty}\int_0^\infty \big[\nabla_v^T r\big(t, x_N(t), u_0(t)\big)-\nabla_v^T r\big(t, x_0(t), u_0(t)\big)\big] \times \big(u_N(t)-u_0(t)\big)\widetilde{\nu}(t)\,dt. \tag{30}$$

Our aim now is to find suitable estimates from below for $l_1, l_2$ and $l_3$.

**Estimate for** $l_1$: We choose a subsequence $\{x_m\}\subset\{x_N\}$ which provides the value of the limit inferior in (28), i.e. we have

$$\liminf_{N\to\infty}\int_0^\infty r\big(t, x_N(t), u_0(t)\big)\widetilde{\nu}(t) = \lim_{m\to\infty}\int_0^\infty r\big(t, x_m(t), u_0(t)\big)\widetilde{\nu}(t). \tag{31}$$

The restrictions of the functions $x_m$ on the interval $[0, 1]$ are elements of weighted Sobolev space $W_p^{1,n}([0,1], \nu|_{[0,1]})$. Then, by the embedding theorem of Rellich–Kondrachov for non-weighted Sobolev spaces, cf. [17], Theorem 6.2, p. 144, and by the positivity and continuity of the function $\nu$ we conclude the existence of continuous representants of the sequence $\{x_m\}$. It means there exist such a subsequence $\{x_{N^1}\}\subset\{x_m\}$ and a function $\hat{x}_1\in(W_p^{1,n}([0,1], \nu|_{[0,1]})\cap C^0[0,1])$ that the subsequence $\{x_{N^1}\}$ converges uniformly on $[0, 1]$ to the function $\hat{x}_1$:

$$x_{N^1} \rightharpoonup \hat{x}_1\big(\text{in } W_p^{1,n}\big([0,1], \nu|_{[0,1]}\big)\big), \tag{32}$$

$$\lim_{N^1\to\infty}\|x_{N^1}-\hat{x}_1\|_{C^0[0,1]} = 0. \tag{33}$$

In general we build the subsequences $\{x_{N^k}\}\subset\{x_{N^{k-1}}\}$, $k = 2, 3, \dots$ satisfying

$$x_{N^k} \rightharpoonup \hat{x}_k\big(\text{in } W_p^{1,n}\big([0,k], \nu|_{[0,k]}\big)\big), \tag{34}$$

$$\lim_{N^k\to\infty}\|x_{N^k}-\hat{x}_k\|_{C^0[0,k]} = 0. \tag{35}$$

By the construction, we have the coincidence $\hat{x}_k(t) = \hat{x}_{k-1}(t)$ on $[0, k-1]$. This leads to the limit function $\hat{x} \in (W^{1,n}_{p,\mathrm{loc}}(\mathbb{R}^+, \nu) \cap C^0(\mathbb{R}^+))$ whose restriction on $[0, k]$ coincides with $\hat{x}_k$. The uniqueness of the weak limit yields $\hat{x}(t) = x_0(t)$ for all $t \in \mathbb{R}^+$.

We now construct a diagonal sequence having following properties: From the first sequence $\{x_{N^1}\}$ we take the first element $x_{N_1^1}$ satisfying

$$\|x_{N_1^1} - \hat{x}_1\|_{C^0[0,1]} \le 1/2, \tag{36}$$

from $\{x_{N^2}\}$ we take the first element $x_{N_2^2}$ with $N_2^2 > N_1^1$

$$\|x_{N_2^2} - \hat{x}_2\|_{C^0[0,2]} \le 1/4, \tag{37}$$

and generally we take from the sequence $\{x_{N^k}\}$ the first element $x_{N_k^k}$ having $N_k^k > N_{k-1}^{k-1}$ and

$$\|x_{N_k^k} - \hat{x}_k\|_{C^0[0,k]} \le 1/2^k. \tag{38}$$

The constructed diagonal sequence $\{x_{N_k^k}\}$ converges pointwise everywhere on $[0, \infty[$ to the limit function $\hat{x} = x_0$. Since the integrand $r$ has nonnegative values, the Fatou lemma, cf. [18], p. 152, can be applied in order to bring the limit under the integral sign:

$$\begin{aligned} l_1 &= \lim_{N_k^k \to \infty} \int_0^\infty r\big(t, x_{N_k^k}(t), u_0(t)\big)\widetilde{\nu}(t)\, dt \\ &\ge \int_0^\infty \liminf_{N_k^k \to \infty} r\big(t, x_{N_k^k}(t), u_0(t)\big)\widetilde{\nu}(t)\, dt. \end{aligned} \tag{39}$$

Together with the estimate (39), the equation (31) and in consequence of the continuity of the function $r(t, \cdot, v)$ for all $(t, v) \in \mathbb{R}^+ \times \mathbb{R}^m$, one obtains the following chain of estimates:

$$\begin{aligned} l_1 &= \liminf_{N \to \infty} \int_0^\infty r\big(t, x_N(t), u_0(t)\big)\widetilde{\nu}(t)\, dt \\ &= \lim_{N_k^k \to \infty} \int_0^\infty r\big(t, x_{N_k^k}(t), u_0(t)\big)\widetilde{\nu}(t)\, dt \\ &\ge \int_0^\infty \liminf_{N_k^k \to \infty} r\big(t, x_{N_k^k}(t), u_0(t)\big)\widetilde{\nu}(t)\, dt \\ &= \int_0^\infty \lim_{N_k^k \to \infty} r\big(t, x_{N_k^k}(t), u_0(t)\big)\widetilde{\nu}(t)\, dt \\ &= L\text{-}\!\int_0^\infty r\big(t, x_0(t), u_0(t)\big)\widetilde{\nu}(t)\, dt. \end{aligned} \tag{40}$$

From this it follows that

$$l_1 \geq L\text{-}\int_0^\infty r\big(t, x_0(t), u_0(t)\big)\widetilde{\nu}(t)\,dt = J_\infty(x_0, u_0). \tag{41}$$

**Estimate for** $l_2$: We choose a subsequence $\{u_s\} \subset \{u_N\}$ which provides the value of the limit inferior in $l_2$, i.e. we have

$$l_2 = \lim_{s\to\infty} \int_0^\infty \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\big(u_s(t) - u_0(t)\big)\widetilde{\nu}(t)\,dt. \tag{42}$$

