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Strong Convergence Theorems for Maximal and Inverse-Strongly Monotone Mappings in Hilbert Spaces and Applications

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Abstract In this paper, we prove two strong convergence theorems for finding a common point of the set of zero points of the addition of an inverse-strongly monotone mapping and a maximal monotone operator and the set of zero points of a maximal monotone operator, which is related to an equilibrium problem in a Hilbert space. Such theorems improve and extend the results announced by Y. Liu (Nonlinear Anal. 71:4852–4861, 2009). As applications of the results, we present well-known and new strong convergence theorems which are connected with the variational inequality, the equilibrium problem and the fixed point problem in a Hilbert space.

Keywords Equilibrium problem · Fixed point · Inverse-strongly monotone mapping · Maximal monotone operator · Resolvent · Strict pseudo-contraction

1 Introduction

The theory of nonexpansive mappings in a Hilbert space is very important because it is applied to convex optimization, the theory of nonlinear evolution equations and others. Browder and Petryshyn [1] introduced a class of nonlinear mappings, called strict pseudo-contractions, which includes the class of nonexpansive mappings. For strict pseudo-contractions, we are interested in finding fixed points of the mappings. We also know the class of inverse-strongly monotone mappings which is related to nonexpansive mappings. For inverse-strongly monotone mappings, we are interested

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in finding zero points of the mappings. On the other hand, the generalized equilibrium problems which are formulated by the Ky Fan inequality have many important applications in optimization problems, variational inequalities, minimax problems, economics and others. Some methods have been proposed for solving the generalized equilibrium problems in Hilbert spaces; see, for example, [2, 3]. Recently, Liu [4] studied strong convergence theorems for strict pseudo-contractions with equilibrium problems in a Hilbert space. We know from [5] that the solutions of equilibrium problems in [4] are written by using the resolvent of a maximal monotone operator with some domain condition. Furthermore, the class of strict pseudo-contractions is related to the class of inverse-strongly monotone mappings in a Hilbert space.

In this paper, motivated by these results, we prove implicit and explicit strong convergence theorems for finding a common point of the set of zero points of the addition of an inverse-strongly monotone mapping and a maximal monotone operator and the set of zero points of a maximal monotone operator which is related to an equilibrium problem in a real Hilbert space. Such theorems improve and extend the results announced by Liu [4]. The limit point of the implicit strong convergence theorem is simply proved by using an implicit contraction mapping which does not appear in other references. Such a unique fixed point of the mapping is used in the proof of the explicit strong convergence theorem. Using this explicit strong convergence theorem, we obtain well-known and new strong convergence theorems in a Hilbert space.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. We also denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [6], for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle \tag{1}$$

and

$$\big\|\lambda x + (1-\lambda)y\big\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2}$$

Furthermore we have, for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{3}$$

Let C be a nonempty, closed and convex subset of H. Let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ is called nonexpansive iff $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [6]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_D x\| \le \|x - y\|$ for all $x \in H$ and $y \in D$. Such P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive, that is,

$$\|P_D x - P_D y\|^2 \le \langle P_D x - P_D y, x - y\rangle$$

for all $x, y \in H$. Furthermore $\langle x - P_D x, y - P_D x\rangle \le 0$ holds for all $x \in H$ and $y \in D$; see [7].

For a positive number $\alpha > 0$, a mapping $A : C \to H$ is called α-inverse-strongly monotone iff

$$\langle x - y, Ax - Ay\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{4}$$

If $A: C \to H$ is α-inverse-strongly monotone, then $\langle x - y, Ax - Ay\rangle \ge 0$ and $\|Ax - Ay\| \le (1/\alpha)\|x - y\|$ for all $x, y \in C$; see, for example, [8, 9] for inverse-strongly monotone mappings.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H iff $\langle x - y, u - v\rangle \ge 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal iff its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}: H \to \mathrm{dom}(B)$, which is called the resolvent of B for r. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know from [10] that

$$A_r x \in BJ_r x, \quad \forall x \in H, r > 0. \tag{5}$$

Let B be a maximal monotone operator on H and let

$$B^{-1}0 = \{x \in H : 0 \in Bx\}.$$

It is known that $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \le \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{6}$$

We also know the following lemma from [5].

Lemma 2.1 *Let H be a real Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds*:

$$\frac{s - t}{s}\langle J_s x - J_t x, J_s x - x\rangle \ge \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2.1, we have

$$\|J_\lambda x - J_\mu x\| \le \big(|\lambda - \mu|/\lambda\big)\|x - J_\lambda x\|$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [7, 11].

To prove our main results, we need the following lemma [12, 13]:

Lemma 2.2 *Let* $\{s_n\}$ *be a sequence of non-negative real numbers, let* $\{\alpha_n\}$ *be a sequence of* $[0, 1]$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *let* $\{\beta_n\}$ *be a sequence of non-negative real numbers with* $\sum_{n=1}^{\infty} \beta_n < \infty$, *and let* $\{\gamma_n\}$ *be a sequence of real numbers with* $\limsup_{n\to\infty} \gamma_n \le 0$. *Suppose that*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \dots$. *Then* $\lim_{n\to\infty} s_n = 0$.

3 Strong Convergence Theorems

In this section, we first prove the following implicit strong convergence theorem of Browder's type [14] in a Hilbert space. Before proving it, we need some definitions. Let H be a Hilbert space. A mapping $g : H \to H$ is a contraction iff there exists $k \in]0, 1[$ such that$\|g(x) - g(y)\| \le k\|x - y\|$ for all $x, y \in H$. We call such g a k-contraction. A linear bounded operator $G : H \to H$ is called strongly positive iff there exists $\overline{\gamma} > 0$ such that $\langle Gx, x\rangle \ge \overline{\gamma}\|x\|^2$ for all $x \in H$. We call such G a strongly positive operator with coefficient $\overline{\gamma} > 0$. Marino and Xu [15] proved the following result.

Lemma 3.1 *Let* H *be a Hilbert space and let* G *be a strongly positive bounded linear self-adjoint operator on* H *with coefficient* $\overline{\gamma} > 0$. *If* $0 < \gamma \le \|G\|^{-1}$, *then* $\|I - \gamma G\| \le 1 - \gamma\overline{\gamma}$.

Theorem 3.1 *Let* H *be a real Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $\alpha > 0$ *and let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and let* B *be a maximal monotone operator on* H. *Let* F *be a maximal monotone operator on* H *such that the domain of* F *is included in* C. *Let* $J_\lambda = (I + \lambda B)^{-1}$ *and* $T_r = (I + rF)^{-1}$ *be the resolvents of* B *and* F *for* $\lambda > 0$ *and* $r > 0$, *respectively. Let* $0 < k < 1$ *and let* g *be a* k*-contraction of* H *into itself. Let* G *be a strongly positive bounded linear self-adjoint operator on* H *with coefficient* $\overline{\gamma} > 0$. *Let* $0 < \gamma < \frac{\overline{\gamma}}{k}$ *and suppose* $(A + B)^{-1}0 \cap F^{-1}0 \neq \emptyset$. *Assume that* $\{\alpha_n\} \subset]0, 1[$, $\{\lambda_n\} \subset]0, \infty[$ *and* $\{r_n\} \subset]0, \infty[$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad 0 < a \le \lambda_n \le 2\alpha, \quad \textit{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then, the following hold:

(i) *For sufficiently large* $n \in \mathbb{N}$, *define* $T_n : H \to H$ *by*

$$T_n x = \alpha_n\gamma g(x) + (I - \alpha_n G)J_{\lambda_n}(I - \lambda_n A)T_{r_n}x, \quad \forall x \in H.$$

Then, T_n *has a unique fixed point* x_n *in* H *and* $\{x_n\}$ *is bounded.*

(ii) *For any nonempty closed convex subset D of H, $P_D(I - G + \gamma g)$ has a unique fixed point z_0 in D. This point $z_0 \in D$ is also a unique solution of the variational inequality*

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in D.$$

In particular, the set $(A + B)^{-1}0 \cap F^{-1}0$ is a nonempty, closed and convex subset of H and $P_{(A+B)^{-1}0 \cap F^{-1}0}(I - G + \gamma g)$ has a unique fixed point z_0 in $(A + B)^{-1}0 \cap F^{-1}0$.

