Dual Semidefinite Programs Without Duality Gaps for a Class of Convex Minimax Programs

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Abstract In this paper, we introduce a new dual program, which is representable as a semidefinite linear programming problem, for a primal convex minimax programming problem, and we show that there is no duality gap between the primal and the dual whenever the functions involved are sum-of-squares convex polynomials. Under a suitable constraint qualification, we derive strong duality results for this class of minimax problems. Consequently, we present applications of our results to robust sum-of-squares convex programming problems under data uncertainty and to minimax fractional programming problems with sum-of-squares convex polynomials. We obtain these results by first establishing sum-of-squares polynomial representations of non-negativity of a convex max function over a system of sum-of-squares convex constraints. The new class of sum-of-squares convex polynomials is an important subclass of convex polynomials and it includes convex quadratic functions and separable convex polynomials. The sum-of-squares convexity of polynomials can numerically be checked by solving semidefinite programming problems whereas numerically verifying convexity of polynomials is generally very hard.

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1 Introduction

The problem of minimizing a maximum of finitely many functions subject to inequality constraints, known as discrete minimax program, arises in many areas of applications in engineering and commerce, as resource allocation and planning problems (see [1, 2] and other references therein). More recently, discrete minimax programs have appeared in robust optimization [3, 4], which is becoming increasingly important in optimization due to the reality of uncertainty in many real-world optimization problems and the importance of finding solutions that are immunized against data uncertainty [3, 5].

On the other hand, the standard convex polynomial program, where we minimize a single convex polynomial subject to convex polynomial inequality constraints, appears, often in the form of a convex quadratic program, in decision-making problems of finance and engineering. Remarkably, there is no duality gap between a primal convex polynomial program and its Lagrangian dual [6]. However, the Lagrangian dual, in general, may not easily be solvable. Recent research has shown that, whenever the functions involved in the primal convex program are *sum-of-squares convex polynomials* (in short, *SOS-convex polynomials* (see Definition 2.1), the convex program enjoys special properties such as exact sum of squares relaxation and zero duality gap between the primal and its dual [7, 8]. The key feature of these results is that the sum of squares relaxation problem or the dual problem is representable as a semidefinite programming problem (SDP). Such a result is of great interest in optimization because semidefinite programs can efficiently be solved by interior-point methods [9], and so the optimal value of the original problem can be found by solving its relaxation or dual problem.

The new class of SOS-convex polynomials from algebraic geometry [7, 10] is an important subclass of convex polynomials and it includes convex quadratic functions and separable convex polynomials. The SOS-convexity of polynomials can numerically be checked by solving semidefinite programming problems, whereas deciding convexity of polynomials is generally very hard [11, 12].

In this paper, we study discrete minimax programs with SOS-convex polynomials and examine the very basic issue of which *discrete minimax programs* can be presented with zero duality gap, where the duals can be represented as semidefinite linear programming problems. We make the following contributions to minimax optimization.

  I. Without any qualifications, we establish dual characterizations of non-negativity of max functions of convex polynomials over a system of convex polynomial inequalities and then derive sum-of-squares polynomial representations of nonnegativity of max functions of SOS-convex polynomials over a system of SOS-convex polynomial inequalities.

 II. Using the sum-of-squares polynomial representations, we introduce a dual program, which is representable as a semidefinite linear programming problem, and show that there is no duality gap between the primal and its dual, whenever the functions involved in the primal program are SOS-convex polynomials. Under a constraint qualification, we prove that strong duality holds between the primal minimax problem and its dual problem. As an application, we prove that the

value of a robust convex programming problem under polytopic data uncertainty is equal to its dual semidefinite program. The significance of our duality theorems is that the value of our primal minimax program can easily be found by solving its dual semidefinite program.

III. Under a constraint qualification, we establish that strong duality continues to hold for SOS-convex minimax fractional programming problems with their corresponding dual semidefinite programs, including minimax linear fractional programming problems, for which the dual semidefinite programs reduce to linear programming problems.

The outline of the paper is as follows. Section 2 provides dual characterizations and representations of non-negativity of max functions of convex polynomials as well as SOS-convex polynomials over a system of inequalities. Section 3 presents zero duality gaps and strong duality results. Section 4 gives applications of our duality results to classes of robust convex optimization problems and minimax fractional programming problems. Appendix provides basic re-formulation of our dual problem as a semidefinite linear programming problem.

2 Dual Characterizations and Representations of Non-negativity

In this section, we present dual characterizations of solvability of inequality systems involving convex as well as SOS-convex polynomials. Firstly, we shall recall a few basic definitions and results which will be needed later in the sequel. We say that a real polynomial f is *sum-of-squares* [13] iff there exist real polynomials f_j, $j = 1, \ldots, s$, such that $f = \sum_{j=1}^{s} f_j^2$. The set of all sum-of-squares real polynomials is denoted by Σ^2, whereas the set consisting of all sum of squares real polynomials with degree at most d is denoted by Σ_d^2. Similarly, we say a matrix polynomial $F \in \mathbb{R}[x]^{n \times n}$ is a SOS-matrix polynomial provided that $F(x) = H(x)H(x)^T$, where $H(x) \in \mathbb{R}[x]^{n \times s}$ is a matrix polynomial for some $s \in \mathbb{N}$. We now introduce the definition of SOS-convex polynomial.

Definition 2.1 [10, 11] A real polynomial f on $\mathbb{R}^n$ is called *SOS-convex* iff the Hessian matrix function $x \mapsto \nabla^2 f(x)$ is a SOS-matrix polynomial.

Clearly, a SOS-convex polynomial is convex. However, the converse is not true. Thus, there exists a convex polynomial, which is not SOS-convex [11]. It is known that any convex quadratic function and any convex separable polynomial is a SOS-convex polynomial. Moreover, a SOS-convex polynomial can be non-quadratic and non-separable. For instance, $f(x) = x_1^8 + x_1^2 + x_1 x_2 + x_2^2$ is a SOS-convex polynomial (see [14]), which is non-quadratic and non-separable.

The following basic known results on convex polynomials will play key roles throughout the paper.

Lemma 2.1 [10, Lemma 8] *Let f be a SOS-convex polynomial. If $f(u) = 0$ and $\nabla f(u) = 0$ for some $u \in \mathbb{R}^n$, then f is a sum-of-squares polynomial.*

Lemma 2.2 ([15, Theorem 3]) *Let $f_0, f_1, \ldots, f_m$ be convex polynomials on $\mathbb{R}^n$. Suppose that $\inf_{x \in C} f_0(x) > -\infty$, where $C := \{x \in \mathbb{R}^n : f_i(x) \le 0, i \in \mathbb{N}_m\} \ne \emptyset$. Then, $\operatorname{argmin}_{x \in C} f_0(x) \ne \emptyset$.*

Corollary 2.1 *Let f be a non-negative SOS-convex polynomial on $\mathbb{R}^n$. Then, f is a sum-of-squares polynomial.*

Proof Assume that f is a non-negative SOS-convex polynomial on $\mathbb{R}^n$. In virtue of Lemma 2.2, we know that $\min_{x \in \mathbb{R}^n} f(x) = f(x^*)$ for some $x^* \in \mathbb{R}^n$. Therefore, $h := f - f(x^*)$ is a non-negative SOS-convex polynomial such that $h(x^*) = 0$ and $\nabla h(x^*) = 0$. By applying Lemma 2.1, we find that h is a sum-of-squares polynomial, so $f - f(x^*) = \sigma$ for some $\sigma \in \Sigma^2$. Therefore, $f = \sigma + f(x^*)$ is a sum-of-squares polynomial since $f(x^*) \ge 0$.   □

Let Δ be the simplex in $\mathbb{R}^r$, that is, $\Delta := \{\delta \in \mathbb{R}^r_+ : \sum_{j=1}^r \delta_j = 1\}$. Note that for $l \in \mathbb{N}$, $\mathbb{N}_l$ is defined by $\mathbb{N}_l = \{1, 2, \ldots, l\}$

Theorem 2.1 (Dual characterization of non-negativity) *Let p_j and g_i be convex polynomials for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$, with $\mathcal{F} := \{x \in \mathbb{R}^n : g_i(x) \le 0, i \in \mathbb{N}_m\}$ a nonempty set. Then, the following statements are equivalent*:

(i) $g_i(x) \le 0, i \in \mathbb{N}_m \Rightarrow \max_{j \in \mathbb{N}_r} p_j(x) \ge 0$.
(ii) $(\forall \varepsilon > 0) \ (\exists \bar{\delta} \in \Delta, \bar{\lambda} \in \mathbb{R}^m_+) \ \sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i + \varepsilon > 0$.

