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Weak Subdifferentials for Set-Valued Mappings

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Abstract The purpose of this paper is to study the weak subdifferential for set-valued mappings, which was introduced by Chen and Jahn (Math. Methods Oper. Res., 48:187–200, 1998). Two existence theorems of weak subgradients for set-valued mappings are obtained. Moreover, some properties of the weak subdifferential for set-valued mappings are derived. Our results improve the corresponding ones in the literature. Some examples are given to illustrate our results.

Keywords Weak subgradient · Set-valued mapping · Contingent derivative · Existence

1 Introduction

It is well known that the subgradient plays an important role in optimization and duality theory. The concept of subgradients for a convex function was considered by Rockafellar [1] in finite-dimensional spaces. In recent years, the concept of subgradients has been generalized to vector-valued mappings and set-valued mappings in abstract spaces by many authors; see [2–8]. In [9], Chen and Craven in-

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troduced the weak subgradient for a vector-valued mapping and discussed the existence of the weak subgradient. Yang [10] generalized the concept introduced by Chen and Craven [9] to set-valued mappings. Chen and Jahn [2] defined another weak subgradient, which is stronger than the weak subgradient introduced by Yang [10]. They also proved the existence of the weak subgradient by the Eidelheit separation theorem. By the Hahn–Banach theorem, Peng et al. [11] proved the existence of the weak subgradient for set-valued mappings introduced by Yang [10]. Recently, Li and Guo [12] proved some existence theorems of two kinds of weak subgradients for set-valued mappings by virtue of a Hahn–Banach extension theorem obtained by Zălinescu [13]. Very recently, Hernandez and Rodriguez-Marin [14] considered the weak subgradient of set-valued mappings introduced by Chen and Jahn [2] and also presented a new notion of the strong subgradient for set-valued mappings. Moreover, they obtained some existence theorems of both subgradients. Note that as mentioned above the assumptions that the cone-convexity of the objective function and the upper semicontinuity of the objective function at a given point are required. This paper is the effort in removing these restrictions.

Motivated by the work reported in [12, 14], in this paper, we consider the weak subdifferential for set-valued mappings, which was introduced by Chen and Jahn [2]. Without any convexity and upper semicontinuity assumptions on objective functions, we prove two existence theorems of weak subgradients for set-valued mappings. Moreover, we derive some properties of the weak subdifferential for set-valued mappings. Our results improve the corresponding ones in [12, 14].

2 Preliminaries

Throughout this paper, let X and Y be two real locally convex topological vector spaces, and $L(X,Y)$ be the set of all linear continuous operators from X into Y. Let $X' := L(X,\mathbb{R})$ and $C \subset Y$ be a proper (i.e. $\{0\} \neq C$ and $C \neq Y$) closed, convex and pointed cone with nonempty interior $\operatorname{int} C$. The origin of X and Y are denoted by 0_X and 0_Y, respectively. Let X^* and Y^* be the topological dual spaces of X and Y, respectively. The dual cone of C is defined by

$$C^* := \{ f \in Y^* : f(x) \geq 0, \text{ for all } x \in C \}.$$

We denote by (Y,C) the ordered topological vector space, where the ordering is induced by C. For any $y_1, y_2 \in Y$, we define the following ordering relations:

$$y_1 < y_2 \;\Leftrightarrow\; y_2 - y_1 \in \operatorname{int} C,$$
$$y_1 \not< y_2 \;\Leftrightarrow\; y_2 - y_1 \notin \operatorname{int} C.$$

The relations $>$ and $\not>$ are defined similarly.

Let $F : X \rightrightarrows Y$ be a set-valued mapping. The domain, graph and epigraph of F are, respectively, defined by

$$\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\},$$
$$\operatorname{Gr} F := \{(x, y) \in X \times Y : x \in \operatorname{dom} F, \ y \in F(x)\},$$
$$\operatorname{epi} F := \{(x, y) \in X \times Y : x \in \operatorname{dom} F, \ y \in F(x) + C\},$$

where the symbol $\emptyset$ denotes the empty set.

Let K be a nonempty subset of X, $F : K \rightrightarrows Y$ be a set-valued mapping. In this paper, we consider the following set-valued optimization problem (in short, SVOP):

$$\min_C F(x), \quad \text{subject to} \quad x \in K.$$

A pair (x_0, y_0) with $x_0 \in K$ and $y_0 \in F(x_0)$ is called a weak efficient solution of (SVOP) iff $(F(K) - y_0) \cap (-\operatorname{int} C) = \emptyset$, where $F(K) := \bigcup_{x \in K} F(x)$.

