

Further Discussion on the Error Bound for Generalized Linear Complementarity Problem over a Polyhedral Cone

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Abstract In this paper, we consider the global error bound for the generalized linear complementarity problem over a polyhedral cone (GLCP). Based on the new transformation of the problem, we establish its global error bound under milder conditions, which improves the result obtained by Sun and Wang (2009) for GLCP by weakening the assumption.

Keywords GLCP · Reformulation · Global error bound

1 Introduction

In recent years, complementarity problems have been extended in many directions via innovative techniques to study a wide class of problems arising in pure and applied sciences. A useful and important generalization is called the generalized nonlinear complementarity problem over a polyhedral cone (GNCP). This problem was introduced first by Andreani et al. [1] in 2001, and further developed by Wang et al. in [2, 3]. The GNCP plays a significant role in economics, operation research, and nonlinear analysis, etc. [4].

The generalized linear complementarity problem over a polyhedral cone (GLCP) is a special case of the GNCP, and some theoretical issues such as the existence of solutions and the numerical solution methods for the GLCP (GNCP) have been

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discussed, e.g., [1–3]. It is well known that the global error bound is an important tool for theoretical analysis and numerical treatment for a mathematic problem [4, 5]. We are here concerned with the global error bound on the distance between a given point and the solution set of the GLCP in terms of some residual functions.

This paper is a follow-up to [6], as in this paper will establish the global error bound estimation of the GLCP under weaker conditions than that needed in [6]. To this end, we first develop some equivalent reformulations of the GLCP under weaker condition than that discussed in [6], and then establish the global error bound for the GLCP, which improves the result obtained in [6]. Compared with the error bound result in [6, 7], the requirement of a non-degenerate solution and the square root term of residual function in the error bound estimation are both removed here.

2 Preliminaries

Let \mathbb{R}^m be a real Euclidean space equipped with the standard inner product, and let mappings $F(x) = Mx + p$, $G(x) = Nx + q$, where $M, N \in \mathbb{R}^{m \times n}$, $p, q \in \mathbb{R}^m$. The generalized linear complementarity problem over a polyhedral cone, abbreviated as GLCP, is to find vector $x^* \in \mathbb{R}^n$ such that

$$F(x^*) \in \mathcal{K}, \quad G(x^*) \in \mathcal{K}^\circ, \quad F(x^*)^\top G(x^*) = 0, \quad (1)$$

where \mathcal{K} is a polyhedral cone in \mathbb{R}^m such that there exist matrices $A \in \mathbb{R}^{s \times m}$, $B \in \mathbb{R}^{t \times m}$ satisfying $\mathcal{K} = \{v \in \mathbb{R}^m \mid Av \geq 0, Bv = 0\}$, and \mathcal{K}° is its dual cone which admits the following form:

$$\mathcal{K}^\circ = \{u \in \mathbb{R}^m \mid u = A^\top \lambda_1 + B^\top \lambda_2, \lambda_1 \in \mathbb{R}_+^s, \lambda_2 \in \mathbb{R}^t\}.$$

Throughout this paper, we denote the solution set of the GLCP by X^* and assume that it is nonempty throughout this paper.

Based on this, we can give the needed assumptions for our analysis.

Assumption 2.1 For the matrices A, M, N involved in the GLCP, we assume that

- (A1) the matrix $M^\top N$ is positive semi-definite;
- (A2) the matrix A^\top has full-column rank.

Under this assumption, matrix A^\top has full-column rank and it has left inverse $(AA^\top)^{-1}A$, which is also its pseudo-inverse of A^\top . On the other hand, the condition that the matrix A^\top has full-column rank is weaker than that the matrix (A^\top, B^\top) has full-column rank discussed in [6].

To end this section, some notations used are in order. The norm $\|\cdot\|$ and $\|\cdot\|_1$ denote the Euclidean 2-norm and 1-norm, respectively. We use \mathbb{R}_+^n to denote the orthogonal nonnegative orthant in \mathbb{R}^n , and use x_+ and x_- to denote the projections of vector x onto \mathbb{R}_+^n and \mathbb{R}_-^n , respectively.

3 Main Result

In the following, we first establish some equivalent reformulations to the GLCP. To this end, the following straightforward result is needed.

Theorem 3.1 *A point $x^* \in \mathbb{R}^n$ is a solution of the GLCP if and only if there exist $\lambda_1^* \in \mathbb{R}^s, \lambda_2^* \in \mathbb{R}^t$, such that*

$$\begin{cases} A(Mx^* + p) \geq 0, \\ B(Mx^* + p) = 0, \\ \lambda_1^* \geq 0, \\ (Mx^* + p)^\top(Nx^* + q) = 0, \\ Nx^* + q = A^\top \lambda_1^* + B^\top \lambda_2^*. \end{cases} \tag{2}$$

From Theorem 3.1, under Assumption 2.1(A2), we can transform the second inequality and the last equality in (2) into a new system in which neither parameter λ_1 nor parameter λ_2 is involved. To this end, we need the following conclusion [8].

Lemma 3.1 *Suppose that the non-homogeneous linear equation system $Hy = b$ is consistent. Then $y = H^+b$ is the solution with the minimum 2-norm, where H^+ is the pseudo-inverse of H .*

Since $X^* \neq \emptyset$, we can establish the following result.

