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Optimal Control of Inclusion and Crack Shapes in Elastic Bodies

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Abstract The paper is concerned with the control of the shape of rigid and elastic inclusions and crack paths in elastic bodies. We provide the corresponding problem formulations and analyze the shape sensitivity of such inclusions and cracks with respect to different perturbations. Inequality type boundary conditions are imposed at the crack faces to provide a mutual nonpenetration between crack faces. Inclusion and crack shapes are considered as control functions and control objectives, respectively. The cost functional, which is based on the Griffith rupture criterion, characterizes the energy release rate and provides the shape sensitivity with respect to a change of the geometry. We prove an existence of optimal solutions.

Keywords Rigid inclusion · Inclusion shape · Crack · Nonpenetration condition · Variational inequality · Control problem

1 Introduction

Crashworthiness and resistance against damage and failure is the driving motivation in recent research on influencing the energy release rate in brittle composite mate-

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rials. In particular, in material sciences, one intends to improve such properties using elastic or rigid fibers or inclusions. The influence of the shapes and the material properties of inclusions or applied boundary forces on crack-tip sensitivities is a challenging mathematical problem that can be denoted as a crack-control problem. Some attempts have been made in the literature. See [1] which initialized this field of research in studying a distributed control problem for the Laplacian with a linear crack, that is, a crack where no nonpenetration condition is assumed to hold. The goal of that paper was to stop the crack propagation under the action of the control. In [2, 3], the problem of crack control has been treated with a nonpenetration condition along the crack and boundary controls. The authors of the very recent paper [4] consider the shape of inclusions with different material properties as controls, but take a linear crack model for a problem in conductivity. See also [5] for examples in mechanical engineering, where sensitivities are typically based on FEM-models. All articles mentioned are concerned with the reduction of the energy release rate. To the best knowledge of the authors, there is no published paper available, which is concerned with shape variations of rigid inclusions in order to influence the energy release rate associated with nonpenetrating cracks. This leads to a problem of shape-optimization in the context of variational inequalities; see [6] for an approach involving obstacles. The maximization of the energy release rate, rather its reduction, is important in some cases, where one wants to release as much energy as possible such that the material does not undergo a global crack. A first attempt toward optimization of the shape of inclusion with respect to maximizing the energy release rate have been reported in [7–11]. However, a rigorous mathematical treatment on the infinite dimensional level is still in its infancy. The corresponding analysis strongly depends on the mathematical modeling of cracks. It is known that classical crack problems in elasticity are characterized by linear boundary conditions imposed at the crack faces. Such a linear approach allows the opposite crack faces to penetrate each other which leads to inconsistency from the practical standpoint. In recent years, a crack modeling with nonpenetration conditions has been under active study. The corresponding theory is characterized by inequality type boundary conditions at the crack faces, and it leads to free boundary value problems. The book [12] contains results for crack models with the nonpenetration conditions for a wide class of constitutive laws. Existence theorems and qualitative properties of solutions in equilibrium problems for elastic bodies with rigid inclusions can be found in [13–18]. Elastic behavior of bodies with cracks is analyzed in the book [19]. As for a differentiability of energy functionals with respect to the crack length in elasticity, we can also mention the papers [20–22]. The problem of crack sensitivities with respect to length changes has been approached in the general setting of the speed method originally designed for shape-optimization problems. We refer the reader to publications [20–24]. In order to describe composite materials, it is necessary to analyze mathematical models of elastic bodies with rigid inclusions and cracks. In such a case, new types of boundary value problems and boundary conditions appear. Problems of the optimal choice of crack shapes and optimal choice of boundaries in elastic bodies with cracks are considered in [7, 8, 23–25].