Let $A \subset Y$. We denote by $\operatorname{WMin} A := \{y \in A : (A - y) \cap -\operatorname{int} C = \emptyset\}$ the set of weak efficient elements of A.

Definition 2.1 [15] Let K be a nonempty subset of X and $x_0 \in \operatorname{cl} K$. The contingent cone $T(K, x_0)$ to K at x_0 is the set of all $h \in X$ for which there exist a net $\{t_\alpha : \alpha \in I\}$ of positive real numbers and a net $\{x_\alpha : \alpha \in I\} \subset K$ such that

$$\lim_\alpha x_\alpha = x_0 \quad \text{and} \quad \lim_\alpha t_\alpha (x_\alpha - x_0) = h.$$

Remark 2.1 From Definition 2.1, we have that $T(K, x_0) \subset \operatorname{clcone}(K - x_0)$ and $T(K, x_0)$ is a closed cone. Moreover, If K is convex, then $T(K, x_0)$ is a closed and convex cone.

Remark 2.2 It is not difficult to see that $h \in T(K, x_0)$ if and only if there exist a net $\{t_\alpha : \alpha \in I\}$ of positive real numbers and a net $\{h_\alpha : \alpha \in I\}$ with $h_\alpha \to h$ such that $t_\alpha h_\alpha \to 0$ and $x_0 + t_\alpha h_\alpha \in K$.

Definition 2.2 [3] Let $F : X \rightrightarrows Y$ be a set-valued mapping. Let $(x_0, y_0) \in \operatorname{Gr} F$. The contingent derivative $DF(x_0, y_0)$ of F at (x_0, y_0) is a set-valued mapping from X to Y defined by

$$\operatorname{Gr}\big(DF(x_0, y_0)\big) := T\big(\operatorname{Gr}(F); (x_0, y_0)\big).$$

Remark 2.3 Let $(x_0, y_0) \in \operatorname{Gr} F$. It is easy to see that

(i) $y \in DF(x_0, y_0)(x)$ if and only if there exist a net $\{(x_\alpha, y_\alpha) : \alpha \in I\} \subset \operatorname{Gr} F$ and a net $\{t_\alpha : \alpha \in I\}$ of positive real numbers such that

$$\lim_\alpha (x_\alpha, y_\alpha) = (x_0, y_0) \quad \text{and} \quad \lim_\alpha t_\alpha (x_\alpha - x_0, y_\alpha - y_0) = (x, y);$$

(ii) the set-valued mapping $DF(x_0, y_0)$ is positively homogeneous with closed graphs;

(iii) [16] $(0, 0) \in \operatorname{Gr}(DF(x_0, y_0))$.

Definition 2.3 [6] Let K be a convex subset of X. A set-valued mapping $F : X \rightrightarrows Y$ is said to be C-convex on K iff, for any $x_1, x_2 \in K$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F\big(\lambda x_1 + (1-\lambda)x_2\big) + C.$$

Remark 2.4 If the set-valued mapping F is C-convex on K, then $F(K) + C$ is a convex set.

Definition 2.4 [17] A set-valued mapping $F : X \rightrightarrows Y$ is said to be compactly approximable at $(x_0, y_0) \in \operatorname{Gr} F$ iff, for each $v_0 \in X$, there exists a set-valued mapping H from X into the set of all nonempty compact subsets of Y, a neighborhood V of x_0 in X, and a function $r :]0, 1[\times X \to]0, +\infty)$ satisfying

(i) $\lim_{(t,v)\to(0^+,v_0)} r(t, v) = 0$;
(ii) for each $v \in V$ and $t \in]0, 1]$,

$$F(x_0 + tv) \subset y_0 + t\big(H(v_0) + r(t, v)B_Y\big),$$

where B_Y is the closed unit ball around the origin of Y.

The following lemma will be used in the sequel which plays an important role in proving our main results.