Lemma 3.2 *Suppose that Assumption 2.1(A2) holds. Then, for any $x \in \mathbb{R}^n$, the following statements are equivalent:*

- (1) *There exist $\lambda_1 \in \mathbb{R}_+^s, \lambda_2 \in \mathbb{R}^t$ such that $Nx + q = A^\top \lambda_1 + B^\top \lambda_2$,*
- (2)
$$\begin{cases} \{-A_L^{-1}B^\top[(A^\top A_L^{-1} - I)B^\top]^\dagger[A^\top A_L^{-1} - I] + A_L^{-1}\}(Nx + q) \geq 0, \\ \{A^\top \{-A_L^{-1}B^\top[(A^\top A_L^{-1} - I)B^\top]^\dagger[A^\top A_L^{-1} - I] + A_L^{-1}\} \\ \quad + B^\top[(A^\top A_L^{-1} - I)B^\top]^\dagger[A^\top A_L^{-1} - I] - I\}(Nx + q) = 0, \end{cases}$$

where $A_L^{-1} = (AA^\top)^{-1}A$.

Proof Set

$$X_1 := \{x \in \mathbb{R}^n \mid Nx + q = A^\top \lambda_1 + B^\top \lambda_2 \text{ for some } \lambda_1 \in \mathbb{R}_+^s, \lambda_2 \in \mathbb{R}^t\},$$

$$X_2 := \left\{ x \in \mathbb{R}^n \mid \begin{cases} \{-A_L^{-1}B^\top[(A^\top A_L^{-1} - I)B^\top]^\dagger[A^\top A_L^{-1} - I] + A_L^{-1}\}(Nx + q) \geq 0, \\ \{A^\top \{-A_L^{-1}B^\top[(A^\top A_L^{-1} - I)B^\top]^\dagger[A^\top A_L^{-1} - I] + A_L^{-1}\} \\ \quad + B^\top[(A^\top A_L^{-1} - I)B^\top]^\dagger[A^\top A_L^{-1} - I] - I\}(Nx + q) = 0. \end{cases} \right\}.$$

Now, we show that these two sets are equal.

First, for any $x \in X_1$, there exist $\lambda_1 \in \mathbb{R}_+^s, \lambda_2 \in \mathbb{R}^t$ such that

$$Nx + q = A^\top \lambda_1 + B^\top \lambda_2. \tag{3}$$

Pre-multiplying (3) by $A_L^{-1} := (AA^\top)^{-1}A$ gives

$$A_L^{-1}(Nx + q) = \lambda_1 + A_L^{-1}B^\top \lambda_2, \tag{4}$$

and combining this with (3), we have

$$\begin{aligned} Nx + q &= A^\top (A_L^{-1}(Nx + q) - A_L^{-1}B^\top \lambda_2) + B^\top \lambda_2 \\ &= A^\top A_L^{-1}(Nx + q) - [A^\top A_L^{-1}B^\top - B^\top] \lambda_2, \end{aligned}$$

i.e.,

$$[A^\top A_L^{-1}B^\top - B^\top] \lambda_2 = [A^\top A_L^{-1} - I](Nx + q).$$

Recalling Lemma 3.1, we further have

$$\lambda_2 = [(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I](Nx + q). \tag{5}$$

Combining this with (4) yields

$$\lambda_1 = \{-A_L^{-1}B^\top [(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I] + A_L^{-1}\}(Nx + q). \tag{6}$$

Using (3), (5), (6), we have

$$\begin{aligned} &\{A^\top \{-A_L^{-1}B^\top [(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I] + A_L^{-1}\} \\ &\quad + B^\top [(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I] - I\}(Nx + q) = 0. \end{aligned} \tag{7}$$

Using the fact that $\lambda_1 \geq 0$, by (6), one has

$$\{-A_L^{-1}B^\top [(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I] + A_L^{-1}\}(Nx + q) \geq 0.$$

Combining this with (7) leads to that $x \in X_2$. This shows that $X_1 \subseteq X_2$.

Second, for any $x \in X_2$, let

$$\begin{aligned} \lambda_1 &= \{-A_L^{-1}B^\top [(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I] + A_L^{-1}\}(Nx + q), \\ \lambda_2 &= \{[(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I]\}(Nx + q). \end{aligned}$$

Then $\lambda_1 \in \mathbb{R}_+^s, \lambda_2 \in \mathbb{R}^t$. From (7), one has

$$\begin{aligned} Nx + q &= A^\top \{-A_L^{-1}B^\top [(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I] + A_L^{-1}\}(Nx + q) \\ &\quad + B^\top \{[(A^\top A_L^{-1} - I)B^\top]^+ [A^\top A_L^{-1} - I]\}(Nx + q) \\ &= A^\top \lambda_1 + B^\top \lambda_2, \end{aligned}$$

i.e., $x \in X_1$. Hence, $X_2 \subseteq X_1$, and the desired result follows. □

Combining this conclusion with Theorem 3.1, we can establish the following equivalent formulation of the GLCP under Assumption 2.1(A2):

$$\begin{cases} AF(x) \geq 0, \\ BF(x) = 0, \\ (F(x))^T G(x) = 0, \\ UG(x) \geq 0, \\ VG(x) = 0, \end{cases} \tag{8}$$

where

$$\begin{aligned} U &= \{-A_L^{-1}B^T[(A^T A_L^{-1} - I)B^T]^+[A^T A_L^{-1} - I] + A_L^{-1}\}, \\ V &= \{A^T\{-A_L^{-1}B^T[(A^T A_L^{-1} - I)B^T]^+[A^T A_L^{-1} - I] + A_L^{-1}\} \\ &\quad + B^T[(A^T A_L^{-1} - I)B^T]^+[A^T A_L^{-1} - I] - I\}. \end{aligned}$$