In view of the growth condition (14) stated in Assumption 2.3 and due to Theorem 25 from [21], p. 59, the Nemytskij operator $N(\cdot,\cdot)$ defined by

$$N(x_0, u_0)(t) = \big|\nabla_v r\big(t, x_0(t), u_0(t)\big)\big| \frac{\widetilde{\nu}(t)}{\nu(t)} \tag{43}$$

maps all the functions from the space $L_p^{n+m}(\mathbb{R}^+, \nu)$ into the space $L_q(\mathbb{R}^+, \nu)$. Particularly, we obtain

$$\big|\nabla_v r\big(\cdot, x_0(\cdot), u_0(\cdot)\big)\big| \frac{\widetilde{\nu}(\cdot)}{\nu(\cdot)} \in L_q\big(\mathbb{R}^+, \nu\big) \tag{44}$$

and can use the function $\nabla_v r(\cdot, x_0(\cdot), u_0(\cdot))\frac{\widetilde{\nu}(\cdot)}{\nu(\cdot)}$ as an inducing element of a linear continuous functional on the space $L_p^m(\mathbb{R}^+, \nu)$. Since $\{u_s\}$ is weakly convergent in $L_p^m(\mathbb{R}^+, \nu)$ to the function $u_0$, we arrive due to the definition of the weak convergence, cf. Definition 5.7 from [19], p. 261 f., at

$$l_2 = \lim_{s\to\infty} \int_0^\infty \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\frac{\widetilde{\nu}(t)}{\nu(t)}\big(u_s(t) - u_0(t)\big)\nu(t)\,dt = 0. \tag{45}$$

**Estimate for** $l_3$: Similarly as in the previous estimate, we choose a subsequence $\{x_q, u_q\} \subset \{x_N, u_N\}$ giving the value of the limit inferior in $l_3$, i.e. we have

$$\begin{aligned} l_3 &= \liminf_{N\to\infty} \int_0^\infty \big[\nabla_v^T r\big(t, x_N(t), u_0(t)\big) - \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\big] \\ &\quad \times \big(u_N(t) - u_0(t)\big)\widetilde{\nu}(t)\,dt \\ &= \lim_{q\to\infty} \int_0^\infty \big[\nabla_v^T r\big(t, x_q(t), u_0(t)\big) - \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\big] \\ &\quad \times \big(u_q(t) - u_0(t)\big)\widetilde{\nu}(t)\,dt. \end{aligned} \tag{46}$$

The subsequence $\{x_q, u_q\}$ will again be indexed by the index $N \in \mathbb{N}$. Since for $1 < p < \infty$ there exists a uniform majorant $\beta \in L_p(\mathbb{R}^+, \nu)$ for the elements of the weakly convergent sequence $\{x_N\}$ such that

$$\big|x_N(t)\big| \leq \beta(t), \quad \forall t \in \mathbb{R}^+, \tag{47}$$

we can apply the growth condition (14) in order to construct a uniform majorant $M : \mathbb{R}^+ \to \mathbb{R}^+$ for the family of functions

$$\left(\nabla_v r\big(t, x_N(t), u_0(t)\big) - \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\right)\frac{\widetilde{\nu}(t)}{\nu(t)}, \quad N \in \mathbb{N}. \tag{48}$$

Thus, the estimate

$$\begin{aligned}
&\left|\left(\nabla_v^T r\big(t, x_N(t), u_0(t)\big) - \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\right)\frac{\widetilde{\nu}(t)}{\nu(t)}\right| \\
&\quad \le \left|\nabla_v^T r\big(t, x_N(t), u_0(t)\big)\frac{\widetilde{\nu}(t)}{\nu(t)}\right| + \left|\nabla_v^T r\big(t, x_0(t), u_0(t)\big)\frac{\widetilde{\nu}(t)}{\nu(t)}\right| \\
&\quad \le A_2(t)\nu(t)^{-1/q} + B_2 \cdot \sum_{i=1}^{n} \left|x_N^i(t)\right|^{p/q} + B_2 \cdot \sum_{k=1}^{m} \left|u_0^k(t)\right|^{p/q} \\
&\qquad + A_2(t)\nu(t)^{-1/q} + B_2 \cdot \sum_{i=1}^{n} \left|x_0^i(t)\right|^{p/q} + B_2 \cdot \sum_{k=1}^{m} \left|u_0^k(t)\right|^{p/q}
\end{aligned} \tag{49}$$

with $A_2 \in L_q(\mathbb{R}^+)$, $B_2 > 0$ holds for any arbitrary number $N \in \mathbb{N}$ and $p \in ]1, \infty[$. The last estimate can be continued in the following way:

$$\begin{aligned}
&\left|\left(\nabla_v r\big(t, x_N(t), u_0(t)\big)^T - \nabla_v r\big(t, x_0(t), u_0(t)\big)^T\right)\frac{\widetilde{\nu}(t)}{\nu(t)}\right| \\
&\quad \le 2A_2(t)\nu(t)^{-1/q} + 2nB_2\beta(t)^{p/q} + 2mB_2\left|u_0(t)\right|^{p/q} := M(t).
\end{aligned} \tag{50}$$

It is obvious that the function $M$ belongs to $L_q(\mathbb{R}^+, \nu)$, since due to Minkowski's inequality we have

$$\begin{aligned}
&\left\{\int_0^\infty \left|2A_2(t)\nu(t)^{-1/q} + 2nB_2\beta(t)^{p/q} + 2mB_2\left|u_0(t)\right|^{p/q}\right|^q \nu(t)\,dt\right\}^{1/q} \\
&\quad \le \left\{\int_0^\infty \left|2A_2(t)\right|^q dt\right\}^{1/q} + \left\{\int_0^\infty \left|2nB_2\beta(t)^{p/q}\right|^q \nu(t)\,dt\right\}^{1/q} \\
&\qquad + \left\{\int_0^\infty \left|2mB_2\left|u_0(t)\right|^{p/q}\right|^q\right\}^{1/q} \nu(t)\,dt \\
&\quad = 2\|A_2\|_{L_q(\mathbb{R}^+)} + 2nB_2\|\beta\|_{L_p(\mathbb{R}^+,\nu)} + 2mB_2\|u_0\|_{L_p(\mathbb{R}^+,\nu)} < \infty.
\end{aligned} \tag{51}$$

The sequence $\{u_N\}$ is bounded in the norm because of its weak convergence. It means the validity of the inequality

$$\|u_N - u_0\|_{L_p(\mathbb{R}^+,\nu)} \le C, \quad N \in \mathbb{N} \tag{52}$$

for some constant $C > 0$. On the other side, the elements of the sequence $\{u_N\}$ are also pointwise bounded, due to the compactness of the set $U$ used in the definition of $\mathbb{U}_p$. Thus, there exists a number $K > 0$ so that

$$\left|u_N(t)\right| \le K \tag{53}$$

holds for all $t \in \mathbb{R}^+$ and all $N \in \mathbb{N}$. Through a similar diagonal selection method used in the estimate for $l_1$, we construct a diagonal subsequence $\{x_{N_k}\} \subset \{x_N\}$ which converges everywhere on $\mathbb{R}^+$ to the function $x_0$. Preparing the final estimate of the term $l_3$ we need several steps.