Lemma 2.1 [13] *Let X, Y be separated locally convex topological vector spaces, $F : X \rightrightarrows Y$ be a C-convex set-valued mapping, $X_0 \subset X$ be a linear subspace and $T_0 \in L(X_0, Y)$. Suppose that* $\operatorname{int}(\operatorname{epi} F) \neq \emptyset$, $X_0 \cap \operatorname{int}(\operatorname{dom} F) \neq \emptyset$, *and $T_0(x) \not> y$ for all $(x, y) \in \operatorname{Gr} F \cap (X_0 \times Y)$. If $T_0(x) = \langle x, x_0^*\rangle y_0$ for every $x \in X_0$ with fixed $x_0^* \in X^*$ and $y_0 \in Y$, then there exists $T \in L(X, Y)$ such that $T|_{X_0} = T_0$ and $T(x) \not> y$ for all $(x, y) \in \operatorname{Gr} F$.*

By Lemma 2.5 in [18], it is easy to prove the following result.

Lemma 2.2 *Let $C \subset Y$ be a closed, convex and pointed cone with* $\operatorname{int} C \neq \emptyset$, *and let S be a nonempty subset of Y. Then, for $y \in Y$,*

$$(S - y) \cap -\operatorname{int} C = \emptyset \quad \Leftrightarrow \quad (S + \operatorname{int} C - y) \cap -\operatorname{int} C = \emptyset.$$

3 Existence of Weak Subgradients

In this section, we establish two existence theorems of weak subgradients for set-valued mappings. Denote $W := Y \backslash (-\operatorname{int} C)$.

Definition 3.1 [2] Let K be a subset of X with $x_0 \in K$. Let $F : K \rightrightarrows Y$ be a set-valued mapping. $T \in L(X, Y)$ is called a weak subgradient of F at x_0 iff

$$F(x) - F(x_0) - T(x - x_0) \subset W, \quad \forall x \in K.$$

The set of all weak subgradients of F at x_0, denoted by $\partial^w F(x_0)$, is called the weak subdifferential of F at x_0.

Theorem 3.1 *Let K be a convex subset of X with* $\operatorname{int} K \neq \emptyset$. *Let $F : K \rightrightarrows Y$ be a set-valued mapping with $F(x) \neq \emptyset$ for any $x \in K$. Let $x_0 \in \operatorname{int} K$ and $y_0 \in F(x_0) \cap$* $\operatorname{WMin} F(K)$. *If the following conditions are satisfied*:

(i) $DF(x_0, y_0)$ *is C-convex on $K - \{x_0\}$*;
(ii) *there exists $a \in Y$ such that $DF(x_0, y_0)(K - x_0) \subset a - \operatorname{int} C$*;
(iii) $F(x) - F(x_0) \subset DF(x_0, y_0)(x - x_0) + C$, $\forall x \in K$;

then, $\partial^w F(x_0) \neq \emptyset$. Moreover, there exists $T \in \partial^w F(x_0)$ such that for every $x \in K$,

$$T(x - x_0) \notin -\operatorname{int} C \quad \Leftrightarrow \quad T(x - x_0) \in C.$$

Proof We define the set-valued mapping $G : K \rightrightarrows Y$ by

$$G(x) := DF(x_0, y_0)(x - x_0).$$

We now prove that G is a C-convex set-valued mapping. Indeed, for any $x_1, x_2 \in K$ and $\lambda \in [0, 1]$, by the C-convexity of $DF(x_0, y_0)$ on $K - \{x_0\}$, we have

$$\begin{aligned} \lambda G(x_1) + (1 - \lambda)G(x_2) &= \lambda DF(x_0, y_0)(x_1 - x_0) + (1 - \lambda)DF(x_0, y_0)(x_2 - x_0) \\ &\subset DF(x_0, y_0)\big(\lambda(x_1 - x_0) + (1 - \lambda)(x_2 - x_0)\big) + C \\ &= G\big(\lambda x_1 + (1 - \lambda)x_2\big) + C, \end{aligned}$$

which implies that G is a C-convex set-valued mapping.

Let

$$M := \big\{(x, y) : x \in K, y \in G(x) + \operatorname{int} C\big\}.$$

Since K is a nonempty convex set and G is C-convex, M is a nonempty convex set. The proof of the theorem is divided into the following three steps.

(I) We prove that $\operatorname{int} M \neq \emptyset$.