Proof (ii) $\Rightarrow$ (i) Suppose that, for each $\varepsilon > 0$, there exist $\bar{\delta} \in \Delta$ and $\bar{\lambda} \in \mathbb{R}^m_+$ such that $\sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i + \varepsilon > 0$. Then, for any $x \in \mathcal{F}$ we have

$$\max_{\delta \in \Delta} \sum_{j=1}^r \delta_j p_j(x) + \varepsilon \ge \sum_{j=1}^r \bar{\delta}_j p_j(x) + \varepsilon \ge \sum_{j=1}^r \bar{\delta}_j p_j(x) + \sum_{i=1}^m \bar{\lambda}_i g_i(x) + \varepsilon > 0.$$

Letting $\varepsilon \to 0$, we see that

$$\max_{j \in \mathbb{N}_r} p_j(x) = \max_{\delta \in \Delta} \sum_{j=1}^r \delta_j p_j(x) \ge 0$$

for all $x \in \mathcal{F}$.

(i) $\Rightarrow$ (ii) Assume that (i) holds. Let $\varepsilon > 0$ be arbitrary and let $f_j := p_j + \varepsilon$ for all $j \in \mathbb{N}_r$. Then, one has

$$\max_{j \in \mathbb{N}_r} f_j(x) = \max_{j \in \mathbb{N}_r} \{p_j(x)\} + \varepsilon > 0 \quad \forall x \in \mathcal{F}.$$

Now, we will show that the set

$$G := \left\{ z = (\underline{z}, \overline{z}) \in \mathbb{R}^{r+m} : \exists x \in \mathbb{R}^n \text{ s.t. } f_j(x) \le \underline{z}_j, j \in \mathbb{N}_r, g_i(x) \le \overline{z}_i, i \in \mathbb{N}_m \right\}$$

is a closed and convex set. As f_j and g_i are all convex polynomials, then G is clearly a convex set. To see that it is closed, let $\{z^k\}_{k \in \mathbb{N}} \subset G$ be such that $\{z^k\} \to z^*$ as $k \to \infty$.

Then, for each $k \in \mathbb{N}$, there exists $x^k \in \mathbb{R}^n$ such that $f_j(x^k) \leq \underline{z}_j^k$ and $g_i(x^k) \leq \overline{z}_i^k$, for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$. Now, consider the convex optimization problem

$$(\bar{P}) \quad \min_{x \in \mathbb{R}^n, u \in \mathbb{R}^{r+m}} \|u - z^*\|^2$$

$$\text{s.t.} \quad f_j(x) - \underline{u}_j \leq 0, \quad j \in \mathbb{N}_r,$$

$$g_i(x) - \overline{u}_i \leq 0, \quad i \in \mathbb{N}_m.$$

Obviously, $0 \leq \inf(\bar{P}) \leq \|z^k - z^*\|^2$ for all $k \in \mathbb{N}$, where $\inf(\bar{P})$ denotes the optimal value of the problem $\inf(\bar{P})$. Since $\|z^k - z^*\|^2 \to 0$ as $k \to \infty$, we get $\inf(\bar{P}) = 0$. Moreover, Lemma 2.2 implies that $\inf(\bar{P})$ is attained, and so, there exists $x^* \in \mathbb{R}^n$ such that $f_j(x^*) \leq \underline{z}_j^*$, $j \in \mathbb{N}_r$, and $g_i(x^*) \leq \overline{z}_i^*$, $i \in \mathbb{N}_m$. So $z^* \in G$, and consequently, G is closed.

Since $\max_{j \in \mathbb{N}_r} f_j(x) > 0$ for all $x \in \mathcal{F}$, $0 \notin G$. Hence, by the strict separation theorem [16, Theorem 1.1.5], there exist $v = (\underline{v}, \overline{v}) \in \mathbb{R}^{r+m} \setminus \{0\}$, $\alpha \in \mathbb{R}$ and $\xi > 0$ such that

$$0 = v^T 0 \leq \alpha < \alpha + \xi \leq \underline{v}^T \underline{z} + \overline{v}^T \overline{z}$$

for all $z \in G$. Since $G + (\mathbb{R}_+^r \times \mathbb{R}_+^m) \subset G$, $\underline{v}_j \geq 0$ and $\overline{v}_i \geq 0$, for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$. Observe that, for each $x \in \mathbb{R}^n$, $(f_1(x), \ldots, f_r(x), g_1(x), \ldots, g_m(x)) \in G$. So, for each $x \in \mathbb{R}^n$,

$$\sum_{j=1}^{r} \underline{v}_j f_j(x) + \sum_{i=1}^{m} \overline{v}_i g_i(x) \geq \alpha + \xi \geq \xi > 0. \tag{1}$$

Now, we claim that $\underline{v} \in \mathbb{R}_+^r \setminus \{0\}$. Otherwise, if $\underline{v} = 0$, then from (1) we get $\sum_{i=1}^{m} \overline{v}_i g_i(\bar{x}) > 0$ for any $\bar{x} \in \mathcal{F}$ (recall that $\mathcal{F}$ is nonempty). Since $g_i(\bar{x}) \leq 0$ and $\overline{v}_i \geq 0$ for all $i \in \mathbb{N}_m$, $\sum_{i=1}^{m} \overline{v}_i g_i(\bar{x}) \leq 0$, which is a contradiction. So, one has $\kappa := \sum_{j=1}^{r} \underline{v}_j > 0$. Therefore, (1) implies that

$$\sum_{j=1}^{r} \bar{\delta}_j f_j(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) \geq \bar{\xi} > 0$$

for all $x \in \mathbb{R}^n$, where $\bar{\delta}_j := \kappa^{-1} \underline{v}_j \geq 0$ for all $j \in \mathbb{N}_r$, $\bar{\lambda}_i := \kappa^{-1} \overline{v}_i \geq 0$ for all $i \in \mathbb{N}_m$, and $\bar{\xi} := \kappa^{-1} \xi > 0$. Since $\sum_{j=1}^{r} \bar{\delta}_j = 1$, we can write

$$\sum_{j=1}^{r} \bar{\delta}_j p_j + \sum_{i=1}^{m} \bar{\lambda}_i g_i + \varepsilon > 0.$$

Thus, the conclusion follows.   □

Let d be the smallest even number such that

$$d \geq \max \left\{ \max_{j \in \mathbb{N}_r} \deg p_j, \max_{i \in \mathbb{N}_m} \deg g_i \right\}.$$

Theorem 2.2 (SOS-Convexity and representation of non-negativity) *Let p_j and g_i be SOS-convex polynomials for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$. Assume that the set $\mathcal{F} := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in \mathbb{N}_m\}$ is nonempty. Then, the following statements are equivalent*:

(i) $g_i(x) \leq 0, i \in \mathbb{N}_m \Rightarrow \max_{j \in \mathbb{N}_r} p_j(x) \geq 0$.
(ii) $(\forall \varepsilon > 0) \ (\exists \bar{\delta} \in \Delta, \bar{\lambda} \in \mathbb{R}_+^m, \bar{\sigma} \in \Sigma_d^2) \ \sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i + \varepsilon = \bar{\sigma}$.