For system (8), by the first equality and the last equality, one has

$$\begin{aligned} &(F(x))^T G(x) \\ &= (F(x))^T \{A^T\{-A_L^{-1}B^T[(A^T A_L^{-1} - I)B^T]^+[A^T A_L^{-1} - I] + A_L^{-1}\} \\ &\quad + B^T[(A^T A_L^{-1} - I)B^T]^+[A^T A_L^{-1} - I]\}G(x) \\ &= [AF(x)]^T \{-A_L^{-1}B^T[(A^T A_L^{-1} - I)B^T]^+[A^T A_L^{-1} - I] + A_L^{-1}\}G(x) \\ &\quad + [BF(x)]^T \{[(A^T A_L^{-1} - I)B^T]^+[A^T A_L^{-1} - I]\}G(x) \\ &= [AF(x)]^T [UG(x)]. \end{aligned} \tag{9}$$

Thus, system (8) can be further written as

$$\begin{cases} AF(x) \geq 0, & BF(x) = 0, \\ (AF(x))^T [UG(x)] = 0, \\ UG(x) \geq 0, & VG(x) = 0. \end{cases} \tag{10}$$

For convenience of discussion, we denote

$$\Omega = \{x \in \mathbb{R}^n \mid BF(x) = 0, VG(x) = 0\}$$

in the following. To proceed, we present the following assumption, which will be needed in the sequel.

Assumption 3.1 For system (10), there exists a point $\hat{x} \in \Omega$ such that

$$AF(\hat{x}) > 0, \quad UG(\hat{x}) > 0.$$

Obviously, Assumption 3.1 is weaker than the condition in Theorem 4.2 of [6]. Now, we consider the following set associated with the GLCP:

$$\mathcal{L}(\epsilon) \triangleq \{x \in \Omega \mid \varphi(x) \leq \epsilon\}, \tag{11}$$

where $\epsilon \geq 0$ and $\varphi(x) = \|(-AF(x))_+\|_1 + \|(-UG(x))_+\|_1 + ((AF(x))^\top(UG(x)))_+$.

Theorem 3.2 *Suppose Assumptions 2.1 and 3.1 hold, and matrix $\begin{pmatrix} A & M \\ U & N \end{pmatrix}$ is of full column rank, then $\mathcal{L}(\epsilon)$ is bounded for any $\epsilon \geq 0$.*

Proof For any $x \in \mathcal{L}(\epsilon)$, we have

$$\begin{aligned} \|(-AF(x))_+\|_1 &\leq \epsilon, & \|(-UG(x))_+\|_1 &\leq \epsilon, \\ ((AF(x))^\top(UG(x)))_+ &\leq \epsilon. \end{aligned} \tag{12}$$

Then, for $i = 1, 2, \dots, n$, one has

$$\min\{(AF(x))_i, (UG(x))_i\} \geq -\epsilon. \tag{13}$$

By Assumption 2.1(A1),

$$\begin{aligned} 0 &\leq (x - \hat{x})^\top M^\top N(x - \hat{x}) \\ &= \langle F(x) - F(\hat{x}), G(x) - G(\hat{x}) \rangle \\ &= F(x)^\top G(x) - F(x)^\top G(\hat{x}) - F(\hat{x})^\top G(x) + F(\hat{x})^\top G(\hat{x}) \\ &= (AF(x))^\top [UG(x)] - (AF(x))^\top [UG(\hat{x})] - (AF(\hat{x}))^\top [UG(x)] \\ &\quad + (AF(\hat{x}))^\top [UG(\hat{x})], \end{aligned} \tag{14}$$

where the last equality is obtained by a similar argument of (9). Certainly, (14) can be written as

$$\begin{aligned} &(AF(x))^\top [UG(\hat{x})] + (AF(\hat{x}))^\top [UG(x)] \\ &\leq (AF(x))^\top [UG(x)] + (AF(\hat{x}))^\top [UG(\hat{x})]. \end{aligned}$$

Using (12) and (13), one has

$$\begin{aligned} &\sum_{[UG(x)]_i > 0} [AF(\hat{x})]_i [UG(x)]_i + \sum_{[AF(x)]_i > 0} [AF(x)]_i [UG(\hat{x})]_i \\ &\leq [AF(x)]^\top [UG(x)] + [AF(\hat{x})]^\top [UG(\hat{x})] \\ &\quad - \left[\sum_{[UG(x)]_i < 0} [AF(\hat{x})]_i [UG(x)]_i + \sum_{[AF(x)]_i < 0} [UG(\hat{x})]_i [AF(x)]_i \right] \\ &\leq ([AF(x)]^\top [UG(x)])_+ + [AF(\hat{x})]^\top [UG(\hat{x})] \end{aligned}$$

$$\begin{aligned}
 & + \epsilon [\| [AF(\hat{x})] \|_1 + \| [UG(\hat{x})] \|_1] \\
 & \leq [AF(\hat{x})]^\top [UG(\hat{x})] + \epsilon [1 + \| [AF(\hat{x})] \|_1 + \| [UG(\hat{x})] \|_1]. \tag{15}
 \end{aligned}$$

We now break up the discussion into two cases.

First, if $[AF(x)]_i < 0$, then by (13), we have $-\epsilon < [AF(x)]_i < 0$.