Suppose that there exists $a \in Y$ such that

$$G(x) \subset a - \operatorname{int} C, \quad \forall x \in K. \tag{1}$$

Let $c \in \operatorname{int} C$ and $y_0 = a + c$. Then, $y_0 - a = c \in \operatorname{int} C$. It follows that there exists a neighborhood U of 0_Y such that

$$U + y_0 - a \subset C. \tag{2}$$

Let $x_0 \in \operatorname{int} K$. Then there exists a neighborhood V of 0_X such that $x_0 + V \subset K$. From (1), for any $x \in x_0 + V$ and $y_x \in G(x)$, there exists $c_x \in \operatorname{int} C$ such that

$$y_x = a - c_x.$$

This fact together with (2) yields

$$U + y_0 - y_x = U + y_0 - a + c_x \subset \operatorname{int} C,$$

which implies that

$$U + y_0 \subset y_x + \operatorname{int} C \subset G(x) + \operatorname{int} C. \tag{3}$$

On the other hand, for any $x \in x_0 + V$,

$$y_0 = a + c = y_x + c_x + c \in G(x) + \operatorname{int} C + \operatorname{int} C \subset G(x) + \operatorname{int} C. \tag{4}$$

Combining (3) and (4) yields

$$(x, y) \in M, \quad \forall x \in x_0 + V, \ \forall y \in U + y_0.$$

It follows that $\operatorname{int} M \neq \emptyset$.

(II) We prove that $(x_0, 0) \notin M$.

Indeed, if $(x_0, 0) \in M$, then $0 \in G(x_0) + \operatorname{int} C$, and so $G(x_0) \cap -\operatorname{int} C \neq \emptyset$. This implies that there exists $c \in \operatorname{int} C$ such that $-c \in DF(x_0, y_0)(0)$. It follows that there exist nets $\{\lambda_\alpha : \alpha \in I\}$ of positive real numbers and $\{(x_\alpha, y_\alpha) : \alpha \in I\} \subset \operatorname{Gr} F$ satisfying

$$\lim_\alpha (x_\alpha, y_\alpha) = (x_0, y_0) \quad \text{and} \quad \lim_\alpha \lambda_\alpha \big[(x_\alpha, y_\alpha) - (x_0, y_0)\big] = (0, -c).$$

Therefore, there exists $\alpha_0 \in I$ such that

$$\lambda_\alpha (y_\alpha - y_0) \in -\operatorname{int} C, \quad \forall \alpha \geq \alpha_0$$

and so

$$y_\alpha - y_0 \in -\operatorname{int} C, \quad \forall \alpha \geq \alpha_0,$$

which contradicts the fact $y_0 \in \operatorname{WMin} F(K)$.

(III) There exists $T \in L(X, Y)$ such that $T \in \partial^w F(x_0)$.

Since M is a nonempty convex set with $\operatorname{int} M \neq \emptyset$ and $(x_0, 0) \notin M$, by the separation theorem of convex sets, there exists $(-x^*, y^*) \in X^* \times Y^* \backslash \{(0, 0)\}$ such that

$$\langle -x^*, x \rangle + \langle y^*, y \rangle \geq \langle -x^*, x_0 \rangle + \langle y^*, 0 \rangle, \quad \forall (x, y) \in M,$$

or equivalently,

$$\langle -x^*, x \rangle + \langle y^*, y \rangle \geq \langle -x^*, x_0 \rangle, \quad \forall (x, y) \in M. \tag{5}$$

We claim that $y^* \neq 0$. In fact, if $y^* = 0$, then $\langle x^*, x \rangle \leq \langle x^*, x_0 \rangle, \forall x \in K$. Since $x_0 \in \operatorname{int} K$, there exists a symmetric neighborhood U of 0_X such that $x_0 + U \subset K$. It follows that

$$\langle x^*, x_0 \pm u \rangle \leq \langle x^*, x_0 \rangle, \quad \forall u \in U.$$

This implies $x^* = 0$, which contradicts that $(-x^*, y^*) \neq (0, 0)$. Therefore, $y^* \neq 0$.