Proof (ii) $\Rightarrow$ (i) Suppose that for each $\varepsilon > 0$, there exist $\bar{\delta} \in \Delta$, $\bar{\lambda} \in \mathbb{R}_+^m$ and $\bar{\sigma} \in \Sigma_d^2$ such that $\sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i + \varepsilon = \bar{\sigma}$. Then, for any $x \in \mathcal{F}$ we have

$$\max_{\delta \in \Delta} \sum_{j=1}^r \delta_j p_j(x) + \varepsilon \geq \sum_{j=1}^r \bar{\delta}_j p_j(x) + \varepsilon = \bar{\sigma}(x) - \sum_{i=1}^m \bar{\lambda}_i g_i(x) \geq 0.$$

Letting $\varepsilon \to 0$, we see that $\max_{j \in \mathbb{N}_r} p_j(x) \geq 0$ for all $x \in \mathcal{F}$.

(i) $\Rightarrow$ (ii) Assume that (i) holds and let $\varepsilon > 0$ arbitrary. Then, by Theorem 2.1, there exist $\bar{\delta} \in \Delta$ and $\bar{\lambda} \in \mathbb{R}_+^m$ such that

$$L := \sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i + \varepsilon > 0.$$

Since p_j and g_i are all SOS-convex polynomials, then L is a (non-negative) SOS-convex polynomial too. Hence, Corollary 2.1 ensures that L is a sum-of-squares polynomial (of degree at most d), that is, there exist $\bar{\sigma} \in \Sigma_d^2$ such that

$$\sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i + \varepsilon = \bar{\sigma}.$$

Thus, the conclusion follows. $\qquad \square$

3 Duality for Minimax Programs with SOS-Convex Polynomials

In this section, we introduce the dual problem for our minimax model problem and establish duality theorems whenever the functions involved are SOS-convex polynomials.

Consider the minimax programming problem

$$(P) \quad \inf_{x \in \mathbb{R}^n} \max_{j \in \mathbb{N}_r} p_j(x)$$

$$\text{s.t.} \quad g_i(x) \leq 0, \quad i \in \mathbb{N}_m, \tag{2}$$

and its associated dual problem

$$(D) \quad \sup \mu$$

$$\text{s.t.} \quad \sum_{j=1}^r \delta_j p_j + \sum_{i=1}^m \lambda_i g_i - \mu \in \Sigma_d^2 \tag{3}$$

$$\delta \in \Delta, \quad \lambda \in \mathbb{R}_+^m, \quad \mu \in \mathbb{R},$$

where p_j and g_i are real polynomials on $\mathbb{R}^n$ for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$ and d is the smallest even number such that $d \geq \max\{\max_{j \in \mathbb{N}_r} \deg p_j, \max_{i \in \mathbb{N}_m} \deg g_i\}$.

Throughout the paper, the set $\mathcal{F} := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in \mathbb{N}_m\}$ is assumed to be nonempty.

It is well known that optimization problems of the form (D) can equivalently be reformulated as semidefinite programming problem [7]. See the appendix for details. For instance, consider the quadratic optimization problem (P^{cq}) where p_j and g_i are all quadratic functions, that is, for all $x \in \mathbb{R}^n$, $p_j(x) = x^T A_j x + a_j^T x + \alpha_j$ and $g_i(x) = x^T C_i x + c_i^T x + \gamma_i$, with $A_j, C_i \in \mathbb{S}^n$, the space of all symmetric $(n \times n)$ matrices, $a_j, c_i \in \mathbb{R}^n$ and $\alpha_j, \gamma_i \in \mathbb{R}$ for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$, that is,

$$(P^{cq}) \quad \inf_{x \in \mathbb{R}^n} \max_{j \in \mathbb{N}_r} x^T A_j x + a_j^T x + \alpha_j$$

$$\text{s.t.} \quad x^T C_i x + c_i^T x + \gamma_i \leq 0, \quad \forall i \in \mathbb{N}_m. \tag{4}$$

In this particular case, the sum-of-squares constraint in its associated dual problem $\sum_{j=1}^{r} \delta_j p_j + \sum_{i=1}^{m} \lambda_i g_i - \mu \in \Sigma_2^2$ is equivalent to $\sum_{j=1}^{r} \delta_j p_j + \sum_{i=1}^{m} \lambda_i g_i - \mu \geq 0$. This, in turn (see [17, p. 163]), is equivalent to

$$\begin{pmatrix} \sum_{j=1}^{r} \delta_j \alpha_j + \sum_{i=1}^{m} \lambda_i \gamma_i - \mu & \frac{1}{2}(\sum_{j=1}^{r} \delta_j a_j^T + \sum_{i=1}^{m} \lambda_i c_i^T) \\ \frac{1}{2}(\sum_{j=1}^{r} \delta_j a_j + \sum_{i=1}^{m} \lambda_i c_i) & \sum_{j=1}^{r} \delta_j A_j + \sum_{i=1}^{m} \lambda_i C_i \end{pmatrix} \succeq 0.$$

Therefore, the dual problem of (P^{cq}) becomes

$$(D^{cq}) \quad \sup \mu$$

$$\text{s.t.} \quad \sum_{j=1}^{r} \delta_j \begin{pmatrix} 2\alpha_j & a_j^T \\ a_j & 2A_j \end{pmatrix} + \sum_{i=1}^{m} \lambda_i \begin{pmatrix} 2\gamma_i & c_i^T \\ c_i & 2C_i \end{pmatrix} - \mu \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \succeq 0, \tag{5}$$

$$\delta \in \Delta, \quad \lambda \in \mathbb{R}_+^m, \quad \mu \in \mathbb{R},$$

which is clearly a semidefinite programming problem.

Lemma 3.1 *Let p_j and g_i be convex polynomials for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$. Then,*

$$\inf(P) = \sup_{\delta \in \Delta, \lambda \in \mathbb{R}_+^m} \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{r} \delta_j p_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\}.$$

Proof Note that, for any $\bar{x} \in \mathcal{F}$, $\bar{\delta} \in \Delta$ and $\bar{\lambda} \in \mathbb{R}_+^m$, one has

$$\max_{j \in \mathbb{N}_r} p_j(\bar{x}) \geq \sum_{j=1}^{r} \bar{\delta}_j p_j(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \geq \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^{r} \bar{\delta}_j p_j(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) \right\}.$$

Therefore, $\inf(P) \geq \sup_{\delta \in \Delta, \lambda \in \mathbb{R}_+^m} \inf_{x \in \mathbb{R}^n} \{\sum_{j=1}^{r} \delta_j p_j(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}$.

To see the reverse inequality, we may assume without any loss of generality that $\inf(P) > -\infty$, otherwise the conclusion follows immediately. Since $\mathcal{F} \neq \emptyset$, we have

$\mu^* := \inf(P) \in \mathbb{R}$. Then, for $\varepsilon > 0$ arbitrary, as $\max_{j \in \mathbb{N}_r}\{p_j(x) - \mu^*\} \geq 0$ for all $x \in \mathcal{F}$, by Theorem 2.1 we see that there exist $\bar{\delta} \in \Delta$ and $\bar{\lambda} \in \mathbb{R}_+^m$ such that $\sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i > \mu^* - \varepsilon$. Consequently,

$$\sup_{\delta \in \Delta, \lambda \in \mathbb{R}_+^m} \inf_{x \in \mathbb{R}^n} \left\{ \sum_{j=1}^r \delta_j p_j(x) + \sum_{i=1}^m \lambda_i g_i(x) \right\} \geq \mu^* - \varepsilon.$$

Since the above inequality holds for any $\varepsilon > 0$, passing to the limit we obtain the desired inequality, which concludes the proof. $\qquad\square$

As a consequence of Lemma 3.1, we derive the following zero duality gap result for (P).

