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A Survey on Fractional-Order Iterative Learning Control

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Abstract In this paper, an overview of fractional-order iterative learning control (FOILC) is presented including main developments of this field since 2001. Many theoretical and experimental results are provided to show the advantages of FOILC such as the improvement of transient and steady-state performances. Some unique characters of fractional-order operators are illustrated to show the new features and techniques of FOILC. A number of unsolved problems are briefly presented.

Keywords Iterative learning control · Fractional calculus · Convergence speed · Transient and steady-state performances · Digital and analog implementations

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1 Introduction

The formal concept of iterative learning control (ILC) was published in 1978 by Uchiyama (in Japanese) and in 1984 by Arimoto et al. (in English) [1]. Some earlier works of ILC can be traced back to 1967 in the US patent 3555252 "Learning control of actuators in control systems" and the multi-pass system analysis in 1974 by Edwards and Owens [2]. The ILC method is a batch process that operates a given objective system repeatedly on a fixed time interval so that the reference signal can be perfectly tracked as the operation repeats. Along with the global Lipschitz condition, the ILC schemes can be easily applied to both nonlinear and linear systems with less prior model knowledge [1–7]. Thus both the theory and applications of ILC gained increasing attention and have been highly developed in the past three decades [8–12]. Many interesting problems of ILC have been discussed in [13–22], and all of them are linked to the "repetition" of the control system, such as the batch process and periodic uncertainties and disturbances, etc. [5–7, 23, 24]. In simple words, an ILC scheme is to improve the system control performance by using the historical data even without a complete knowledge of the system to be controlled [3, 8, 9, 21, 22].

The FOILC is a relatively new topic in ILC, which can be traced back to 2001 [25]. In [25], a D^α-type fractional-order learning law is proposed and the convergence conditions are analyzed in frequency domain, which becomes an important technique in FOILC. The high-pass characteristic of the D^α term $((j\omega)^\alpha)$ is applied to compromise the low-pass characteristic of the controlled system. Allowing for the practical applications, the implementation of $(j\omega)^\alpha$ is discussed as well. Besides, given a manipulator model, it is verified that the optimal ILC scheme is pointed towards a fractional-order one [25]. Since then, many fractional-order ILC problems are presented aiming at enhancing the performance of ILC scheme for various systems [26–29]. Particularly, in recent years, a number of new questions have emerged in FOILC such as the time domain analysis, the applications to various fractional-order systems, the tuning and auto-tuning rules, the combination with robust feedback control strategies, etc. [2]. The strategy of FOILC addresses the fractional-order systems and/or the fractional-order learning laws. In earlier references, it is proved that the convergence conditions of FOILC are exactly the same with the corresponding ILC cases if the fractional-order system and the fractional-order learning law share the same fractional-order α [27]. The good news is that the FOILC can be directly applied to the more complicated fractional-order systems using the same convergence conditions. Many nice results are derived under the knowledge of α. However, some researchers point out that the above strategy may be impractical, because the accurate identification of the system order α is not easy in some circumstances. Thus the order-adaptive problems are raised, and several early discussions can be found in [12, 25]. The rejection of the order uncertainty or the order disturbance in both time and frequency domains become a specific topic in FOILC. But, as far as the authors know, the knowledge of adaptive control cannot be directly applied to these problems so that new control strategies should be further investigated in this field. Several papers and their references are cited here to illustrate some other recent developments of FOILC [27, 30]. It can be seen from the above literatures that the FOILC not only retains the advantages of the classical ILC but also offers potential for better performances in a variety of complex physical processes.

In this survey, a summary of FOILC in the past 12 years is organized as follows. Some mathematical preliminaries of fractional calculus are introduced in Sect. 2. The main idea of FOILC is generalized in Sect. 3, which is further divided into three parts: the linear time-invariant FOILC (Sect. 4), order-dependent FOILC (Sect. 5) and order-independent FOILC (Sect. 6). Some applications of FOILC are discussed with citations, and a number of unsolved problems for FOILC are presented briefly in Sect. 7. The conclusions are shown in Sect. 8.

2 Preliminaries

In this section, some mathematical preliminaries, such as Laplace transform, convolution, Riemann–Liouville and Caputo fractional-order operators, and Mittag–Leffler functions, are presented.

