J Optim Theory Appl (2013) 159:473–488
DOI 10.1007/s10957-013-0352-2

Predictor–Corrector Methods for General Regularized Nonconvex Variational Inequalities

Qamrul Hasan Ansari · Javad Balooee

Received: 17 January 2013 / Accepted: 30 May 2013 / Published online: 18 June 2013

Abstract This paper is devoted to the study of a new class of nonconvex variational inequalities, named general regularized nonconvex variational inequalities. By using the auxiliary principle technique, a new modified predictor–corrector iterative algorithm for solving general regularized nonconvex variational inequalities is suggested and analyzed. The convergence of the iterative algorithm is established under the partially relaxed monotonicity assumption. As a consequence, the algorithm and results presented in the paper overcome incorrect algorithms and results existing in the literature.

Keywords General regularized nonconvex variational inequalities · Nonconvex sets · Predictor–corrector iterative algorithm · Prox-regularity · Convergence analysis

1 Introduction

In the last three decades, the theory of variational inequalities has been extensively studied in the literature because of its applications to optimization, game theory, mechanics, and engineering sciences. This field is experiencing an explosive growth in both theory and applications. Several numerical methods have been developed for

Q.H. Ansari (✉)
Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
e-mail: qhansari@gmail.com

Q.H. Ansari
Department of Mathematics & Statistics, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia

J. Balooee
Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran
e-mail: javad.balooee@gmail.com

solving variational inequalities and related optimization problems. These methods include the projection method and its various forms, the linear approximation method, the descent and Newton's method, the method based on the auxiliary principle technique, etc. For applications, numerical methods, and other aspects of variational inequalities, see, for example, [1–4] and the references therein.

It is worth to mention that most of the results regarding the existence and iterative approximation of solutions to variational inequalities have been investigated and considered so far with restriction to the case where the underlying set is convex. Recently, the concept of a convex set has been generalized in many directions, which have potential and important applications in various fields. It is well known that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases; for more details, see, for example, [5–9].

Very recently, Noor [10] introduced and considered a new class of variational inequalities, so-called general nonconvex variational inequalities, and claimed that the elements of the aforesaid class are equivalent to the fixed point problems. He used this alternative equivalent formulation to suggest and analyze some iterative algorithms for solving general nonconvex variational inequalities. He also studied the convergence analysis of the suggested iterative algorithms under certain conditions.

This paper is devoted to the study of a new class of nonconvex variational inequalities, termed general regularized nonconvex variational inequalities (GRNVI). It gives the correct form of the general nonconvex variational inequalities studied by Noor [10]. We establish the equivalence between GRNVI and a fixed point problem. By utilizing the auxiliary principle technique, we also consider and analyze a new class of predictor–corrector iterative algorithms for solving GRNVI. The convergence of these methods is established under the partially relaxed monotonicity assumption. The algorithm and results overcome incorrect algorithms and results presented in [10].

2 Preliminaries and Basic Facts

Throughout the paper, unless otherwise specified, we use the following notation, terminology, and assumptions. Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let K be a nonempty and closed subset of $\mathcal{H}$. We denote by $d_K(.)$ the usual distance function from a point to a set K, that is, $d_K(u) := \inf_{v \in K} \|u - v\|$.

Definition 2.1 For a given point $u \in \mathcal{H}$, a point $v \in K$ is called *a projection of u onto K* iff $d_K(u) = \|u - v\|$. The set of all such projections is denoted by $P_K(u)$, that is,

$$P_K(u) := \left\{ v \in K : d_K(u) = \|u - v\| \right\}.$$

Definition 2.2 The *proximal normal cone* of K at a point $u \in K$ is given by

$$N_K^P(u) := \left\{ \xi \in \mathcal{H} : \exists \alpha > 0 \text{ so that } u \in P_K(u + \alpha \xi) \right\}.$$

The next lemma gives the characterization of the proximal normal cone.

Lemma 2.1 [6, Proposition 1.1.5] *Let K be a nonempty closed subset of $\mathcal{H}$. Then, $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha = \alpha(\xi, u) > 0$ such that $\langle \xi, v - u\rangle \le \alpha\|v - u\|^2$ for all $v \in K$.*

Definition 2.3 [11] Let $f : \mathcal{H} \to \mathbb{R}$ be locally Lipschitz near a point x. The *Clarke's directional derivative* of f at x in the direction v, denoted by $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in $\mathcal{H}$ and t is a positive scalar.

We note that f° enjoys nice properties, but in the merely differentiable case f° does not shrink to the classic Fréchet derivative.

The *tangent cone* to K at a point $x \in K$, denoted by $T_K(x)$, is defined by

$$T_K(x) := \{v \in \mathcal{H} : d_K^\circ(x; v) = 0\}.$$

The *normal cone* to K at $x \in K$, denoted by $N_K(x)$, is defined by

$$N_K(x) := \{\xi \in \mathcal{H} : \langle \xi, v\rangle \le 0 \text{ for all } v \in T_K(x)\}.$$

The *Clarke normal cone*, denoted by $N_K^C(x)$, is defined by $N_K^C(x) := cl(conv)[N_K^P(x)]$, where $cl(conv)[S]$ denotes the closure of the convex hull of S.

Clearly, $N_K^P(x) \subseteq N_K^C(x)$. Note that $N_K^C(x)$ is a closed and convex cone, whereas $N_K^P(x)$ is convex, but may not be closed. For further details of this topic, we refer to [6, 8, 11] and the references therein.

