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High-Order $\mathcal{D}^\alpha$-Type Iterative Learning Control for Fractional-Order Nonlinear Time-Delay Systems

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Abstract This paper presents a high-order $\mathcal{D}^\alpha$-type iterative learning control (ILC) scheme for a class of fractional-order nonlinear time-delay systems. First, a discrete system for $\mathcal{D}^\alpha$-type ILC is established by analyzing the control and learning processes, and the ILC design problem is then converted to a stabilization problem for this discrete system. Next, by introducing a suitable norm and using a generalized Gronwall–Bellman Lemma, the sufficiency condition for the robust convergence with respect to the bounded external disturbance of the control input and the tracking errors is obtained. Finally, the validity of the method is verified by a numerical example.

Keywords Fractional-order · Nonlinear time-delay system · Iterative learning control · Generalized Gronwall–Bellman lemma

1 Introduction

Fractional differential calculus [1, 2], an old mathematical topic from the 17th century, has recently attracted a rapid growth in the number of applications. It was found that many systems in interdisciplinary fields could be elegantly described with the help of fractional derivatives and integrals [3, 4]. Also, fractional-order controllers have so far been implemented to enhance the robustness and the performance of the control systems [5–7].

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Iterative learning control (ILC) is an approach for improving the transient performance of systems that operate repetitively over a fixed time interval [8, 9]. Owing to its simplicity and effectiveness, ILC has been found to be a good alternative in many areas and applications (see, for instance, [10, 11] and the referenced therein). In recent years, the application of ILC to the fractional-order systems has become a new topic [4, 12–14]. The authors in [12] were the first to propose the $\mathcal{D}^{\alpha}$-type ILC algorithm in frequency domain. For fractional-order linear systems described in the state space form, the convergence conditions are derived in [5]. In [13], the asymptotic stability of P$\mathcal{D}^{\alpha}$-type ILC for a fractional-order linear time invariant (LTI) system was investigated. The convergence condition of open-loop P-type ILC for fractional-order nonlinear system was studied in [14].

It should be noted that the higher-order learning algorithms are the ones in which the information from past cycles, not just from the last cycle, is taken advantage of. As a result, developing higher-order learning algorithms can lead to better performance in terms of both robustness and convergence rate [11, 15, 16]. The key idea of the presented method was to use past information of more than one to update the current adaptation learning law.

In this paper, we investigated a high-order $\mathcal{D}^{\alpha}$-type ILC updating law design method for a class of fractional-order nonlinear time-delay systems. The rest of this paper is organized as follows. In Sect. 2, the fractional derivative and some preliminaries are presented. The high-order $\mathcal{D}^{\alpha}$-type ILC scheme as well as the convergence condition for fractional-order systems is discussed in Sect. 3. MATLAB/SIMULINK results are shown in Sect. 4. Finally, some conclusions are drawn in Sect. 5.

2 Fractional Derivative and Preliminaries

In this section, some basic definitions and properties (for more details see [1]) are introduced, which will be used in the following sections.

Definition 2.1 The definition of fractional integral is described by

$$ {}_{t_0}\mathcal{D}_t^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0, $$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Definition 2.2 The Riemann–Liouville derivative is defined as

$$ {}_{t_0}^{RL}\mathcal{D}_t^{q} f(t) := \mathcal{D}_{t_0}^{m} \mathcal{D}_t^{q-m} f(t), \quad q \in [m-1, m), $$

and the Caputo derivative is

$$ {}_{t_0}^{C}\mathcal{D}_t^{q} f(t) := {}_{t_0}\mathcal{D}_t^{q-m} \mathcal{D}^{m} f(t), \quad q \in [m-1, m), $$

where $m \in \mathbb{Z}^{+}$, D^{m} is the classical m-order derivative.

Definition 2.3 [1] The two-parameter Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \quad \beta > 0.$$

Property 2.1 [1] The fractional-order differentiation or integral of Mittag–Leffler function is

$${}_{t_0}\mathcal{D}_t^{\rho}\left[t^{\beta-1}E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)\right] = t^{\beta-\rho-1}E_{\alpha,\beta-\rho}\left(\lambda t^{\alpha}\right),$$

where $\rho < \beta$, $\mathcal{D}$ denotes either the Riemann–Liouville or Caputo fractional-order operator.