The paper is organized as follows. In Sect. 2, we provide some basic notations and discuss the underlying equilibrium problem with crack γ. In Sect. 3, we introduce shape variations with respect to the rigid inclusion via the speed-method. We prove continuous dependence of the solution with respect to such shape variations. For a given shape of the rigid inclusion with support disjoint from the crack, we consider the shape derivative of the potential energy with respect to the crack length—for a straight crack in Sect. 4 and for curvilinear cracks in Sect. 5. This provides a formula for the energy release rate. We prove that the optimal control problem which consists in maximizing the energy release rate has a solution. Optimality conditions remain open. Finally, in Sect. 6, we relax the rigidity of the inclusion and discuss elastic inclusions which, as a stiffness parameter tends to zero, recovers the rigid situation. A numerical treatment of this approach is under way.

2 Formulation of the Equilibrium Problem

Let $\Omega \subset \mathbb{R}^2, \omega \subset \mathbb{R}^2$ be bounded domains with smooth boundaries $\Gamma, \partial\omega$, respectively, and let $\partial\omega \cap \Gamma = \emptyset, \omega \subset \Omega$. We assume that $\gamma_, \subset \Omega$ be a smooth curve without self-intersections such that $\bar{\gamma}_, \subset \Omega, \gamma_, \cap \bar{\omega} = \emptyset$. Denote by $n(x) = (n_1(x), n_2(x))$ the exterior unit normal vector to $\partial\omega$, $x \in \partial\omega$, by $\nu = (\nu_1, \nu_2)$ a smooth normal field to $\gamma_,$, and set $\Omega_\gamma = \Omega \setminus \bar{\gamma}_,$; see Fig. 1.

In the sequel, the domain Ω_γ represents a region filled with elastic material, containing the rigid inclusion ω and the crack $\gamma_,$. The latter is composed of the two crack faces $\gamma_,^+$ and $\gamma_,^-$. On $\Omega_{\gamma_,}$, we consider an elasticity problem for the displacement fields u on $\Omega_{\gamma_,}$ with nonlinear boundary conditions on the crack faces $\gamma_,^\pm$, which prevent the mutual penetration between $\gamma_,^\pm$. For the precise formulation, we keep in mind that the notion of a rigid inclusion ω allows only displacement fields $\rho =: u|_\omega$ in the set $R(\omega)$ of infinitesimal rigid displacements on ω, where

$$R(\omega) = \{\rho(x) := d(x_2, -x_1) + (c_1, c_2) \mid c_1, c_2, d \in \mathbb{R}, x \in \omega\}.$$

In what follows, displacements of the elastic body found at the crack faces $\gamma_,^+, \gamma_,^-$ are different in general. Hence, we denote by $[v] := v^+ - v^-$ the jump of a function v on $\gamma_,$, where $v^\pm$ is the trace (the boundary value) of the function v on the crack face $\gamma^\pm$, respectively.

Now the problem formulation for the body with elastic and rigid parts $\Omega_\gamma \setminus \bar{\omega}$ and ω, respectively, and the crack $\gamma_,$ reads as follows. For given external forces $f \in$

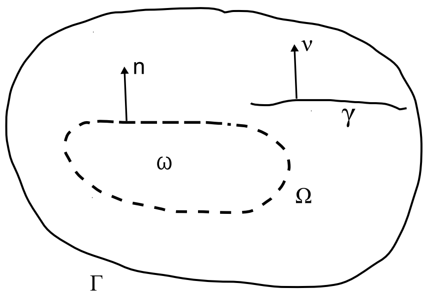

Fig. 1 Elastic body with rigid inclusion ω and crack $\gamma_,$

$C^1(\overline{\Omega})^2$ acting on the body Ω, we want to find a displacement field $u = (u_1, u_2)$ defined on Ω_γ, together with an infinitesimal rigid motion $\rho_0 \in R(\omega)$ such that

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega_{\gamma_,} \setminus \bar{\omega}, \tag{1}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_{\gamma_,} \setminus \bar{\omega}, \tag{2}$$

$$u = \rho_0 \quad \text{in } \omega, \tag{3}$$

$$u = 0 \quad \text{on } \Gamma, \tag{4}$$

$$-\int_{\partial\omega} \sigma n \cdot \rho = \int_\omega f \cdot \rho \quad \forall \rho \in R(\omega), \tag{5}$$

$$[u]\nu \geq 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \cdot [u]\nu = 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0 \text{ on } \gamma_,. \tag{6}$$