Theorem 3.1 (Zero duality gap) *Let p_j and g_i be SOS-convex polynomials for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$. Then, $\inf(P) = \sup(D)$.*

Proof For any feasible points $\bar{x} \in \mathcal{F}$ and $\bar{\delta} \in \Delta$, $\bar{\lambda} \in \mathbb{R}_+^m$ and $\bar{\mu} \in \mathbb{R}$ such that $\sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i - \bar{\mu} = \bar{\sigma} \in \Sigma_d^2$,

$$\sum_{j=1}^r \bar{\delta}_j (p_j(\bar{x}) - \bar{\mu}) = \sum_{j=1}^r \bar{\delta}_j p_j(\bar{x}) - \bar{\mu} = \bar{\sigma}(\bar{x}) - \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) \geq 0.$$

Then, there exists $j_0 \in \mathbb{N}_r$ such that $p_{j_0}(\bar{x}) - \bar{\mu} \geq 0$, and so, $\bar{\mu} \leq \max_{j \in \mathbb{N}_r} p_j(\bar{x})$. Thus, $\sup(D) \leq \inf(P)$.

To see the reverse inequality, we may assume without any loss of generality that $\inf(P) > -\infty$, otherwise the conclusion follows immediately. Since $\mathcal{F} \neq \emptyset$, we have $\mu^* := \inf(P) \in \mathbb{R}$. Then, as a consequence of Lemma 3.1, for $\varepsilon > 0$ arbitrary we have

$$\sup_{\delta \in \Delta, \lambda \in \mathbb{R}_+^m, \mu \in \mathbb{R}} \left\{ \mu : \sum_{j=1}^r \delta_j p_j + \sum_{i=1}^m \lambda_i g_i - \mu \geq 0 \right\} \geq \mu^* - \varepsilon.$$

As p_j and g_i are all SOS-convex polynomials, then $L := \sum_{j=1}^r \delta_j p_j + \sum_{i=1}^m \lambda_i g_i - \mu$ is a SOS-convex polynomial too. So, by Corollary 2.1, L is non-negative if and only if $L \in \Sigma_d^2$. Hence, $\mu^* - \varepsilon \leq \sup(D)$. Since the previous inequality holds for any $\varepsilon > 0$, passing to the limit we get $\mu^* \leq \sup(D)$, which concludes the proof. $\qquad\square$

Corollary 3.1 *Let f and g_i be SOS-convex polynomials for all $i \in \mathbb{N}_m$. Then,*

$$\inf_{x \in \mathbb{R}^n} \left\{ f(x) : g_i(x) \leq 0, i \in \mathbb{N}_m \right\} = \sup_{\lambda \in \mathbb{R}_+^m, \mu \in \mathbb{R}} \left\{ \mu : f + \sum_{i=1}^m \lambda_i g_i - \mu \in \Sigma_d^2, \right\}$$

where d is the smallest even number such that $d \geq \max\{\deg f, \max_{i \in \mathbb{N}_m} \deg g_i\}$.

Proof It is a straightforward consequence of Theorem 3.1 by taking $r = 1$. $\qquad\square$

Remark 3.1 It is worth observing that the conclusion of Theorem 3.1 can be also obtained from Corollary 3.1. To see this, let us consider the following SOS-convex polynomial optimization problem associated with (P):

$$(P_e) \quad \inf_{(x,z)\in\mathbb{R}^n\times\mathbb{R}} z$$

$$\text{s.t.} \quad p_j(x) - z \leq 0, \quad \forall j \in \mathbb{N}_r, \tag{6}$$

$$g_i(x) \leq 0, \quad \forall i \in \mathbb{N}_m.$$

Then, $\inf(P) = \inf(P_e)$. Corollary 3.1 gives us that $\inf(P_e) = \sup(D_e)$, where the dual problem (D_e) is given by

$$\sup_{\substack{\mu\in\mathbb{R},\sigma\in\Sigma^2 \\ \lambda\in\mathbb{R}_+^m,\delta\in\mathbb{R}_+^r}} \left\{ \mu : z + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{j=1}^r \delta_j\left(p_j(x) - z\right) - \mu = \sigma(x,z), \forall (x,z) \in \mathbb{R}^{n+1} \right\}$$

The constraint in the above dual problem can be written as follows:

$$z\left(1 - \sum_{j=1}^r \delta_j\right) + \sum_{j=1}^r \delta_j p_j(x) + \sum_{i=1}^m \lambda_i g_i(x) - \mu = \sigma(x,z) \tag{7}$$

for all $(x,z) \in \mathbb{R}^{n+1}$. Let $\gamma := 1 - \sum_{j=1}^r \delta_j$ and fix $x = x_0 \in \mathbb{R}^n$. Following the arguments employed in [18], we claim that $\gamma = 0$. If $\gamma > 0$, letting $z \in \mathbb{R}$ small enough we find that the right-hand side of (7) becomes negative, which contradicts the fact that $\sigma(x_0,z) \geq 0$ for all $z \in \mathbb{R}$. On the other hand, if $\gamma < 0$, letting $z \in \mathbb{R}$ large enough, we see that the right-hand side of (7) becomes negative, which implies again a contradiction. Consequently, $\gamma = 0$ and σ does not depend on z. So, the problem (D_e) collapses to problem (D) introduced in (3).

We now see that whenever the Slater condition,

$$\left\{ x \in \mathbb{R}^n : g_i(x) < 0, i \in \mathbb{N}_m \right\} \neq \emptyset,$$

is satisfied, strong duality between (P) and (D) holds.

Theorem 3.2 (Strong duality) *Let p_j and g_i be SOS-convex polynomials for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$. If the Slater condition holds, then $\inf(P) = \max(D)$.*

Proof Let $f := \max_{j\in\mathbb{N}_r} p_j$ and $\mu^* := \inf(P) \in \mathbb{R}$. Thus, since the Slater condition is fulfilled, by the usual convex programming duality and the convex-concave minimax theorem, we get

$$\mu^* \leq \inf(P) = \inf_{x\in\mathbb{R}^n} \left\{ f(x) : g_i(x) \leq 0, i \in \mathbb{N}_m \right\}$$

$$= \max_{\lambda\in\mathbb{R}_+^m} \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \lambda_i g_i(x) \right\}$$

$$= \max_{\lambda \in \mathbb{R}_+^m} \inf_{x \in \mathbb{R}^n} \max_{\delta \in \Delta} \left\{ \sum_{i=1}^r \delta_j p_j(x) + \sum_{i=1}^m \lambda_i g_i(x) \right\}$$

$$= \max_{\lambda \in \mathbb{R}_+^m, \delta \in \Delta} \inf_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^r \delta_j p_j(x) + \sum_{i=1}^m \lambda_i g_i(x) \right\}.$$

Hence, there exist $\bar{\lambda} \in \mathbb{R}_+^m$ and $\bar{\delta} \in \Delta$ such that

$$L := \sum_{j=1}^r \bar{\delta}_j p_j + \sum_{i=1}^m \bar{\lambda}_i g_i - \mu^* \geq 0.$$

As p_j and g_i are all SOS-convex polynomials, L is a (non-negative) SOS-convex polynomial too, and consequently, in virtue of Corollary 2.1, L is a sum-of-squares polynomial (of degree at most d). Hence, $(\bar{\delta}, \bar{\lambda}, \mu^*)$ is a feasible point of (D), so $\mu^* \leq \sup(D)$. Since weak duality always holds, we conclude $\inf(P) = \max(D)$.   □

Recall the minimax quadratic programming problem (P^{cq}) introduced in (4) and its dual problem (D^{cq}) given in (5). Note that the set of all $(n \times n)$ positive semidefinite matrices is denoted by $\mathbb{S}_+^n$.