Lemma 2.1 *If the function $f(t,x)$ is continuous, then the initial value problem*

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\alpha}x(t) = f(t,x(t)), & 0 < \alpha < 1, \\ x(t_0) = x(0) \end{cases}$$

is equivalent to the following nonlinear Volterra integral equation:

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f\big(s, x(s)\big)\, ds,$$

and its solutions are continuous [17]. *The initial value problem*:

$$\begin{cases} {}_{t_0}^{RL}\mathcal{D}_t^{\alpha}x(t) = f(t,x(t)), & 0 < \alpha < 1, \\ {}_{t_0}^{RL}\mathcal{D}_t^{\alpha-1}x(t_0) = x(0) \end{cases}$$

is equivalent to the following nonlinear Volterra integral equation [18]:

$$x(t) = \frac{x(0)}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f\big(s, x(s)\big)\, ds.$$

Lemma 2.2 (Generalized Gronwall Inequality, [14]) *Let $u(t)$ be a continuous function on $t \in [0,T]$ and let $v(t-\tau)$ be continuous and nonnegative on the triangle $0 \le \tau \le T$. Moreover, let $w(t)$ be a positive continuous and non-decreasing function on $t \in [0,T]$. If*

$$u(t) \le h(t) + \int_0^t v(t-\tau)u(\tau)\, d\tau, \quad t \in [0,T],$$

then

$$u(t) \le w(t) e^{\int_0^t v(t-\tau)\, d\tau}, \quad t \in [0,T],$$

Throughout this paper, the 2-norm for the n-dimensional vector $w = (w_1, w_2, \dots, w_n)$ and the matrix $A_{n\times n}$ is defined as $\|w\| := \sqrt{\sum_{i=1}^{n} w_i^2}$, $\|A\| := \sqrt{\lambda_{\max}(A^T A)}$, respectively. The λ-norm for n-vector-valued function $h(t):[0,T]\to\mathbb{R}^n$ is defined as

$$\|h(t)\|_\lambda := \sup_{t\in[0,T]}\{e^{-\lambda t}\|h(t)\|\},\quad \lambda>0,$$

while the (λ,ξ)-norm for m-vector-valued function $g_k(t):[0,T]\to\mathbb{R}^m$, $k\in\{0,1,2,\dots\}$ is defined as

$$\|g_k(t)\|_{(\lambda,\xi)} := \sup_{t\in[0,T]}\{e^{-\lambda t}\|g_k(t)\|\xi^k\},\quad \lambda>0,$$

where $\|\cdot\|$ can be chosen as any kind of norm.

3 High-Order $\mathcal{D}^\alpha$-Type ILC for Fractional-Order Nonlinear Time-Delay Systems

Consider the following fractional-order nonlinear time-delay system:

$$\begin{cases}\mathcal{D}_t^\alpha x_k(t) = f(x_k(t), x_k(t-\tau), t) + Bu_k(t),\\ y_k(t) = Cx_k(t) + D\mathcal{D}_t^{-\alpha}u_k(t),\end{cases}\tag{1}$$

where $k\in\{0,1,2,\dots\}$, $t\in[0,T]$, $0<\alpha<1$.

$$\begin{aligned}&\|f(x_k(t)), f(x_k(t-\tau),t) - f(\bar{x}_k(t)), f(\bar{x}_k(t-\tau),t)\|\\ &\quad\le a\|x_k(t)-\bar{x}_k(t)\| + a_1\|x_k(t-\tau)-\bar{x}_k(t-\tau)\|,\end{aligned}$$

$x_k(t)\in\mathbb{R}^n$ is the state of the plant, and $u_k(t)\in\mathbb{R}^m$ and $y_k(t)\in\mathbb{R}^m$ are the control input and output, respectively. A, A_1, B, C and D are constant system matrices with appropriate dimensions, τ is a pure delay and with the associated function of the initial state: $x_k(t)=\psi(t), -\tau\le t\le 0$. $\psi(t)$ is a given continuous function on $[-\tau,0]$. $\mathcal{D}_t^\alpha$ denotes either Caputo derivative or Riemann–Liouville derivative of order α. (If one denotes the Riemann–Liouville derivative, the additional condition $\mathcal{D}_t^{\alpha-1}x_k(0)=x(0)$ is needed.)