Here, $u = (u_1, u_2)$ is the displacement field in Ω_γ, $\sigma = \{\sigma_{ij}\}$ is the stress tensor; $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the strain tensor: $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), i, j = 1, 2$; and $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$, is a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad i, j, k, l = 1, 2, a_{ijkl} = \text{const},$$

$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2 \quad \forall \xi_{ji} = \xi_{ij}, c_0 = \text{const} > 0.$$

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices.

Relation (1) is the equilibrium equation, and (2) represents the Hooke's law; $\sigma = \sigma(u)$ is defined from (2); $\sigma n = (\sigma_{1j}n_j, \sigma_{2j}n_j)$, $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$; $f \cdot v = f_i v_i$.

The boundary conditions (6) imposed at the crack faces were analyzed in many works; see, for example, [12, 19]. In particular, the first condition of (6) provides a mutual non-penetration between the crack faces. As for the boundary condition (5), see [14, 25].

In order to provide a variational formulation describing an equilibrium state for the described structure with the rigid inclusion ω and the crack $\gamma_,$ we introduce the Sobolev space

$$H^1_\Gamma(\Omega_{\gamma_,}) = \left\{v \in H^1(\Omega_{\gamma_,}) \mid v = 0 \text{ on } \Gamma\right\}$$

and the set K of admissible displacements,

$$K := \left\{v \in H^1_\Gamma(\Omega_\gamma)^2 \mid [v]\nu \geq 0 \text{ on } \gamma_,;\ v|_\omega \in R(\omega)\right\}.$$

Let

$$\Pi(v) := \frac{1}{2}\int_{\Omega_\gamma \setminus \bar{\omega}} \sigma(v) : \varepsilon(v) - \int_{\Omega_\gamma} f \cdot v$$

be the energy functional. Here, $\sigma(v):\varepsilon(v):=\sigma_{ij}(v)\varepsilon_{ij}(v)$. Consider the minimization problem:

$$\text{Find} \quad u\in K \quad \text{s.t.} \quad \Pi(u)=\inf_{v\in K}\Pi(v). \tag{7}$$

This problem has a unique solution satisfying the following variational inequality, i.e., there exists a unique

$$u\in K, \tag{8}$$

$$\int_{\Omega_\gamma\setminus\bar{\omega}}\sigma(u):\varepsilon(v-u)-\int_{\Omega_\gamma} f\cdot(v-u)\geq 0 \quad \forall v\in K. \tag{9}$$

Note that in the first integral of (9) we can, in fact, integrate over $\Omega_{\gamma,}$ formally assuming that the Hooke's law (2) holds in ω. We should also remark that the solution of (8)–(9) exists also for $f=(f_1,f_2)\in L^2(\Omega)$. The additional regularity of f will be used in the further analysis.

Problem formulations (1)–(6) and (8)–(9) are equivalent, which means that we can derive (8)–(9) from (1)–(6), and conversely, (1)–(6) follows from (8)–(9), provided that the solutions are smooth enough.