2.1 Laplace Transform

The Laplace transform of $f(t)$ is defined as [31]: $f(s) = L\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\,\mathrm{d}t$, where $f(t)$ is piecewise continuous on every finite interval in $[0, \infty)$ satisfying $|f(t)| \le Me^{at}$ for all $t \in [0, \infty[$, $s > a$ and a sufficiently large constant $M > 0$.

2.2 Convolution

The convolution defined according to the above Laplace transform is shown as [31]: $f(t) * g(t) = \int_{0^-}^{t} f(t-\tau)g(\tau)\,\mathrm{d}\tau = \int_{0^-}^{t} f(\tau)g(t-\tau)\,\mathrm{d}\tau$, where $f(t)$ and $g(t)$ are integrable functions on $[0^-, t]$.

2.3 Fractional Calculus

Fractional calculus plays an important role in modern sciences [29, 32, 33]. In this paper, we use both Riemann–Liouville and Caputo fractional operators. The definition of the Riemann–Liouville fractional integral with $\alpha \in]0, 1[$ is

$$ {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, $$

where $f(t)$ is an integrable function, ${}_{t_0}D_t^{-\alpha}$ is the fractional integral of order α on $[t_0, t]$, and $\Gamma(\cdot)$ denotes the Gamma function. For an arbitrary real number p, the Riemann–Liouville and Caputo fractional derivatives/integrals are respectively defined as

$$ {}_{t_0}^{RL}D_t^{p} f(t) = \frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} \left[{}_{t_0}D_t^{-([p]-p+1)} f(t) \right] $$

and

$$ {}_{t_0}^{C}D_t^{p} f(t) = {}_{t_0}D_t^{-([p]-p+1)} \left[\frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} f(t) \right], $$

where $[p]$ stands for the integer part of p, ${}^{RL}D$ and ${}^{C}D$ are Riemann–Liouville (RL) and Caputo (C) fractional operators, respectively.

2.4 Mittag–Leffler Function

The Mittag–Leffler function, which is the fundamental solution of fractional differential equations, has the following form: $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, where $\alpha > 0$ and $z \in \mathbb{C}^{n\times n}$. The Mittag–Leffler function in two parameters is defined as $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$, where $\alpha, \beta > 0$ and $z \in \mathbb{C}^{n\times n}$ [32].

Lemma 2.1 *For any $\alpha, \beta > 0$ and $A \in \mathbb{C}^{n\times n}$, $L\{t^{\beta-1}E_{\alpha,\beta}(At^\alpha)\} = s^{\alpha-\beta}(s^\alpha I_n - A)^{-1}$, where $|s| > \|A\|^{1/\alpha}$.*

Proof The proof can be found in [27, 32]. □

Lemma 2.2 *The following equivalent relationship holds for any $A \in \mathbb{C}^{n\times n}$, $t \in \mathbb{R}$ and $\alpha > 0$:*

$$E_\alpha\big(At^\alpha\big) = t^\alpha A E_{\alpha,1+\alpha}\big(At^\alpha\big) + I_n.$$

Proof By using the definition of Mittag–Leffler function, we have

$$t^\alpha E_{\alpha,1+\alpha}\big(At^\alpha\big)A + I_n = At^\alpha \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(k\alpha+1+\alpha)} + I_n = E_\alpha\big(At^\alpha\big).$$ □

3 Fractional-Order Iterative Learning Control

The fractional-order iterative learning control is dedicated to improve the transient and steady-state performances of ILC strategy for both integer order and fractional-order systems. It includes but is not limited to the applications of FOILC to integer order systems, and the applications of FOILC to fractional-order systems, etc. As a summary, a generalized FOILC scheme can be illustrated as follows.

Given a fractional-order nonlinear state-space system

$$x^{(\alpha)}(t) = f(t, x, u), \tag{1}$$

$$y(t) = g(t, x, u), \tag{2}$$

where all the variables and parameters have proper dimensions, the initial condition is well-defined, $\alpha \in]0, 1[$, and $(\cdot)$ denotes either the RL or the C fractional-order derivative. Moreover, it is always required that both f and g satisfy the global Lipschitz conditions with respect to u, and the uniqueness and existence conditions are guaranteed for the above system. When $\alpha = 1$, it is corresponding to the integer order systems.