In this paper, the following high-order $\mathcal{D}^\alpha$-type ILC updating law is considered:

$$u_{k+1}(t) = \Lambda u_k(t) + u_{kh}(t) + \Gamma\mathcal{D}_t^\alpha e_k(t),\quad k\in\{1,2,\dots\},\tag{2}$$

where

$$u_{kh}(t) = \begin{cases}\sum_{i=1}^N \Lambda_i u_{k-i}(t),\quad \sum_{i=1}^N\Lambda_i = I-\Lambda, & k\in\{N+1,N+2,\dots\},\\ 0, & k\in\{1,2,\dots,N\},\end{cases}$$

and $t\in[0,T]$, $0<\alpha<1$, $e_k(t)=y_d(t)-y_k(t)$ denotes the tracking error, Γ, Λ_i and Λ are unknown gain matrices to be determined.

For fractional-order nonlinear time-delay system (1) under the $\mathcal{D}^\alpha$-type ILC updating law (2), we have the following Lemmas.

Lemma 3.1 *Let* $\Delta u_k(t) := u_k(t) - u_{k-1}(t)$, $\Delta x_k(t) := x_k(t) - x_{k-1}(t)$, $\Delta f_k(t) := f_k(\cdot) - f_{k-1}(\cdot)$ *and*

$$Q_k(t) := \begin{bmatrix} \mathcal{D}_t^\alpha e_k(t) \\ \Delta u_k(t) \end{bmatrix}, \qquad G := \begin{pmatrix} I - (CB+D)\Gamma & (I-\Lambda)(CB+D) \\ \Gamma & (\Lambda - 1)I \end{pmatrix},$$

$$F_k(t) := \begin{bmatrix} -C\Delta f_{k+1}(t) + (CB+D)\sum_{i=1}^N \Lambda_i \sum_{j=1}^{i-1} \Delta u_{k-j}(t) \\ -\sum_{i=1}^N \Lambda_i \sum_{j=1}^{i-1} \Delta u_{k-j}(t) \end{bmatrix},$$

then

$$Q_{k+1}(t) = GQ_k(t) + F_k(t), \quad k \ge N. \tag{3}$$

Proof It follows from (2) that, for $k \ge N$,

$$u_{k+1}(t) = \Lambda u_k(t) + \sum_{i=1}^N \Lambda_i u_{k-i}(t) + \Gamma \mathcal{D}_t^\alpha e_k(t). \tag{4}$$

Noting that $\sum_{i=1}^N \Lambda_i = I - \Lambda$, it can easily be shown that

$$u_{k+1}(t) - u_k(t) = (\Lambda - I)\Delta u_k(t) - \sum_{i=1}^N \Lambda_i \sum_{j=1}^{i-1} \Delta u_{k-j}(t) + \Gamma \mathcal{D}_t^\alpha e_k(t). \tag{5}$$

Since $e_{k+1}(t) - e_k(t) = -(y_{k+1}(t) - y_k(t))$, then, from (1), one has

$$\mathcal{D}_t^\alpha e_{k+1}(t) - \mathcal{D}_t^\alpha e_k(t) = -C\Delta f_{k+1}(t) - (CB+D)\Delta u_{k+1}(t). \tag{6}$$

Taking into account (5), it yields

$$\begin{aligned} \mathcal{D}_t^\alpha e_{k+1}(t) = {}& \big[I - (CB+D)\Gamma\big]\mathcal{D}_t^\alpha e_k(t) - (\Lambda - I)(CB+D)\Delta u_k(t) \\ & - C\Delta f_{k+1}(t) - (CB+D)\sum_{i=1}^N \Lambda_i \sum_{j=1}^{i-1} \Delta u_{k-j}(t). \end{aligned} \tag{7}$$

Therefore, from (5) and (7), one gets

$$\begin{bmatrix} \mathcal{D}_t^\alpha e_{k+1}(t) \\ \Delta u_{k+1}(t) \end{bmatrix} = G \begin{bmatrix} \mathcal{D}_t^\alpha e_k(t) \\ \Delta u_k(t) \end{bmatrix} + F_k(t). \tag{8}$$