(iii) *The sequence $\{x_n\}$ converges strongly to $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$, where $\{z_0\} = VI((A + B)^{-1}0 \cap F^{-1}0, G - \gamma g)$.*

Proof Let us prove (i). For sufficiently large $n \in \mathbb{N}$, define $T_n : H \to H$ by

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n A) T_{r_n} x, \quad \forall x \in H.$$

From $\alpha_n \to 0$, we have $\alpha_n \leq \|G\|^{-1}$. Then we have from Lemma 3.1 that for any $x, y \in H$,

$$\begin{aligned} \|T_n x - T_n y\| &= \big\| \alpha_n \gamma g(x) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n A) T_{r_n} x \\ &\qquad - \big\{ \alpha_n \gamma g(y) + (I - \alpha_n G) J_{\lambda_n}(I - \lambda_n A) T_{r_n} y \big\} \big\| \\ &\leq \alpha_n \gamma \big\| g(x) - g(y) \big\| \\ &\qquad + \|I - \alpha_n G\| \big\| J_{\lambda_n}(I - \lambda_n A) T_{r_n} x - J_{\lambda_n}(I - \lambda_n A) T_{r_n} y \big\| \\ &\leq \alpha_n \gamma k \|x - y\| + (1 - \alpha_n \overline{\gamma}) \big\| (I - \lambda_n A) T_{r_n} x - (I - \lambda_n A) T_{r_n} y \big\| \\ &\leq \alpha_n \gamma k \|x - y\| + (1 - \alpha_n \overline{\gamma}) \|T_{r_n} x - T_{r_n} y\| \\ &\leq \alpha_n \gamma k \|x - y\| + (1 - \alpha_n \overline{\gamma}) \|x - y\| \\ &= (\alpha_n \gamma k + 1 - \alpha_n \overline{\gamma}) \|x - y\| \\ &= \big(1 - \alpha_n (\overline{\gamma} - \gamma k)\big) \|x - y\|. \end{aligned}$$

Since $0 < 1 - \alpha_n(\overline{\gamma} - \gamma k) < 1$, T_n is a $(1 - \alpha_n(\overline{\gamma} - \gamma k))$-contraction of H into itself and hence T_n has a unique fixed point x_n in H. Next, we show that $\{x_n\}$ is bounded. Let $u \in (A + B)^{-1}0 \cap F^{-1}0$. Using $u = \alpha_n G u + u - \alpha_n G u$, we have, for all $n \in \mathbb{N}$,

$$\begin{aligned} \|x_n - u\| &= \|T_n x_n - \alpha_n G u - u + \alpha_n G u\| \\ &= \big\| \alpha_n \big(\gamma g(x_n) - G u \big) + (I - \alpha_n G) \big(J_{\lambda_n}(I - \lambda_n A) T_{r_n} x_n - u \big) \big\| \\ &\leq \alpha_n \big\| \gamma g(x_n) - G u \big\| + \|I - \alpha_n G\| \big\| J_{\lambda_n}(I - \lambda_n A) T_{r_n} x_n - u \big\| \\ &\leq \alpha_n \gamma k \|x_n - u\| + \alpha_n \big\| \gamma g(u) - G u \big\| + (1 - \alpha_n \overline{\gamma}) \|x_n - u\|. \end{aligned}$$

Thus we have $\alpha_n(\overline{\gamma} - \gamma k)\|x_n - u\| \leq \alpha_n \|\gamma g(u) - G u\|$ and hence

$$(\overline{\gamma} - \gamma k) \|x_n - u\| \leq \big\| \gamma g(u) - G u \big\|.$$

So, we have $\|x_n - u\| \leq \frac{\|\gamma g(u) - G u\|}{\overline{\gamma} - \gamma k}$. This implies that $\{x_n\}$ is bounded.

Let us prove (ii). Since $g : H \to H$ is a k-contraction and G is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$, we have, for any $x, y \in H$,

$$\begin{aligned}&\big\langle x - y, (G - \gamma g)x - (G - \gamma g)y\big\rangle \\ &\quad = \big\langle x - y, G(x - y)\big\rangle - \langle x - y, \gamma gx - \gamma gy\rangle \\ &\quad \geq \overline{\gamma}\|x - y\|^2 - \gamma k\|x - y\|^2 \\ &\quad = (\overline{\gamma} - \gamma k)\|x - y\|^2.\end{aligned}$$

Then $G - \gamma g : H \to H$ is a $(\overline{\gamma} - \gamma k)$-strongly monotone operator. Furthermore, taking a positive number μ with $\mu(\|G\| + \gamma k)^2 < 2(\overline{\gamma} - \gamma k)$ and $2\mu(\overline{\gamma} - \gamma k) < 1$, we have, for any $x, y \in H$,

$$\begin{aligned}&\big\|x - \mu(G - \gamma g)x - \big(y - \mu(G - \gamma g)y\big)\big\|^2 \\ &\quad = \|x - y\|^2 - 2\big\langle x - y, \mu(G - \gamma g)x - \mu(G - \gamma g)y\big\rangle \\ &\qquad + \big\|\mu(G - \gamma g)x - \mu(G - \gamma g)y\big\|^2 \\ &\quad \leq \|x - y\|^2 - 2\mu(\overline{\gamma} - \gamma k)\|x - y\|^2 \\ &\qquad + \mu^2\big(\|G\|^2 + 2\|G\|\gamma k + (\gamma k)^2\big)\|x - y\|^2 \\ &\quad = \big\{1 - 2\mu(\overline{\gamma} - \gamma k) + \mu^2(\|G\| + \gamma k)^2\big\}\|x - y\|^2 \\ &\quad = \big(1 - \mu\big\{2(\overline{\gamma} - \gamma k) - \mu(\|G\| + \gamma k)^2\big\}\big)\|x - y\|^2\end{aligned}$$

and

$$0 < 1 - \mu\big\{2(\overline{\gamma} - \gamma k) - \mu(\|G\| + \gamma k)^2\big\} < 1.$$

So, $I - \mu(G - \gamma g)$ is a contraction of H into itself and hence $P_D(I - \mu(G - \gamma g))$ is also a contraction of D into itself. Thus there exists a unique point $z_0 \in D$ such that $z_0 = P_D(I - \mu(G - \gamma g))z_0$. We also have, for $w \in D$,

$$\begin{aligned}&w = P_D\big(I - \mu(G - \gamma g)\big)w \\ &\quad \Longleftrightarrow \quad \big\langle w - \mu(G - \gamma g)w - w, w - q\big\rangle \geq 0, \quad \forall q \in D \\ &\quad \Longleftrightarrow \quad \big\langle (G - \gamma g)w, w - q\big\rangle \leq 0, \quad \forall q \in D \\ &\quad \Longleftrightarrow \quad \langle w - Gw + \gamma gw - w, w - q\rangle \geq 0, \quad \forall q \in D \\ &\quad \Longleftrightarrow \quad w = P_D(I - G + \gamma g)w.\end{aligned}$$

Thus $VI(D, G - \gamma g) = \{z_0\}$. Next, we show that $(A + B)^{-1}0 \cap F^{-1}0$ is closed and convex. Since F is a maximal monotone operator, we know from [6] that $F^{-1}0$ is closed and convex. Furthermore, we know from [16] that for any $\lambda > 0$,

$$w \in (A + B)^{-1}0 \quad \Longleftrightarrow \quad w \in F\big(J_\lambda(I - \lambda A)\big).$$

If $0 < \lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive and then $J_\lambda(I - \lambda A)$ is nonexpansive. Thus $F(J_\lambda(I - \lambda A))$ is closed and convex and so is $(A + B)^{-1}0$. Therefore, $(A + B)^{-1}0 \cap F^{-1}0$ is closed and convex. Thus $P_{(A+B)^{-1}0\cap F^{-1}0}(I - G + \gamma g)$ has a unique fixed point z_0 in $(A + B)^{-1}0 \cap F^{-1}0$.