3 Shape Perturbation for the Rigid Inclusion

In this section, we are going to prove the continuous dependence of the solution to the problem (1)–(6) on the shape of the rigid inclusion ω. To this end and for the sake of simplicity, consider a function $\xi\in H_0^2(0,1)$, and assume that a part of the boundary $\partial\omega$ be described as a graph of the function $x_2=\lambda\xi(x_1)$, $x_1\in\,]0,1[$, with a small parameter λ. The domain ω corresponding to the parameter λ is denoted by ω^λ, with corresponding boundary $\partial\omega^\lambda$. Furthermore, we require that $\gamma\cap\bar{\omega}^\lambda=\emptyset$ for $|\lambda|\leq\lambda_0$. We aim at proving the continuity of the mapping $\lambda\to u^\lambda$, where u^λ is the displacement field for the problem (1)–(6), with ω replaced with ω^λ, for a given parameter λ. First, we formulate the equilibrium problem for a given parameter λ. We have to find functions $u^\lambda=(u_1^\lambda,u_2^\lambda)$, $\rho_0^\lambda\in R(\omega^\lambda)$, $\sigma^\lambda=\{\sigma_{ij}^\lambda\}$, $i,j=1,2$, defined, respectively, in $\Omega_{\gamma,}\,\omega^\lambda$, $\Omega_\gamma\setminus\bar{\omega}^\lambda$ such that

$$-\operatorname{div}\sigma^\lambda=f \quad \text{in } \Omega_{\gamma,}\setminus\bar{\omega}^\lambda, \tag{10}$$

$$\sigma^\lambda-A\varepsilon\big(u^\lambda\big)=0 \quad \text{in } \Omega_{\gamma,}\setminus\bar{\omega}^\lambda, \tag{11}$$

$$u^\lambda=\rho_0^\lambda \quad \text{in } \omega; \qquad u^\lambda=0 \quad \text{on } \Gamma, \tag{12}$$

$$-\int_{\partial\omega^\lambda}\sigma^\lambda n^\lambda\cdot\rho=\int_{\omega^\lambda} f\cdot\rho \quad \forall\rho\in R\big(\omega^\lambda\big), \tag{13}$$

$$\big[u^\lambda\big]\nu\geq 0,\ \big[\sigma_\nu^\lambda\big]=0,\ \sigma_\nu^\lambda\leq 0,\ \sigma_\tau^\lambda=0,\ \sigma_\nu^\lambda\cdot\big[u^\lambda\big]\nu=0 \text{ on } \gamma_,, \tag{14}$$

where n^λ is the exterior unit normal vector to $\partial\omega^\lambda$. The problem (10)–(14), for a given λ, can be written in variational form again. The set of admissible displacements is now

$$K^\lambda = \left\{v \in H^1_\Gamma(\Omega_\gamma)^2 \mid [v]\nu \geq 0 \text{ on } \gamma,; v|_{\omega^\lambda} \in R\left(\omega^\lambda\right)\right\},$$

and we consider the minimization problem

$$\text{Find} \quad u \in K^\lambda \quad \text{s.t.} \quad \left\{\frac{1}{2}\int_{\Omega_\gamma \setminus \bar{\omega}^\lambda} \sigma(v) : \varepsilon(v) - \int_{\Omega_\gamma} f \cdot v\right\} \to \min. \tag{15}$$

As mentioned above, we have

$$\int_{\Omega_\gamma \setminus \bar{\omega}^\lambda} \sigma(v) : \varepsilon(v) = \int_{\Omega_\gamma} \sigma(v) : \varepsilon(v) \quad \text{for } v \in K^\lambda,$$

the minimization problem (15) differs from (7) just by the set of admissible functions. Problem (15) again has a unique solution u^λ satisfying the variational inequality

$$u^\lambda \in K^\lambda, \tag{16}$$

$$\int_{\Omega_\gamma \setminus \bar{\omega}^\lambda} \sigma\left(u^\lambda\right) : \varepsilon\left(v - u^\lambda\right) - \int_{\Omega_\gamma} f \cdot \left(v - u^\lambda\right) \geq 0 \quad \forall v \in K^\lambda. \tag{17}$$

Note that the inequality (17) implies

$$\int_{\Omega_\gamma} \sigma\left(u^\lambda\right) : \varepsilon\left(u^\lambda\right) = \int_{\Omega_\gamma} f \cdot u^\lambda, \tag{18}$$

and consequently, we have the following estimate which holds uniformly in λ:

$$\left\|u^\lambda\right\|_{H^1_\Gamma(\Omega_\gamma)^2} \leq c. \tag{19}$$

Hence, if $\lambda_n \to 0$ for some sequence (λ_n), we obtain $u_0 \in H^1_\Gamma(\Omega_\gamma)$ such that

$$u^{\lambda_n} \to u_0 \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2 \tag{20}$$

for a suitable subsequence again denoted by u^{λ_n}. It will turn out later (see Theorem 3.1) that u_0 is the unique solution to the variational inequality (8), hence (9) is independent of the approximating sequence λ_n.