The main objective of FOILC is to search the desired control input $u_d(t)$ by using the recursive algorithm

$$u_{k+1}(t) = \mathcal{L}\big\{u_k(t), e_i(t)\big\} \quad (k = 0, 1, 2, \ldots), \tag{3}$$

where $t \in [0, t_f]$, $i \in \{1, 2, \ldots, k+1\}$, $e_i(t) = y_d(t) - y_i(t)$, $y_d(t)$ is the desired system output and $\mathcal{L}$ is a linear or nonlinear operator, so that, if the system is repetitive and the convergence conditions are met, we will finally have

$$\lim_{k\to\infty} u_k(t) = u_d(t). \tag{4}$$

Remark 3.1 In the above discussions, if $i = k+1$, it means that the error of the current iteration is included in the scheme, i.e. the FOILC scheme is a combination of feedforward and feedback loops, which improves the robustness of the control system and guarantees the perfect tracking simultaneously. Various fractional-order operators can be included in $\mathcal{L}\{\cdot, \cdot\}$. Moreover, if information from more than one previous iterations $(u_k, u_{k-1}, \ldots)$ is included in (3), it is pointing towards the higher-order FOILC scheme, and two fundamental references are cited in here [34–36]. Lastly, the initial control input $u_0(t)$ can be arbitrarily chosen, theoretically. However, a better $u_0(t)$ will surely lead to a faster convergence speed. Therefore, the optimization of the initial control input is a practically important problem in FOILC.

4 Linear Time-Invariant FOILC

In this section, some theoretical methods of the classical ILC are extended to FOILC parallelly. Particularly, the linear time-invariant cases of FOILC can be analyzed by using the fractional-order linear operators and the fractional-order transfer functions.

Theorem 4.1 *For the system $y_k = T_s u_k$, and the linear time-invariant FOILC*

$$u_{k+1} = T_u u_k + T_e(y_d - y_k), \tag{5}$$

where T_s, T_u and T_e are all fractional-order linear operators, it can be proved that

$$\lim_{k\to\infty} u_k(t) = u_d(t) \quad \bigl(t \in [0, t_f]\bigr),$$

if

$$\|T_u - T_e T_s\|_{\text{op}} < 1, \tag{6}$$

where $u_d(t)$ is the desired control input, and $\|\cdot\|_{\text{op}}$ denotes the operator norm. Moreover, it can be proved that

$$\lim_{k\to\infty} e_k(t) = \bigl[I - T_s(I - T_u + T_e T_s)^{-1} T_e\bigr] y_d(t),$$

where $e_k(t) = y_d(t) - y_k(t)$.

Proof The proof is the same as the corresponding ILC cases by using the fixed-point theorem [37, 38]. □

Remark 4.1 It can be easily seen from Theorem 4.1 that, if $T_u = I$, we can finally arrive at $\lim_{k\to\infty} e_k(t) = 0$ for all $t \in [0, t_f]$. Otherwise, the error $\lim_{k\to\infty} e_k(t) \neq 0$ over the entire interval.

To date, most FOILC schemes are linear ones. Thus it is common to analyze it in frequency domain, even it is a finite time problem [19, 26]. Given a control system $G(s)$, the input–output relationship at the kth iteration can be described by

$$Y_k(s) = G(s)U_k(s), \tag{7}$$

where $y_k(0) = y_d(0)$, $y_d(t)$ being the desired output. Moreover, let the learning law in the Laplace domain be

$$U_{k+1}(s) = U_k(s) + \gamma H(s)E_k(s), \tag{8}$$

where γ is a scalar learning gain, and $H(s)$ is the learning compensator in the Laplace domain.[1] It follows from (7) and (8) that

$$E_{k+1}(s) = \big[1 - \gamma G(s)H(s)\big]E_k(s),$$

where the commutative property of γ and $G(s)$ should be satisfied in the Laplace domain. It can be proved that the convergence condition of the above ILC scheme is [39, 40]:

$$\big\| I - \gamma G(j\omega)H(j\omega)\big\|_\infty < 1, \tag{9}$$

where $s = j\omega$ and $\|\cdot\|_\infty$ denotes the H_∞-norm.

It should be noted that if γ is a time-varying constant or more generally an output of a dynamic system, (9) does not hold any more. Moreover, in reality, many systems cannot be fully recovered to the desired initial state $y_d(0)$, so that the perturbed initial conditions should be considered in these cases. Some further discussions will be shown in Sect. 7.

Remark 4.2 It can be seen from the operator and frequency domain discussion of FOILC that the classical linear ILC schemes are special cases of FOILC, i.e. the FOILC should have better, at least equivalent, performance than ILC, theoretically. Moreover, the convergence conditions of FOILC remain the same as ILC, which are shown in the previous discussions of this section. However, either the fractional-order system or the fractional operator makes the analysis of FOILC more complicated and therefore some novel methods and techniques should be explored.