**Step 1**: Choose an arbitrary $\epsilon > 0$. Since the integral

$$\|M\|^q_{L_q(X,\nu)} = \int_X \left|M(t)\right|^q \nu(t)\,dt \tag{54}$$

is an absolutely continuous function with respect to $X$, for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for all sets $X$ with $\mu_\nu(X) < \delta$ the inequality

$$\|M\|_{L_q(X,\nu)} < \frac{\epsilon}{2C} \tag{55}$$

holds.

**Step 2**: Find a number $\delta = \delta(\epsilon)$ with (55).

**Step 3**: Set $\epsilon_1 := \min\{\epsilon, \delta(\epsilon)\}$.

**Step 4**: Since the integrand $r$ was assumed to be continuously differentiable in the second and the third variables and due to the positivity and continuity of $\frac{\widetilde{\nu}}{\nu}$, the sequence $\{\nabla_\upsilon r(t, x_{N_k}(t), u_0(t))\frac{\widetilde{\nu}(t)}{\nu(t)}\}_{k=1}^\infty$ converges to $\nabla_\upsilon r(t, x_0(t), u_0(t))\frac{\widetilde{\nu}(t)}{\nu(t)}$ pointwise everywhere on $\mathbb{R}^+$. The application of Jegorow Theorem, cf. [19], Theorem 3.5, p. 250, to the sequence $\{\nabla_\upsilon r(t, x_{N_k}(t), u_0(t))\frac{\widetilde{\nu}(t)}{\nu(t)}\}$ implies for $\epsilon_1 > 0$ from the previous step the existence of a set $M_{\epsilon_1}$ satisfying $\mu_\nu(\mathbb{R}^+ \backslash M_{\epsilon_1}) < \epsilon_1 < \delta(\epsilon)$ on which this sequence converges uniformly to $\nabla_\upsilon r(t, x_0(t), u_0(t))\frac{\widetilde{\nu}(t)}{\nu(t)}$. Thus, for any $\epsilon > 0$ one is able to find such a number $N_0 \in \mathbb{N}$ that for all $N_k \ge N_0$ and for all $t \in M_{\epsilon_1}$ the inequality

$$\left|\nabla_\upsilon^T r\big(t, x_{N_k}(t), u_0(t)\big) - \nabla_\upsilon^T r\big(t, x_0(t), u_0(t)\big)\right| \frac{\widetilde{\nu}(t)}{\nu(t)} < \frac{\epsilon_1}{4KV} < \frac{\epsilon}{4KV} \tag{56}$$

is true, where

$$V = L\text{-}\int_0^\infty \nu(t)\,dt.$$

The sequence $\{x_{N_k}\}$ will again be numerated by the index $N$.

**Step 5**: We now estimate for $p \in ]1, \infty[$ using the Hölder's inequality, as well as the inequalities (50), (52), (53), (55), (56):

$$\begin{aligned}
&\left|\int_0^\infty \big[\nabla_\upsilon^T r\big(t, x_N(t), u_0(t)\big) - \nabla_\upsilon^T r\big(t, x_0(t), u_0(t)\big)\big]\big(u_N(t) - u_0(t)\big)\widetilde{\nu}(t)\,dt\right| \\
&\quad \le \int_0^\infty \left|\nabla_\upsilon^T r\big(t, x_N(t), u_0(t)\big) - \nabla_\upsilon^T r\big(t, x_0(t), u_0(t)\big)\right| \frac{\widetilde{\nu}(t)}{\nu(t)} \\
&\quad\quad \times \left|u_N(t) - u_0(t)\right| \nu(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{M_{\epsilon_1}} \left|\nabla_v^T r\big(t, x_N(t), u_0(t)\big) - \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\right| \frac{\widetilde{\nu}(t)}{\nu(t)} \\
&\quad \times \left|u_N(t) - u_0(t)\right| \nu(t)\, dt \\
&\quad + \int_{\mathbb{R}^+ \setminus M_{\epsilon_1}} \left|\nabla_v^T r\big(t, x_N(t), u_0(t)\big) - \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\right| \frac{\widetilde{\nu}(t)}{\nu(t)} \\
&\quad \times \left|u_N(t) - u_0(t)\right| \nu(t)\, dt \\
&\le \int_{M_{\epsilon_1}} \frac{\epsilon}{4KV} 2K \nu(t)\, dt + \int_{\mathbb{R}^+ \setminus M_{\epsilon_1}} \left|M(t)\right| \left|u_N(t) - u_0(t)\right| \nu(t)\, dt \\
&\le \frac{\epsilon}{4KV} 2K \int_0^\infty \nu(t)\, dt + \|M\|_{L_q(\mathbb{R}^+ \setminus M_{\epsilon_1}, \nu)} \cdot \|u_N - u_0\|_{L_p(\mathbb{R}^+, \nu)} \\
&< \frac{\epsilon}{2KV} \cdot KV + \frac{\epsilon}{2C} \cdot C = \epsilon.
\end{aligned} \tag{57}
$$

Consequently, we arrive at

$$
\begin{aligned}
l_3 = \lim_{N \to \infty} \int_0^\infty &\left[\nabla_v^T r\big(t, x_N(t), u_0(t)\big) - \nabla_v^T r\big(t, x_0(t), u_0(t)\big)\right] \\
&\times \big(u_N(t) - u_0(t)\big) \widetilde{\nu}(t)\, dt = 0.
\end{aligned} \tag{58}
$$

Summarizing, we obtain together with (27), (41), (45), (58) the inequality

$$
l = \liminf_{n \to \infty} J_\infty(x_N, u_N) \ge l_1 + l_2 + l_3 \ge J_\infty(x_0, u_0), \tag{59}
$$

which proves the weak lower semicontinuity of the functional $J_\infty$ from (7) on the set $\mathbb{X}_p \times \mathbb{U}_p$. □

*Remark 3.2* The integrability of the weight function $\nu$ plays an important role in the growth conditions of Assumptions 2.2 and 2.3 which in turn allow to assure the continuity of corresponding Nemytskij operators mapping between weighted Lebesgue spaces. Therefore, the condition $L\text{-}\int_0^\infty \nu(t)\, dt < \infty$ is essential and cannot be omitted. The stronger "decay at infinity"-condition

$$
\lim_{t \to \infty} \frac{\nu(t + \epsilon)}{\nu(t)} = 0, \quad \forall \epsilon > 0, \tag{60}
$$

assumed in [13] in order to assure the compactness of the embedding

$$
W_p^{1,n}\big(\mathbb{R}^+, \nu\big) \hookrightarrow L_p^n\big(\mathbb{R}^+, \nu\big), \tag{61}
$$

could be avoided here. Instead of this we assumed a weaker condition posed in (19). Moreover, the above embedding is not compact without the condition (60), since the latter is a necessary one as well, cf. [16, 17].