Corollary 3.2 *Let* $A_j, C_i \in \mathbb{S}_+^n$, $a_j, c_i \in \mathbb{R}^n$, *and* $\alpha_j, \gamma_i \in \mathbb{R}$ *for all* $j \in \mathbb{N}_r$ *and* $i \in \mathbb{N}_m$. *If there exists* $\bar{x} \in \mathbb{R}^n$ *such that* $\bar{x}^T C_i \bar{x} + c_i^T \bar{x} + \gamma_i < 0$ *for all* $i \in \mathbb{N}_m$, *then* $\inf(P^{cq}) = \max(D^{cq})$.

Proof As $A_j, C_i \in \mathbb{S}_+^n$ for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$, all the quadratic functions involved in (P^{cq}) are convex. Hence, since the Slater condition holds and any convex quadratic function is a SOS-convex polynomial, by applying Theorem 3.2 we get $\inf(P^{cq}) = \max(D^{cq})$.   □

Remark 3.2 (Attainment of the optimal value) For the problem (P) introduced in (2), note that, if $f := \max_{j \in \mathbb{N}_r} p_j$ (which is not a polynomial, in general) is bounded from below on the nonempty set $\mathcal{F}$, then f attains its minimum on $\mathcal{F}$. In other words, if $\inf(P) \in \mathbb{R}$, then there exists $x^* \in \mathcal{F}$ such that $f(x^*) = \min(P)$. To see this, let consider again the convex polynomial optimization problem (P_e) introduced in (6). Let $\mathcal{F}_e$ be the (nonempty) feasible set of (P_e). Observe that $x_0 \in \mathcal{F}$ implies $(x_0, z_0) \in \mathcal{F}_e$ for all $z_0 \geq f(x_0)$, and conversely, $(x_0, z_0) \in \mathcal{F}_e$ implies $x_0 \in \mathcal{F}$. Moreover, one has $\inf(P) = \inf(P_e)$. Thus, Lemma 2.2 can be applied to problem (P_e) and then, there exists $(x^*, z^*) \in \mathcal{F}_e$ such that $z^* = \min(P_e)$. Since $z^* \leq z$ for all $(x, z) \in \mathcal{F}_e$ and $(x, f(x)) \in \mathcal{F}_e$ for all $x \in \mathcal{F}$, then we get

$$z^* \leq f(x) \quad \forall x \in \mathcal{F}. \tag{8}$$

On the other hand, as $(x^*, z^*) \in \mathcal{F}_e$ we get $x^* \in \mathcal{F}$ and

$$f(x^*) \leq z^*. \tag{9}$$

Combining (8) and (9), we conclude $f(x^*) \leq f(x)$ for all $x \in \mathcal{F}$, and so, x^* is a minimizer of (P).

Recall that the subdifferential of the (convex) function f at $x \in \mathbb{R}^n$ is defined to be the set

$$\partial f(x) := \{v \in \mathbb{R}^n : f(y) \geq f(x) + v^T(y - x), \ \forall y \in \text{dom } f\}.$$

For a convex set $C \subset \mathbb{R}^n$, the normal cone of C of at $x \in C$ is given by

$$N_C(x) := \{v \in \mathbb{R}^n : v^T(y - x) \leq 0, \ \forall y \in C\}.$$

We will say that the *normal cone condition* holds for $\mathcal{F}$ at $x \in \mathcal{F}$, provided that

$$N_{\mathcal{F}}(x) = \left\{\sum_{i=1}^m \lambda_i \nabla g_i(x) : \lambda \in \mathbb{R}_+^m, \sum_{i=1}^m \lambda_i g_i(x) = 0\right\}.$$

It is known that the Slater condition guarantees the normal cone condition (see [19, Sect. 2.2]). However, the converse statement is not true in general. The following example[1] shows this fact.

Example 3.1 Let $g_1(x_1, x_2) := x_1 + x_2^2$, $g_2(x_1, x_2) := -x_1 + x_2^2$, $g_3(x_1, x_2) := x_1^2 + x_2$ and $g_4(x_1, x_2) := x_1^2 - x_2$, for all $x \in \mathbb{R}^2$. Then, it is easy to check that each g_i is a SOS-convex polynomial and $\mathcal{F} := \{x \in \mathbb{R}^2 : g_i(x) \leq 0, i \in \mathbb{N}_4\} = \{(0, 0)\}$. So, $N_{\mathcal{F}}(x) = \mathbb{R}^2$ for all $x \in \mathcal{F}$. Since, for all $x \in \mathcal{F}$,

$$\left\{\sum_{i=1}^4 \lambda_i \nabla g_i(x) : \lambda \in \mathbb{R}_+^4, \sum_{i=1}^4 \lambda_i g_i(x) = 0\right\} = \bigcup_{\lambda \in \mathbb{R}_+^4} (\lambda_1 - \lambda_2, \lambda_3 - \lambda_4) = \mathbb{R}^2,$$

then the normal condition holds. On the other hand, it is easy to see that the Slater condition fails.

Theorem 3.3 (Min-max duality) *Let p_j and g_i be SOS-convex polynomials for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$. Let $x^* \in \mathcal{F}$ be an optimal solution of (P) and assume that the normal cone condition for $\mathcal{F}$ at x^* holds. Then, $\min(P) = \max(D)$.*

Proof Let $f := \max_{j \in \mathbb{N}_r} p_j$ and $\mu^* := \min(P) \in \mathbb{R}$. If $x^* \in \mathcal{F}$ is an optimal solution of (P), that is, $f(x^*) = \mu^*$, then by optimality conditions we have $0 \in \partial f(x^*) + N_{\mathcal{F}}(x^*)$. As a consequence of the normal cone condition for $\mathcal{F}$ at x^* and [20, Proposition 2.3.12], we get

$$0 = \sum_{j=1}^r \bar{\delta}_j \nabla p_j(x^*) + \sum_{i=1}^m \bar{\lambda}_i \nabla g_i(x^*)$$

for some $\bar{\lambda} \in \mathbb{R}_+^m$ with $\bar{\lambda}_i g_i(x^*) = 0$ for all $i \in \mathbb{N}_m$, and $\bar{\delta} \in \Delta$ with $\bar{\delta}_j = 0$ for those $j \in \mathbb{N}_r$ such that $p_j(x^*) \neq \mu^*$. Note that the polynomial

$$L := \sum_{j=1}^{r} \bar{\delta}_j p_j + \sum_{i=1}^{m} \bar{\lambda}_i g_i - \mu^*$$

satisfies $L(x^*) = 0$ and $\nabla L(x^*) = 0$. Moreover, L is a SOS-convex polynomial since p_j and g_i are all SOS-convex polynomials. Then, as a consequence of Lemma 2.1, L is a sum-of-squares polynomial (of degree at most d). Then, $(\bar{\delta}, \bar{\lambda}, \mu^*)$ is a feasible point of (D), so $\mu^* \leq \sup(D)$. Since weak duality always holds, we conclude $\min(P) = \max(D)$. ∎

The following simple example illustrates the above min-max duality theorem.

Example 3.2 Consider the optimization problem

$$(P_1) \quad \min_{x \in \mathbb{R}} \{ \max\{2x^4 - x, 5x^2 + x\} : x \geq -2 \}.$$

It is easy to check that $x^* = 0$ is a minimizer of (P_1) and $\min(P_1) = 0$. The corresponding dual problem of (P_1) is

$$(D_1) \quad \max_{\delta \geq 0, \lambda \geq 0, \mu \in \mathbb{R}} \{ \mu : \delta(2x^4 - x) + (1-\delta)(5x^2 + x) - \lambda(x+2) - \mu \in \Sigma_4^2 \}.$$

As $x^4 + \frac{5}{2}x^2 \in \Sigma_4^2$, $\delta = \frac{1}{2}$, $\lambda = 0$ and $\mu = 0$ is a feasible point of (D_1). So, $\sup(D_1) \geq 0$. On the other hand, the sum-of-squares constraint in (D_1) gives us $-2\lambda - \mu \geq 0$. Consequently, $\mu \leq -2\lambda \leq 0$, which implies $\max(D) = 0$.