5 Order-Dependent FOILC

Motivated by the above frequency domain results, it is straightforward to convert the FOILC schemes back to the time domain, which naturally leads to the convolution

[1]If either $G(s)$ or $H(s)$ is related to s^ν, where ν is a non-integer constant, then (8) becomes a FOILC learning law. Moreover, in this case $T_u = I$, so that $\lim_{k\to\infty} e_k(t) = 0$ for all $t \in [0, t_f]$.

operators. Particularly, the fractional-order RL and C operators defined by the convolutions play fundamental roles in FOILC. Most of FOILC schemes are P-, D^α-, PD^α- and $PI^\beta D^\alpha$-type FOILC schemes, etc. [28, 41–49], where D denotes either the RL or C fractional-order operator. These linear FOILC schemes have been successfully applied to the linear, nonlinear, affine and time-delay systems, etc. [28, 41–49].

For simplicity, let the fractional-order nonlinear system be

$$y^{(\alpha)}(t) = f\big(t, y(t), u(t)\big), \tag{10}$$

where $\alpha \in]0, 1[$, the initial conditions are properly defined, f is global Lipschitz with respect to u, and the uniqueness and existence conditions are satisfied. A typical first-order open-loop $PI^\beta D^\alpha$-type FOILC learning law can be written as

$$u_{k+1}(t) = u_k(t) + K_p e_k(t) + K_i e_k^{(-\beta)}(t) + K_d e_k^{(\alpha)}(t), \tag{11}$$

where $\beta \in]0, 1]$ and the α in (11) is exactly the same as the one in (10), i.e. it is system order dependant.

In [41, 46], the P-type ILC is applied to the fractional-order nonlinear time-delay systems by using the generalized Gronwall–Bellman lemma. However, the results deserve further study, because the convergence conditions are related to the final time t_f, and the learning speed may not be fast by using the proportional term only.

Fortunately, the above issues of P-type FOILC can be almost ignored in the D^α-type FOILC [25, 44, 47] and PD^α-type FOILC [28, 42, 45, 48, 49], which attracted much attention in FOILC. Especially, the PD^α-type FOILC has become a hot spot, because, for example, the convergence conditions of D^α-type and PD^α-type FOILC schemes are exactly the same as the corresponding ILC cases, and the P term can efficiently improve the transient performance of FOILC although it does not influence the convergence [47].

Several $PI^\beta D^\alpha$-type FOILC schemes are illustrated in [43]. Note that here, either the ILC or FOILC is typically a kind of integral along the iteration axis so that there is a trade-off of using or not using the I^β terms [50].

In the above-mentioned works, a commonly used relationship should be introduced regarding the λ-norm and fractional-order systems.

Lemma 5.1 *For the fractional-order nonlinear systems* (10) *and an arbitrary positive constant* $q > 1/\alpha$, *suppose* $\|\frac{\partial f}{\partial u}\|_\infty \le \gamma \|u\|_\infty$, *there exists a large enough* λ *satisfying*

$$\|e\|_\lambda \le O\big(\lambda^{-1/q}\big)\|\delta u\|_\lambda,$$

where

$$O\big(\lambda^{-1/q}\big) = \frac{\gamma(1 - e^{-q\lambda T})T^{\frac{q\alpha-1}{q}}\Gamma^{\frac{q-1}{q}}(\frac{q\alpha-1}{q-1})}{(q\lambda)^{1/q}\Gamma(\alpha) - c(1 - e^{-q\lambda T})T^{\frac{q\alpha-1}{q}}\Gamma^{\frac{q-1}{q}}(\frac{q\alpha-1}{q-1})}.$$

Proof A proof can be found in [47, 48]. □

Lastly, some time domain results of FOILC are summarized as follows. Firstly, the advantages of the time domain analysis of FOILC include but are not limited to:

- The P and D^{α} terms in the FOILC learning laws can be tuned to accelerate the convergence speed. Better knowledge of the system model can lead to better performance of the FOILC scheme.
- Given the system order α, the convergence condition of the PD^{α}-type FOILC is exactly the same with the corresponding ILC scheme, which extends the applications of FOILC to a great extent.