Second, if $[AF(x)]_i > 0$, combining (15) with $[UG(\hat{x})]_i > 0$, we have

$$\begin{aligned}
 0 & < [AF(x)]_i \\
 & = ([UG(\hat{x})]_i)^{-1} [UG(\hat{x})]_i [AF(x)]_i \\
 & < ([UG(\hat{x})]_i)^{-1} \left(\sum_{[UG(x)]_j > 0} [AF(\hat{x})]_j [UG(x)]_j \right. \\
 & \quad \left. + \sum_{[AF(x)]_j > 0} [UG(\hat{x})]_j [AF(x)]_j \right) \\
 & \leq ([UG(\hat{x})]_i)^{-1} ([AF(\hat{x})]^\top [UG(\hat{x})] + \epsilon [1 + \| [AF(\hat{x})] \|_1 + \| [UG(\hat{x})] \|_1]),
 \end{aligned}$$

which shows that $[AF(x)]_i$ is bounded w.r.t. x .

Similar to discussion above, we can also prove that $[UG(x)]_i$ is bounded w.r.t. x .

Since the matrix $\begin{pmatrix} AM \\ UN \end{pmatrix}$ is of column full rank, we know that

$$\begin{aligned}
 \|x\| & = \left\| \begin{pmatrix} AM \\ UN \end{pmatrix}_L^{-1} \begin{pmatrix} AM \\ UN \end{pmatrix} x \right\| \\
 & = \left\| \begin{pmatrix} AM \\ UN \end{pmatrix}_L^{-1} \left[\begin{pmatrix} AM \\ UN \end{pmatrix} x + \begin{pmatrix} Ap \\ Uq \end{pmatrix} - \begin{pmatrix} Ap \\ Uq \end{pmatrix} \right] \right\| \\
 & = \left\| \begin{pmatrix} AM \\ UN \end{pmatrix}_L^{-1} \left[\begin{pmatrix} AF(x) \\ UG(x) \end{pmatrix} - \begin{pmatrix} Ap \\ Uq \end{pmatrix} \right] \right\| \\
 & \leq \left\| \begin{pmatrix} AM \\ UN \end{pmatrix}_L^{-1} \right\| \left\| \begin{pmatrix} AF(x) \\ UG(x) \end{pmatrix} \right\| + \left\| \begin{pmatrix} AM \\ UN \end{pmatrix}_L^{-1} \begin{pmatrix} Ap \\ Uq \end{pmatrix} \right\|,
 \end{aligned}$$

where $\begin{pmatrix} AM \\ UN \end{pmatrix}_L^{-1}$ is left inverse of the matrix $\begin{pmatrix} AM \\ UN \end{pmatrix}$. So $\mathcal{L}(\epsilon)$ is bounded for any $\epsilon \geq 0$. \square

Lemma 3.3 *Given constant $\mu > 0$, for any $\|x\| \leq \mu, x \in \mathbb{R}^n$, then there exists constant $v > 0$ such that*

$$\varphi(x) \leq vr(x),$$

where $r(x) = \|\min\{AF(x), UG(x)\}\|$.

Proof Using the fact that

$$|(-a)_+| \leq |\min\{a, b\}|, \quad \forall a, b \in \mathbb{R},$$

we know that there exists constant $c_1 > 0$ such that

$$\|(-AF(x))_+\|_1 + \|(-UG(x))_+\|_1 \leq 2c_1 \|\min\{AF(x), UG(x)\}\|. \quad (16)$$

For any $x \in \mathbb{R}^n$ with $\|x\| \leq \mu$, there exists constant $c_2 > 0$ such that

$$\|\max\{AF(x), UG(x)\}\| \leq c_2.$$

Therefore, by the definition of $\varphi(x)$, we have

$$\begin{aligned} \varphi(x) &\leq 2c_1 \|\min\{AF(x), UG(x)\}\| + [(AF(x))^T (UG(x))]_+ \\ &\leq 2c_1 \|\min\{AF(x), UG(x)\}\| \\ &\quad + \|\min\{AF(x), UG(x)\}\| \cdot \|\max\{AF(x), UG(x)\}\| \\ &\leq \nu \|\min\{AF(x), UG(x)\}\|, \end{aligned}$$

where $\nu = \max\{2c_1, c_2\}$, the first inequality is by (16), and the second inequality follows from the fact that

$$(a + b)_+ \leq a_+ + b_+, \quad \text{and} \quad (cd)_+ \leq \|\min\{c, d\}\| \|\max\{c, d\}\|, \quad \forall a, b, c, d \in \mathbb{R}.$$

□

By Lemma 3.3 and Theorem 3.2, we immediately obtain the following conclusion.

Lemma 3.4 *Suppose that the hypothesis of Theorem 3.2 holds, then the set*

$$X_\varepsilon := \{x \in \Omega \mid r(x) \leq \varepsilon, \varepsilon > 0\}$$

is bounded.

Based on Lemma 3.4, we can establish the following error bound of the GLCP in X_ε .

Lemma 3.5 *Suppose that the hypothesis of Theorem 3.2 holds. Then there exists constant $\tau > 0$ such that*

$$\text{dist}(x, X^*) \leq \tau r(x), \quad \forall x \in X_\varepsilon, \quad (17)$$

where $\text{dist}(x, X^*)$ denotes the distance between the point x and the solution set X^* , and $r(x)$ defined in Lemma 3.3.

Proof The proof uses a similar technique to that of Corollary 3.2 in [9]. For completeness, we include it.