## 4 An Existence Theorem

**Theorem 4.1** *Let $1 < p < \infty$. We assume that Assumptions 2.1–2.4 are satisfied and that there exists at least one admissible pair of the problem $(P_\infty)$. Furthermore, let the functions $A$ and $B$ from (15) be uniformly Lipschitz continuous in $x(t)$ and the control set $U$ be convex. Then, there exists an optimal solution of the problem $(P_\infty)$.*

*Proof* In order to prove this theorem, we want to use the generalized Weierstraß theorem which says that a weakly lower semicontinuous functional $J = J(x, u)$ defined on a weakly compact set $\mathcal{A}$ reaches its lower bound, i.e. there exists such a pair $(x^*, u^*) \in \mathcal{A}$ that

$$J\big(x^*, u^*\big) = \inf_{(x,u)\in\mathcal{A}} J(x, u). \tag{62}$$

Due to Assumptions 2.1–2.3, we now can apply Theorem 3.1 from the previous section and make a conclusion about the weak lower semicontinuity of the functional $J_\infty$. It now remains to prove the weak compactness of the admissible set

$$\mathcal{A} = \big\{(x, u) \in W_p^{1,n}\big(\mathbb{R}^+, \nu\big) \times L_p^m\big(\mathbb{R}^+, \nu\big) \mid \text{(9)–(12) are satisfied}\big\}, \tag{63}$$

which can be done by showing the weak closedness and boundedness of the set $\mathcal{A}$; cf. [22], p. 264. Therefore, we proceed in two steps.

**Step 1.** We prove that the set $\mathcal{A}$ is closed with respect to weak convergence. For this purpose we will need four lemmas.

**Lemma 4.1** *Let the sequence $\{u_N\}$ with elements from $L_p^m(\mathbb{R}^+, \nu)$ converge weakly to the function $u_0$. Further, let the inclusion*

$$u_N(t) \in U \tag{64}$$

*be satisfied for every $N \in \mathbb{N}$ almost everywhere on $\mathbb{R}^+$. $U$ denotes a nonempty compact convex set in $\mathbb{R}^m$. Then, the inclusion*

$$u_0(t) \in U \tag{65}$$

*holds almost everywhere on $\mathbb{R}^+$.*

*Proof* We denote

$$\mathcal{A}_u := \big\{u \in L_p^m\big(\mathbb{R}^+, \nu\big) \mid u(t) \in U \text{ a.e. on } \mathbb{R}^+\big\} \tag{66}$$

and prove the closedness and convexity of the set $\mathcal{A}_u$. Consider a sequence $\{\widetilde{u}_N\} \subset \mathcal{A}_u$ which converges in the $L_p^m(\mathbb{R}^+, \nu)$-norm to some function $\widetilde{u}_0$. Using the scheme illustrating connections between different convergence types, see [23], p. 446 f., and the fact that

$$\mu_\nu\big(\mathbb{R}^+\big) = \int_0^\infty d\mu(t) = \int_0^\infty \nu(t)\,dt < \infty, \tag{67}$$

we can select a subsequence $\{\widetilde{u}_{N_k}\}$ of the sequence $\{\widetilde{u}_N\}$ which converges to $\widetilde{u}_0$ $\mu_\nu$-a.e. and consequently $\mu$-a.e. on $\mathbb{R}^+$, since the sets of measure zero are the same for both Lebesgue and $\mu_\nu$ measures. It means, for all $t \in \mathbb{R}^+ \backslash \mathcal{N}$, that one has the pointwise convergence

$$\widetilde{u}_{N_k}(t) \rightarrow \widetilde{u}_0(t), \tag{68}$$

where the set $\mathcal{N}$ is a set of Lebesgue measure zero. Further, we define the set

$$B := \bigcup_{k=1}^{\infty} \left\{ t : \widetilde{u}_{N_k}(t) \notin U \right\}, \tag{69}$$

which is a set of Lebesgue measure zero due to the sigma additivity of Lebesgue measure. It means that the set $\mathcal{N} \cup B$ is of Lebesgue measure zero as well, and for all $t \in \mathbb{R}^+ \backslash (\mathcal{N} \cup B)$ the convergence

$$\widetilde{u}_{N_k}(t) \rightarrow \widetilde{u}_0(t) \tag{70}$$

remains valid. Since for all $t \in \mathbb{R}^+ \backslash (\mathcal{N} \cup B)$ and for all $N_k$ the inclusion $\widetilde{u}_{N_k}(t) \in U$ holds true, we obtain by using the compactness of the set $U$ and the convergence stated in (70) that the inclusion

$$\widetilde{u}_0(t) \in U, \quad \forall t \in \mathbb{R}^+ \backslash (\mathcal{N} \cup B) \tag{71}$$

is satisfied, which is the same as the condition (65), and the closedness of $\mathcal{A}_u$ is proved. To verify the convexity of the set $\mathcal{A}_u$, we consider a convex combination $\lambda u_1 + (1-\lambda) u_2, \lambda \in [0, 1]$ of the functions $u_1 \in \mathcal{A}_u, u_2 \in \mathcal{A}_u$ and show that it belongs to the set $\mathcal{A}_u$ as well. It holds $\lambda u_1 + (1-\lambda) u_2 \in L_p^m(\mathbb{R}^+, \nu)$, since $L_p^m(\mathbb{R}^+, \nu)$ is a vector space and $\forall t \in \mathbb{R}^+ \backslash (\mathcal{N}_{u_1} \cup \mathcal{N}_{u_2})$

$$\lambda u_1(t) + (1-\lambda) u_2(t) \in U, \tag{72}$$

where $\mathcal{N}_{u_1} \cup \mathcal{N}_{u_2}$ is a set of Lebesgue measure zero. Thus, the convexity of the set $\mathcal{A}_u$ is proved as well. We now can apply Theorem 3.3.8 from [24], p. 108., which says that the weak limit of a weak convergent sequence $\{u_N\}$ with elements from a closed convex subset $V$ of a normed space $X$ lies in the same subset $V$. This completes the proof of the lemma. □