Let us prove (iii). Put $y_n = J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n$ and $u_n = T_{r_n}x_n$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, $\{y_n\}$ and $\{u_n\}$ are bounded. Furthermore, $\{g(x_n)\}$ and $\{Gx_n\}$ are also bounded. Let $z \in (A + B)^{-1}0 \cap F^{-1}0$. We note that

$$\|y_n - z\| = \left\| J_{\lambda_n}(I - \lambda_n A)u_n - z\right\| \leq \|u_n - z\| \tag{7}$$

and

$$\begin{aligned}\|u_n - y_n\| &\leq \|u_n - x_n\| + \|x_n - y_n\| \\ &= \|u_n - x_n\| + \left\|\alpha_n\gamma g(x_n) + (I - \alpha_n G)y_n - y_n\right\| \\ &= \|u_n - x_n\| + \alpha_n\left\|\gamma g(x_n) - Gy_n\right\|.\end{aligned} \tag{8}$$

Using (6), we have

$$\begin{aligned}2\|u_n - z\|^2 &= 2\|T_{r_n}x_n - T_{r_n}z\|^2 \\ &\leq 2\langle x_n - z, u_n - z\rangle \\ &= \|x_n - z\|^2 + \|u_n - z\|^2 - \|u_n - x_n\|^2\end{aligned}$$

and hence

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{9}$$

Then we have from (1) and (9)

$$\begin{aligned}\|x_n - z\|^2 &= \|(I - \alpha_n G)(y_n - z) + \alpha_n\big(\gamma g(x_n) - Gz\big)\|^2 \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|y_n - z\|^2 + 2\alpha_n\big\langle\gamma g(x_n) - Gz, x_n - z\big\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|u_n - z\|^2 + 2\alpha_n\big\langle\gamma g(x_n) - Gz, x_n - z\big\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\big(\|x_n - z\|^2 - \|x_n - u_n\|^2\big) \\ &\quad + 2\alpha_n\gamma k\|x_n - z\|^2 + 2\alpha_n\left\|\gamma g(z) - Gz\right\|\|x_n - z\| \\ &= \big\{1 - 2\alpha_n(\overline{\gamma} - \gamma k) + \alpha_n^2\overline{\gamma}^2\big\}\|x_n - z\|^2 \\ &\quad - (1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 + 2\alpha_n\left\|\gamma g(z) - Gz\right\|\|x_n - z\| \\ &\leq \|x_n - z\|^2 + \alpha_n^2\overline{\gamma}^2\|x_n - z\|^2 - (1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 \\ &\quad + 2\alpha_n\left\|\gamma g(z) - Gz\right\|\|x_n - z\|\end{aligned}$$

and hence

$$(1 - \alpha_n\overline{\gamma})^2\|x_n - u_n\|^2 \leq \alpha_n^2\overline{\gamma}^2\|x_n - z\|^2 + 2\alpha_n\left\|\gamma g(z) - Gz\right\|\|x_n - z\|.$$

From $\alpha_n \to 0$, we have

$$\|x_n - u_n\| \to 0. \tag{10}$$

Then we have from (8)

$$\|y_n - u_n\| \to 0. \tag{11}$$

Take $\lambda_0 \in [a, 2\alpha]$. Putting $z_n = (I - \lambda_n A)u_n$, we have from Lemma 2.1

$$\begin{aligned}
&\big\| J_{\lambda_0}(I - \lambda_0 A)u_n - y_n \big\| \\
&\quad \le \big\| J_{\lambda_0}(I - \lambda_0 A)u_n - J_{\lambda_0}(I - \lambda_n A)u_n \big\| + \big\| J_{\lambda_0}(I - \lambda_n A)u_n - y_n \big\| \\
&\quad \le \big\| (I - \lambda_0 A)u_n - (I - \lambda_n A)u_n \big\| + \big\| J_{\lambda_0} z_n - J_{\lambda_n} z_n \big\| \\
&\quad \le |\lambda_n - \lambda_0| \, \|Au_n\| + \frac{|\lambda_n - \lambda_0|}{\lambda_0} \|J_{\lambda_0} z_n - z_n\|.
\end{aligned} \tag{12}$$

Furthermore, we have

$$\big\| J_{\lambda_0}(I - \lambda_0 A)u_n - u_n \big\| \le \big\| J_{\lambda_0}(I - \lambda_0 A)u_n - y_n \big\| + \|y_n - u_n\|. \tag{13}$$

We will use these inequalities (12) and (13) later. We know from (ii) that there exists a unique $z_0 \in (A + B)^{-1}0 \cap F^{-1}0$ such that

$$\big\langle (G - \gamma g)z_0, q - z_0 \big\rangle \ge 0, \quad \forall q \in (A + B)^{-1}0 \cap F^{-1}0.$$

In order to show that $x_n \to z_0$, it suffices to show that if $\{x_{n_i}\}$ is any subsequence of $\{x_n\}$, then we can find a subsequence of $\{x_{n_i}\}$ converging strongly to z_0. Since $\{x_{n_i}\}$ is bounded and $\{\lambda_{n_i}\} \subset [a, 2\alpha]$, without loss of generality there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup w$ and $\lambda_{n_{i_j}} \to \lambda_0$ for some $\lambda_0 \in [a, 2\alpha]$. From $x_n - u_n \to 0$, we have $u_{n_{i_j}} \rightharpoonup w$. Since $\{u_{n_{i_j}}\} \subset C$ and C is closed and convex, we have $w \in C$. Using $\lambda_{n_{i_j}} \to \lambda_0$ and (12), we have

$$\big\| J_{\lambda_0}(I - \lambda_0 A)u_{n_{i_j}} - y_{n_{i_j}} \big\| \to 0.$$

Furthermore we have from $\|y_{n_{i_j}} - u_{n_{i_j}}\| \to 0$ and (13) that

$$\big\| J_{\lambda_0}(I - \lambda_0 A)u_{n_{i_j}} - u_{n_{i_j}} \big\| \to 0.$$

Since $J_{\lambda_0}(I - \lambda_0 A)$ is nonexpansive, we have $w = J_{\lambda_0}(I - \lambda_0 A)w$ and hence $w \in (A + B)^{-1}0$. We show $w \in F^{-1}0$. Since F is a maximal monotone operator, we have from (5) that $A_{r_{n_{i_j}}} x_{n_{i_j}} \in F T_{r_{n_{i_j}}} x_{n_{i_j}}$, where A_r is the Yosida approximation of F for $r > 0$. Furthermore we have, for any $(u, v) \in F$,

$$\left\langle u - u_{n_{i_j}}, v - \frac{x_{n_{i_j}} - u_{n_{i_j}}}{r_{n_{i_j}}} \right\rangle \ge 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_{i_j}} \rightharpoonup w$ and $x_{n_{i_j}} - u_{n_{i_j}} \to 0$, we have

$$\langle u - w, v \rangle \ge 0.$$

Since F is a maximal monotone operator, we have $0 \in Fw$ and hence $w \in F^{-1}0$. Thus we have $w \in (A+B)^{-1}0 \cap F^{-1}0$. On the other hand, we have

$$x_n - z_0 = \alpha_n\big(\gamma g(x_n) - Gz_0\big) + (I - \alpha_n G)(y_n - z_0).$$