In 1995, Clarke et al. [7] introduced a nonconvex set, called a *proximally smooth set*. Subsequently, it has been investigated by Poliquin et al. [8] but under the name of uniformly prox-regular set. Such kinds of sets are used in many nonconvex applications in optimization, economic models, dynamical systems, differential inclusions, etc. For further details and applications, we refer to [12–14] and the references therein. This class of nonconvex sets seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumption.

Definition 2.4 [7, 8] For a given $r \in]0, +\infty]$, a subset K_r of $\mathcal{H}$ is said to be *normalized uniformly prox-regular* (or *uniformly r-prox-regular*) iff every nonzero proximal normal to K_r can be realized by an r-ball, that is, for all $\bar{x} \in K_r$ and all $0 \ne \xi \in N_{K_r}^P(\bar{x})$,

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \le \frac{1}{2r}\|x - \bar{x}\|^2, \quad \text{for all } x \in K_r.$$

The class of normalized uniformly prox-regular sets includes the class of convex sets, p-convex sets [15], $C^{1,1}$ submanifolds of $\mathcal{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and several other nonconvex sets [7].

Lemma 2.2 [7] *A closed set $K \subseteq \mathcal{H}$ is convex iff it is uniformly r-prox-regular for every $r > 0$.*

If $r = +\infty$, then in view of Definition 2.4 and Lemma 2.2, the uniform r-prox-regularity of K_r is equivalent to the convexity of K_r. That is, for $r = +\infty$, we set $K_r = K$.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel.

Proposition 2.1 [7, 8] *Let $r > 0$ and K_r be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Let $U(r) = \{u \in \mathcal{H} : d_{K_r}(u) < r\}$. Then, the following assertions hold*:

(a) *For all $x \in U(r)$, $P_{K_r}(x) \neq \emptyset$.*
(b) *For all $r' \in]0, r[$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $U(r') = \{u \in \mathcal{H} : d_{K_r}(u) < r'\}$.*
(c) *The proximal normal cone is closed as a set-valued mapping.*

As a direct consequent of part (c) of Proposition 2.1, for any uniformly r-prox-regular subset K_r of $\mathcal{H}$, we have $N^C_{K_r}(x) = N^P_{K_r}(x)$. Therefore, we define $N_{K_r}(x) := N^C_{K_r}(x) = N^P_{K_r}(x)$ for such a class of sets.

The union of two disjoint intervals $[a, b]$ and $[c, d]$ is uniformly r-prox-regular with $r = \frac{c-b}{2}$ [5, 6, 8]. The finite union of disjoint intervals is also uniformly r-prox-regular and r depends on the distances between the intervals.

3 Formulations and Basic Comments

This section is devoted to the investigation and analysis of the general nonconvex variational inequalities introduced in [10] and we point out that all the results in [10] are incorrect.

Let K_r be an uniformly r-prox-regular set and suppose that $T, g : \mathcal{H} \to \mathcal{H}$ are two different nonlinear operators. Recently, Noor [10] introduced and considered the following general nonconvex variational inequality:

Find $u \in \mathcal{H}$ such that $g(u) \in K_r$ and

$$\langle \rho T u, g(v) - g(u) \rangle + \gamma \| g(v) - g(u) \|^2 \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K_r, \tag{1}$$

where $\rho, \gamma > 0$ are arbitrarily fixed constants.

He considered several special cases of the above mentioned problem. He has also claimed that the problem (1) is equivalent to finding $u \in \mathcal{H}$ with $g(u) \in K_r$ such that

$$0 \in \rho T u + g(u) - g(u) + N^P_{K_r}\big(g(u)\big),$$

or equivalently,

$$0 \in \rho T u + N^P_{K_r}\big(g(u)\big), \tag{2}$$

where $N^P_{K_r}(g(u))$ denotes the normal cone of K_r at $g(u)$ in the sense of nonconvex analysis, see p. 2957 in [10].

The equivalence between the two problems (1) and (2) has a key role in proposing algorithms and in getting the results in [10], and plays a crucial and basic role in [10]. Indeed, all the results in [10] have been obtained based on the equivalence between the two problems (1) and (2). Unfortunately, the problems (1) and (2) are not equivalent.

In view of Definition 2.4,

$$0 \neq \xi \in N^P_{K_r}(u) \quad \Leftrightarrow \quad \langle \xi, v-u\rangle \leq \frac{\|\xi\|}{2r}\|v-u\|^2, \quad \forall v \in K_r.$$

Hence, for each $u \in \mathcal{H}$ with $g(u) \in K_r$, if $Tu \neq 0$, we have

$$\begin{aligned} &0 \in \rho Tu + N^P_{K_r}\big(g(u)\big) \quad \Leftrightarrow \\ &\big\langle -\rho Tu, v-g(u)\big\rangle \leq \frac{\rho\|Tu\|}{2r}\big\|v-g(u)\big\|^2, \quad \forall v \in K_r. \end{aligned}$$

If $Tu = 0$, clearly $0 \in \rho Tu + N^P_{K_r}(g(u))$, because the zero vector always belongs to any normal cone. Therefore, the two problems (1) and (2) are equivalent if and only if the problem (1) is equivalent to the problem of finding $u \in \mathcal{H}$ with $g(u) \in K_r$ such that

$$\big\langle \rho Tu, v-g(u)\big\rangle + \frac{\rho\|Tu\|}{2r}\big\|v-g(u)\big\|^2 \geq 0, \quad \forall v \in K_r. \tag{3}$$

However, the two problems (1) and (3) are not necessarily equivalent. To prove this claim, it is enough to show that the problem (1) is not equivalent to the problem of finding $u \in \mathcal{H}$ with $g(u) \in K_r$ such that

$$\big\langle \rho Tu, v-g(u)\big\rangle + \lambda\big\|v-g(u)\big\|^2 \geq 0, \quad \forall v \in K_r, \tag{4}$$

where $\lambda > 0$ is an arbitrary constant.