**Lemma 4.2** *We are given a sequence $\{x_N\}$ with elements from $W_p^{1,n}(\mathbb{R}^+, \nu)$ which converges weakly to $x_0 \in W_p^{1,n}(\mathbb{R}^+, \nu)$, $1 < p < \infty$. Further, we assume that*

$$x_N(0) = x_0, \quad \forall N \in \mathbb{N}. \tag{73}$$

*Then, the equality*

$$x_0(0) = x_0 \tag{74}$$

*is also true.*

*Proof* We choose an arbitrary interval $[0, \tau]$ and apply the Rellich–Kondrachov theorem, see [17], p. 144, on this closed interval. It implies the existence of a subsequence $\{x_{N_k}\} \subset \{x_N\}$ converging uniformly on $[0, \tau]$ to $x_0$. This in turn yields the convergence at the point $t = 0$, which means

$$x_0 = \lim_{N_k \to \infty} x_0 = \lim_{N_k \to \infty} x_{N_k}(0) = x_0(0) \tag{75}$$

and the lemma is proved. □

**Lemma 4.3** *We are given sequences of functions* $\{x_N\} \subset W_p^{1,n}(\mathbb{R}^+, \nu)$ *and* $\{u_N\} \subset L_p^m(\mathbb{R}^+, \nu)$ *which converge weakly to* $x_0 \in W_p^{1,n}(\mathbb{R}^+, \nu)$ *and* $u_0 \in L_p^m(\mathbb{R}^+, \nu)$ *respectively,* $1 < p < \infty$. *Further, for all* $N \in \mathbb{N}$ *let the equation*

$$\dot{x}_N(t) = A\big(t, x_N(t)\big) + B\big(t, x_N(t)\big)u_N(t) \tag{76}$$

*be satisfied almost everywhere on* $\mathbb{R}^+$ *and let* $u_N$ *satisfy the inclusion* (12). *The functions* $A : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $B : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ *are assumed to be continuous in the first variable and uniformly Lipschitz continuous in the second. Then, the limit pair* $(x_0, u_0)$ *satisfies the differential equation*

$$\dot{x}_0(t) = A\big(t, x_0(t)\big) + B\big(t, x_0(t)\big)u_0(t) \tag{77}$$

*almost everywhere on* $\mathbb{R}^+$.

*Proof* The restrictions of the functions $x_N$ on the interval $[0, 1]$ are elements of $W_p^{1,n}([0, 1], \nu)$, while the restrictions of the admissible controls $u_N$ on the interval $[0, 1]$ are elements of $L_\infty^m([0, 1], \nu)$. Moreover, the sequence $\{u_N\}$, as a sequence of admissible controls, is bounded in the norm of the space $L_\infty^m([0, 1], \nu) = [L_1^m([0, 1], \nu)]^*$. Due to Theorem 3, p. 234 from [23] and the separability of $L_1^m([0, 1], \nu)$, there exists a subsequence of $\{u_N\}$ which converges weakly* in the space $L_\infty^m([0, 1], \nu)$. This subsequence will again be denoted by $\{u_N\}$. Furthermore, by the embedding theorem of Rellich–Kondrachov for non-weighted spaces, cf. [17], Theorem 6.2, p. 144, and by the positivity and continuity of the density function $\nu$, we conclude the existence of continuous representants of the sequence $\{x_N\}$. That means the existence of subsequences $\{x_{N^1}, u_{N^1}\} \subset \{x_N, u_N\}$ and of functions $\hat{x}_1 \in (W_p^{1,n}([0, 1], \nu) \cap C^0[0, 1])$, $\hat{u}_1 \in L_\infty^m([0, 1], \nu)$ such that

$$x_{N^1} \rightharpoonup \hat{x}_1\big(\text{in } W_p^{1,n}([0, 1], \nu)\big), \qquad u_{N^1} \rightharpoonup^* \hat{u}_1\big(\text{in } L_\infty^m([0, 1], \nu)\big), \tag{78}$$

$$\dot{\hat{x}}_1(t) = A\big(t, \hat{x}_1(t)\big) + B\big(t, \hat{x}_1(t)\big)\hat{u}_1(t) \quad \text{a.e. on } [0, 1], \tag{79}$$

$$\lim_{N^1 \to \infty} \|x_{N^1} - \hat{x}_1\|_{C^0[0,1]} = 0. \tag{80}$$

Through analogous argumentation, step-by-step we construct for all $k \in \mathbb{N}$, $k = 2, 3, \ldots$ subsequences $\{x_{N^k}, u_{N^k}\} \subset \{x_{N^{k-1}}, u_{N^{k-1}}\}$ satisfying

$$x_{N^k} \rightharpoonup \hat{x}_k\big(\text{in } W_p^{1,n}([0,k],\nu)\big), \qquad u_{N^k} \rightharpoonup^* \hat{u}_k\big(\text{in } L_\infty^m([0,k],\nu)\big), \tag{81}$$

$$\dot{\hat{x}}_k(t) = A\big(t, \hat{x}_k(t)\big) + B\big(t, \hat{x}_k(t)\big)\hat{u}_k(t) \quad \text{a.e. on } [0,k], \tag{82}$$

$$\lim_{N^k \to \infty} \|x_{N^k} - \hat{x}_k\|_{C^0[0,k]} = 0. \tag{83}$$

By the construction, we have $\hat{x}_k(t) = \hat{x}_{k-1}(t)$ for all $t \in [0, k-1]$ and $\hat{u}_k(t) = \hat{u}_{k-1}(t)$ for almost all $t \in [0, k-1]$. This leads to the limit functions

$$\hat{x} \in \big(W_{p,\text{loc}}^{1,n}(\mathbb{R}^+, \nu) \cap C^0(\mathbb{R}^+)\big), \qquad \hat{u} \in L_{\infty,\text{loc}}^m(\mathbb{R}^+, \nu), \tag{84}$$

whose restrictions on $[0,k]$ coincide with $\hat{x}_k$ and $\hat{u}_k$, respectively. The uniqueness of the weak limit yields $\hat{x}(t) = x_0(t)$ for all $t \in \mathbb{R}^+$ and $\hat{u}(t) = u_0(t)$ for almost all $t \in \mathbb{R}^+$.