Now we consider a transformation of the independent variables in Ω_γ:

$$\begin{aligned} &y = x + \lambda V(x), \quad x, y \in \Omega_\gamma, \\ &V = \left(V^1, V^2\right), \quad V^1(x) = 0,\ V^2(x) = \xi(x_1)\theta(x), \end{aligned} \tag{21}$$

where $\theta(x)$ is a smooth function with a compact support in Ω_γ, disjoint from a neighborhood of the crack, equal to 1 in a small neighborhood of the graph

$x_2 = \lambda\xi(x_1)$, $|\lambda| \le \lambda_0$. In this case, we assume $\xi(x_1) = 0$ outside of $[0, 1]$. The transformation (21) can be written in the form

$$y = \Phi(\lambda, x), \quad x, y \in \Omega_\gamma; \qquad \Phi = \left(\Phi^1, \Phi^2\right). \tag{22}$$

Clearly, for a small λ, the transformation (22) is one-to-one. The inverse to (22) mapping is denoted by

$$x = \Psi(\lambda, y), \quad x, y \in \Omega_\gamma; \qquad \Psi = \left(\Psi^1, \Psi^2\right). \tag{23}$$

It is convenient to introduce notations for the transformed stress and strain tensors. For a given $w \in H^1_\Gamma(\Omega_\gamma)^2$ we put

$$\Sigma_{ij}(\Psi; w) := a_{ijkl} E_{kl}(\Psi; w),$$

$$E_{ij}(\Psi; w) := \frac{1}{2}\left(w_{i,k}\Psi^k_{,j} + w_{j,k}\Psi^k_{,i}\right), \quad i, j = 1, 2.$$

By $J(\lambda)$, we denote the Jacobian of the mapping $x \to \Phi(\lambda, x)$,

$$J(\lambda) = \left|\frac{\partial\Phi(\lambda, x)}{\partial x}\right|.$$

From (21), (23), it follows

$$\Psi(\lambda, y) = y - \lambda V(y) + r(\lambda), \qquad \left\|r(\lambda)\right\|_{W^{1,\infty}(\Omega_\gamma)^2} = o(\lambda). \tag{24}$$

Also, by (21), the following expansions hold:

$$J(\lambda) = 1 + \lambda\,\mathrm{div}(V), \tag{25}$$

$$f_i\left(\Phi(\lambda)\right) = f_i + \lambda f_{i,j} V^j + r^i_1(\lambda),$$
$$\left\|r^i_1(\lambda)\right\|_{L^2(\Omega_{\gamma,})} = o(\lambda), \quad i = 1, 2. \tag{26}$$

Denote $u_\lambda(x) = u^\lambda(\Phi(\lambda, x))$, $x \in \Omega_{\gamma,}$ and change the independent variables in (16), (17). This yields

$$u_\lambda \in K_\lambda, \tag{27}$$

$$\int_{\Omega_\gamma\setminus\bar{\omega}^0} J(\lambda)\Sigma_{ij}\left(\Psi(\lambda); u_\lambda\right)E_{ij}\left(\Psi(\lambda); \bar{v}_\lambda - u_\lambda\right)$$
$$\ge \int_{\Omega_\gamma} J(\lambda) f_i\left(\Phi(\lambda)\right)(\bar{v}_{\lambda i} - u_{\lambda i}) \quad \forall \bar{v}_\lambda \in K_\lambda. \tag{28}$$