The following example illustrates that the inequality (1) does not imply the inequality (4).

Example 3.1 Let $\mathcal{H} = \mathbb{R}$ and K_r be the union of two disjoint intervals $[0,\alpha]$ and $[\beta,\varrho]$ where $0 < \alpha < \beta < \varrho$. Then, $K_r = [0,\alpha] \cup [\beta,\varrho]$ is a r-prox-regular set with $r = \frac{\beta-\alpha}{2}$. Define $T, g : \mathcal{H} \to \mathcal{H}$ by

$$Tx = \theta e^{sx} \quad \text{and} \quad g(x) = l, \quad \forall x \in \mathcal{H},$$

where $s, \theta, l \in \mathbb{R}$, $\theta < 0$ and $l \in [0,\alpha]$ are arbitrary but fixed. Take $u = \alpha$ and let $\rho, \gamma > 0$ and $\lambda \in]0, \frac{-\rho\theta e^{s\alpha}}{\varrho}[$ be arbitrary but fixed. Then, we have

$$\big\langle \rho Tu, g(v)-g(u)\big\rangle + \gamma\big\|g(v)-g(u)\big\|^2 = \rho\theta e^{s\alpha}(l-l) + \gamma(l-l)^2 = 0, \quad \forall v \in \mathcal{H}.$$

Since $g(v) \in K_r$, for all $v \in \mathcal{H}$, it follows that

$$\big\langle \rho Tu, g(v)-g(u)\big\rangle + \gamma\big\|g(v)-g(u)\big\|^2 \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K_r.$$

However, it is obvious that $(v-l)(\rho\theta e^{s\alpha}+\lambda(v-l))<0$, for all $v\in[\beta,\varrho]$, that is,

$$\langle \rho Tu, v-g(u)\rangle+\lambda\|v-g(u)\|^2<0, \quad \forall v\in[\beta,\varrho].$$

Hence, the inequality

$$\langle \rho Tu, v-g(u)\rangle+\lambda\|v-g(u)\|^2\geq 0,$$

that is, the inequality (4) cannot hold for all $v\in K_r$.

It is mentioned in [10] that the projection operator P_{K_r} has the following characterization for the uniformly prox-regular set K_r.

Lemma 3.1 [10, Lemma 2.3] *Let K_r be a prox-regular and closed set in $\mathcal{H}$. Then, for a given $z\in\mathcal{H}$, $u\in K_r$ satisfies the inequality*

$$\langle u-z, v-u\rangle+\gamma\|v-u\|^2\geq 0, \quad \forall v\in K_r,$$

iff

$$u=P_{K_r}[z],$$

where P_{K_r} is the projection of $\mathcal{H}$ onto the prox-regular set K_r.

By a careful reading, we found that there is a fatal error in the proof of Lemma 3.1. Indeed, in the proof of Lemma 3.1, the author used the following equivalence:

$$\langle u-z, v-u\rangle+\gamma\|v-u\|^2\geq 0, \quad \forall v\in K_r \quad \Leftrightarrow \quad u-z\in N^P_{K_r}(u), \tag{5}$$

where $z\in\mathcal{H}$ and $u\in K_r$.

For all $z\in\mathcal{H}$ and $u\in K_r$ such that $u\neq z$, it follows from Definition 2.4 that

$$\begin{aligned} u-z\in N^P_{K_r}(u) \quad &\Leftrightarrow \quad \left\langle \frac{u-z}{\|u-z\|}, v-u\right\rangle\leq\frac{1}{2r}\|v-u\|^2, \quad \forall v\in K_r \\ &\Leftrightarrow \quad \langle z-u, v-u\rangle+\frac{\|u-z\|}{2r}\|v-u\|^2\geq 0, \quad \forall v\in K_r. \end{aligned} \tag{6}$$

Obviously, if $u=z$, then $u-z\in N^P_{K_r}(u)$.

From (5), (6), and the above fact, it follows that the following equivalence must be satisfied for all $z\in\mathcal{H}$ and $u\in K_r$:

$$\begin{aligned} &\langle u-z, v-u\rangle+\gamma\|v-u\|^2\geq 0, \quad \forall v\in K_r \\ \Leftrightarrow \quad &\langle z-u, v-u\rangle+\frac{\|u-z\|}{2r}\|v-u\|^2\geq 0, \quad \forall v\in K_r. \end{aligned} \tag{7}$$

The following example illustrates that the equivalence (7) is not necessarily true.

Example 3.2 Let $\mathcal{H}=\mathbb{R}$ and K_r be the same as in Example 3.1 such that $\varrho<2\beta-\alpha$. Take $u=\varrho$ and $z=\varrho+2$ and let $\gamma>0$ be arbitrary but fixed. Then, we have

$$\langle u-z, v-u\rangle+\gamma\|v-u\|^2=-2(v-\varrho)+\gamma(v-\varrho)^2\geq 0, \quad \forall v\in K_r.$$

However, taking $v=\beta$ and by using $\varrho<2\beta-\alpha$, it follows that

$$\begin{aligned}
&\langle z-u, v-u\rangle+\frac{\|u-z\|}{2r}\|v-u\|^2\\
&=2(\beta-\varrho)+\frac{1}{r}(\beta-\varrho)^2\\
&=(\beta-\varrho)\left(2+\frac{1}{r}(\beta-\varrho)\right)<0,
\end{aligned}$$

that is, the inequality

$$\langle z-u, v-u\rangle+\frac{\|u-z\|}{2r}\|v-u\|^2\geq 0,$$

cannot hold for all $v\in K_r$.