However, there are also some issues in the above time-dependent FOILC in applications:

- In the above-mentioned references, the system order α is supposed to be known in advance so that the FOILC can be designed according to the exact value of α. For example, suppose the system order is α, the order of a D^{α}-type FOILC is α as well. However, the accurate identification of α could be very hard, such as an irrational value identification. The misidentification of α may lead to bad performance, or even divergence, of the FOILC schemes [27].
- In some references [41, 46], the order-unknown FOILC is discussed for fractional-order nonlinear systems. Nevertheless, the final time t_f is included in the convergence conditions. Thus, it is still impractical to use such kind of FOILC scheme, although the system order can be unknown.
- One of the known issues of λ-norm is that the tracking errors can get worse in some iterations even if their λ-norms are monotonically convergent. This happens in both ILC and FOILC [51].

It should be highlighted that the α-dependent FOILC schemes are still very meaningful for many reasons. For instance, the exact knowledge of the fractional-order system provides a possible way to derive the optimal ILC scheme. Particularly, to our best knowledge, the FOILC always performs better than the conventional ILC for fractional-order systems [25, 27].

6 Order-Independent FOILC

The original motivations of the order-independent FOILC were from the first FOILC paper [25], in which a number of advantages of FOILC schemes are proved or illustrated. Besides, there are more tuning parameters in FOILC schemes, such as the fractional orders of the learning law, so that a better performance can be expected. Moreover, the FOILC has a larger learnable band in frequency domain, i.e. better performance can be arrived on certain range of frequencies [26, 52]. Some other results are summarized as follows.

6.1 Order Adaptation

The order-adaptive FOILC was first proposed in [25], and it is verified by using a single joint manipulator. It is shown that “the most notable improvement of FOILC is that the learning convergence gets more and more monotonic, and much faster convergence speed when fractional-order derivative (a D^γ-type FOILC) is applied."

A theoretical proof of the D^γ-type FOILC for state-space systems can be found in [27], in which the system order α is unknown and the order γ is an undetermined constant. It was shown that the fastest tracking speed is pointed towards $\alpha = \gamma$ in terms of the λ-norm. In [47], a non-smooth tracking reference is illustrated to show that the overshoots at the non-smooth points get larger even if the λ-norms monotonically decrease. Moreover, it was also shown that the D^γ-type FOILC may diverge for $\gamma > \alpha$, which partially explains the experimental results of [25].

Nevertheless, if other conditions are satisfied, $\gamma \le \alpha$ is sufficient but not necessary for the convergence of FOILC [25]. Besides, to allow for the non-smooth cases, a low-pass type FOILC is proposed in [30], which can filter the high frequency noises and guarantee the convergence despite the system order α. Both the digital and analog implementations of the FOILC compensators are considered in it as well. Particularly, the fractional-order RC circuits are shown to be an analog implementation of fractional-order learning laws [53].

6.2 Half-order FOILC

In [26], a half-order FOILC is proposed by applying a 0.5th order derivative in forward time and a −0.5th order integral in the backward time, which is equivalent to a unit-gain (all-pass) non-casual shifter. It has a wider learnable band to ensure higher tracking accuracy, and the learning speed will not be sacrificed at any frequency due to the unit-gain property.

6.3 Band-stop FOILC

It is well known that a low-pass type ILC compensator will lead to slow learning speed at high frequencies, and a D- or D^α-type ILC shows slow learning speed at low frequencies. Thus the previous unit-gain design has become an important technique in ILC [26]. For example, the combination of a low-pass filter and a differentiator can guarantee the learning speed at both low and high frequencies, i.e. the development of the band-stop type FOILC.

In [30], the classical $PI^\beta D^\alpha$-type FOILC is extended to a generalized one:

$$u_{k+1}(t) = u_k(t) + \tilde{\mathcal{L}}\{e_k(t)\}, \tag{12}$$

where $\tilde{\mathcal{L}}$ denotes an arbitrary bounded linear operator. It should be noted that, to date, almost all the single-term FOILC schemes without feedback terms are special cases of (12), and therefore the advantages of classical ILC schemes, and the order adaptiveness, etc., can be inherited. Particularly, it allows the design of FOILC with multi-terms of fractional-order compensators so that the low-pass and high-pass filters can be cascaded together [30, 49]. For example, let the FOILC learning law be

$$u_{k+1}(t) = u_k(t) + \Gamma\phi(t) * e^{(\alpha)}(t), \tag{13}$$

where

$$\phi(t)=\frac{a}{\lambda}t^{\beta-1}E_{\beta,\beta}\left(-\frac{at^{\beta}}{\lambda}\right)+\frac{\mathrm{d}}{\mathrm{d}t}E_{\tilde{\beta}}\left(-\frac{ct^{\tilde{\beta}}}{b}\right),$$