Note that $0 \in DF(x_0, y_0)(0)$. This fact together with (5) yields $\langle y^*, c\rangle > 0$, $\forall c \in \operatorname{int} C$. And so $\langle y^*, c\rangle \geq 0$, $\forall c \in C$, that is $y^* \in C^*$. Then there exists some $c_0 \in \operatorname{int} C$ with $\langle y^*, c_0\rangle = 1$. We now define a mapping $T : X \to Y$ by

$$T(x) := \langle x^*, x\rangle c_0, \quad \forall x \in K - \{x_0\}.$$

Obviously, T is linear and continuous. Next we prove that for this mapping T satisfying

$$F(x) - F(x_0) - T(x - x_0) \subset W, \quad \forall x \in K.$$

We now prove that

$$G(x) - T(x - x_0) \subset W, \quad \forall x \in K.$$

By Lemma 2.2, we only need to prove that

$$\big(G(x) + \operatorname{int} C - T(x - x_0)\big) \cap -\operatorname{int} C = \emptyset, \quad \forall x \in K.$$

Suppose by contradiction that there exist $x \in K$ and $y \in G(x) + \operatorname{int} C$ such that

$$y - T(x - x_0) \in -\operatorname{int} C.$$

Because of $y^* \in C^* \backslash \{0\}$, we have

$$0 > \langle y^*, y - T(x - x_0)\rangle = \langle y^*, y\rangle - \langle x^*, x - x_0\rangle\langle y^*, c_0\rangle = \langle y^*, y\rangle - \langle x^*, x - x_0\rangle,$$

which contradicts (5). Therefore, by condition (iii), $T \in \partial^w F(x_0)$. Finally, for every $x \in K$, we have

$$\begin{aligned} T(x - x_0) \notin -\operatorname{int} C \ &\Leftrightarrow\ \langle x^*, x - x_0\rangle c_0 \notin -\operatorname{int} C \ \Leftrightarrow\ \langle x^*, x - x_0\rangle \geq 0 \\ &\Leftrightarrow\ T(x - x_0) \in C. \end{aligned}$$

This completes the proof. □

Remark 3.1 In [14], Hernandez and Modriguez-Marin obtained the existence theorem of weak subgradients for set-valued mappings. The assumptions that $F(x_0)$ is upper bounded and F is upper semicontinuous at x_0 are required in [14]. However, Theorem 3.1 does not require these assumptions. The following example is given to illustrate the case that Theorem 3.1 is applicable, but Theorem 4.1 of [14] is not applicable.

Example 3.1 Let $X = Y = \mathbb{R}$, $K = \mathbb{R}$, $C = \{y : y \geq 0\}$, and let

$$F(x) = \begin{cases} \{0\}, & \text{if } x \leq 0, \\ \{0, 1\}, & \text{if } x > 0. \end{cases}$$

Let $(x_0, y_0) = (0, 0)$. Then,

$$T\big(\operatorname{Gr}(F); (0, 0)\big) = \big\{(x, 0) : x \in \mathbb{R}\big\}.$$

It is easy to see that the assumptions of Theorem 3.1 are satisfied. Obviously, $0 \in \partial^w F(0)$. However, Theorem 4.1 in [14] is not applicable because F is not upper semicontinuous at x_0.

We now give a sufficient condition, which guarantees the assumption (i) in Theorem 3.1 holds.

Proposition 3.1 *Let K be a convex subset of X and $F : K \rightrightarrows Y$ be a C-convex set-valued mapping. Let $(x_0, y_0) \in \operatorname{Gr} F$. If F is compactly approximable at (x_0, y_0), then $DF(x_0, y_0)$ is C-convex.*

Proof Since F is compactly approximable at (x_0, y_0), by Proposition 2.2 in [19],

$$D(F + C)(x_0, y_0)(x) = D(F)(x_0, y_0)(x) + C, \quad \forall x \in X.$$

Since F is C-convex, $\operatorname{epi} F$ is a convex set. It follows that $T(\operatorname{epi} F; (x_0, y_0))$ is a convex set. And so $D(F + C)(x_0, y_0)$ is convex. Therefore, $DF(x_0, y_0)$ is C-convex. This completes the proof. □

Theorem 3.2 *Let X and Y be separated locally convex topological vector spaces, and K be a convex subset of X with $\operatorname{int} K \neq \emptyset$. Let $F : K \rightrightarrows Y$ be a set-valued mapping with $F(x) \neq \emptyset$ for any $x \in K$. Let $x_0 \in \operatorname{int} K$ and $y_0 \in F(x_0)$. If the following conditions are satisfied:*

(i) *$DF(x_0, y_0)$ is C-convex on $K - \{x_0\}$;*
(ii) *$DF(x_0, y_0)(0) \cap -\operatorname{int} C = \emptyset$;*
(iii) *$F(x) - F(x_0) \subset DF(x_0, y_0)(x - x_0) + C$, $\forall x \in K$;*

then, $\partial^w F(x_0) \neq \emptyset$.