The proof is complete. □

Lemma 3.2 *Denote that* $b := \|B\|$, $c := \|C\|$, *and* $M_1 := e^{\frac{aT^\alpha + a_1[(T-\tau)^\alpha + \tau^\alpha]}{\Gamma(\alpha+1)}}$, $M_2 := (\|\Gamma\| + \|\Lambda - I\|)$, $h := (\frac{a+a_1 e^{-\lambda\tau}}{\lambda^\alpha})bcM_1$, *then*

$$\|F_k(t)\|_{(\lambda,\xi)} < hM_2\|Q_k\|_{(\lambda,\xi)} + \sum_{i=1}^{N}\big((h+1)\|\Lambda_i\| + \|(CB+D)\Lambda_i\|\big)\sum_{j=1}^{i-1}\xi^j\|Q_{k-j}(t)\|_{(\lambda,\xi)}. \tag{9}$$

Proof It follows the definition of $F_k(t)$ that

$$\|F_k(t)\| \le ca\|\Delta x_{k+1}(t)\| + ca_1\|\Delta x_{k+1}(t-\tau)\| + \sum_{i=1}^{N}\big(\|(CB+D)\Lambda_i\| + \|\Lambda_i\|\big)\sum_{j=1}^{i-1}\|\Delta u_{k-j}(t)\|. \tag{10}$$

On the other hand, from Lemma 2.1 and in accordance with the property of the fractional-order $0 < \alpha < 1$, we have

$$\Delta x_{k+1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\{(\Delta f_{k+1}(s) + B\Delta u_{k+1}(s)\big\}\,ds. \tag{11}$$

Therefore, if $t \in [0, \tau]$, then

$$\|\Delta x_{k+1}(t)\| \le \frac{a}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\Delta x_{k+1}(s)\|\,ds + \frac{b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\Delta u_{k+1}(s)\|\,ds. \tag{12}$$

If $t \in [\tau, T]$, then

$$\begin{aligned}\|\Delta x_{k+1}(t)\| &\le \frac{a}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\Delta x_{k+1}(s)\|\,ds \\ &\quad + \frac{a_1}{\Gamma(\alpha)}\int_0^\tau (t-s)^{\alpha-1}\|\Delta x_{k+1}(s-\tau)\|\,ds \\ &\quad + \frac{a_1}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|\Delta x_{k+1}(s-\tau)\|\,ds \\ &\quad + \frac{b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\Delta u_{k+1}(s)\|\,ds \\ &\le \frac{a}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\Delta x_{k+1}(s)\|\,ds \\ &\quad + \frac{a_1}{\Gamma(\alpha)}\int_0^t |t-\tau-s|^{\alpha-1}\|\Delta x_{k+1}(s)\|\,ds\end{aligned}$$

$$+\frac{b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|\Delta u_{k+1}(s)\big\|\,ds. \tag{13}$$

After combining (12) and (13), it yields, for any $t \in [0, T]$,

$$\begin{aligned}\big\|\Delta x_{k+1}(t)\big\| \le{}& \frac{a}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|\Delta x_{k+1}(s)\big\|\,ds\\ &+\frac{a_1}{\Gamma(\alpha)}\int_0^t |t-\tau-s|^{\alpha-1}\big\|\Delta x_{k+1}(s)\big\|\,ds\\ &+\frac{b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}\,ds\big\|\Delta u_{k+1}(t)\big\|_\lambda. \end{aligned}\tag{14}$$

Noting that

$$\frac{b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}\,ds = bt^\alpha E_{1,1+\alpha}(\lambda t),$$

it follows from the Property 2.1 that, for $\lambda > 0$,

$$\frac{dt^\alpha E_{1,1+\alpha}(\lambda t)}{dt} = t^{\alpha-1}E_{1,\alpha}(\lambda t) > 0.$$

Therefore,

$$\begin{aligned}h(t) ={}& \frac{b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}\,ds\big\|\Delta u_{k+1}(t)\big\|_\lambda\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}\,ds\big\|\Delta w_{k+1}(t)\big\|_\lambda\end{aligned}$$

is an increasing function. Setting

$$v(t-s) = \frac{a}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{a_1}{\Gamma(\alpha)}|t-\tau-s|^{\alpha-1},$$

it can be proved that, for all $t \in [0, T]$,

$$e^{\int_0^t v(t-s)\,ds} \le e^{\frac{aT^\alpha + a_1[(T-\tau)^\alpha+\tau^\alpha]}{\Gamma(\alpha+1)}} := M_1. \tag{15}$$