Assume that the theorem is false. Then there exist $\varepsilon_0 > 0$, a positive sequence $\{\tau_k\}$, and a sequence $\{x^k\} \subseteq X_{\varepsilon_0}$ such that $\tau_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\text{dist}(x^k, X^*) > \tau_k r(x^k). \quad (18)$$

Hence,

$$\frac{r(x^k)}{\text{dist}(x^k, X^*)} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{19}$$

Since X_{ε_0} is bounded and $r(x)$ is continuous, by (19), we have $r(x^k) \rightarrow 0$ as $k \rightarrow \infty$. Since X_{ε_0} is bounded again and $\{x^k\} \subseteq X_{\varepsilon_0}$, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $\lim_{k_i \rightarrow \infty} x^{k_i} = \bar{x} \in X_{\varepsilon_0}$ with $r(\bar{x}) = 0$. Hence $\bar{x} \in X^*$. By (19), we further obtain

$$\lim_{k_i \rightarrow \infty} \frac{r(x^{k_i})}{\|x^{k_i} - \bar{x}\|} \leq \lim_{k_i \rightarrow \infty} \frac{r(x^{k_i})}{\text{dist}(x^{k_i}, X^*)} = 0. \tag{20}$$

On the other hand, since $AF(x)$ and $UG(x)$ are both polynomial functions with powers 1, respectively, from $x^{k_i} \rightarrow \bar{x}$ and $r(x^{k_i}) \rightarrow 0$, we know that, for all sufficiently large k_i ,

$$\frac{r(x^{k_i})}{\|x^{k_i} - \bar{x}\|} \geq \theta,$$

for some positive number θ , this contradicts (20). Thus, (17) holds. □

Based on Lemmas 3.4 and 3.5, we have the following conclusion.

Theorem 3.3 *Suppose that the hypothesis of Theorem 3.2 holds. Then there exists constant $\rho > 0$ such that*

$$\text{dist}(x, X^*) \leq \rho r(x), \quad \forall x \in \Omega.$$

Proof Assume that the conclusion is false. Then, for any $\rho_k > 0$ such that $\rho_k \rightarrow \infty$ as $k \rightarrow \infty$, there exist $x^k \in \Omega$ and $\bar{x} \in X^*$, such that

$$\|x^k - \bar{x}\| > \rho_k r(x^k). \tag{21}$$

Now, we claim that there exist $k_0 > 0$ and $\varepsilon_0 > 0$ such that

$$r(x^k) > \varepsilon_0, \quad \forall k > k_0. \tag{22}$$

In fact, if not, for any $\varepsilon \in (0, 1)$, and for any $k_0 > 0$, there exists $\bar{k} > k_0$ such that $r(x^{\bar{k}}) \leq \varepsilon$. Then set $\Theta_\varepsilon = \{x \in \Omega | r(x) \leq \varepsilon\}$ is bounded. Combining this with Lemma 3.5, there exists constant $\tau \geq 0$, such that for $x^{\bar{k}} \in \Theta_\varepsilon$, there exists $\bar{x}(x^{\bar{k}}) \in X^*$ with that $\|x^{\bar{k}} - \bar{x}(x^{\bar{k}})\| \leq \tau r(x^{\bar{k}})$. Combining this with (21), for $x^{\bar{k}}$, and $\bar{x}(x^{\bar{k}}) \in X^*$, we have

$$\frac{\tau}{\rho_{\bar{k}}} \|x^{\bar{k}} - \bar{x}(x^{\bar{k}})\| > \tau r(x^{\bar{k}}) \geq \|x^{\bar{k}} - \bar{x}(x^{\bar{k}})\|,$$

i.e. $(\tau/\rho_{\bar{k}}) > 1$. Letting $\bar{k} \rightarrow \infty$ yields $(\tau/\rho_{\bar{k}}) < 1$. This is a contradiction. Hence (22) holds.

By (21) and (22), we have $\|x^k\| > \rho_k \varepsilon_0 - \|\bar{x}\|$, i.e., $\|x^k\| \rightarrow \infty$ as $k \rightarrow \infty$.

Let $y^k = \frac{x^k}{\|x^k\|}$, then there exists a subsequence $\{y^{k_i}\}$ of $\{y^k\}$, such that $y^{k_i} \rightarrow \bar{y}$ with $\|\bar{y}\| = 1$ as $k_i \rightarrow \infty$. Thus, for subsequence $\{x^{k_i}\}$ with $\|x^{k_i}\| \rightarrow \infty$ as $i \rightarrow \infty$, if

$$\left[- \begin{pmatrix} AF(x^{k_i})/\|x^{k_i}\| \\ UG(x^{k_i})/\|x^{k_i}\| \end{pmatrix} \right]_+ \rightarrow 0 \tag{23}$$

does not hold, then there exists an index j such that

$$\liminf_{k_i \rightarrow \infty} \begin{pmatrix} AF(x^{k_i})/\|x^{k_i}\| \\ UG(x^{k_i})/\|x^{k_i}\| \end{pmatrix}_j < 0.$$

In this case, based on definition of $\varphi(x^{k_i})/\|x^{k_i}\|$ in (11), we have

$$\liminf_{k_i \rightarrow \infty} \varphi(x^{k_i})/\|x^{k_i}\| > 0. \tag{24}$$

If the sequence $\{x^{k_i}\}$ satisfies the condition (23), then

$$\lim_{k_i \rightarrow \infty} \begin{pmatrix} AF(x^{k_i})/\|x^{k_i}\| \\ UG(x^{k_i})/\|x^{k_i}\| \end{pmatrix} = \begin{pmatrix} AM \\ UN \end{pmatrix} \bar{y}.$$