So we have

$$\begin{aligned}\|x_n - z_0\|^2 &= \alpha_n\big\langle \gamma g(x_n) - Gz_0, x_n - z_0\big\rangle + \big\langle (I - \alpha_n G)(y_n - z_0), x_n - z_0\big\rangle \\ &\le \alpha_n\big\langle \gamma g(x_n) - Gz_0, x_n - z_0\big\rangle + \|I - \alpha_n G\|\|y_n - z_0\|\|x_n - z_0\| \\ &\le \alpha_n\big\langle \gamma g(x_n) - Gz_0, x_n - z_0\big\rangle + (1 - \alpha_n\overline{\gamma})\|x_n - z_0\|^2.\end{aligned}$$

Then we have

$$\alpha_n\overline{\gamma}\|x_n - z_0\|^2 \le \alpha_n\big\langle \gamma g(x_n) - Gz_0, x_n - z_0\big\rangle$$

and hence

$$\begin{aligned}\|x_n - z_0\|^2 &\le \frac{1}{\overline{\gamma}}\big\langle \gamma g(x_n) - Gz_0, x_n - z_0\big\rangle \\ &= \frac{1}{\overline{\gamma}}\big\langle \gamma g(x_n) - \gamma g(z_0) + \gamma g(z_0) - Gz_0, x_n - z_0\big\rangle \\ &\le \frac{1}{\overline{\gamma}}\gamma k\|x_n - z_0\|^2 + \frac{1}{\overline{\gamma}}\big\langle \gamma g(z_0) - Gz_0, x_n - z_0\big\rangle.\end{aligned}$$

This implies that

$$\|x_n - z_0\|^2 \le \frac{\langle \gamma g(z_0) - Gz_0, x_n - w\rangle}{\overline{\gamma} - \gamma k}.$$

In particular we have

$$\|x_{n_{i_j}} - z_0\|^2 \le \frac{\langle \gamma g(z_0) - Gz_0, x_{n_{i_j}} - z_0\rangle}{\overline{\gamma} - \gamma k}.$$

Since $x_{n_{i_j}} \rightharpoonup w$, $w \in (A+B)^{-1}0 \cap F^{-1}0$ and

$$\big\langle (G - \gamma g)z_0, q - z_0\big\rangle \ge 0, \quad \forall q \in (A+B)^{-1}0 \cap F^{-1}0,$$

we have

$$\begin{aligned}\limsup_{j\to\infty}\|x_{n_{i_j}} - z_0\|^2 &\le \lim_{j\to\infty}\frac{\langle \gamma g(z_0) - Gz_0, x_{n_{i_j}} - z_0\rangle}{\overline{\gamma} - \gamma k} \\ &= \frac{\langle \gamma g(z_0) - Gz_0, w - z_0\rangle}{\overline{\gamma} - \gamma k} \\ &\le 0.\end{aligned}$$

Thus $x_{n_{i_j}} \to z_0$. Then $\{x_n\}$ converges strongly to $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ such that

$$\langle (G-\gamma g)z_0, q-z_0\rangle \geq 0, \quad \forall q \in (A+B)^{-1}0 \cap F^{-1}0.$$

We also know that this z_0 is a unique fixed point of $P_{(A+B)^{-1}0\cap F^{-1}0}(I-G+\gamma g)$. This completes the proof. □

Compare the proof of Theorem 3.1(ii) with the proof in [4]. We prove simply that $P_D(I-G+\gamma g)$ has a unique fixed point by using another contraction mapping which is different from $P_D(I-G+\gamma g)$. Using this result, we prove Theorem 3.1(iii). Next, we prove a strong convergence theorem of Halpern's type [17] in a Hilbert space; see also [18].

Theorem 3.2 *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H and let B be a maximal monotone operator on H. Let F be a maximal monotone operator on H such that the domain of F is included in C. Let $J_\lambda = (I+\lambda B)^{-1}$ and $T_r = (I+rF)^{-1}$ be the resolvents of B and F for $\lambda > 0$ and $r > 0$, respectively. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \frac{\overline{\gamma}}{k}$ and suppose $(A+B)^{-1}0 \cap F^{-1}0 \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I-\alpha_n G)J_{\lambda_n}(I-\lambda_n A)T_{r_n}x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset]0,1[$, $\{\lambda_n\} \subset]0,\infty[$ and $\{r_n\} \subset]0,\infty[$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty, \quad 0 < a \leq \lambda_n \leq 2\alpha,$$

$$\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty, \quad \liminf_{n\to\infty} r_n > 0, \quad \text{and} \quad \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $(A+B)^{-1}0 \cap F^{-1}0$, where $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ is a unique fixed point of $P_{(A+B)^{-1}0\cap F^{-1}0}(I-G+\gamma g)$. This point $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ is also a unique solution of the variational inequality

$$\langle (G-\gamma g)z_0, q-z_0\rangle \geq 0, \quad \forall q \in (A+B)^{-1}0 \cap F^{-1}0.$$

Proof Put $u_n = T_{r_n}x_n$ and $y_n = J_{\lambda_n}(I-\lambda_n A)T_{r_n}x_n$ for all $n \in \mathbb{N}$. Let $z \in (A+B)^{-1}0 \cap F^{-1}0$. Then, we have from $z = T_{r_n}z$ and $z = J_{\lambda_n}(I-\lambda_n A)z$ that

$$\begin{aligned} \|y_n - z\| &= \big\| J_{\lambda_n}(I-\lambda_n A)T_{r_n}x_n - z\big\| \\ &= \big\| J_{\lambda_n}(I-\lambda_n A)T_{r_n}x_n - J_{\lambda_n}(I-\lambda_n A)T_{r_n}z\big\| \\ &\leq \big\| (I-\lambda_n A)T_{r_n}x_n - (I-\lambda_n A)T_{r_n}z\big\| \\ &\leq \|T_{r_n}x_n - T_{r_n}z\| \\ &\leq \|x_n - z\|. \end{aligned} \tag{14}$$

Since $x_{n+1}=\alpha_n\gamma g(x_n)+(I-\alpha_n G)y_n$ and $z=\alpha_n Gz+z-\alpha_n Gz$, we have

$$\begin{aligned}\|x_{n+1}-z\| &= \big\|\alpha_n\big(\gamma g(x_n)-Gz\big)+(I-\alpha_n G)(y_n-z)\big\| \\ &\le \alpha_n\big\|\gamma g(x_n)-Gz\big\|+\|I-\alpha_n G\|\|x_n-z\| \\ &\le \alpha_n\gamma k\|x_n-z\|+\alpha_n\big\|\gamma g(z)-Gz\big\|+(1-\alpha_n\overline{\gamma})\|x_n-z\| \\ &= \big\{1-\alpha_n(\overline{\gamma}-\gamma k)\big\}\|x_n-z\|+\alpha_n\big\|\gamma g(z)-Gz\big\| \\ &= \big\{1-\alpha_n(\overline{\gamma}-\gamma k)\big\}\|x_n-z\|+\alpha_n(\overline{\gamma}-\gamma k)\frac{\|\gamma g(z)-Gz\|}{\overline{\gamma}-\gamma k}.\end{aligned}$$