Taking into account (15) and applying Lemma 2.2 to (14), one obtains

$$\big\|\Delta x_{k+1}(t)\big\| \le \frac{bM_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}\,ds\big\|\Delta u_{k+1}(t)\big\|_\lambda, \tag{16}$$

and

$$\big\|\Delta x_{k+1}(t-\tau)\big\| \le \frac{bM_1}{\Gamma(\alpha)}\int_0^{t-\tau} (t-\tau-s)^{\alpha-1}e^{\lambda s}\,ds\big\|\Delta u_{k+1}(t)\big\|_\lambda. \tag{17}$$

From (10), (16) and (17), it yields

$$\begin{aligned}\|F_k(t)\| &\le \frac{abcM_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}\,ds\,\|\Delta u_{k+1}(t)\|_\lambda \\ &\quad+\frac{a_1bcM_1}{\Gamma(\alpha)}\int_0^{t-\tau}(t-\tau-s)^{\alpha-1}e^{\lambda s}\,ds\,\|\Delta u_{k+1}(t)\|_\lambda \\ &\quad+\sum_{i=1}^{N}\bigl(\|(CB+D)\Lambda_i\|+\|\Lambda_i\|\bigr)\sum_{j=1}^{i-1}\|\Delta u_{k-j}(t)\|. \end{aligned}\tag{18}$$

Multiplying both sides of (18) by $e^{-\lambda t}$ and taking the λ-norm, one has

$$\begin{aligned}\|F_k(t)\|_\lambda &\le \frac{abcM_1e^{-\lambda t}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}\,ds\,\|\Delta u_{k+1}(t)\|_\lambda \\ &\quad+\frac{a_1bcM_1e^{-\lambda t}}{\Gamma(\alpha)}\int_0^{t-\tau}(t-\tau-s)^{\alpha-1}e^{\lambda s}\,ds\,\|\Delta u_{k+1}(t)\|_\lambda \\ &\quad+\sum_{i=1}^{N}\bigl(\|(CB+D)\Lambda_i\|_\lambda+\|\Lambda_i\|_\lambda\bigr)\sum_{j=1}^{i-1}\|\Delta u_{k-j}(t)\|_\lambda. \end{aligned}\tag{19}$$

Note that

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}\,ds &\overset{t-s=w}{=\!=\!=} \int_0^t w^{\alpha-1}e^{\lambda(t-w)}\,dw = e^{\lambda t}\int_0^t w^{\alpha-1}e^{-\lambda w}\,dw \\ &\overset{\lambda w=s}{=\!=\!=} \frac{e^{\lambda t}}{\lambda^\alpha}\int_0^{\lambda t} s^{\alpha-1}e^{-s}\,ds < \frac{e^{\lambda t}}{\lambda^\alpha}\Gamma(\alpha), \end{aligned}\tag{20}$$

and

$$\int_0^{t-\tau}(t-\tau-s)^{\alpha-1}e^{\lambda s}\,ds < \frac{e^{\lambda t-\lambda\tau}}{\lambda^\alpha}\Gamma(\alpha).\tag{21}$$

From (19)–(21), it yields, for any $t\in[0,T]$,

$$\begin{aligned}\|F_k(t)\|_\lambda &< \left(\frac{a+a_1e^{-\lambda\tau}}{\lambda^\alpha}\right)bcM_1\|\Delta u_{k+1}(t)\|_\lambda \\ &\quad+\sum_{i=1}^{N}\bigl(\|(CB+D)\Lambda_i\|+\|\Lambda_i\|\bigr)\sum_{j=1}^{i-1}\|\Delta u_{k-j}(t)\|_\lambda. \end{aligned}\tag{22}$$

Moreover, it follows from (5) that

$$\Delta\|u_{k+1}(t)\|_\lambda \le \bigl(\|(\Lambda-I)\|+\|\Gamma\|\bigr)\|Q_k\|_\lambda+\sum_{i=1}^{N}\|\Lambda_i\|\sum_{j=1}^{i-1}\|\Delta u_{k-j}(t)\|_\lambda.\tag{23}$$

As a result, one obtains from (22) and (23) that

$$\begin{aligned}\big\|F_k(t)\big\|_\lambda &\le hM_2\|Q_k\|_\lambda \\ &\quad+\sum_{i=1}^{N}\big((h+1)\|\Lambda_i\|+\big\|(CB+D)\Lambda_i\big\|\big)\sum_{j=1}^{i-1}\big\|Q_{k-j}(t)\big\|_\lambda. \qquad (24)\end{aligned}$$