Clearly, $\begin{pmatrix} AM \\ UN \end{pmatrix} \bar{y} \neq 0$ and $\begin{pmatrix} AM \\ UN \end{pmatrix} \bar{y} \geq 0$ by condition (23). By Assumption 2.1(A1), using the technique used in (14) again, we have

$$\begin{aligned} 0 &\leq 2(x^{k_i} - \hat{x})^\top M^\top N(x^{k_i} - \hat{x}) \\ &= \langle F(x^{k_i}) - F(\hat{x}), G(x^{k_i}) - G(\hat{x}) \rangle + \langle G(x^{k_i}) - G(\hat{x}), F(x^{k_i}) - F(\hat{x}) \rangle \\ &= (F(x^{k_i}))^\top G(x^{k_i}) + (G(x^{k_i}))^\top F(x^{k_i}) - (F(x^{k_i}))^\top G(\hat{x}) - (G(x^{k_i}))^\top F(\hat{x}) \\ &\quad - (F(\hat{x}))^\top G(x^{k_i}) - (G(\hat{x}))^\top F(x^{k_i}) + (F(\hat{x}))^\top G(\hat{x}) + (G(\hat{x}))^\top F(\hat{x}) \\ &= [AF(x^{k_i})]^\top [UG(x^{k_i})] + [UG(x^{k_i})]^\top [AF(x^{k_i})] - [AF(x^{k_i})]^\top [UG(\hat{x})] \\ &\quad - [UG(x^{k_i})]^\top [AF(\hat{x})] - [AF(\hat{x})]^\top [UG(x^{k_i})] \\ &\quad - [UG(\hat{x})]^\top [AF(x^{k_i})] + [AF(\hat{x})]^\top [UG(\hat{x})] + [UG(\hat{x})]^\top [AF(\hat{x})] \\ &= \begin{pmatrix} AF(x^{k_i}) \\ UG(x^{k_i}) \end{pmatrix}^\top \begin{pmatrix} UG(x^{k_i}) \\ AF(x^{k_i}) \end{pmatrix} + \begin{pmatrix} AF(\hat{x}) \\ UG(\hat{x}) \end{pmatrix}^\top \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix} \\ &\quad - \begin{pmatrix} AF(x^{k_i}) \\ UG(x^{k_i}) \end{pmatrix}^\top \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix} - \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix}^\top \begin{pmatrix} AF(x^{k_i}) \\ UG(x^{k_i}) \end{pmatrix}, \tag{25} \end{aligned}$$

where the deduction of the second equality uses a similar technique to that of (9). By (25), we obtain

$$\begin{aligned} & \begin{pmatrix} AF(x^{k_i}) \\ UG(x^{k_i}) \end{pmatrix}^\top \begin{pmatrix} UG(x^{k_i}) \\ AF(x^{k_i}) \end{pmatrix} \\ & \geq \begin{pmatrix} AF(x^{k_i}) \\ UG(x^{k_i}) \end{pmatrix}^\top \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix} + \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix}^\top \begin{pmatrix} AF(x^{k_i}) \\ UG(x^{k_i}) \end{pmatrix} \\ & \quad - \begin{pmatrix} AF(\hat{x}) \\ UG(\hat{x}) \end{pmatrix}^\top \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix}. \end{aligned} \tag{26}$$

Dividing both sides of (26) by $\|x^{k_i}\|$ gives

$$\begin{aligned} & \liminf_{k_i \rightarrow \infty} 2 \{ [AF(x^{k_i})]^\top [UG(x^{k_i})] \}_+ / \|x^{k_i}\| \\ & \geq \liminf_{k_i \rightarrow \infty} 2 [AF(x^{k_i})]^\top [UG(x^{k_i})] / \|x^{k_i}\| \\ & = \liminf_{k_i \rightarrow \infty} \begin{pmatrix} AF(x^{k_i}) \\ UG(x^{k_i}) \end{pmatrix}^\top \begin{pmatrix} UG(x^{k_i}) \\ AF(x^{k_i}) \end{pmatrix} / \|x^{k_i}\| \\ & \geq \liminf_{k_i \rightarrow \infty} \left\{ \begin{pmatrix} AF(x^{k_i}) / \|x^{k_i}\| \\ UG(x^{k_i}) / \|x^{k_i}\| \end{pmatrix}^\top \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix} + \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix}^\top \begin{pmatrix} AF(x^{k_i}) / \|x^{k_i}\| \\ UG(x^{k_i}) / \|x^{k_i}\| \end{pmatrix} \right\} \\ & \quad - \liminf_{k_i \rightarrow \infty} \left\{ \begin{pmatrix} AF(\hat{x}) \\ UG(\hat{x}) \end{pmatrix}^\top \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix} \right\} / \|x^{k_i}\| \\ & = 2 \begin{pmatrix} AM\bar{y} \\ UN\bar{y} \end{pmatrix}^\top \begin{pmatrix} UG(\hat{x}) \\ AF(\hat{x}) \end{pmatrix} > 0, \end{aligned} \tag{27}$$

where the last inequality follows from the properties of $\begin{pmatrix} AM \\ UN \end{pmatrix} \bar{y}$ and Assumption 3.1. Based on definition of $\varphi(x^{k_i}) / \|x^{k_i}\|$ in (11) again, we can obtain (24).

Dividing both sides of (21) by $\|x^{k_i}\|$, taking (24) into consideration, and letting $k_i \rightarrow \infty$ yield

$$1 = \lim_{k_i \rightarrow \infty} \frac{\|x^{k_i} - \bar{x}\|}{\|x^{k_i}\|} \geq \lim_{k_i \rightarrow \infty} \frac{\rho_{k_i} r(x^{k_i})}{\|x^{k_i}\|} \geq \lim_{k_i \rightarrow \infty} \rho_{k_i} v^{-1} \frac{\varphi(x^{k_i})}{\|x^{k_i}\|} \rightarrow \infty,$$

where the second inequality is by Lemma 3.3. This is contradiction, and the desired result follows. □

To establish a global error bound for GLCP, we give the following result from [10] on the error bound for a polyhedral cone.