Putting

$$K=\max\left\{\frac{\|\gamma g(z)-Gz\|}{\overline{\gamma}-\gamma k},\|x_1-z\|\right\},$$

we have $\|x_n-z\|\le K$ for all $n\in\mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$ and $\{y_n\}$ are bounded. Since

$$\begin{aligned}x_{n+2}-x_{n+1} &= \alpha_{n+1}\gamma g(x_{n+1})+(I-\alpha_{n+1}G)y_{n+1}-\big(\alpha_n\gamma g(x_n)+(I-\alpha_n G)y_n\big) \\ &= \alpha_{n+1}\gamma g(x_{n+1})-\alpha_{n+1}\gamma g(x_n)+\alpha_{n+1}\gamma g(x_n)-\alpha_n\gamma g(x_n) \\ &\quad +(I-\alpha_{n+1}G)y_{n+1}-(I-\alpha_{n+1}G)y_n \\ &\quad +(I-\alpha_{n+1}G)y_n-(I-\alpha_n G)y_n,\end{aligned}$$

we have

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le \alpha_{n+1}\gamma k\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\gamma\big\|g(x_n)\big\| \\ &\quad +(1-\alpha_{n+1}\overline{\gamma})\|y_{n+1}-y_n\|+|\alpha_{n+1}-\alpha_n|\|Gy_n\| \\ &\le \alpha_{n+1}\gamma k\|x_{n+1}-x_n\| \\ &\quad +(1-\alpha_{n+1}\overline{\gamma})\|y_{n+1}-y_n\|+2|\alpha_{n+1}-\alpha_n|M_1,\end{aligned}$$

where $M_1=\sup\{\gamma\|g(x_n)\|+\|Gy_n\|:n\in\mathbb{N}\}$. Putting $z_n=(I-\lambda_n A)T_{r_n}x_n$, we have from Lemma 2.1 that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \big\|J_{\lambda_{n+1}}(I-\lambda_{n+1}A)T_{r_{n+1}}x_{n+1}-J_{\lambda_n}(I-\lambda_n A)T_{r_n}x_n\big\| \\ &\le \big\|J_{\lambda_{n+1}}(I-\lambda_{n+1}A)T_{r_{n+1}}x_{n+1}-J_{\lambda_{n+1}}(I-\lambda_{n+1}A)T_{r_n}x_n\big\| \\ &\quad +\big\|J_{\lambda_{n+1}}(I-\lambda_{n+1}A)T_{r_n}x_n-J_{\lambda_{n+1}}(I-\lambda_n A)T_{r_n}x_n\big\| \\ &\quad +\big\|J_{\lambda_{n+1}}(I-\lambda_n A)T_{r_n}x_n-J_{\lambda_n}(I-\lambda_n A)T_{r_n}x_n\big\| \\ &\le \big\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\big\|+\big\|(I-\lambda_{n+1}A)T_{r_n}x_n-(I-\lambda_n A)T_{r_n}x_n\big\| \\ &\quad +\big\|J_{\lambda_{n+1}}z_n-J_{\lambda_n}z_n\big\| \\ &\le \big\|T_{r_{n+1}}x_{n+1}-T_{r_{n+1}}x_n\big\|+\big\|T_{r_{n+1}}x_n-T_{r_n}x_n\big\| \\ &\quad +\big\|(I-\lambda_{n+1}A)T_{r_n}x_n-(I-\lambda_n A)T_{r_n}x_n\big\|+\big\|J_{\lambda_{n+1}}z_n-J_{\lambda_n}z_n\big\|\end{aligned}$$

$$\begin{aligned}
&\le \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\|T_{r_{n+1}}x_n - x_n\| \\
&\quad + |\lambda_{n+1}-\lambda_n|\|AT_{r_n}x_n\| + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J_{\lambda_{n+1}}z_n - z_n\| \\
&\le \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{b}\|T_{r_{n+1}}x_n - x_n\| \\
&\quad + |\lambda_{n+1}-\lambda_n|\|AT_{r_n}x_n\| + \frac{|\lambda_{n+1}-\lambda_n|}{a}\|J_{\lambda_{n+1}}z_n - z_n\| \\
&\le \|x_{n+1}-x_n\| + |r_{n+1}-r_n|M_2 + 2|\lambda_{n+1}-\lambda_n|M_2,
\end{aligned}$$

where

$$M_2 = \sup\left\{\frac{1}{b}\|T_{r_{n+1}}x_n - x_n\| + \frac{1}{a}\|J_{\lambda_{n+1}}z_n - z_n\| + \|AT_{r_n}x_n\| : n \in \mathbb{N}\right\}$$

and $0 < b \le r_n$ for all $n \in \mathbb{N}$. Thus we have

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\le \alpha_{n+1}\gamma k\|x_{n+1}-x_n\| + 2|\alpha_{n+1}-\alpha_n|M_1 \\
&\quad + (1-\alpha_{n+1}\overline{\gamma})\|y_{n+1}-y_n\| \\
&\le \alpha_{n+1}\gamma k\|x_{n+1}-x_n\| + 2|\alpha_{n+1}-\alpha_n|M_1 \\
&\quad + (1-\alpha_{n+1}\overline{\gamma})\bigl\{\|x_{n+1}-x_n\| + |r_{n+1}-r_n|M_2 + 2|\lambda_{n+1}-\lambda_n|M_2\bigr\} \\
&\le \bigl\{1-\alpha_{n+1}(\overline{\gamma}-\gamma k)\bigr\}\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|M_3 \\
&\quad + |r_{n+1}-r_n|M_3 + |\lambda_{n+1}-\lambda_n|M_3 \\
&\le \bigl\{1-\alpha_{n+1}(\overline{\gamma}-\gamma k)\bigr\}\|x_{n+1}-x_n\| \\
&\quad + \bigl(|\alpha_{n+1}-\alpha_n| + |r_{n+1}-r_n| + |\lambda_{n+1}-\lambda_n|\bigr)M_3,
\end{aligned}$$

where $M_3 = 2M_1 + 2M_2$. Using Lemma 2.2, we obtain

$$\|x_{n+2}-x_{n+1}\| \to 0. \tag{15}$$

We also have from $x_{n+1} = \alpha_n\gamma g(x_n) + (I-\alpha_n G)y_n$ that

$$\begin{aligned}
\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1}-y_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|\gamma g(x_n) - \alpha_n Gy_n\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1}-x_n\| \to 0$, we get

$$y_n - x_n \to 0. \tag{16}$$

As in the proof of Theorem 3.1, we have

$$\|u_n - z\|^2 \le \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{17}$$

Then we have from (1) and (17) that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|(I-\alpha_n G)(y_n-z)+\alpha_n\big(\gamma g(x_n)-Gz\big)\|^2\\
&\le (1-\alpha_n\overline{\gamma})^2\|y_n-z\|^2+2\alpha_n\langle\gamma g(x_n)-Gz, x_{n+1}-z\rangle\\
&\le (1-\alpha_n\overline{\gamma})^2\|u_n-z\|^2+2\alpha_n\langle\gamma g(x_n)-Gz, x_{n+1}-z\rangle\\
&\le (1-\alpha_n\overline{\gamma})^2\big(\|x_n-z\|^2-\|x_n-u_n\|^2\big)\\
&\quad +2\alpha_n\gamma k\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\big\|\gamma g(z)-Gz\big\|\|x_{n+1}-z\|\\
&= \big\{1-2\alpha_n\overline{\gamma}+\alpha_n^2\overline{\gamma}^2\big\}\|x_n-z\|^2-(1-\alpha_n\overline{\gamma})^2\|x_n-u_n\|^2\\
&\quad +2\alpha_n\gamma k\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\big\|\gamma g(z)-Gz\big\|\|x_{n+1}-z\|\\
&\le \|x_n-z\|^2+\alpha_n^2\overline{\gamma}^2\|x_n-z\|^2-(1-\alpha_n\overline{\gamma})^2\|x_n-u_n\|^2\\
&\quad +2\alpha_n\gamma k\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\big\|\gamma g(z)-Gz\big\|\|x_{n+1}-z\|
\end{aligned}$$