Proof Let $S = K - \{x_0\}$. We define the set-valued mapping $G : X \rightrightarrows Y$ by

$$G(x) := DF(x_0, y_0)(x), \quad \forall x \in S.$$

Similarly to the proof of Theorem 3.1, we can prove that G is C-convex on S.

By condition (ii), $G(0) \cap -\operatorname{int} C = \emptyset$. It follows that

$$0 \not> y, \quad \forall y \in G(0). \tag{6}$$

We next consider the special subspace $X_0 = \{0\}$ and $T_0(0) := 0$. Since $x_0 \in \operatorname{int} K$ and $F(x) \neq \emptyset$ for any $x \in K$, it is easy to see that

$$\operatorname{int}(\operatorname{epi} G) \neq \emptyset, \qquad X_0 \cap \operatorname{int}(\operatorname{dom} G) \neq \emptyset.$$

From (6), we have

$$T_0(x) \not> y, \quad \forall (x, y) \in \operatorname{Gr} G \cap (X_0 \times Y) = \{(0, y) : y \in G(0)\}.$$

By Lemma 2.1, there exists $T \in L(X, Y)$ such that

$$T(x) \not> y, \quad \forall (x, y) \in \operatorname{Gr} G$$

and so

$$T(x) \notin DF(x_0, y_0)(x) + \operatorname{int} C, \quad \forall x \in S.$$

It follows that

$$T(x - x_0) \notin DF(x_0, y_0)(x - x_0) + \operatorname{int} C, \quad \forall x \in K,$$

or

$$DF(x_0, y_0)(x - x_0) - T(x - x_0) \subset W, \quad \forall x \in K,$$

which, together with condition (iii), yields

$$\begin{aligned} F(x) - F(x_0) - T(x - x_0) &\subset DF(x_0, y_0)(x - x_0) - T(x - x_0) + C \\ &\subset W, \quad \forall x \in K. \end{aligned}$$

This implies $T \in \partial^w F(x_0)$. This completes the proof. □

Remark 3.2 In [12, Theorem 3.2], Li and Guo obtained a existence theorem of weak subgradients by using similar proof methods. It is important to note that our assumptions are different from the ones used in [12]. First, the condition that F is C-convex has been relaxed because we consider C-convexity of the contingent derivative of F instead of F. Second, the assumptions that $F(x_0) - C$ is convex and $F(x_0) \cap (F(x_0) - \operatorname{int} C) = \emptyset$ are required in [12], but Theorem 3.2 does not require these assumptions.

Remark 3.3 In [2, Theorem 7] and [11, Theorem 4.1], the authors derived some existence theorems of weak subgradients for set-valued mappings. The assumptions that $-F(x_0)$ is minorized, F is C-convex and upper semicontinuous at x_0 are required in [2, 11]. However, Theorem 3.2 does not require these assumptions.

Now, we give an example to illustrate Theorem 3.2.

Example 3.2 Let $X = Y = \mathbb{R}$, $K = \mathbb{R}$, $C = \{y : y \geq 0\}$, and let

$$F(x) = \begin{cases} \{0\}, & \text{if } x \leq 0; \\ \{2, -x\}, & \text{if } x > 0. \end{cases}$$

Let $(x_0, y_0) = (0, 0)$. Then,

$$T\big(\operatorname{Gr}(F); (0, 0)\big) = \big\{(x, 0) : x \leq 0\big\} \cup \big\{(x, -y) : x = y, x > 0\big\}.$$

It is easy to see that the assumptions of Theorem 3.2 are satisfied. Obviously, $0 \in \partial^w F(0)$. However, Theorem 3.2 in [12], Theorem 7 in [2] and Theorem 4.1 in [11] are not applicable since F is not C-convex on K. Indeed, letting $x_1 = -4$, $x_2 = 2$ and $\lambda = \frac{1}{2}$, we have

$$\lambda F(x_1) + (1 - \lambda) F(x_2) = \{-1, 1\} \not\subseteq F\big(\lambda x_1 + (1 - \lambda) x_2\big) + \mathbb{R}_+ = \mathbb{R}_+.$$

4 Properties of Weak Subgradients

In this section, we obtain some properties of weak subgradients for set-valued mappings.