Applying the (λ,ξ)-norm to (24) yields (9), which completes the proof. □

Theorem 3.1 *For the fractional-order nonlinear time-delay system* (1) *and a given reference* $y_d(t)$, *suppose that* $y_d(0)=y_k(0)$ *and*

$$\sum_{j=1}^{N}\big(\|\Lambda_j\|+\big\|(CB+D)\Lambda_j\big\|\big)=c_1<1, \qquad (25)$$

$$\rho\big\{G(t)\big\}\le\bar\rho<1, \qquad (26)$$

where $\rho\{G(t)\}$ *is the spectral radius of* G, $\bar\rho$ *is a constant, then, for all* $t\in[0,T]$, *and arbitrary initial input satisfying* $u_{-i}(t)=0$, $i=1,2,\dots,N$, *the high-order* $\mathcal{D}^\alpha$*-type ILC updating law* (2) *guarantees that* $\{u_k(t)\}$ *is uniformly convergent, and*

$$\lim_{k\to\infty}y_k(t)=y_d(t). \qquad (27)$$

Proof It follows from (3) that, for $k>N$,

$$Q_k(t)=G^{k-N}Q_N(t)+\sum_{i=N}^{k-1}G^{k-i-1}F_i(t). \qquad (28)$$

Therefore, for $k>N$,

$$\big\|Q_k(t)\big\|\le\bar\rho^{k-N}\big\|Q_N(t)\big\|+\sum_{i=N}^{k-1}\bar\rho^{k-i-1}\big\|F_i(t)\big\|. \qquad (29)$$

Noting that $0\le\bar\rho<1$ and $c_1<1$ by assumption, there exist a constant $\xi>1$ and a sufficiently large λ such that $\bar\rho\xi<1$, and

$$0<\hat h=\frac{1}{1-\bar\rho\xi}\big[N\xi^{N+1}(c_1+c_2h)+\xi hM_2\big]<1, \qquad (30)$$

where $c_2=\sum_{j=1}^{N}\|\Lambda_j\|$, M_2 and h as defined in Lemma 3.2.

For the above λ and ξ, multiplying both sides of (29) by $e^{-\lambda t}\xi^k$ and taking the (λ,ξ)-norm, it yields

$$\begin{aligned}&\big(\big\|Q_k(t)\big\|\xi^k\big)e^{-\lambda t}\\ &\quad\le\bar\rho^{-N}\big\|Q_N(t)\big\|_\lambda+\sum_{i=N}^{k-1}(\bar\rho\xi)^{k-i-1}\xi\big\|F_i(t)\big\|_{(\lambda,\xi)}. \qquad (31)\end{aligned}$$

Now, it follows from (9) that (31) gives

$$
\begin{aligned}
&\big(\|Q_k(t)\|\xi^k\big)e^{-\lambda t}\\
&\quad\le \bar\rho^{-N}\big\|Q_N(t)\big\|_\lambda+\sum_{i=N}^{k-1}(\bar\rho\xi)^{k-i-1}\xi hM_2\|Q_k\|_{(\lambda,\xi)}\\
&\qquad+\sum_{i=N}^{k-1}(\bar\rho\xi)^{k-i-1}\xi\sum_{j=1}^{N}\big((h+1)\|\Lambda_j\|+\big\|(CB+D)\Lambda_j\big\|\big)\sum_{s=1}^{j-1}\xi^s\big\|Q_{k-s}(t)\big\|_{(\lambda,\xi)}\\
&\quad<\bar\rho^{-N}\big\|Q_N(t)\big\|_\lambda+\frac{1}{1-\rho\xi}\xi hM_2\sup_{1\le i\le k}\|Q_i\|_{(\lambda,\xi)}\\
&\qquad+\frac{1}{1-\bar\rho\xi}\sum_{j=1}^{N}\big((h+1)\|\Lambda_j\|+\big\|(CB+D)\Lambda_j\big\|\big)\cdot N\xi^{N+1}\sup_{1\le i\le k}\|Q_i\|_{(\lambda,\xi)}\\
&\quad<\bar\rho^{-N}\big\|Q_N(t)\big\|_\lambda+\frac{1}{1-\bar\rho\xi}\big[N\xi^{N+1}(c_1+c_2h)+\xi hM_2\big]\sup_{1\le i\le k}\|Q_i\|_{(\lambda,\xi)}\\
&\quad=\bar\rho^{-N}\big\|Q_N(t)\big\|_\lambda+\hat h\sup_{1\le i\le k}\|Q_i\|_{(\lambda,\xi)}.
\end{aligned}
\tag{32}
$$