and hence

$$\begin{aligned}
&(1-\alpha_n\overline{\gamma})^2\|x_n-u_n\|^2\\
&\quad\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n^2\overline{\gamma}^2\|x_n-z\|^2\\
&\qquad +2\alpha_n\gamma k\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\big\|\gamma g(z)-Gz\big\|\|x_{n+1}-z\|\\
&\quad\le \|x_n-x_{n+1}\|\big(\|x_n-z\|+\|x_{n+1}-z\|\big)+\alpha_n^2\overline{\gamma}^2\|x_n-z\|^2\\
&\qquad +2\alpha_n\gamma k\|x_n-z\|\|x_{n+1}-z\|+2\alpha_n\big\|\gamma g(z)-Gz\big\|\|x_{n+1}-z\|.
\end{aligned}$$

From $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$, we have

$$\|x_n-u_n\|\to 0. \tag{18}$$

Then we have from (16) and (18) that

$$\|y_n-u_n\|\le\|y_n-x_n\|+\|x_n-u_n\|\to 0. \tag{19}$$

From $\sum_{n=1}^{\infty}|\lambda_n-\lambda_{n+1}|<\infty$, we find that $\{\lambda_n\}$ is a Cauchy sequence. So, we have $\lambda_n\to\lambda_0\in[a,2\alpha]$. Since $u_n=T_{r_n}x_n$, $z_n=(I-\lambda_n A)u_n$ and $y_n=J_{\lambda_n}(I-\lambda_n A)T_{r_n}x_n$, we have from Lemma 2.1 that

$$\begin{aligned}
&\big\|J_{\lambda_0}(I-\lambda_0 A)u_n-y_n\big\|\\
&\quad=\big\|J_{\lambda_0}(I-\lambda_0 A)u_n-J_{\lambda_n}(I-\lambda_n A)u_n\big\|\\
&\quad=\big\|J_{\lambda_0}(I-\lambda_0 A)u_n-J_{\lambda_0}(I-\lambda_n A)u_n\\
&\qquad +J_{\lambda_0}(I-\lambda_n A)u_n-J_{\lambda_n}(I-\lambda_n A)u_n\big\|\\
&\quad\le\big\|(I-\lambda_0 A)u_n-(I-\lambda_n A)u_n\big\|+\|J_{\lambda_0}z_n-J_{\lambda_n}z_n\|\\
&\quad\le|\lambda_0-\lambda_n|\|Au_n\|+\frac{|\lambda_0-\lambda_n|}{\lambda_0}\|J_{\lambda_0}z_n-z_n\|\to 0.
\end{aligned}\tag{20}$$

We also have from (19) and (20) that

$$\left\|u_n - J_{\lambda_0}(I - \lambda_0 A)u_n\right\| \le \|u_n - y_n\| + \left\|y_n - J_{\lambda_0}(I - \lambda_0 A)u_n\right\| \to 0. \tag{21}$$

We will use (20) and (21) later. From Theorem 3.1, we can take a unique solution $z_0 \in (A+B)^{-1}0 \cap F^{-1}0$ of the variational inequality

$$\left\langle (G - \gamma g)z_0, q - z_0\right\rangle \ge 0, \quad \forall q \in (A+B)^{-1}0 \cap F^{-1}0.$$

We show that $\limsup_{n\to\infty}\langle (G-\gamma g)z_0, x_n - z_0\rangle \ge 0$. Put

$$l = \limsup_{n\to\infty}\left\langle (G - \gamma g)z_0, x_n - z_0\right\rangle.$$

Without loss of generality, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $l = \lim_{i\to\infty}\langle (G-\gamma g)z_0, x_{n_i} - z_0\rangle$ and $\{x_{n_i}\}$ converges weakly to some point $w \in H$. From $\|x_n - u_n\| \to 0$, we also find that $\{u_{n_i}\}$ converges weakly to $w \in C$. On the other hand, from $\lambda_n \to \lambda_0 \in [a, 2\alpha]$ we have $\lambda_{n_i} \to \lambda_0 \in [a, 2\alpha]$. Using (20), we have

$$\left\| J_{\lambda_0}(I - \lambda_0 A)u_{n_i} - y_{n_i}\right\| \to 0.$$

Furthermore, from (21) we have

$$\left\|u_{n_i} - J_{\lambda_0}(I - \lambda_0 A)u_{n_i}\right\| \to 0.$$

Since $J_{\lambda_0}(I - \lambda_0 A)$ is a nonexpansive mapping of C into H, we have from [19, Lemma 4.1] that $w = J_{\lambda_0}(I - \lambda_0 A)w$. This means that $0 \in Aw + Bw$. As in the proof of Theorem 3.1, we can also show $w \in F^{-1}0$. Thus we have $w \in (A+B)^{-1}0 \cap F^{-1}0$. So, we have

$$l = \lim_{i\to\infty}\left\langle (G - \gamma g)z_0, x_{n_i} - z_0\right\rangle = \left\langle (G - \gamma g)z_0, w - z_0\right\rangle \ge 0.$$

Since $x_{n+1} - z_0 = \alpha_n(\gamma g(x_n) - Gz_0) + (I - \alpha_n G)(y_n - z_0)$, we find from (1) that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\le (1 - \alpha_n\overline{\gamma})^2\|y_n - z_0\|^2 + 2\left\langle \alpha_n\left(\gamma g(x_n) - Gz_0\right), x_{n+1} - z_0\right\rangle \\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - z_0\|^2 + 2\alpha_n\left\langle \gamma g(x_n) - Gz_0, x_{n+1} - z_0\right\rangle \\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - z_0\|^2 + 2\alpha_n\gamma k\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\left\langle \gamma g(z_0) - Gz_0, x_{n+1} - z_0\right\rangle \\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - z_0\|^2 + \alpha_n\gamma k\left(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\right) \\
&\quad + 2\alpha_n\left\langle \gamma g(z_0) - Gz_0, x_{n+1} - z_0\right\rangle \\
&= \left\{(1 - \alpha_n\overline{\gamma})^2 + \alpha_n\gamma k\right\}\|x_n - z_0\|^2 \\
&\quad + \alpha_n\gamma k\|x_{n+1} - z_0\|^2 + 2\alpha_n\left\langle \gamma g(z_0) - Gz_0, x_{n+1} - z_0\right\rangle
\end{aligned}$$