Now, we still have to show the equality (82) for all $k \in \mathbb{N}$. Similarly as in [25], Lemma 2.3, p. 223, we multiply the equation (76) by an arbitrary test function $\phi \in C^{0,\infty}[0,k]$, i.e. an infinitely many times differentiable function with compact support on $[0,k]$, and derive the equation

$$\int_0^k \phi(t)\big[\dot{x}_{N^k}(t) - A\big(t, x_{N^k}(t)\big) - B\big(t, x_{N^k}(t)\big)u_{N^k}(t)\big]\, dt = 0 \tag{85}$$

for all $t \in [0,k]$. Integrating the left hand side of the last equation by parts we obtain for any $\phi \in C^{0,\infty}[0,k]$

$$\phi(t)x_{N^k}(t)\big|_0^k - \int_0^k \dot{\phi}(t)x_{N^k}(t)\, dt$$
$$= \int_0^k \phi(t)\big(A\big(t, x_{N^k}(t)\big) + B\big(t, x_{N^k}(t)\big)u_{N^k}(t)\big)\, dt, \tag{86}$$

which is the same as

$$\int_0^k \dot{\phi}(t)x_{N^k}(t)\, dt + \int_0^k \phi(t)\big(A\big(t, x_{N^k}(t)\big) + B\big(t, x_{N^k}(t)\big)u_{N^k}(t)\big)\, dt = 0. \tag{87}$$

The uniform convergence of $\{x_{N^k}\}$ on the interval $[0,k]$ implies

$$\lim_{k\to\infty} \int_0^k \dot{\phi}(t)x_{N^k}(t)\, dt = \int_0^k \dot{\phi}(t)\hat{x}_k(t)\, dt. \tag{88}$$

Next we consider

$$\left| \int_0^k \phi(t)A\big(t, x_{N^k}(t)\big)\, dt - \int_0^k \phi(t)A\big(t, \hat{x}_k(t)\big)\, dt \right|$$
$$\leq \int_0^k \big|\phi(t)\big| \cdot \big|A\big(t, x_{N^k}(t)\big) - A\big(t, \hat{x}_k(t)\big)\big|\, dt \tag{89}$$

and using the Lipschitz continuity of the function $A(t,\cdot)$, uniform with respect to $t$, we continue the previous estimate

$$\int_0^k \left|\phi(t)\right| \cdot \left|A\big(t, x_{N^k}(t)\big) - A\big(t, \hat{x}_k(t)\big)\right| dt$$
$$\leq L_1 \|\phi\|_{C^0} \cdot \int_0^k \left|x_{N^k}(t) - \hat{x}_k(t)\right| dt \to 0, \quad N^k \to \infty, \tag{90}$$

which holds due to the uniform convergence of $\{x_{N^k}\}$ to the function $\hat{x}_k$. In the next step we estimate

$$\left|\int_0^k \phi(t) B\big(t, x_{N^k}(t)\big) u_{N^k}(t)\, dt - \int_0^k \phi(t) B\big(t, \hat{x}_k(t)\big) \hat{u}_k(t)\, dt\right|$$
$$\leq \int_0^k \left|\phi(t)\right| \cdot \left|B\big(t, x_{N^k}(t)\big) - B\big(t, \hat{x}_k(t)\big)\right| \left|u_{N^k}(t)\right| dt$$
$$+ \left|\int_0^k \phi(t) B\big(t, \hat{x}_k(t)\big)\big(u_{N^k}(t) - \hat{u}_k(t)\big)\, dt\right|$$
$$\leq L_2 \cdot K \|\phi\|_{C^0} \cdot \int_0^k \left|x_{N^k}(t) - \hat{x}_k(t)\right| dt$$
$$+ \left|\int_0^k \phi(t) B\big(t, \hat{x}_k(t)\big)\big(u_{N^k}(t) - \hat{u}_k(t)\big)\, dt\right| \tag{91}$$

due to the uniform Lipschitz continuity of the function $B(t,\cdot)$. Taking the weak* convergence of $u_{N^k} \rightharpoonup^* \hat{u}_k$ in $L_\infty^m([0,k],\nu)$ into account one has for all $\phi(\cdot)B(\cdot) \in L_1^{n\times m}[0,k]$ the componentwise convergence

$$\int_0^k \phi(t) B(t) u_{N^k}(t)\, dt \to \int_0^k \phi(t) B(t) \hat{u}_k(t)\, dt, \quad N^k \to \infty \tag{92}$$

and, in particular, the convergence

$$\int_0^k \phi(t) B\big(t, \hat{x}_k(t)\big) u_{N^k}(t)\, dt \to \int_0^k \phi(t) B\big(t, \hat{x}_k(t)\big) \hat{u}_k(t)\, dt, \quad N^k \to \infty, \tag{93}$$

since the inclusion $B(\cdot, \hat{x}_k(\cdot)) \in L_q^{n\times m}(\mathbb{R}^+,\nu) \subset L_1^{n\times m}(\mathbb{R}^+,\nu) \subset L_1^{n\times m}[0,k]$ holds true because of (16) and the properties of $\phi(\cdot)$ as a test function. Having the uniform convergence of $\{x_{N^k}\}$, estimate (91) and convergence (93) in mind we obtain

$$\lim_{N^k\to\infty} \int_0^k \phi(t) B\big(t, x_{N^k}(t)\big) u_{N^k}(t)\, dt = \int_0^k \phi(t) B\big(t, \hat{x}_k(t)\big) \hat{u}_k(t)\, dt. \tag{94}$$

Using (88), (90) and (94) we pass to the limit $N^k \to \infty$ in (87), i.e. we have

$$\int_0^k \dot{\phi}(t) \hat{x}_k(t)\, dt + \int_0^k \phi(t) A\big(t, \hat{x}_k(t)\big)\, dt + \int_0^k \phi(t) B\big(t, \hat{x}_k(t)\big) \hat{u}_k(t)\, dt = 0 \tag{95}$$

for all test functions $\phi \in C^{0,\infty}[0,k]$. Through the integration by parts in the first term one has

$$\int_0^k \phi(t)\bigl\{\dot{\hat{x}}_k(t) - A\bigl(t,\hat{x}_k(t)\bigr) - B\bigl(t,\hat{x}_k(t)\bigr)\hat{u}_k(t)\bigr\}\,dt = 0, \quad \forall\phi \in C^{0,\infty}[0,k]. \tag{96}$$

Consequently, the equation

$$\dot{\hat{x}}_k(t) = A\bigl(t,\hat{x}_k(t)\bigr) + B\bigl(t,\hat{x}_k(t)\bigr)\hat{u}_k(t) \tag{97}$$

is satisfied almost everywhere on $[0,k]$, and therefore, the equation (82) is proved for all $k \in \mathbb{N}$.