Therefore,

$$
\sup_{1\le i\le k}\big\|Q_i(t)\big\|_{(\lambda,\xi)}<\rho^{-N}\big\|Q_N(t)\big\|_\lambda+\hat h\sup_{1\le i\le k}\big\|Q_i(t)\big\|_{(\lambda,\xi)}. \tag{33}
$$

Hence,

$$
\sup_{1\le i\le k}\big\|Q_i(t)\big\|_{(\lambda,\xi)}<\frac{\rho^{-N}}{1-\hat h}\big\|Q_N(t)\big\|_\lambda. \tag{34}
$$

Note that

$$
\big\|Q_k(t)\big\|=\xi^{-k}e^{\lambda t}\big(\big\|Q_k(t)\big\|\xi^k\big)e^{-\lambda t}\le\xi^{-k}e^{\lambda t}\sup_{1\le i\le k}\big\|Q_i(t)\big\|_{(\lambda,\xi)}. \tag{35}
$$

Consequently, one obtains from (34) and (35)

$$
\big\|Q_k(t)\big\|\le\frac{\rho^{-N}e^{\lambda T}}{(1-\hat h)\xi^k}\big\|Q_N(t)\big\|_\lambda=\frac{r}{\xi^k}, \tag{36}
$$

where $r=\frac{\rho^{-N}e^{\lambda T}}{1-\hat h}\|Q_N(t)\|_\lambda$. It follows from $\xi>1$ and (36) that

$$
\lim_{k\to\infty}\big\|Q_k(t)\big\|=0,\qquad t\in[0,T]. \tag{37}
$$

Therefore, for all $t \in [0, T]$, we have

$$\lim_{k\to\infty} \Delta u_k(t) = 0, \qquad \lim_{k\to\infty} \mathcal{D}_t^\alpha e_k(t) = 0. \tag{38}$$

Furthermore, it follows from the initial conditions that $\{u_k(t)\}$ is uniformly robust convergent, and $\lim_{k\to\infty} y_k(t) = y_d(t)$. The proof is complete. □

Corollary 3.1 *For fractional-order linear time-delay system*

$$\begin{cases} \mathcal{D}_t^\alpha x_k(t) = Ax_k(t) + A_1 x_k(t-\tau) + Bu_k(t), \\ y_k(t) = Cx_k(t) + D\mathcal{D}_t^{-\alpha} u_k(t), \end{cases} \tag{39}$$

where $k \in \{0, 1, 2, \ldots\}, t \in [0, T], \alpha \in (0, 1)$, *and a given reference* $y_d(t)$, *suppose that* $y_d(0) = y_k(0)$ *and* $\rho\{G(t)\} \le \bar{\rho} < 1$, *then, for all* $t \in [0, T]$, *and arbitrary initial input satisfying* $u_{-1}(t) = u_0(t)$, *the second-order* $\mathcal{D}^\alpha$*-type ILC updating law*

$$u_{k+1}(t) = \Lambda u_k(t) + (1 - \Lambda)u_{k-1}(t) + \Gamma \mathcal{D}_t^\alpha e_k(t), \tag{40}$$

guarantees that $\{u_k(t)\}$ *is uniformly convergent, and* $\lim_{k\to\infty} y_k(t) = y_d(t)$.

Corollary 3.2 *For the fractional-order linear system*

$$\begin{cases} \mathcal{D}_t^\alpha x_k(t) = Ax_k(t) + Bu_k(t), \\ y_k(t) = Cx_k(t) + D\mathcal{D}_t^{-\alpha} u_k(t), \end{cases} \tag{41}$$

and a given reference $y_d(t)$, *suppose that* $y_d(0) = y_k(0)$ *and*

$$\rho\big(I - (CB + D)\Lambda\big) < 1, \tag{42}$$

then, for all $t \in [0, T]$, *and arbitrary initial input satisfying* $u_0(t)$, $\mathcal{D}^\alpha$*-type ILC updating law*

$$u_{k+1}(t) = u_k(t) + \Lambda \mathcal{D}_t^\alpha e_k(t), \tag{43}$$

guarantees that $\{u_k(t)\}$ *is uniformly convergent, and* $\lim_{k\to\infty} y_k(t) = y_d(t)$.