We now present the correct version of Lemma 3.1 and establish that the projection operator P_{K_r} has the following characterization for the uniformly r-prox-regular set K_r.

Lemma 3.2 *Let K_r be a r-prox-regular and closed set in $\mathcal{H}$. Then, for a given $z\in\mathcal{H}$, $u\in K_r$ $(u\neq z)$ satisfies the inequality*

$$\langle u-z, v-u\rangle+\frac{\|u-z\|}{2r}\|v-u\|^2\geq 0, \quad \forall v\in K_r, \tag{8}$$

iff

$$u=P_{K_r}[z],$$

where P_{K_r} is the projection of $\mathcal{H}$ onto K_r.

Proof Let $u\in K_r$, $u\neq z$, be a solution of (8). Then, by using Definition 2.4, we have

$$\begin{aligned}
&\left\langle \frac{z-u}{\|z-u\|}, v-u\right\rangle\leq\frac{1}{2r}\|v-u\|^2, \quad \forall v\in K_r\\
&\Leftrightarrow\quad z-u\in N^P_{K_r}(u)\\
&\Leftrightarrow\quad z\in\left(I+N^P_{K_r}\right)(u)\\
&\Leftrightarrow\quad u=P_{K_r}[z],
\end{aligned}$$

where I is the identity operator and we used the well-known fact that $P_{K_r}=(I+N^P_{K_r})^{-1}$. □

It should be pointed out that, for $u=z$, we have

$$0=z-u\in N^P_{K_r}(u) \quad\Leftrightarrow\quad z\in\big(I+N^P_{K_r}\big)(u) \quad\Leftrightarrow\quad u=P_{K_r}[z].$$

It is mentioned in [10] that the problem (1) is equivalent to a fixed point problem.

Lemma 3.3 [10, Lemma 3.1] *$u\in\mathcal{H}$ with $g(u)\in K_r$ is a solution of the problem* (1) *iff*

$$g(u)=P_{K_r}\big[g(u)-\rho Tu\big], \tag{9}$$

where P_{K_r} is the projection of $\mathcal{H}$ onto the uniformly r-prox-regular set K_r.

Indeed, Lemma 3.3 is deduced by utilizing Lemma 3.2. It is also asserted in [10] that one can prove the equivalence between problem (1) and fixed point problem (9) by using the equivalence between the problems (1) and (2). However, we have seen that the statement of Lemma 3.2 is incorrect and problems (1) and (2) are not necessarily equivalent.

Even without considering to the above fact, according to Proposition 2.1, for any $r'\in]0,r[$, the projection of points in $U(r')=\{u\in\mathcal{H}:d_{K_r}(u)<r'\}$ onto the set K_r exists and is unique, that is, for any $x\in U(r')$, the set $P_{K_r}(x)$ is nonempty and singleton. Equation (9) and Proposition 2.1 imply that the point $g(u)-\rho Tu$ should be belonged to $U(r')$, for some $r'\in]0,r[$. Unfortunately, it is not necessarily true. Indeed, (9) is not necessarily well-defined. If $g(u)\in K_r$ and $\rho<\frac{r'}{1+\|Tu\|}$, for some $r'\in]0,r[$, then we have

$$\begin{aligned} d_{K_r}\big(g(u)-\rho Tu\big)&=\inf_{v\in K_r}\big\|g(u)-\rho Tu-v\big\| \\ &\le\big\|g(u)-\rho Tu-g(u)\big\|=\rho\|Tu\| \\ &<\frac{r'\|Tu\|}{1+\|Tu\|}<r'. \end{aligned}$$

Therefore, $g(u)-\rho Tu\in U(r')$, that is, the set $P_{K_r}[g(u)-\rho Tu]$ is nonempty and singleton. Hence, in the statement of Lemma 3.3, the constant ρ should be satisfied $\rho<\frac{r'}{1+\|Tu\|}$, for some $r'\in]0,r[$.

In view of the above arguments, the statement of Lemma 3.3 is incorrect. The aforesaid lemma is the main tool to construct Algorithms 3.1–3.5 in [10]. Therefore, Algorithms 3.1–3.5 in [10] are not valid.

Noor [10] suggested and analyzed the following predictor–corrector-type algorithm for solving the problem (1).

Algorithm 3.1 [10, Algorithm 3.6] For a given $u_0\in\mathcal{H}$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} &\big\langle\rho Tu_n+g(w_n)-g(u_n),g(v)-g(w_n)\big\rangle+\gamma\big\|g(v)-g(w_n)\big\|^2\ge 0, \\ &\quad\forall g(v)\in K_r, \end{aligned}$$

$$\langle \rho T w_n + g(y_n) - g(w_n), g(v) - g(y_n) \rangle + \gamma \| g(v) - g(y_n) \|^2 \geq 0,$$
$$\forall g(v) \in K_r,$$
$$\langle \rho T y_n + g(u_{n+1}) - g(y_n), g(v) - g(u_{n+1}) \rangle + \gamma \| g(v) - g(u_{n+1}) \|^2 \geq 0,$$
$$\forall g(v) \in K_r,$$

where γ is a constant.