and hence

$$
\begin{aligned}
\|x_{n+1}-z_0\|^2 \le{}& \frac{1-2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2+\alpha_n\gamma k}{1-\alpha_n\gamma k}\|x_n-z_0\|^2 \\
&+\frac{2\alpha_n}{1-\alpha_n\gamma k}\big\langle \gamma g(z_0)-Gz_0, x_{n+1}-z_0\big\rangle \\
={}& \left(1-\frac{2(\overline{\gamma}-\gamma k)\alpha_n}{1-\alpha_n\gamma k}\right)\|x_n-z_0\|^2+\frac{(\alpha_n\overline{\gamma})^2}{1-\alpha_n\gamma k}\|x_n-z_0\|^2 \\
&+\frac{2\alpha_n}{1-\alpha_n\gamma k}\big\langle \gamma g(z_0)-Gz_0, x_{n+1}-z_0\big\rangle \\
={}& \left(1-\frac{2(\overline{\gamma}-\gamma k)\alpha_n}{1-\alpha_n\gamma k}\right)\|x_n-z_0\|^2+\frac{\alpha_n\cdot\alpha_n\overline{\gamma}^2}{1-\alpha_n\gamma k}\|x_n-z_0\|^2 \\
&+\frac{2\alpha_n}{1-\alpha_n\gamma k}\big\langle \gamma g(z_0)-Gz_0, x_{n+1}-z_0\big\rangle \\
={}& (1-\beta_n)\|x_n-z_0\|^2 \\
&+\beta_n\left(\frac{\alpha_n\overline{\gamma}^2\|x_n-z_0\|^2}{2(\overline{\gamma}-\gamma k)}+\frac{1}{\overline{\gamma}-\gamma k}\big\langle \gamma g(z_0)-Gz_0, x_{n+1}-z_0\big\rangle\right),
\end{aligned}
\tag{22}
$$

where $\beta_n=\frac{2(\overline{\gamma}-\gamma k)\alpha_n}{1-\alpha_n\gamma k}$. Since $\sum_{n=1}^{\infty}\beta_n=\infty$, we have from Lemma 2.2 and (22) that $x_n\to z_0$, where $z_0=P_{(A+B)^{-1}0\cap F^{-1}0}(I-G+\gamma g)z_0$. This completes the proof. □

4 Applications

In this section, using Theorem 3.2, we obtain new strong convergence theorems for in a Hilbert space. Let H be a Hilbert space and let f be a proper lower semicontinuous convex function of H into $]-\infty,\infty]$. Then, the subdifferential ∂f of f is defined as follows:

$$\partial f(x):=\big\{z\in H: f(x)+\langle z,y-x\rangle\le f(y),\ \forall y\in H\big\}$$

for all $x\in H$. From Rockafellar [20], we know that ∂f is a maximal monotone operator. Let C be a nonempty, closed and convex subset of H and let i_C be the indicator function of C. Then i_C is a proper lower semicontinuous and convex function on H. So, we can define the resolvent J_λ of ∂i_C for $\lambda>0$, i.e.,

$$J_\lambda x=(I+\lambda\partial i_C)^{-1}x$$

for all $x\in H$. We have, for any $\lambda>0$, $x\in H$ and $u\in C$,

$$
\begin{aligned}
u = J_\lambda x \quad &\Longleftrightarrow \quad x \in u + \lambda \partial i_C u \quad \Longleftrightarrow \quad x \in u + \lambda N_C u \\
&\Longleftrightarrow \quad \frac{1}{\lambda}\langle x - u, v - u\rangle \le 0, \quad \forall v \in C \\
&\Longleftrightarrow \quad \langle x - u, v - u\rangle \le 0, \quad \forall v \in C \\
&\Longleftrightarrow \quad u = P_C x,
\end{aligned}
$$

where $N_C u$ is the normal cone to C at u, i.e.,

$$N_C u := \bigl\{z \in H : \langle z, v - u\rangle \le 0,\ \forall v \in C\bigr\}.$$

Let $f : C \times C \to \mathbb{R}$ be a bifunction and let A be a mapping of C into H. A generalized equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle \ge 0, \quad \forall y \in C. \tag{23}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f, A)$, i.e.,

$$EP(f, A) = \bigl\{\hat{x} \in C : f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle \ge 0,\ \forall y \in C\bigr\}.$$

In the case of $A = 0$, $EP(f, A)$ is denoted by $EP(f)$. In the case of $f = 0$, $EP(f, A)$ is also denoted by $VI(C, A)$. This is the set of solutions of the variational inequality for A.

Using Theorem 3.2, we prove a strong convergence theorem for inverse-strongly monotone operators in a Hilbert space.

Theorem 4.1 *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $0 < k < 1$ and let g be a k-contraction of H into itself and let G be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Take γ with $0 < \gamma < \frac{\overline{\gamma}}{k}$ and suppose $VI(C, A) \neq \emptyset$. Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n G) P_C (I - \lambda_n A) P_C x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset]0, 1[$ and $\{\lambda_n\} \subset]0, \infty[$ satisfy

$$
\begin{gathered}
\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty, \\
0 < a \le \lambda_n \le 2\alpha, \quad \textit{and} \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.
\end{gathered}
$$

Then, the sequence $\{x_n\}$ converges strongly to a point z_0 of $VI(C, A)$, where $z_0 \in VI(C, A)$ is a unique fixed point of $P_{VI(C,A)}(I - G + \gamma g)$. This point $z_0 \in VI(C, A)$ is also a unique solution of the variational inequality

$$\bigl\langle (G - \gamma g) z_0, q - z_0 \bigr\rangle \ge 0, \quad \forall q \in VI(C, A).$$

Proof Put $B = F = \partial i_C$ in Theorem 3.2. Then, we have, for $\lambda_n > 0$ and $r_n > 0$,

$$J_{\lambda_n} = T_{r_n} = P_C.$$

Furthermore, we have $(\partial i_C)^{-1}0 = C$ and $(A + \partial i_C)^{-1}0 = VI(C, A)$. In fact, we have, for $z \in C$,

$$\begin{aligned} z \in (A + \partial i_C)^{-1}0 \quad &\Longleftrightarrow \quad 0 \in Az + \partial i_C z \\ &\Longleftrightarrow \quad 0 \in Az + N_C z \\ &\Longleftrightarrow \quad \langle -Az, v - z\rangle \le 0, \quad \forall v \in C \\ &\Longleftrightarrow \quad \langle Az, v - z\rangle \ge 0, \quad \forall v \in C \\ &\Longleftrightarrow \quad z \in VI(C, A). \end{aligned}$$

Thus we obtain the desired result by Theorem 3.2. □

Let C be a nonempty, closed and convex subset of H. Then, $U : C \to H$ is called a widely strict pseudo-contraction iff there exists $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + r\big\|(I - U)x - (I - U)y\big\|^2, \quad \forall x, y \in C.$$

We call such U a widely r-strict pseudo-contraction. If $0 \le r < 1$, then U is a strict pseudo-contraction [1]. Furthermore, if $r = 0$, then U is nonexpansive. Conversely, let $T : C \to H$ be a nonexpansive mapping and define $U : C \to H$ by $U = \frac{1}{1+n}T + \frac{n}{1+n}I$ for all $x \in C$ and $n \in \mathbb{N}$. Then U is a widely $(-n)$-strict pseudo-contraction. In fact, from the definition of U, it follows that $T = (1 + n)U - nI$. Since T is nonexpansive, we have, for any $x, y \in C$,

$$\big\|(1 + n)Ux - nx - \big((1 + n)Uy - ny\big)\big\|^2 \le \|x - y\|^2$$

and hence

$$\|Ux - Uy\|^2 \le \|x - y\|^2 - n\big\|(I - U)x - (I - U)y\big\|^2.$$

Using Theorem 3.2, we obtain an extension of Zhou's strong convergence theorem [21] in a Hilbert space.

Theorem 4.2 *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $r \in \mathbb{R}$ with $r < 1$ and let U be a widely r-strict pseudo-contraction of C into H such that $F(U) \neq \emptyset$. Let $u \in C$, $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C\big\{(1 - t_n)U + t_n I\big\}x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset]0, 1[$ and $\{t_n\} \subset] - \infty, 1[$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$$

$$r \le t_n \le b < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |t_n - t_{n+1}| < \infty.$$

Then, the sequence $\{x_n\}$ *converges strongly to a point* z_0 *of* $F(U)$, *where* $z_0 = P_{F(U)}u$.