Remark 3.1 Note that the convergence analysis of ILC updating law (43) for fractional-order linear system (41) has been investigated in [4], in which the convergence condition is

$$\big\|I - (CB + D)\Lambda\big\| < 1. \tag{44}$$

Since $\rho(I - (CB + D)\Lambda) \le \|I - (CB + D)\Lambda\|$, the convergence condition (42) is less conservative than the condition (44).

4 Numerical Example

Consider the fractional-order linear time-delay system (39) with the Caputo derivative (fractional order $\alpha = 0.85$),

$$\begin{cases} A = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0.5 \end{bmatrix}, \\ B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0, \end{cases} \tag{45}$$

$t \in [0, 1]$, $\tau = 0.5$ and $\psi(t) = [0 \quad 1]^T$, $-0.5 \le t < 0$. Let the reference and external disturbance be

$$y_d(t) = \begin{bmatrix} 12t^2(1-t) \\ \sin(3\pi t) \end{bmatrix}, \qquad w_k(t) = [0.1 \sin t \quad 0.2 \cos t]^T, \quad t \in [0, 1],$$

respectively. We apply the second-order $\mathcal{D}^\alpha$-type ILC updating law

$$u_{k+1}(t) = 0.9u_k(t) + 0.1u_{k-1}(t) + (CB)^{-1}\mathcal{D}_t^\alpha e_k(t).$$

with the initial control be $u_{-1}(t) = u_0(t) = 0$. In this case, it can be calculated that $\rho(G) = 0.3702 < 1$. The simulation results are shown in Figs. 1, 2, and 3. Figures 1 and 2 show the system output $y_k(t)$ (solid) of the first five iterations and the referenced trajectory $y_d(t)$ (dotted), while Fig. 3 shows the 2-norm of the tracking errors in the first eight iterations. It can be seen that the output is capable of approaching the desired trajectory accurately within few iterations.

5 Concluding Remarks

In this paper, a high-order $\mathcal{D}^\alpha$-type ILC scheme for fractional-order nonlinear time-delay systems was investigated. By using the generalized Gronwall–Bellman Lemma, the convergence condition was derived. The validity of the proposed method was verified by a numerical example.

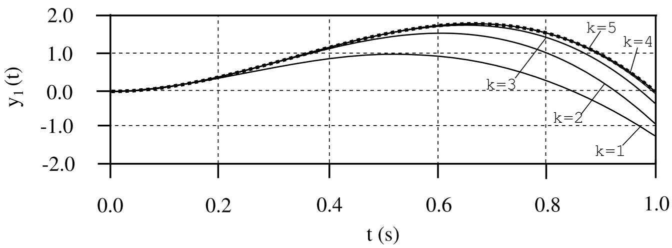

Fig. 1 The tracking performance of the system output ($y_1^k(t)$: *solid*, $y_1^d(t)$: *dotted*)

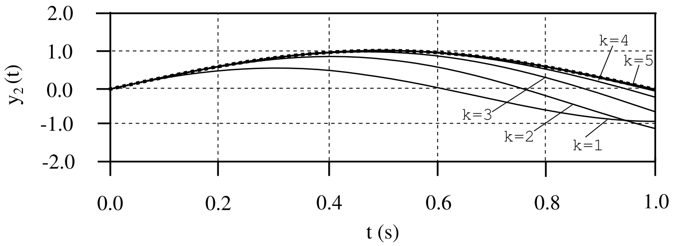

Fig. 2 The tracking performance of the system output ($y_2^k(t)$: *solid*, $y_2^d(t)$: *dotted*)

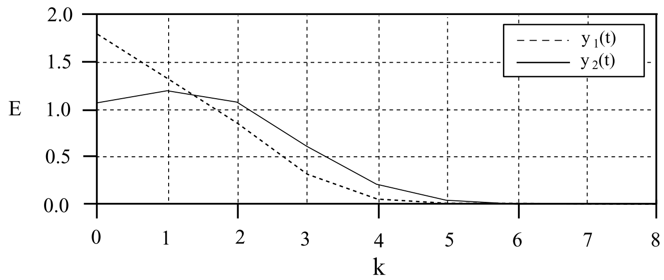

Fig. 3 The 2-norm of the tracking errors in each iteration

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