Definition 3.1 [10, Definition 2.4] An operator $T : \mathcal{H} \to \mathcal{H}$ with respect to an arbitrary operator g is said to be:

(a) *g-monotone* iff

$$\langle Tu - Tv, g(u) - g(v) \rangle \geq 0, \quad \forall u, v \in \mathcal{H};$$

(b) *partially relaxed strongly g-monotone* iff there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, g(z) - g(v) \rangle \geq -\alpha \| g(u) - g(z) \|^2, \quad \forall u, v, z \in \mathcal{H}.$$

Remark 3.1 By a careful reading of the results in [10], we found that there is a minor mistake in the context of Definition 2.4 in [10]. In fact, partially relaxed strongly g-pseudomonotone must be replaced by partially relaxed strongly g-monotone, as we have done in part (b) of Definition 3.1.

Noor [10] presented the following result which has a key role in the study of the convergence analysis of Algorithm 3.1.

Theorem 3.1 [10, Theorem 3.1] *Let $u \in \mathcal{H}$ with $g(u) \in K_r$ be a solution of* (1) *and let u_{n+1} be the approximate solution obtained from Algorithm* 3.1. *If the operator T is partially relaxed strongly g-monotone with constant $\alpha > 0$, then*

$$(1 - 4\gamma) \| g(u_{n+1}) - g(u) \|^2 \leq \| g(y_n) - g(u) \|^2 - (1 - 2\alpha\rho) \| g(u_{n+1}) - g(y_n) \|^2, \tag{10}$$

$$(1 - 4\gamma) \| g(y_n) - g(u) \|^2 \leq \| g(w_n) - g(u) \|^2 - (1 - 2\alpha\rho) \| g(y_n) - g(w_n) \|^2, \tag{11}$$

$$(1 - 4\gamma) \| g(w_n) - g(u) \|^2 \leq \| g(u_n) - g(u) \|^2 - (1 - 2\alpha\rho) \| g(w_n) - g(u_n) \|^2. \tag{12}$$

Noor [10] used Theorem 3.1 and asserted that the sequence $\{u_n\}$ generated by Algorithm 3.1 is strongly convergent to a solution of the problem (1).

Theorem 3.2 [10, Theorem 3.2] *Let $u \in \mathcal{H}$ with $g(u) \in K_r$ be a solution of* (1) *and let u_{n+1} be the approximate solution obtained from Algorithm* 3.1. *Let $\mathcal{H}$ be a finite dimensional space and assume that g^{-1} exists. If $0 < \rho < \frac{1}{2\alpha}$ and $\gamma < \frac{1}{4}$, then* $\lim_{n\to\infty} u_n = u$.

It should be pointed out that Theorem 3.1 is a main tool to establish the statement of Theorem 3.2. Therefore, the statement of Theorem 3.2 is not necessarily true. After a careful reading of the proof of Theorem 3.2, we discovered that there is a fatal error (p. 2960, line 3). In fact, it is asserted in [10] that by using the inequalities (10)–(12) in Theorem 3.1, one can conclude that

$$\left\|g(u_{n+1}) - g(\bar{u})\right\| \leq \left\|g(u_n) - g(\bar{u})\right\|, \tag{13}$$

that is, the sequence $\{\|g(u_n) - g(\bar{u})\|\}$ is nonincreasing. But, applying the inequalities (10)–(12), what we obtain is the following inequality:

$$\left\|g(u_{n+1}) - g(\bar{u})\right\| \leq \frac{1}{(1-4\gamma)\sqrt{1-4\gamma}}\left\|g(u_n) - g(\bar{u})\right\|, \tag{14}$$

not the inequality (13). In view of the assumption $\gamma < \frac{1}{4}$ in the statement of Theorem 3.2, the inequality (14) does not guarantee that the sequence $\{\|g(u_n) - g(\bar{u})\|\}$ is nonincreasing, whereas to prove the statement of Theorem 3.2, the sequence $\{\|g(u_n) - g(\bar{u})\|\}$ must be necessarily nonincreasing, and without this fact, one cannot prove the statement of Theorem 3.2.

4 Main Results

In this section, to overcome the problems in [10], we consider a new class of nonconvex variational inequalities instead of the general nonconvex variational inequalities of type (1). Furthermore, the correct versions of the results in [10] are presented.

Let $T : K_r \to \mathcal{H}$ and $g : K_r \to K_r$ be two nonlinear single-valued operators. To overcome the problems in [10], we consider the following problem instead of the problem (1): Find $u \in K_r$ such that

$$\left\langle Tu, g(v) - g(u)\right\rangle + \frac{\|Tu\|}{2r}\left\|g(v) - g(u)\right\|^2 \geq 0, \quad \forall v \in K_r, \tag{15}$$

which is called the *general regularized nonconvex variational inequality* (GRNVI). In the sequel, we denote the set of all solutions of the problem (15) by GRNVI(T, g, K_r).

If $r = \infty$, that is, $K_r = K$, the closed convex set in $\mathcal{H}$, and $g \equiv I$, then the problem (15) collapses to the classical variational inequality.

In the next proposition, the equivalence between the two problems (15) and (2) is established.

Proposition 4.1 *Let T and g be the same as in the problem* (15). *Then, the problem* (15) *is equivalent to the problem* (2).

Proof Let $u \in K_r$ be a solution of the problem (15). If $Tu = 0$, then $0 \in Tu + N^P_{K_r}(g(u))$ because the zero vector always belongs to any normal cone. If $Tu \neq 0$, then for all $v \in K_r$, we have

$$\left\langle -Tu, g(v) - g(u)\right\rangle \leq \frac{\|Tu\|}{2r}\left\|g(v) - g(u)\right\|^2.$$

Lemma 2.1 implies that $-Tu \in N^P_{K_r}(g(u))$, and so

$$0 \in Tu + N^P_{K_r}\big(g(u)\big),$$

that is, $u \in K_r$ is a solution of the problem (2).