We now construct a diagonal sequence having the following properties: From the first sequence $\{x_{N^1}, u_{N^1}\}$ we take the first pair $\{x_{N_1^1}, u_{N_1^1}\}$ satisfying

$$\|x_{N_1^1} - \hat{x}_1\|_{C^0[0,1]} \le 1/2, \tag{98}$$

from $\{x_{N^2}, u_{N^2}\}$ we take the first pair $\{x_{N_2^2}, u_{N_2^2}\}$ with $N_2^2 > N_1^1$

$$\|x_{N_2^2} - \hat{x}_2\|_{C^0[0,2]} \le 1/4, \tag{99}$$

and generally we select from the sequence $\{x_{N^k}, u_{N^k}\}$ the first pair $\{x_{N_k^k}, u_{N_k^k}\}$ having $N_k^k > N_{k-1}^{k-1}$ and

$$\|x_{N_k^k} - \hat{x}_k\|_{C^0[0,k]} \le 1/2^k. \tag{100}$$

The diagonal sequence $\{x_{N_k^k}\}$ converges on the whole half-axis $[0,\infty[$ pointwise to the limit function $\hat{x} = x_0$. From this we can deduce the result of Lemma 4.2, the condition $x_0(0) = x_0$, as well. □

The state constraint (11) remains satisfied as $N$ tends to infinity, as the next lemma shows.

**Lemma 4.4** *Let $\{x_N\}$ be a sequence converging weakly to the function $x_0$ in the weighted Sobolev space $W_p^{1,n}(\mathbb{R}^+, \nu)$. Furthermore, let all of functions $x_N$ satisfy the inequality*

$$\bigl|x_N(t)\bigr| \le \beta(t) \quad \text{a.e. on } \mathbb{R}^+. \tag{101}$$

*Then, the limit function $x_0$ satisfies the same inequality*

$$\bigl|x_0(t)\bigr| \le \beta(t) \quad \text{a.e. on } \mathbb{R}^+. \tag{102}$$

*Proof* Suppose the inequality (102) is violated, which means there exists such a set $F \subset \mathbb{R}^+$ of positive Lebesgue measure $\mu(F) > 0$ that for all $t \in F$ the inequality

$$\bigl|x_0(t)\bigr| - \beta(t) \ge K_1 > 0 \tag{103}$$

holds true. Further, there exists an interval $[0, T]$ with $\mu(F \cap [0, T]) > 0$. We denote $F_1 := F \cap [0, T]$. Since the embedding $W_p^{1,n}([0, T], \nu) \to L_p^m([0, T], \nu)$ is compact, cf. the Rellich–Kondrachov theorem, Theorem 6.2, [17], p. 144, one can find a subsequence of the weakly convergent sequence $\{x_N|_{[0,T]}\}$ of the weighted Sobolev space $W_p^{1,n}([0, T], \nu)$ which is strong convergent in the weighted Lebesgue space $L_p^n([0, T], \nu)$. This subsequence will again be indexed by $N \in \mathbb{N}$. Then we select a subsequence $\{x_{N^k}\} \subset \{x_N|_{[0,T]}\}$ which converges to $x_0$ almost everywhere on $[0, T]$ and, consequently, on $F_1$. This is possible due to the scheme provided in [23], p. 446, and the continuity of the density function $\nu$. Further, we estimate for large enough $N^k \in \mathbb{N}$ and almost all $t \in F_1$:

$$\begin{aligned} |x_{N^k}| - \beta(t) &= \left|x_{N^k}(t)\right| - \left|x_0(t)\right| + \left|x_0(t)\right| - \beta(t) \\ &\geq \left|x_{N^k}(t)\right| - \left|x_0(t)\right| + K_1 \geq K_1 - \epsilon = K_2 > 0 \end{aligned} \tag{104}$$

which contradicts to the inequality (101). The last estimate is true because of the continuity of the Euclidean norm function $|\cdot|$ and the inclusion $F_1 \subset [0, T]$, which guarantees the uniform convergence of the sequence $\{x_{N^k}|_{F_1}\}$ on $F_1$ with the exception of a set of Lebesgue measure zero. □

This result together with three previous lemmas completes the proof of the weak closedness of the feasible set $\mathcal{A}$. It remains to show the boundedness of this set what we do in the second step.

**Step 2.** The set of all admissible controls is bounded since the inclusion $u(t) \in U$ holds almost everywhere on $\mathbb{R}^+$, where $U$ is a nonempty compact convex subset of $\mathbb{R}^m$, and it yields the existence of such a constant $K > 0$ that for almost all $t \in \mathbb{R}^+$ the inequality

$$\left|u(t)\right| \leq K \tag{105}$$

remains valid. It in turn implies

$$\|u\|^p_{L_p^m(\mathbb{R}^+,\nu)} = \int_0^\infty \left|u(t)\right|^p \nu(t)\,dt \leq K^p V = \text{const.}, \tag{106}$$

whereby

$$V = \int_0^\infty \nu(t)\,dt < \infty. \tag{107}$$

The application of the growth condition (17) and of the inequality (105) allows to derive the following estimate for the derivative of the state trajectory $x$:

$$\begin{aligned} \|\dot{x}\|^p_{L_p^n(\mathbb{R}^+,\nu)} &= \int_0^\infty \left|\dot{x}(t)\right|^p \nu(t)\,dt \\ &= \int_0^\infty \left|f\big(t, x(t), u(t)\big)\right|^p \nu(t)\,dt \end{aligned}$$

$$\begin{aligned}
&\le \int_0^\infty \bigl(C_1 + C_2|x(t)| + C_3|x(t)||u(t)| + C_4|u(t)|\bigr)^p \nu(t)\,dt \\
&\le \int_0^\infty \bigl(C_1 + (C_2 + C_3K)|x(t)| + C_4K\bigr)^p \nu(t)\,dt \\
&\le 2^p(C_1 + C_4K)^p \int_0^\infty \nu(t)\,dt + 2^p(C_2 + C_3K)^p \int_0^\infty |\beta(t)|^p \nu(t)\,dt \\
&\le 2^p(C_1 + C_4K)^p V + 2^p(C_2 + C_3K)^p \|\beta\|^p_{L_p^n(\mathbb{R}^+,\nu)} \\
&= \text{const.}
\end{aligned} \tag{108}$$

Due to condition (11), we have

$$\begin{aligned}
\|x\|^p_{L_p^n(\mathbb{R}^+,\nu)} &= \int_0^\infty |x(t)|^p \nu(t)\,dt \le \int_0^\infty |\beta(t)|^p \nu(t)\,dt = \|\beta\|^p_{L_p^n(\mathbb{R}^+,\nu)} \\
&= \text{const.}
\end{aligned} \tag{109}$$

The estimate (108) was made by using the elementary inequality

$$(a+b)^p \le 2^p\bigl(a^p + b^p\bigr), \quad a, b > 0,\ p \ge 1. \tag{110}$$

Thus, the boundedness of the admissible set $\mathcal{A}$ is proved and therefore the weak compactness of the set $\mathcal{A}$. The proof of the theorem is completed. □

*Remark 4.1*

(a) The condition (11) is essential for establishing the existence of an optimal solution, since, together with the other assumptions, it guarantees both the weak lower semicontinuity of the functional $J_\infty$ and the boundedness of the feasible set, which is otherwise not necessarily given.