$\beta, \tilde{\beta} \in]0,1[$, and

$$\mathcal{L}\{\phi(t)\}=\frac{b}{b+\lambda s^{\beta}}+\frac{as^{\tilde{\beta}}}{c+as^{\tilde{\beta}}}.$$

Note that the above two terms are fractional-order low-pass and high-pass filters, respectively. Moreover, given arbitrary bounds of the system order $\alpha \in]0,1]$, the α in (13) can be absorbed in β and $\tilde{\beta}$ so that the order adaptiveness can be achieved as well.

Remark 6.1 Given the linear fractional-order systems

$$y^{(\beta)}(t)=-\frac{b}{\lambda}y(t)+\frac{b}{\lambda}x(t),$$
$$y^{(\tilde{\beta})}(t)=-\frac{c}{a}y(t)+x^{(\tilde{\beta})}(t),$$

it can be easily proved that the transfer functions of the above two equations are

$$Y(s)=\frac{b}{b+\lambda s^{\beta}}X(s),$$
$$Y(s)=\frac{as^{\tilde{\beta}}}{c+as^{\tilde{\beta}}}X(s).$$

Both of them can be easily realized by using the idea of the fractional-order element networks, or the fractional-order circuits [53]. On the other hand, the digital implementations of the above two equations can be found in [54].

7 Notes and Open Problems of FOILC

A number of interesting works in FOILC have been proposed since [25]. Nevertheless, for future developments, some notes and unsolved problems are briefly presented as follows.

- In many circumstances, the heredity (memory) of fractional-order system makes the system non-repeatable at the first several iterations, which is pretty like the precondition in material science.
- The initial condition of a fractional-order system is history-dependent, i.e. the whole history of the system can influence the current state. Thus, the formal Laplace transform can be replaced by the bilateral form $\int_{-\infty}^{+\infty} e^{-st} f(t)\,\mathrm{d}t$, or replaced by the Fourier transform. Several related references are [55–59].

- The optimization of the time-varying gains of FOILC. The trade-off here is that the accurate system identification is unnecessary for the ILC schemes; however, the precise knowledge of system parameters is indeed helpful to enhance the leaning speed. Thus, based on proper filtering methods, the time-varying FOILC based on the different filtering methods can be widely used to optimize the FOILC scheme.
- The order-adaptive FOILC-based iterative learning identification methods, especially the system order identification, will be meaningful works to improve the performances of FOILC and ILC schemes. Besides, the fractional-order iterative learning identification is also suitable for variable-order systems.
- The optimal initial control input and the iteration-varying reference case of a fractional-order system deserve special attention in practice.
- The discrete FOILC scheme will be widely used in the digital computers so that the users can benefit from the supervector formulation, and analyze and design the FOILC schemes systematically.
- The power-law and non-Gaussian (heavy-tail) phenomena are widely seen in reality. Fractional-order operators are expected to improve the performance of ILC schemes.
- The combination of partial differential equations and FOILC will be an interesting topic, and a possible application is the remote robotic surgery, where biomechanical systems should be modeled as fractional-order ones and the control tasks are usually repetitive. Besides, the combination of networked control with FOILC is also urgently required [60].
- The development of nonlinear FOILC learning laws can be considered if there are substantial advantages. The well-known homomorphic filtering technique is a possible way to be applied.
- New mathematical tools are required to extend the applicability of ILC and FOILC, such as the geometric and algebra methods, etc.
- The reliable implementation of FOILC is a key point for its widespread real world applications. Nowadays most of the digital implementations are actually the integer-order approximations in Z, Laplace or Fourier domain. Nevertheless, the wide existence of fractional-order elements [61, 62] can surely provide rich resources for the analog implementations of FOILC.
- The single-term FOILC schemes can be seen as special cases of the corresponding higher-order ones; however, it is still unclear how to conclude the advantages and applicability of using a single term or information from more than one previous iterations.

8 Conclusions

In this paper, the present status of FOILC since 2001 is summarized including the discussion of convergence analysis, transient and steady-state performances, and implementations, etc. Some open problems are briefly presented with a hope to attract attention from other fields as mathematics, engineering, science, medicine, etc.

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