4 Applications to Robust Optimization and Rational Programs

In this section, we provide applications of our duality theorems to robust SOS-convex programming problems under data uncertainty and to rational programming problems.

Let us consider the following optimization program with the data uncertainty in the constraints and in the objective function:

$$(UP) \quad \inf_{x \in \mathbb{R}^n} \{ f_0(x, v_0) : f_i(x, v_i) \leq 0, \ \forall i = 1, \ldots, k \},$$

where, for each $i \in \{0\} \cup \mathbb{N}_k$, v_i is an uncertain parameter and $v_i \in \mathcal{V}_i$ for some $\mathcal{V}_i \subset \mathbb{R}^{n_i}$. The robust counterpart of (UP) is given by

$$(RP) \quad \inf_{x \in \mathbb{R}^n} \sup_{v_0 \in \mathcal{V}_0} \{ f_0(x, v_0) \ : \ f_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \forall i = 1, \ldots, k \}.$$

Theorem 4.1 (Finite data uncertainty) *Let $f_i(\cdot, v_i)$ be a SOS-convex polynomial for each $v_i \in \mathcal{V}_i := \{v_i^1, \ldots, v_i^{s_i}\}$ and each $i \in \{0\} \cup \mathbb{N}_k$ and let $r := s_0$. Assume that*

there exists $\bar{x} \in \mathbb{R}^n$ such that $f_i(\bar{x}, v_i^j) < 0$ for all $j \in \mathbb{N}_{s_i}$ and $i \in \mathbb{N}_k$. Then, one has $\inf(RP) = \max(RD)$ *where*

$$(RD) \quad \sup \mu$$

$$s.t. \quad \sum_{l=1}^{r} \delta_l f_0(\cdot, v_0^l) + \sum_{i=1}^{k} \sum_{j=1}^{s_i} \lambda_i^j f_i(\cdot, v_i^j) - \mu \in \Sigma_t^2 \qquad (10)$$

$$\delta \in \Delta, \quad \lambda_i \in \mathbb{R}_+^{s_i} \ (\forall i \in \mathbb{N}_k), \quad \mu \in \mathbb{R},$$

and t is the smallest even number such that

$$t \geq \max \left\{ \max_{l \in \mathbb{N}_r} \deg f_0(\cdot, v_0^l), \max_{i \in \mathbb{N}_k} \max_{j \in \mathbb{N}_{s_i}} \deg f_i(\cdot, v_i^j) \right\}.$$

Proof It is easy to see that problem (*RP*) is equivalent to

$$(RP_e) \quad \inf \max_{j \in \mathbb{N}_r} \left\{ f_0(x, v_0^j) \ : \ f_i(x, v_i^j) \leq 0, \ \forall j \in \mathbb{N}_{s_i}, \forall i = 1, \ldots, k \right\}.$$

Since the Slater condition holds, we conclude $\inf(RP_e) = \max(RD)$ by applying Theorem 3.2. $\qquad \square$

Theorem 4.2 (Polytopic data uncertainty) *Suppose that, for each $i \in \{0\} \cup \mathbb{N}_k$, $x \mapsto f_i(x, v_i)$ is a SOS-convex polynomial for each $v_i \in \mathcal{V}_i := \text{conv}\{v_i^1, \ldots, v_i^{s_i}\}$ with $r := s_0$, and $v_i \mapsto f_i(x, v_i)$ is affine for each $x \in \mathbb{R}^n$. Assume that there exists $\bar{x} \in \mathbb{R}^n$ such that $f_i(\bar{x}, v_i^j) < 0$ for all $j \in \mathbb{N}_{s_i}$ and $i \in \mathbb{N}_k$. Then, one has $\inf(RP) = \max(RD)$, where the problem (RD) is defined in (10).*

Proof Let $i \in \mathbb{N}_k$. As $f_i(x, \cdot)$ is affine for each $x \in \mathbb{R}^n$, then $f_i(x, v_i) \leq 0$ for all $v_i \in \mathcal{V}_i := \text{conv}\{v_i^1, \ldots, v_i^{s_i}\}$ if and only if $f_i(x, v_i^j) \leq 0$ for all $j \in \mathbb{N}_{s_i}$. Moreover, we see that

$$\sup_{v_0 \in \mathcal{V}_0} f_0(x, v_0) = \max_{j \in \mathbb{N}_r} f_0(x, v_0^j).$$

Hence, problem (*RP*) is equivalent to (*RP_e*). Reasoning as in the proof of the above theorem, we conclude $\inf(RP) = \max(RD)$. $\qquad \square$

Now, consider the following minimax rational programming problem,

$$(\mathcal{P}) \quad \inf_{x \in \mathbb{R}^n} \left\{ \max_{j \in \mathbb{N}_r} \frac{p_j(x)}{q(x)} : g_i(x) \leq 0, i \in \mathbb{N}_m \right\},$$

where p_j, for $j \in \mathbb{N}_r$, q, and g_i, for $i \in \mathbb{N}_m$, are real polynomials on $\mathbb{R}^n$, and for each $j \in \mathbb{N}_r$, $p_j(x) \geq 0$ and $q(x) > 0$ over the feasible set $\mathcal{F}$, which is assumed to be nonempty. This is a generalization of problem (*P*) introduced in (2). For related minimax fractional programs, see [21, 22]. Minimax fractional programs often appear in resource allocation and planning problems of management science where the

objective function in their optimization problems involve ratios such as cost or profit in time, return on capital and earnings per share (see [23]).

We associate with $(\mathcal{P})$ the following SDP dual problem

$$(\mathcal{D}) \quad \sup \mu$$

$$\text{s.t.} \quad \sum_{j=1}^{r} \delta_j p_j + \sum_{i=1}^{m} \lambda_i g_i - \mu q \in \Sigma_d^2 \tag{11}$$

$$\delta \in \Delta, \quad \lambda \in \mathbb{R}_+^m, \quad \mu \in \mathbb{R},$$

where d is the smallest even number such that

$$d \geq \max \left\{ \deg q, \max_{j \in \mathbb{N}_r} \deg p_j, \max_{i \in \mathbb{N}_m} \deg g_i \right\}.$$

It is worth noting that, in general, problem $(\mathcal{P})$ may not attain its optimal value when it is finite, even when $r = 1$. To see this, consider the rational programming problem $(\mathcal{P}_1) \inf_{x \in \mathbb{R}} \{ \frac{1}{x} : 1 - x \leq 0 \}$. Obviously, $\inf(\mathcal{P}_1) = 0$, however, for any feasible point x, one has $\frac{1}{x} > 0$. Thus, the optimal value of $(\mathcal{P}_1)$ is not attained.