(b) The existence result of the above theorem remains valid if the growth conditions in Assumptions 2.2 and 2.3 are satisfied only for all

$$(t, \xi, v) \in \mathbb{R}^+ \times \mathbb{R}^n \times U. \tag{111}$$

## 5 Application to a Resource Allocation Model

We consider the following infinite horizon optimal control problem of resource allocation. Minimize the integral functional:

$$L\text{-}\int_0^\infty e^{-\rho t} x(t)\bigl(u(t) - 1\bigr)\,dt \to \min! \tag{112}$$

with respect to all pairs

$$(x, u) \in W_2^1\bigl(\mathbb{R}^+, e^{-\alpha^* t}\bigr) \times L_2\bigl(\mathbb{R}^+, e^{-\alpha^* t}\bigr), \tag{113}$$

satisfying the conditions

$$\dot{x}(t) = x(t)u(t) \quad \text{a.e. on } \mathbb{R}^+, \quad x(0) = x_0 > 0, \tag{114}$$

$$\big|x(t)\big| < Ce^{\alpha t}, \quad C \geq x_0, \quad 0 < 2\alpha < \alpha^* < \rho, \tag{115}$$

$$0 \leq u(t) \leq 1, \tag{116}$$

where the discount rate $\rho$ satisfies $0 < \rho < 1$. This model was introduced in [26] and belongs to the class of problems $(P_\infty)$, which becomes clear if one defines the function $\beta$ as $\beta(t) = Ce^{\alpha t}$ and notices that $\beta \in L_2(\mathbb{R}^+, e^{-\alpha^* t})$. We refer to it as to the adapted resource allocation model. It should be mentioned that without posing the state constraint (115) no solution of this problem exists; cf. [26]. We verify whether the problem (112)–(116) satisfies all the conditions of the existence theorem proved before.

- Assumption 2.1 is satisfied, since $r(t, \xi, v) = \xi(v - 1)$ is convex and continuous differentiable in $\xi$ and $v$. Continuity is obviously given with respect to all variables.
- According to Assumption 2.2 there must exist a function $A_1 \in L_1(\mathbb{R}^+)$ as well as a positive constant $B_1$ so that for all $(t, \xi, v) \in \mathbb{R}^+ \times \mathbb{R} \times U$ and $p = 2$ the growth condition

$$\big|\xi(v-1)\big| \leq A_1(t)e^{\rho t} + B_1\big(|\xi|^2 + |v|^2\big)e^{(\rho-\alpha^*)t} \tag{117}$$

  is valid. This is the case, if $A(t) = e^{-\rho t}$ and $B = 2$ due to the inequality

$$\big|\xi(v-1)\big| \leq 2|\xi| \leq 2\big(1 + |\xi|^2\big) \leq 2e^{-\rho t}e^{\rho t} + 2\big(|\xi|^2 + |v|^2\big)e^{(\rho-\alpha^*)t} \tag{118}$$

  and the control restriction (116).
- Assumption 2.3 is fulfilled with $A_2 \equiv 0$, $B_2 = 1$ for $p = q = 2$.
- Having $A(t, x(t)) \equiv 0$ and $B(t, x(t)) = x(t)$ we verify the conditions (17) and (18). The first is satisfied with constants $C_1 = 1$, $C_2 = 0$, $C_3 = 1$, $C_4 = 0$ and $p = q = 2$. The second one is true, since $|B_{11}(t, \xi)| = |\xi|$, and we set $A_{311} \equiv 0$, $B_{311} = 1$.
- The uniform Lipschitz continuity of $B(t, \cdot)$ and the convexity and closedness of the control set $U = [0, 1]$ are obvious.
- The feasible set is not empty, because $(x, u) = (x_0 e^{\alpha t}, \alpha)$ is an admissible process.

Thus, all the conditions of the existence theorem are fulfilled which implies the existence of an optimal solution for the adapted resource allocation problem. The optimal solution is

$$x^*(t) = \begin{cases} x_0 e^{(1-\alpha)\tau} e^t, & t < \tau, \\ Ce^{\alpha t}, & t \geq \tau, \end{cases} \qquad u^*(t) = \begin{cases} 1, & t < \tau, \\ \alpha, & t \geq \tau \end{cases} \tag{119}$$

with the switching point $\tau = \frac{1}{1-\alpha} \ln\{\frac{C}{x_0}\}$.

## 6 Conclusions

In this paper, we succeeded in proving an existence theorem for a special setting of infinite horizon control problems, which involves an integral functional with integral in Lebesgue sense and an integrand, which is convex in control, as well as affine-linear in control dynamics. The problem statement contains weighted Sobolev and weighted Lebesgue spaces, which allows the application of the necessary optimality conditions in the form of Pontryagin's Type Maximum Principle formulated in the same spaces, see [27], for calculating the optimal solution. However, the class of problems for which the existence of optimal solutions is established in [6] allows also nonconvex integrands. In order to generalize the present existence result for problems with nonconvex integrands some relaxation techniques can be applied.

For guaranteeing the existence of an optimal solution, we made use of the additional pointwise state constraint. For certain types of the state equation, it is possible to determine such a majorant that this state constraint is satisfied automatically. In other cases, the form of the state constraint can be derived directly from the modeling background. However, in some economic models it is not clear how the pointwise state constraint should be properly posed. Any artificial choice of the majorant may have an essential impact on the optimal solution, which seems to be rather restrictive. Therefore, our future research will be focused on weakening the pointwise state constraint, e.g. by an isoperimetric state constraint, and establishing existence theorems for the resulting optimal control problem using the techniques provided by weighted functional spaces.

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