Theorem 4.1 *Let K be a convex subset of X and $x_0 \in K$. Let $F : K \rightrightarrows Y$ be a C-convex set-valued mapping with nonempty values and $F(x_0) - C$ is convex. If $T \in \partial^w F(x_0)$, then there exists $y^* \in C^* \backslash \{0\}$ such that*

$$\langle y^*, y - y_1 - T(x - x_0) \rangle \geq 0, \quad \forall (x, y) \in \operatorname{Gr} F,\ \forall y_1 \in F(x_0).$$

Proof Let $T \in \partial^w F(x_0)$. Then

$$\big(F(x) - F(x_0) - T(x - x_0)\big) \cap - \operatorname{int} C = \emptyset, \quad \forall x \in K.$$

This implies that

$$\big(F(x) + C - F(x_0) - T(x - x_0)\big) \cap - \operatorname{int} C = \emptyset, \quad \forall x \in K.$$

We define a set-valued mapping $G : K \rightrightarrows Y$ by

$$G(x) := F(x) - F(x_0).$$

Since F is C-convex and $F(x_0) - C$ is convex, for any $x_1, x_2 \in K$ and $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda G(x_1) + (1-\lambda) G(x_2) &= \lambda F(x_1) - \lambda F(x_0) + (1-\lambda) F(x_2) - (1-\lambda) F(x_0) \\ &\subset F\big(\lambda x_1 + (1-\lambda) x_2\big) + C - F(x_0) + C \\ &\subset G\big(\lambda x_1 + (1-\lambda) x_2\big) + C. \end{aligned}$$

It follows that G is a C-convex set-valued mapping. Note that T is a linear operator, then

$$\bigcup_{x \in K} \big(F(x) + C - F(x_0) - T(x - x_0)\big)$$

is a convex set. By the separation theorem of convex sets, there exists $y^* \in Y^* \backslash \{0\}$ such that

$$\langle y^*, y + c - y_1 - T(x - x_0) \rangle \geq 0, \quad \forall x \in K, y \in F(x), y_1 \in F(x_0), c \in C. \tag{7}$$

We claim that

$$\langle y^*, c \rangle \geq 0, \quad \forall c \in C.$$

In fact, if there exists $c_0 \in C$ such that $\langle y^*, c_0 \rangle < 0$, then by letting $x = x_0$ and $y = y_1$ in (7), we have $\langle y^*, c_0 \rangle \geq 0$. This gives a contradiction. Thus, $y^* \in C^* \backslash \{0\}$. Letting $c = 0$ in (7), we get the conclusion. This completes the proof. □

Theorem 4.2 *Let K be a nonempty subset of X and $x_0 \in K$. Let $F : K \rightrightarrows Y$ be a set-valued mapping with nonempty values. If $\partial^w F(x_0) \neq \emptyset$, then $\partial^w F(x_0)$ is a closed set.*

Proof Suppose by contradiction that there exists a net $\{T_\alpha : \alpha \in I\} \subset \partial^w F(x_0)$ such that $T_\alpha \to T$, but $T \notin \partial^w F(x_0)$. Thus, there exist $\overline{x} \in K$, $\overline{y} \in F(\overline{x})$ and $y_0 \in F(x_0)$ such that

$$\overline{y} - y_0 - T(\overline{x} - x_0) \in -\operatorname{int} C.$$

Note that

$$\overline{y} - y_0 - T_\alpha(\overline{x} - x_0) \to \overline{y} - y_0 - T(\overline{x} - x_0).$$

It follows that there exists $\alpha_0 \in I$ such that

$$\overline{y} - y_0 - T_\alpha(\overline{x} - x_0) \in -\operatorname{int} C, \quad \forall \alpha \geq \alpha_0,$$

which contracts the fact $T_\alpha \in \partial^w F(x_0)$. This completes the proof. □

Remark 4.1 Note that we prove Theorem 4.2 in locally convex topological vector spaces. But a similar result has been proved by Li and Guo [12] in normed spaces.

5 Conclusions

In this paper, we have proved two existence theorems of weak subgradients for set-valued mappings. These two results improve meaningfully the corresponding results obtained by Hernandez and Rodriguez-Marin [14] and Li and Guo [12], respectively. Moreover, two properties of the weak subdifferential for set-valued mappings are derived. It would be interesting to consider the calculations of sum mapping and composed mapping for weak subdifferentials as well as applications to set-valued optimization problems. This may be the topic of some of our forthcoming papers.

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