Theorem 4.3 (Strong duality for minimax rational programs) *Let p_j, g_i and $-q$ be SOS-convex polynomials for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$, such that $p_j(x) \geq 0$ and $q(x) > 0$ for all $x \in \mathcal{F}$. If the Slater condition holds, then*

$$\inf(\mathcal{P}) = \max(\mathcal{D}).$$

Proof Note that, for any $\mu \in \mathbb{R}_+$, one has $\inf(\mathcal{P}) \geq \mu$ if and only if $\inf(P_\mu) \geq 0$, where

$$(P_\mu) \quad \inf_{x \in \mathcal{F}} \max_{j \in \mathbb{N}_r} \{ p_j(x) - \mu q(x) \}. \tag{12}$$

By the assumption, $\inf(\mathcal{P})$ is finite. So, it follows easily that $\mu^* := \inf(\mathcal{P}) \in \mathbb{R}_+$ and then $\inf(P_{\mu^*}) \geq 0$. Since, for each $j \in \mathbb{N}_r$, $p_j - \mu^* q$ is a SOS-convex polynomial and the Slater condition holds, by Theorem 3.2, we have $\inf(P_{\mu^*}) = \max(D_{\mu^*})$, where

$$(D_{\mu^*}) \quad \sup \theta$$

$$\text{s.t.} \quad \sum_{j=1}^{r} \delta_j p_j + \sum_{i=1}^{m} \lambda_i g_i - \mu^* q - \theta \in \Sigma_d^2 \tag{13}$$

$$\delta \in \Delta, \quad \lambda \in \mathbb{R}_+^m, \quad \theta \in \mathbb{R}.$$

As $\max(D_{\mu^*}) = \inf(P_{\mu^*}) \geq 0$, there exist $\bar{\delta} \in \Delta$, $\bar{\lambda} \in \mathbb{R}_+^m$ and $\bar{\theta} \in \mathbb{R}_+$ such that

$$\sum_{j=1}^{r} \bar{\delta}_j p_j + \sum_{i=1}^{m} \bar{\lambda}_i g_i - \mu^* q \in (\bar{\theta} + \Sigma_d^2) \subset \Sigma_d^2.$$

Therefore, $(\bar{\delta}, \bar{\lambda}, \mu^*)$ is a feasible point of $(\mathcal{D})$, so $\mu^* \leq \sup(\mathcal{D})$. Since weak duality always holds, we conclude $\inf(\mathcal{P}) = \max(\mathcal{D})$. $\qquad \square$

Let us consider the particular problem ($\mathcal{P}^{cq}$) where p_j, q and g_i are all quadratic functions; that is, $p_j(x) = x^T A_j x + a_j^T x + \alpha_j, q(x) = x^T B x + b^T x + \beta$ and $g_i(x) = x^T C_i x + c_i^T x + \gamma_i$ for all $x \in \mathbb{R}^n$, with $A_j, B, C_i \in \mathbb{S}^n$, $a_j, b, c_i \in \mathbb{R}^n$ and $\alpha_j, \beta, \gamma_i \in \mathbb{R}$ for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$; that is,

$$(\mathcal{P}^{cq}) \quad \inf_{x \in \mathbb{R}^n} \max_{j \in \mathbb{N}_r} \frac{x^T A_j x + a_j^T x + \alpha_j}{x^T B x + b^T x + \beta}$$
$$\text{s.t.} \quad x^T C_i x + c_i^T x + \gamma_i \leq 0, \quad i \in \mathbb{N}_m.$$

Assume that $p_j(x) \geq 0$ and $q(x) > 0$ over the feasible set. The dual problem of ($\mathcal{P}^{cq}$) is given by

$$(\mathcal{D}^{cq}) \quad \sup \mu$$
$$\text{s.t.} \quad \sum_{j=1}^{r} \delta_j \begin{pmatrix} 2\alpha_j & a_j^T \\ a_j & 2A_j \end{pmatrix} + \sum_{i=1}^{m} \lambda_i \begin{pmatrix} 2\gamma_i & c_i^T \\ c_i & 2C_i \end{pmatrix} - \mu \begin{pmatrix} 2\beta & b^T \\ b & 2B \end{pmatrix} \succeq 0,$$
$$\delta \in \Delta, \quad \lambda \in \mathbb{R}_+^m, \quad \mu \in \mathbb{R},$$

which is clearly a semidefinite programming problem.

Corollary 4.1 *Let consider the problem* ($\mathcal{P}^{cq}$) *such that* $A_j, -B, C_i \in \mathbb{S}_+^n$ *for all* $j \in \mathbb{N}_r$ *and* $i \in \mathbb{N}_m$. *If there exists* $\bar{x} \in \mathbb{R}^n$ *such that* $\bar{x}^T C_i \bar{x} + c_i^T \bar{x} + \gamma_i < 0$ *for all* $i \in \mathbb{N}_m$, *then*

$$\inf(\mathcal{P}^{cq}) = \max(\mathcal{D}^{cq}).$$

Proof Note that the sum-of-squares constraint in its associated dual problem $\sum_{j=1}^{r} \delta_j p_j + \sum_{i=1}^{m} \lambda_i g_i - \mu q \in \Sigma_2^2$ is equivalent to $\sum_{j=1}^{r} \delta_j p_j + \sum_{i=1}^{m} \lambda_i g_i - \mu q \geq 0$. This is further equivalent to

$$\begin{pmatrix} \sum_{j=1}^{r} \delta_j \alpha_j + \sum_{i=1}^{m} \lambda_i \gamma_i - \mu\beta & \frac{1}{2}(\sum_{j=1}^{r} \delta_j a_j^T + \sum_{i=1}^{m} \lambda_i c_i^T - \mu b^T) \\ \frac{1}{2}(\sum_{j=1}^{r} \delta_j a_j + \sum_{i=1}^{m} \lambda_i c_i - \mu b) & \sum_{j=1}^{r} \delta_j A_j + \sum_{i=1}^{m} \lambda_i C_i - \mu B \end{pmatrix} \succeq 0.$$

So, our dual problem ($\mathcal{D}$) collapses to ($\mathcal{D}^{cq}$). Since the Slater condition holds and any convex quadratic function is a SOS-convex polynomial, by applying Theorem 4.3 we get $\inf(\mathcal{P}^{cq}) = \max(\mathcal{D}^{cq})$. $\qquad\square$

Corollary 4.2 *Let* p, g_i *and* $-q$ *be SOS-convex polynomials for all* $i \in \mathbb{N}_m$, *such that* $p(x) \geq 0$ *and* $q(x) > 0$ *for all* $x \in \mathcal{F}$. *If the Slater condition holds, then*

$$\inf_{x \in \mathcal{F}} \frac{p(x)}{q(x)} = \max_{\mu \in \mathbb{R}, \lambda \in \mathbb{R}_+^m} \left\{ \mu : p + \sum_{i=1}^{m} \lambda_i g_i - \mu q \in \Sigma_k^2 \right\}$$

where k *is the smallest even number such that* $k \geq \max\{\deg p, \deg q, \max_{i \in \mathbb{N}_m} \deg g_i\}$.

Proof It is a straightforward consequence of Theorem 4.3 when $r = 1$. $\qquad\square$

Next we show that the non-negativity of the polynomials p_j can be dropped whenever q is an affine function.

Corollary 4.3 *Let p_j and g_i be SOS-convex polynomials for each $j \in \mathbb{N}_r$ and each $i \in \mathbb{N}_m, b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that $b^T x + \beta > 0$ for all $x \in \mathcal{F}$. If the Slater condition holds, then*

$$\inf_{x \in \mathcal{F}} \max_{j \in \mathbb{N}_r} \frac{p_j(x)}{b^T x + \beta} = \max_{\substack{\mu \in \mathbb{R} \\ \delta \in \Delta, \lambda \in \mathbb{R}_+^m}} \left\{ \mu : \sum_{j=1}^{r} \delta_j p_j + \sum_{i=1}^{m} \lambda_i g_i - \mu \left(b^T(\cdot) + \beta \right) \in \Sigma_d^2 \right\}.$$

Proof The proof follows the same line of arguments as the proof of Theorem 4.3, except that, in the case $q(x) := b^T x + \beta$ for all $x \in \mathbb{R}^n$, all polynomials $p_j - \mu^* q$ are SOS-convex without the non-negativity of all p_j's, and therefore, of μ^*. □