Proof Put $B = F = \partial i_C$ and $A = I - U$ in Theorem 3.2. Furthermore, put $g(x) = u$ and $G(x) = x$ for all $x \in H$. Then, we can take $\overline{\gamma} = \frac{1}{2}$. Since $\|g(x) - g(y)\| = 0 \le \frac{1}{3}\|x - y\|$ for all $x, y \in H$, we can take $k = \frac{1}{3}$ and hence set $\gamma = 1$. Putting $a = 1 - b$, $\lambda_n = 1 - t_n$ and $2\alpha = 1 - r$ in Theorem 3.2, we get from $r \le t_n \le b < 1$ that $0 < a \le \lambda_n \le 2\alpha$,

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| = \sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$$

and

$$I - \lambda_n A = I - (1 - t_n)(I - U) = (1 - t_n)U + t_n I.$$

Furthermore, we have, for $z \in C$,

$$\begin{aligned} z \in (A + \partial i_C)^{-1}0 \quad &\Longleftrightarrow \quad 0 \in Az + \partial i_C z \\ &\Longleftrightarrow \quad 0 \in z - Uz + N_C z \\ &\Longleftrightarrow \quad Uz - z \in N_C z \\ &\Longleftrightarrow \quad \langle Uz - z, v - z\rangle \le 0, \quad \forall v \in C \\ &\Longleftrightarrow \quad P_C U z = z. \end{aligned}$$

Since $F(U) \neq \emptyset$, we find, as in the proof of [21, Fact 3], that $F(P_C U) = F(U)$. We also have $z_0 = P_{F(U)}(I - G + \gamma g)z_0 = P_{F(U)}(z_0 - z_0 + 1 \cdot u) = P_{F(U)}u$. Thus, we obtain the desired result by Theorem 3.2. □

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f\bigl(tz + (1 - t)x, y\bigr) \le f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Then, we know the following lemma which appears implicitly in Blum and Oettli [22].

Lemma 4.1 (Blum and Oettli) *Let* C *be a nonempty, closed and convex subset of* H *and let* f *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)–(A4). *Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

The following lemma was also given in Combettes and Hirstoaga [23].

Lemma 4.2 *Assume that* $f : C \times C \to \mathbb{R}$ *satisfies* (A1)–(A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows*:

$$T_r x := \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. *Then*, *the following hold*:

(1) T_r *is single-valued*;
(2) T_r *is a firmly nonexpansive mapping*, *i.e.*, *for all* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(f)$;
(4) $EP(f)$ *is closed and convex.*

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 4.1 and 4.2, Takahashi, Takahashi and Toyoda [5] obtained the following lemma. See [24] for a more general result.

Lemma 4.3 *Let* H *be a Hilbert space and let* C *be a nonempty*, *closed and convex subset of* H. *Let* $f : C \times C \to \mathbb{R}$ *satisfy* (A1)–(A4). *Let* A_f *be a set-valued mapping of* H *into itself defined by*

$$A_f x := \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ *and* A_f *is a maximal monotone operator such that* $\operatorname{dom}(A_f) \subset C$. *Furthermore*, *for any* $x \in H$ *and* $r > 0$, *the resolvent* T_r *of* f *coincides with the resolvent of* A_f, *i.e.*,

$$T_r x = (I + rA_f)^{-1}x.$$

Using Lemma 4.3 and Theorem 3.2, we obtain the following result which generalizes Liu's strong convergence theorem [4].

Theorem 4.3 *Let* H *be a real Hilbert space and let* C *be a nonempty*, *closed and convex subset of* H. *Let* $r \in \mathbb{R}$ *with* $r < 1$ *and let* U *be a widely* r*-strict pseudo-contraction of* C *into* H *and let* f *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)–(A4). *Let* T_r *be the resolvent of* f *for* $r > 0$. *Let* $0 < k < 1$ *and let* g *be a* k*-contraction of* H *into itself. Let* G *be a strongly positive bounded linear self-adjoint operator on* H *with coefficient* $\overline{\gamma} > 0$. *Let* $0 < \gamma < \frac{\overline{\gamma}}{k}$ *and suppose* $F(U) \cap EP(f) \neq \emptyset$. *Let* $x_1 = x \in H$ *and let* $\{x_n\} \subset H$ *be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n G)\{(1 - t_n)U + t_n I\}T_{r_n} x_n$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset]0, 1[$, $\{t_n\} \subset]-\infty, 1[$ *and* $\{r_n\} \subset]0, \infty[$ *satisfy*

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty, \quad r \le t_n \le b < 1,$$

$$\sum_{n=1}^{\infty} |t_n - t_{n+1}| < \infty, \quad \liminf_{n\to\infty} r_n > 0, \quad \text{and} \quad \sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty.$$

Then, the sequence $\{x_n\}$ *converges strongly to a point* z_0 *of* $F(U) \cap EP(f)$, *where* $z_0 \in F(U) \cap EP(f)$ *is a unique fixed point of* $P_{F(U)\cap EP(f)}(I - G + \gamma g)$. *This point* $z_0 \in F(U) \cap EP(f)$ *is also a unique solution of the variational inequality*

$$\langle (G - \gamma g)z_0, q - z_0 \rangle \ge 0, \quad \forall q \in F(U) \cap EP(f).$$

Proof For the bifunction $f : C \times C \to \mathbb{R}$, we can define A_f in Lemma 4.3. Putting $A = I - U$, $Bx = 0$ for all $\in H$ and $F = A_f$ in Theorem 3.2, we obtain from Lemma 4.3 that $J_{\lambda_n} = I$ for all $\lambda_n > 0$ and $T_{r_n} = (I + r_n A_f)^{-1}$ for all $r_n > 0$. As in the proof of Theorem 4.2, the sequence $\{t_n\}$ and U are changed in $\{\lambda_n\}$ and A. We have also from Lemma 4.3 that

$$EP(f) = (A_f)^{-1}0 = F^{-1}0.$$

Furthermore, we have, for $z \in C$,

$$z \in (A + B)^{-1}0 \iff z \in F(U).$$

So, we obtain the desired result by Theorem 3.2. □

Remark 4.1 We note that two assumptions $0 \le r$ and $\lim_{n\to\infty} t_n = b$ in Liu's theorem [4] do not appear in Theorem 4.3.

5 Concluding Remarks

(1) We cannot directly prove that the mapping $P_D(I - G + rg)$ in Theorem 3.1 is a contraction. We proved that the mapping has a unique fixed point by using another contraction which is different from the mapping. Then we showed two strong convergence theorems (Theorems 3.1 and 3.2) by using this result. It seems that such methods are new.

(2) The domain of the maximal monotone operator A_f in Lemma 4.3, which is deduced from an equilibrium problem, is included in C. Thus the maximal monotone operator F in Theorems 3.1 and 3.2 covers the equilibrium problem. Our methods for the resolvents of the maximal monotone operator F, which are used in the proofs of Theorems 3.1 and 3.2, are more general than methods for solving the equilibrium problem.

(3) Since the class of inverse-strongly monotone mappings contains strict pseudo-contractions, our two theorems are general and useful.

(4) For the fixed point problem of nonself-mappings, we use generally the metric projections. For such a problem, we used the resolvents of a maximal monotone operator B in Theorems 3.1 and 3.2. Consequently, we solve the problem of finding a zero point of the addition of an inverse-strongly monotone mapping and a maximal monotone operator.

(5) Our results (Theorems 3.1 and 3.2) are also used for finding a common fixed point of two commuting nonexpansive mappings defined on a bounded, closed and convex subset of a Hilbert space.

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