Conversely, if $u \in K_r$ is a solution of the problem (2), then by Definition 2.4, $u \in K_r$ is a solution of the problem (15). □

We now present the correct version of Lemma 3.3. Indeed, by using the projection operator technique and Proposition 4.1, we establish the equivalence between GRNVI (15) and the fixed point problem (9). Here we would like to point out that one can prove the equivalence between the problem (15) and the fixed point problem (9) by utilizing Lemma 3.2.

Lemma 4.1 *Let T and g be the same as in the problem* (15) *such that g be an onto operator. Then $u \in K_r$ is a solution of the problem* (15) *iff u satisfies equation* (9), *provided $\rho \leq \frac{r'}{1+\|Tu\|}$, for some $r' \in]0, r[$.*

Proof Let $u \in K_r$ be a solution of the problem (15). Since $\rho \leq \frac{r'}{1+\|Tu\|}$, for some $r' \in]0, r[$, it follows that (9) is well-defined. Then, by using Proposition 4.1, we have

$$\begin{aligned} 0 \in Tu + N^P_{K_r}\big(g(u)\big) \quad &\Leftrightarrow \quad g(u) - \rho Tu \in g(u) + N^P_{K_r}\big(g(u)\big) \\ &\Leftrightarrow \quad g(u) - \rho Tu \in \big(I + N^P_{K_r}\big)\big(g(u)\big) \\ &\Leftrightarrow \quad g(u) = P_{K_r}\big[g(u) - \rho Tu\big], \end{aligned}$$

where I is the identity operator and we used the well-known fact that $P_{K_r} = (I + N^P_{K_r})^{-1}$. □

For a given $u \in K_r$, we now consider the following auxiliary general regularized nonconvex variational inequality problem: Find a unique $w \in K_r$ such that

$$\big\langle \rho Tu + g(w) - g(u), g(v) - g(u)\big\rangle + \frac{\rho\|Tu\|}{2r}\big\|g(v) - g(u)\big\|^2 \geq 0, \quad \forall v \in K_r, \tag{16}$$

where $\rho > 0$ is a constant. We observe that, if $w = u$, then clearly w is a solution of GRNVI (15).

On the basis of this observation, we suggest and analyze the following three step predictor–corrector-type algorithm for solving GRNVI (15).

Algorithm 4.1 (Modified Predictor–Corrector Algorithm) Let the operators T and g be the same as in GRNVI (15). For a given $u_0 \in K_r$, compute the approximate solution $u_{n+1} \in K_r$ of GRNVI (15) by the following iterative schemes:

$$\big\langle \rho Tu_n + g(w_n) - g(u_n), g(v) - g(w_n)\big\rangle + \frac{\rho\|Tu_n\|}{2r}\big\|g(v) - g(w_n)\big\|^2 \geq 0, \quad \forall v \in K_r \tag{17}$$

$$\left\langle \rho T w_n + g(y_n) - g(w_n), g(v) - g(y_n) \right\rangle + \frac{\rho \|T w_n\|}{2r} \left\| g(v) - g(y_n) \right\|^2 \geq 0,$$
$$\forall v \in K_r, \tag{18}$$

$$\left\langle \rho T y_n + g(u_{n+1}) - g(y_n), g(v) - g(u_{n+1}) \right\rangle + \frac{\rho \|T y_n\|}{2r} \left\| g(v) - g(u_{n+1}) \right\|^2 \geq 0,$$
$$\forall v \in K_r, \tag{19}$$

where $\rho > 0$ is the same as in the problem (16) and $n \geq 0$.

Lemma 4.2 *For all $u, v \in \mathcal{H}$, we have*

$$2\langle u, v \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2.$$

Definition 4.1 Let $T : K_r \to \mathcal{H}$ and $g : K_r \to K_r$ be two nonlinear single-valued operators. Then, T is said to be

(a) *g-monotone* iff

$$\left\langle Tu - Tv, g(u) - g(v) \right\rangle \geq 0, \quad \forall u, v \in K_r;$$

(b) *κ-strongly g-monotone* iff there exists a constant $\kappa > 0$ such that

$$\left\langle Tu - Tv, g(u) - g(v) \right\rangle \geq \kappa \left\| g(u) - g(v) \right\|^2, \quad \forall u, v \in K_r;$$

(c) *partially (α, β)-relaxed g-monotone* iff there exist two constants $\alpha, \beta > 0$ such that

$$\left\langle Tu - Tv, g(w) - g(v) \right\rangle \geq -\alpha \left\| g(u) - g(w) \right\|^2 + \beta \left\| g(w) - g(v) \right\|^2,$$
$$\forall u, v, w \in K_r.$$

We remark that, if $w = u$, then partially (α, β)-relaxed g-monotone is exactly β-strongly g-monotone of the operator T. If $g \equiv I$, parts (a)–(c) of Definition 4.1 reduce to the definitions of monotonicity, κ-strong monotonicity, and partially (α, β)-relaxed monotonicity of the operator T, respectively.

We first present the following result, which plays a key role in the study of the convergence analysis of Algorithm 4.1.