For the particular problem

$$\left(\mathcal{P}^l \right) \quad \inf_{x \in \mathbb{R}^n} \left\{ \max_{j \in \mathbb{N}_r} \frac{a_j^T x + \alpha_j}{b^T x + \beta} \ : \ c_i^T x + \gamma_i \leq 0, \ i \in \mathbb{N}_m \right\},$$

the corresponding dual problem can be stated as the following linear programming problem:

$$\left(\mathcal{D}^l \right) \quad \max \mu$$

$$\text{s.t.} \quad \sum_{j=1}^{r} \delta_j a_j + \sum_{i=1}^{m} \lambda_i c_i - \mu b = 0, \tag{14}$$

$$\sum_{j=1}^{r} \delta_j \alpha_j + \sum_{i=1}^{m} \lambda_i \gamma_i - \mu \beta \geq 0, \tag{15}$$

$$\delta \in \Delta, \quad \lambda \in \mathbb{R}_+^m, \quad \mu \in \mathbb{R}.$$

Corollary 4.4 *Let $\alpha_j, \beta, \gamma_i \in \mathbb{R}$ and $a_j, b, c_i \in \mathbb{R}^n$ for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$. Assume that $b^T x + \beta > 0$ for all feasible point x of $\mathcal{P}^l$. Then,*

$$\inf(\mathcal{P}^l) = \max(\mathcal{D}^l).$$

Proof By applying Corollary 4.3, we find that $\inf(\mathcal{P}^l)$ equals

$$\max_{\substack{\mu \in \mathbb{R} \\ \delta \in \Delta, \lambda \in \mathbb{R}_+^m}} \left\{ \mu : \sum_{j=1}^{r} \delta_j \left(a_j^T x + \alpha_j \right) + \sum_{i=1}^{m} \lambda_i \left(c_i^T x + \gamma_i \right) - \mu \left(b^T x + \beta \right) \in \Sigma_2^2 \right\}.$$

Since the sum-of-squares constraint in the above dual problem is equivalent to (14) and (15), we conclude $\inf(\mathcal{P}^l) = \max(\mathcal{D}^l)$. □

If a minimizer x^* of $(\mathcal{P})$ is known, then the Slater condition in Theorem 4.3 can be replaced by a weaker condition in order to derive strong duality between $(\mathcal{P})$ and $(\mathcal{D})$.

Theorem 4.4 *Let p_j, g_i and $-q$ be SOS-convex polynomials for all $j \in \mathbb{N}_r$ and $i \in \mathbb{N}_m$, such that $p_j(x) \geq 0$ and $q(x) > 0$ for all $x \in \mathcal{F}$. Let $x^* \in \mathcal{F}$ be an optimal solution of $(\mathcal{P})$ and assume that the normal cone condition for $\mathcal{F}$ at x^* holds. Then, $\min(\mathcal{P}) = \max(\mathcal{D})$.*

Proof Let $\mu^* := \min(\mathcal{P}) \in \mathbb{R}_+$. Note that $(\mathcal{P})$ has optimal solution x^* with optimal value μ^* if and only if x^* is an optimal solution of (P_{μ^*}) with optimal value 0 (cf. [22, Lemma 2.3]), where (P_{μ^*}) is stated in (12). Since, for each $j \in \mathbb{N}_r$, $p_j - \mu^* q$ is a SOS-convex polynomial and the normal cone condition for $\mathcal{F}$ at x^* holds, by Theorem 3.3 we have $\min(P_{\mu^*}) = \max(D_{\mu^*})$, where (D_{μ^*}) has been stated in (13). As $\max(D_{\mu^*}) = 0$, there exist $\bar{\delta} \in \Delta$ and $\bar{\lambda} \in \mathbb{R}_+^m$ such that

$$\sum_{j=1}^{r} \bar{\delta}_j p_j + \sum_{i=1}^{m} \bar{\lambda}_i g_i - \mu^* q \in \Sigma_d^2.$$

Therefore, $(\bar{\delta}, \bar{\lambda}, \mu^*)$ is a feasible point of $(\mathcal{D})$, so $\mu^* \leq \sup(\mathcal{D})$. Since weak duality always holds, we conclude $\min(\mathcal{P}) = \max(\mathcal{D})$. □

5 Conclusions

In this paper, we have shown that various classes of minimax programs involving sum-of-squares convex (SOS-convex) polynomials exhibit zero duality gaps with their corresponding dual programs without any qualifications. The significance of the results is that the dual programs are representable as semidefinite programs which can efficiently be solved by interior-point methods. So, the optimal values of the original minimax problems can be found by solving their dual programs. Under constraint qualifications, we further derived strong duality results where the dual programs attain their optimal values. We applied our results to obtain corresponding results for robust SOS-convex programs under polytopic data uncertainty and to minimax fractional programs including minimax fractional programs involving quadratic functions.

Appendix: SDP Representations of Dual Programs

Finally, for the sake of completeness, we show how our dual problem $(\mathcal{D})$ given in (11) can be represented by a semidefinite linear programming problem. To this aim, let us recall some basic facts on the relationship between sum-of-squares polynomials and semidefinite programming problems.

We denote by $\mathbb{S}^n$ the space of symmetric $n \times n$ matrices. For any $A, B \in \mathbb{S}^n$, we write $A \succeq 0$ if and only if A is positive semidefinite, and $\langle A, B \rangle$ stands for trace(AB).

Let $\mathbb{S}_+^n := \{A \in \mathbb{S}^n : A \succeq 0\}$ be the closed and convex cone of positive semidefinite $n \times n$ (symmetric) matrices. The space of all real polynomials on $\mathbb{R}^n$ with degree d is denoted by $\mathbb{R}_d[x_1, \ldots, x_n]$ and its canonical basis is given by

$$y(x) \equiv \left(x^\alpha\right)_{|\alpha| \leq d} := \left(1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_2^2, \ldots, x_n^2, \ldots, x_1^d, \ldots, x_n^d\right)^T,$$

which has dimension $e(d, n) := \binom{n+d}{d}$, and $\alpha \in (\{0\} \cup \mathbb{N})^n$ is a multi-index such that $|\alpha| := \sum_{i=1}^n \alpha_i$. Let $\mathcal{N} := \{\alpha \in (\{0\} \cup \mathbb{N})^n : |\alpha| \leq d\}$. Thus, if f is a polynomial on $\mathbb{R}^n$ with degree at most d, one has

$$f(x) = \sum_{\alpha \in \mathcal{N}} f_\alpha x^\alpha.$$

Assume that d is an even number, and let $k := d/2$. Then, according to [7, Proposition 2.1], f is a sum-of-squares polynomial if and only if there exists $Q \in \mathbb{S}_+^{e(k,n)}$ such that $f(x) = y(x)^T Q y(x)$. By writing $y(x)y(x)^T = \sum_{\alpha \in \mathcal{N}} B_\alpha x^\alpha$ for appropriate matrices $(B_\alpha) \subset \mathbb{S}^{e(k,n)}$, one finds that f is a sum-of-squares polynomial if and only if there exists $Q \in \mathbb{S}_+^{e(k,n)}$ such that $\langle Q, B_\alpha \rangle = f_\alpha$ for all $\alpha \in \mathcal{N}$.

Using the above characterization, we see that our dual problem $(\mathcal{D})$ can be equivalently rewritten as the following semidefinite programming problem:

$(\mathcal{SD})$ $\sup \mu$

$$\text{s.t.} \quad \sum_{j=1}^r \delta_j (p_j)_\alpha + \sum_{i=1}^m \lambda_i (g_i)_\alpha - \mu q_\alpha = \langle Q, B_\alpha \rangle, \quad \forall \alpha \in \mathcal{N},$$

$$\sum_{j=1}^r \delta_j = 1,$$

$$\delta \in \mathbb{R}_+^r, \quad \lambda \in \mathbb{R}_+^m, \quad \mu \in \mathbb{R}, \quad Q \in \mathbb{S}_+^{e(k,n)}.$$

Letting $q_\alpha = 1$ for $\alpha = (0, \ldots, 0)$ and $q_\alpha = 0$ otherwise, we get the SDP representation for problem (D) in (3).

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