Lemma 3.6 For polyhedral cone $P = \{x \in \mathbb{R}^n \mid D_1x = d_1, D_2x \leq d_2\}$ with $D_1 \in \mathbb{R}^{l \times n}, D_2 \in \mathbb{R}^{m \times n}, d_1 \in \mathbb{R}^l$ and $d_2 \in \mathbb{R}^m$, there exists constant $c > 0$ such that

$$\text{dist}(x, P) \leq c[\|D_1x - d_1\| + \|(D_2x - d_2)_+\|], \quad \forall x \in \mathbb{R}^n.$$

Now, we arrive at the position to state our main results.

Theorem 3.4 Suppose that the hypothesis of Theorem 3.2 holds. Then there exists constant $\eta > 0$ such that

$$\text{dist}(x, X^*) \leq \eta\{\|B(Mx + p)\| + \|V(Nx + q)\| + r(x)\}, \quad \forall x \in \mathbb{R}^n. \tag{28}$$

Proof For any $x \in \mathbb{R}^n$, there exists a vector $\bar{x} \in \Omega$ such that

$$\|x - \bar{x}\| = \text{dist}(x, \Omega).$$

Using the fact that

$$\min\{a, b\} = a - P_{\mathbb{R}_+}(a - b), \quad \forall a, b \in \mathbb{R},$$

one has

$$\begin{aligned} & \|r(x) - r(\bar{x})\| \\ &= \|\min\{AF(x), UG(x)\} - \min\{AF(\bar{x}), UG(\bar{x})\}\| \\ &= \|[AF(x) - P_{\mathbb{R}_+^s}(AF(x) - UG(x))] - [AF(\bar{x}) - P_{\mathbb{R}_+^s}(AF(\bar{x}) - UG(\bar{x}))]\| \\ &\leq \|AF(x) - AF(\bar{x})\| + \|P_{\mathbb{R}_+^s}(AF(x) - UG(x)) - P_{\mathbb{R}_+^s}(AF(\bar{x}) - UG(\bar{x}))\| \\ &\leq \|AF(x) - AF(\bar{x})\| + \|(AF(x) - UG(x)) - (AF(\bar{x}) - UG(\bar{x}))\| \\ &\leq 2\|AF(x) - AF(\bar{x})\| + \|UG(x) - UG(\bar{x})\| \\ &\leq 2\|AM\|\|x - \bar{x}\| + \|UN\|\|x - \bar{x}\| \\ &= (2\|AM\| + \|UN\|)\|x - \bar{x}\| \\ &= (2\|AM\| + \|UN\|)\text{dist}(x, \Omega), \end{aligned}$$

where the second inequality is by non-expanding property of projection operator. Thus,

$$\|r(\bar{x})\| \leq \|r(x)\| + (2\|AM\| + \|UN\|)\text{dist}(x, \Omega). \tag{29}$$

Combining Theorem 3.3 with (29), we have

$$\begin{aligned} \text{dist}(x, X^*) &\leq \text{dist}(x, \Omega) + \text{dist}(\bar{x}, X^*) \\ &\leq \text{dist}(x, \Omega) + \rho\|r(\bar{x})\| \\ &\leq \text{dist}(x, \Omega) + \rho(\|r(x)\| + (2\|AM\| + \|UN\|)\text{dist}(x, \Omega)) \\ &\leq \rho\|r(x)\| + [\rho(2\|AM\| + \|UN\|) + 1]\text{dist}(x, \Omega) \\ &\leq \eta(\|r(x)\| + \|B(Mx + p)\| + \|V(Nx + q)\|), \end{aligned}$$

where the last inequality is by Lemma 3.6, and

$$\eta = \max\{\rho, c[\rho(2\|AM\| + \|UN\|) + 1]\},$$

c is a constant. □

In the end of this section, we present two examples to compare the conditions in Theorem 3.4 in this paper and that in Theorem 4.2 in [6].

Example 3.1 For the matrices A, B, M, N, p, q involved in the GLCP, we set

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & N &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, & p = q &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ A &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & B &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It is easy to see that the solution set $X^* = \{0\}$. Obviously, $M^T N$ is positive semi-definite. By definition of U, V in (8), one has

$$U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $x(\epsilon) := (0, \epsilon)^T, 0 \leq \epsilon \leq 1$, we have

$$\frac{\|x(\epsilon) - 0\|}{\|Bx(\epsilon)\| + \|VNx(\epsilon)\| + \|\min\{Ax(\epsilon), U(Nx(\epsilon) + q)\}\|} = \frac{\epsilon}{0 + 0 + \sqrt{5}\epsilon} \rightarrow \frac{1}{\sqrt{5}}$$

as $\epsilon \rightarrow 0$. Thus, the conclusion in Theorem 3.4 in this paper provides a global error bound for the GLCP. However, since matrix (A^T, B^T) has not full-column rank, thus, the error bound of Theorems 4.1 and 4.2 in [6] is not valid.