Here, K_λ is the image of K^λ under the transformation $v \to v \circ \Phi(\lambda)$. Notice that $K^0 = K_0$. Due to (24), for a given $w \in H^1_\Gamma(\Omega_\gamma)^2$, we have

$$\begin{aligned}
&E_{ij}\big(\Psi(\lambda); w\big) = \varepsilon_{ij}(w) - \lambda E_{ij}(V; w) + r_2^{ij}(\lambda, w),\\
&\big\|r_2^{ij}(\lambda, w)\big\|_{L^2(\Omega_\gamma)} \le h_1^{ij}(\lambda)\|w\|_{H^1_\Gamma(\Omega_\gamma)^2},\\
&0 \le h_1^{ij}(\lambda) = o(\lambda), \quad i, j = 1, 2.
\end{aligned} \tag{29}$$

Moreover, by (25), (29), for given $w, \bar{w} \in H^1_\Gamma(\Omega_\gamma)^2$ the following expansion holds:

$$\begin{aligned}
&\int_{\Omega_\gamma \setminus \bar{\omega}^0} J(\lambda)\Sigma_{ij}\big(\Psi(\lambda); w\big)E_{ij}\big(\Psi(\lambda); \bar{w}\big)\\
&\quad = \int_{\Omega_\gamma \setminus \bar{\omega}^0} \big\{\sigma_{ij}(w)\varepsilon_{ij}(\bar{w}) + \lambda S(V; w, \bar{w}) + r_3(\lambda, w, \bar{w})\big\},\\
&\big\|r_3(\lambda, w, \bar{w})\big\|_{L^1(\Omega_\gamma)} \le h_2(\lambda)\|w\|_{H^1_\Gamma(\Omega_\gamma)^2}\|\bar{w}\|_{H^1_\Gamma(\Omega_\gamma)^2}, \quad 0 \le h_2(\lambda) = o(\lambda),
\end{aligned} \tag{30}$$

with

$$S(V; w, \bar{w}) = \operatorname{div} V \cdot \sigma_{ij}(w)\varepsilon_{ij}(\bar{w}) - \sigma_{ij}(w)E_{ij}(V; \bar{w}) - \sigma_{ij}(\bar{w})E_{ij}(V; w).$$

By the estimate (19), we have uniformly in λ

$$\|u_\lambda\|_{H^1_\Gamma(\Omega_\gamma)^2} \le c.$$

Choosing the same sequence of parameters λ_n as in (20) we obtain[1]

$$u_{\lambda_n} \to u_0 \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2. \tag{31}$$

We use the expansion (30) in (28) and take into account Lemma 3.1 proved below. According to this lemma, the test functions in (28) can be taken as

$$\bar{v}_\lambda = \bar{v} + w_\lambda, w_\lambda \to 0 \quad \text{strongly in } H^1_\Gamma(\Omega_\gamma)^2,$$

for any given $\bar{v} \in K_0$. This allows us to pass to the limit in (28), as $\lambda_n \to 0$, by using (31) which implies

$$u_0 \in K_0, \tag{32}$$

$$\int_{\Omega_\gamma \setminus \bar{\omega}^0} \sigma(u_0) : \varepsilon(\bar{v} - u_0) - \int_{\Omega_\gamma} f \cdot (\bar{v} - u_0) \ge 0 \quad \forall \bar{v} \in K_0. \tag{33}$$

[1] Taking the same sequence of parameters λ_n as in (20), one can easily see that the weak limits in (20) and (31) coincide.