Lemma 4.3 *Let $u \in K_r$ be a solution of* GRNVI (15) *and $\{u_n\}$ be the sequence of approximate solutions of* GRNVI (15) *generated by Algorithm 4.1. Suppose that $\{w_n\}$ and $\{y_n\}$ are two sequences defined in Algorithm 4.1 such that the sequences $\{Tu_n\}$, $\{Tw_n\}$, and $\{Ty_n\}$ are bounded. If T is partially $(\alpha, \frac{\beta}{2r})$-relaxed g-monotone with $\beta = \|Tu\| + \sup\{\|Tu_n\|, \|Tw_n\|, \|Ty_n\| : n \geq 0\}$, then*

$$\left\| g(w_n) - g(u) \right\|^2 \leq \left\| g(u_n) - g(u) \right\|^2 - (1 - 2\alpha\rho) \left\| g(w_n) - g(u_n) \right\|^2, \tag{20}$$

$$\left\| g(y_n) - g(u) \right\|^2 \leq \left\| g(w_n) - g(u) \right\|^2 - (1 - 2\alpha\rho) \left\| g(y_n) - g(w_n) \right\|^2, \tag{21}$$

$$\left\| g(u_{n+1}) - g(u) \right\|^2 \leq \left\| g(y_n) - g(u) \right\|^2 - (1 - 2\alpha\rho) \left\| g(u_{n+1}) - g(y_n) \right\|^2. \tag{22}$$

Proof Since $u \in K_r$ is a solution of GRNVI (15), it follows that

$$\langle \rho T u, g(v) - g(u) \rangle + \frac{\rho \|Tu\|}{2r} \|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in K_r, \tag{23}$$

where ρ is the same as in the problem (16). Taking $v = w_n$ in (23) and $v = u$ in (17), we have

$$\langle \rho T u, g(w_n) - g(u) \rangle + \frac{\rho \|Tu\|}{2r} \|g(w_n) - g(u)\|^2 \geq 0 \tag{24}$$

and

$$\langle \rho T u_n + g(w_n) - g(u_n), g(u) - g(w_n) \rangle + \frac{\rho \|Tu_n\|}{2r} \|g(u) - g(w_n)\|^2 \geq 0. \tag{25}$$

Applying (24) and (25), we obtain

$$\begin{aligned} &\langle \rho T u_n - \rho T u + g(w_n) - g(u_n), g(u) - g(w_n) \rangle \\ &\quad + \frac{\rho \|Tu_n\|}{2r} \|g(u) - g(w_n)\|^2 + \frac{\rho \|Tu\|}{2r} \|g(u) - g(w_n)\|^2 \\ &= \langle g(w_n) - g(u_n), g(u) - g(w_n) \rangle + \langle \rho T u_n - \rho T u, g(u) - g(w_n) \rangle \\ &\quad + \frac{\rho \|Tu_n\|}{2r} \|g(u) - g(w_n)\|^2 + \frac{\rho \|Tu\|}{2r} \|g(u) - g(w_n)\|^2 \geq 0, \end{aligned}$$

which leads to

$$\begin{aligned} \langle g(w_n) - g(u_n), g(u) - g(w_n) \rangle &\geq \rho \langle T u_n - T u, g(w_n) - g(u) \rangle \\ &\quad - \frac{\rho (\|Tu_n\| + \|Tu\|)}{2r} \|g(u) - g(w_n)\|^2. \end{aligned} \tag{26}$$

Since T is partially $(\alpha, \frac{\beta}{2r})$-relaxed g-monotone with

$$\beta = \|Tu\| + \sup\{\|Tu_n\|, \|Tw_n\|, \|Ty_n\| : n \geq 0\},$$

it follows from (26) that

$$\begin{aligned} \langle g(w_n) - g(u_n), g(u) - g(w_n) \rangle &\geq -\alpha\rho \|g(w_n) - g(u_n)\|^2 + \frac{\rho\beta}{2r} \|g(u) - g(w_n)\|^2 \\ &\quad - \frac{\rho (\|Tu_n\| + \|Tu\|)}{2r} \|g(u) - g(w_n)\|^2 \\ &\geq -\alpha\rho \|g(w_n) - g(u_n)\|^2. \end{aligned} \tag{27}$$

By using Lemma 4.2 and (27), we get

$$\begin{aligned} &\langle g(w_n) - g(u_n), g(u) - g(w_n) \rangle \\ &= \frac{1}{2} \left(\|g(u) - g(u_n)\|^2 - \|g(u_n) - g(w_n)\|^2 - \|g(u) - g(w_n)\|^2 \right) \\ &\geq -\alpha\rho \|g(w_n) - g(u_n)\|^2, \end{aligned}$$

which leads to

$$\|g(w_n)-g(u)\|^2 \le \|g(u_n)-g(u)\|^2 - (1-2\alpha\rho)\|g(w_n)-g(u_n)\|^2,$$

the required result (20).

Taking $v=y_n$ in (23) and $v=u$ in (18), we have

$$\langle \rho Tu, g(y_n)-g(u)\rangle + \frac{\rho\|Tu\|}{2r}\|g(y_n)-g(u)\|^2 \ge 0, \tag{28}$$

and

$$\langle \rho Tw_n + g(y_n)-g(w_n), g(u)-g(y_n)\rangle + \frac{\rho\|Tw_n\|}{2r}\|g(u)-g(y_n)\|^2 \ge 0. \tag{29}$$

Applying (28) and (29), and the partially $(\alpha, \frac{\beta}{2r})$-relaxed g-monotonicity of T, we deduce that

$$\|g(y_n)-g(u)\|^2 \le \|g(w_n)-g(u)\|^2 - (1-2\alpha\rho)\|g(y_n)-g(w_n)\|^2,$$

the required result (21).