Example 3.2 For the matrices A, B, M, N, p, q involved in the GLCP, we set

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & N &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, & p = q &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ A &= (1, 0), & B &= (1, 1). \end{aligned}$$

It is easy to see that the solution set $X^* = \{0\}$. Hence it has no non-degenerate solution. Obviously, $M^T N$ is positive semi-definite. By definition of U, V in (8), one has

$$U = (1, -1), \quad V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $x(\epsilon) := (\epsilon, -\epsilon)^T, 0 \leq \epsilon \leq 1$, we have

$$\frac{\|x(\epsilon) - 0\|}{\|Bx(\epsilon)\| + \|VNx(\epsilon)\| + \|\min\{Ax(\epsilon), U(Nx(\epsilon) + q)\}\|} = \frac{\sqrt{2}\epsilon}{0 + 0 + \epsilon} \rightarrow \sqrt{2}$$

as $\epsilon \rightarrow 0$. Thus, Theorem 3.4 in this paper provides a global error bound for the GLCP.

On the other hand, it is obvious that the matrix (A^\top, B^\top) has full-column rank. From Theorem 4.2 in [6], we can obtain

$$Q = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad (I, 0)(QQ^\top)^{-1}Q = (1, -1),$$

$$\bar{Q} = Q^\top(QQ^\top)^{-1}Q - I_m = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using Theorem 4.2 in [6], for $x(\epsilon)$, we have

$$\begin{aligned} A(Mx(\epsilon) + p) &= (1, 0)x(\epsilon) = \epsilon, \\ B(Mx(\epsilon) + p) &= (1, 1)x(\epsilon) = 0, \\ (I, 0)(QQ^\top)^{-1}Q(Nx(\epsilon) + q) &= (1, -3)x(\epsilon) = 4\epsilon, \\ \bar{Q}(Nx(\epsilon) + q) &= 0, \\ (Mx(\epsilon) + p)^\top(Nx(\epsilon) + q) &= 4\epsilon^2. \end{aligned}$$

i.e.,

$$\frac{\|x(\epsilon) - 0\|}{\{[Mx(\epsilon) + p]^\top[Nx(\epsilon) + q]\}_+} = \frac{\sqrt{2}\epsilon}{4\epsilon^2} \rightarrow +\infty$$

as $\epsilon \rightarrow 0$. Thus, Theorem 4.2 in [6] fails in providing an error bound for this GLCP.

Overall, Theorem 4.2 in [6] cannot provide an error bound for these two problems as the problems has no non-degenerate solutions, while Theorem 3.4 in this paper can do.

In the end of this section, we will discuss the hypotheses used in our main results. First, without the requirement of a non-degenerate solution, the square root term in the error bound estimation is removed, as stated in Theorem 3.4 in this paper. Hence, the error estimation becomes more practical than that obtained in Theorem 4.1 in [6] as we may establish quick convergence rate of the Newton-type method for solving the GLCP [2] under the error bound assumption obtained in this paper instead of the nonsingular assumption [11]. Second, when GLCP reduces to the classical linear complementarity problems (LCP), Assumption 2.1(A1) coincides with the assumption in [7], and Assumption 2.1(A2) and the condition that the matrix $\begin{pmatrix} A & M \\ U & N \end{pmatrix}$ is of column full rank are automatically satisfied for LCP. Hence, Assumption 3.1 is weaker than the condition that the LCP has a non-degenerate solution required in Theorem 2.6 of [7].

4 Conclusions and Discussions

In this paper, we established a global error bound on the generalized linear complementarity problems over a polyhedral cone, which improves the result obtained in [6] by weakening the assumption. Surely, under milder conditions, we may established

global error bounds for GLCP with the mapping being nonmonotone, and may use the error bound estimation to establish quick convergence rate of the Newton-type method for solving the GLCP in [2] instead of the nonsingular assumption just as was done for nonlinear equations in [11]. This is a topic for future research.

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References

1. Andreani, R., Friedlander, A., Santos, S.A.: On the resolution of the generalized nonlinear complementarity problem. *SIAM J. Optim.* **12**, 303–321 (2001)
2. Wang, Y.J., Ma, F.M., Zhang, J.Z.: A nonsmooth L–M method for solving the generalized nonlinear complementarity problem over a polyhedral cone. *Appl. Math. Optim.* **52**(1), 73–92 (2005)
3. Zhang, X.Z., Ma, F.M., Wang, Y.J.: A Newton-type algorithm for generalized linear complementarity problem over a polyhedral cone. *Appl. Math. Comput.* **169**, 388–401 (2005)
4. Facchinei, F., Pang, J.S.: *Finite-Dimensional Variational Inequality and Complementarity Problems*. Springer, New York (2003)
5. Pang, J.S.: Error bounds in mathematical programming. *Math. Program.* **79**, 299–332 (1997)
6. Sun, H.C., Wang, Y.J., Qi, L.Q.: Global error bound for the generalized linear complementarity problem over a polyhedral cone. *J. Optim. Theory Appl.* **142**, 417–429 (2009)
7. Mangasarian, O.L.: Error bounds for nondegenerate monotone linear complementarity problems. *Math. Program.* **48**, 437–445 (1990)
8. Horn, R.A., Johnson, C.R.: *Topics in Matrix Analysis*. Cambridge University Press, Cambridge (1991)
9. Xiu, N.H., Zhang, J.Z.: Global projection-type error bound for general variational inequalities. *J. Optim. Theory Appl.* **112**(1), 213–228 (2002)
10. Hoffman, A.J.: On the approximate solutions of linear inequalities. *J. Res. Natl. Bur. Stand.* **49**(4), 263–265 (1952)
11. Yamashita, N., Fukushima, M.: On the rate of convergence of the Levenberg–Marquardt method. *Computing, Suppl.* **15**, 237–249 (2001)