Now, taking $v=u_{n+1}$ in (23) and $v=u$ in (19), we have

$$\langle \rho Tu, g(u_{n+1})-g(u)\rangle + \frac{\rho\|Tu\|}{2r}\|g(u_{n+1})-g(u)\|^2 \ge 0 \tag{30}$$

and

$$\langle \rho Ty_n + g(u_{n+1})-g(y_n), g(u)-g(u_{n+1})\rangle + \frac{\rho\|Ty_n\|}{2r}\|g(u)-g(u_{n+1})\|^2 \ge 0. \tag{31}$$

In a similar way, from (30) and (31), and the partially $(\alpha, \frac{\beta}{2r})$-relaxed g-monotonicity of T, it follows that

$$\|g(u_{n+1})-g(u)\|^2 \le \|g(y_n)-g(u)\|^2 - (1-2\alpha\rho)\|g(u_{n+1})-g(y_n)\|^2,$$

the required result (22). This completes the proof. □

In the next theorem, by utilizing Lemma 4.3, the convergence of the iterative sequence generated by Algorithm 4.1 to a solution of GRNVI (15) is demonstrated.

Theorem 4.1 *Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $T: K_r \to \mathcal{H}$ and $g: K_r \to K_r$ be continuous single-valued operators such that g is invertible. Suppose that all the conditions of Lemma 4.3 hold and* GRNVI$(T,g,K_r)\neq\emptyset$. *If $\rho \in]0, \frac{1}{2\alpha}[$, then for any given $u_0 \in K_r$, the iterative sequence $\{u_n\}$ generated by Algorithm 4.1 converges strongly to a solution $\hat{u}$ of* GRNVI (15).

Proof Let $u \in K_r$ be a solution of GRNVI (15). By using the inequalities (20)–(22), it follows that the sequence $\{\|g(u_n)-g(u)\|\}$ is nonincreasing, and hence, the sequence

$\{g(u_n)\}$ is bounded. Since g is invertible, it follows that the sequence $\{u_n\}$ is also bounded. Moreover, by (20)–(22), we have

$$(1-2\alpha\rho)\bigl(\|g(w_n)-g(u_n)\|^2+\|g(y_n)-g(w_n)\|^2+\|g(u_{n+1})-g(y_n)\|^2\bigr)$$
$$\le \|g(u_n)-g(u)\|^2-\|g(u_{n+1})-g(u)\|^2.$$

From the above inequality it follows that

$$\sum_{n=0}^{\infty}(1-2\alpha\rho)\bigl(\|g(w_n)-g(u_n)\|^2+\|g(y_n)-g(w_n)\|^2+\|g(u_{n+1})-g(y_n)\|^2\bigr)$$
$$\le \|g(u_0)-g(u)\|^2,$$

which implies that $\|g(w_n)-g(u_n)\|\to 0$, $\|g(y_n)-g(w_n)\|\to 0$ and $\|g(u_{n+1})-g(y_n)\|\to 0$, as $n\to\infty$. Let $\hat{u}$ be a cluster point of $\{u_n\}$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i}\to\hat{u}$, as $i\to\infty$. Since g is continuous, it follows that $g(u_{n_i})\to g(\hat{u})$, as $i\to\infty$, consequently, $g(w_{n_i})\to g(\hat{u})$ and $g(y_{n_i})\to g(\hat{u})$, as $i\to\infty$. By (17), we have

$$\bigl\langle\rho Tu_{n_i}+g(w_{n_i})-g(u_{n_i}),g(v)-g(w_{n_i})\bigr\rangle+\frac{\rho\|Tu_{n_i}\|}{2r}\|g(v)-g(w_{n_i})\|^2\ge 0,$$
$$\forall v\in K_r. \tag{32}$$

By the continuity of T and g, letting $i\to\infty$ in (32), we obtain

$$\bigl\langle T\hat{u},g(v)-g(\hat{u})\bigr\rangle+\frac{\|T\hat{u}\|}{2r}\|g(v)-g(\hat{u})\|^2\ge 0,\quad \forall v\in K_r,$$

that is, $\hat{u}\in K_r$ is a solution of GRNVI (15). Now, Lemma 4.3 guarantees that

$$\|g(u_{n+1})-g(\hat{u})\|\le\|g(u_n)-g(\hat{u})\|,\quad \forall n\ge 0. \tag{33}$$

From (33), it follows that $g(u_n)\to g(\hat{u})$, as $n\to\infty$. Since g is continuous and invertible, it follows that $u_n\to\hat{u}$, as $n\to\infty$, that is, the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$. This completes the proof. □

5 Summary and Conclusion

In this paper, we have introduced a so called class of general regularized nonconvex variational inequalities (GRNVI) which presents a correct version of general nonconvex variational inequalities considered in [10]. We have presented the equivalence between GRNVI and a fixed point problem. This equivalence also corrects the equivalent formulation used in [10]. By using the auxiliary principle technique, we have suggested and analyzed a new class of predictor–corrector methods for solving GRNVI. The convergence of iterative methods is studied under the partially relaxed monotonicity assumption. As a consequence, the algorithm and results presented in the paper overcome incorrect algorithms and results existing in the literature.

Acknowledgements The research part of the first author was done during his visit to KFUPM, Dhahran, Saudi Arabia. The first author expresses his thanks to KFUPM for providing excellent research facilities during his stay in KFUPM.

The authors thank Prof. F. Giannessi for his constructive comments which contributed to the improvement of the present paper.

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