Thus, u_0 solves the original problem. In order to see the inclusion (32) we note that $u^\lambda|_{\omega^\lambda} \in R(\omega^\lambda)$ and by (31), $u^{\lambda_n} \to u_0$ a.e. in Ω_γ. Thus $u_\lambda \to \rho_0$ a.e. in ω^0 with $\rho_0 \in R(\omega^0)$, which implies $u_0 \in K_0$. Hence, $u_0 = u$ is the unique solution of the minimization problem (8), (9), and we may as well assume

$$u^\lambda \to u_0, \qquad u_\lambda \to u_0 \quad \text{weakly in } H^1_\Gamma(\Omega_{\gamma,})^2, \text{ as } \lambda \to 0.$$

In fact, we can prove strong convergence of u^λ to u_0. Indeed, inequality (33) yields

$$\int_{\Omega_\gamma} \sigma(u_0) : \varepsilon(u_0) = \int_{\Omega_\gamma} f \cdot u_0. \tag{34}$$

Consequently, by (18), (20), (34), we have

$$\int_{\Omega_\gamma} \sigma\big(u^\lambda\big) : \varepsilon\big(u^\lambda\big) \to \int_{\Omega_\gamma} \sigma(u_0) : \varepsilon(u_0)$$

which implies

$$\big\|u^\lambda\big\|_{H^1_\Gamma(\Omega_\gamma)^2} \to \|u_0\|_{H^1_\Gamma(\Omega_\gamma)^2}.$$

Thus, the strong convergence of u^λ to u_0 in the space $H^1_\Gamma(\Omega_\gamma)^2$ follows. Therefore, we have proved the following result.

Theorem 3.1 *Let u^λ, u_0 be the solutions of the minimization problems* (16), (17), *and* (8), (9), *respectively. Then u^λ converges to u_0 strongly in $H^1_\Gamma(\Omega_{\gamma,})^2$ as $\lambda \to 0$.*

Now we should establish the statement used in the proof of Theorem 3.1

Lemma 3.1 *For any fixed $\bar{v} \in K_0$ there exists a sequence $\bar{v}_\lambda \in K_\lambda$ such that, as $\lambda \to 0$,*

$$\bar{v}_\lambda \to \bar{v} \quad \textit{strongly in } H^1_\Gamma(\Omega_\gamma)^2.$$

Proof We take any fixed $\bar{v} \in K_0$, thus $\bar{v} = \rho$ on ω^0; $\rho \in R(\omega^0)$. Consider domains ω^λ, $|\lambda| \le \lambda_0$. Next, we choose $w_0 \in H^1_\Gamma(\Omega_\gamma)^2$ such that $w_0 = \rho$ in ω^λ, $|\lambda| \le \lambda_0$, $[w_0]\nu \ge 0$ on $\gamma_,$. Hence,

$$\bar{v} - w_0 = 0 \quad \text{in } \omega^0.$$

Clearly, there exists a sequence $\bar{w}_\lambda \in H^1_\Gamma(\Omega_\gamma)^2$ such that $\bar{w}_\lambda = 0$ in ω^λ, $|\lambda| \le \lambda_0$, $[\bar{w}_\lambda]\nu \ge 0$ on $\gamma_,$, and

$$\bar{w}_\lambda \to \bar{v} - w_0 \quad \text{strongly in } H^1_\Gamma(\Omega_\gamma)^2.$$

We set $\bar{v}^\lambda = \bar{w}_\lambda + w_0$; then we have

$$\bar{v}^\lambda \to \bar{v} \quad \text{strongly in } H^1_\Gamma(\Omega_\gamma)^2,$$

and moreover $\bar{v}^\lambda = \rho$ on ω^λ, as $|\lambda| \le \lambda_0$. Then for the functions $\bar{v}_\lambda(x) = \bar{v}^\lambda(\Phi(\lambda, x))$, we obtain as $\lambda \to 0$

$$\bar{v}_\lambda \to \bar{v} \quad \text{strongly in } H^1_\Gamma(\Omega_\gamma)^2,$$

and the proof is complete since $\bar{v}_\lambda \in K_\lambda$ by the definition. □

Remark 3.1 We remark that the choice of the perturbation of the set ω is taken for the sake of simplicity. The more general case of changing the entire boundary of ω can be handled similarly.

4 Optimal Control of the Rigid Inclusion Shape

For the sake of simplicity, in this section, we assume that the crack $\gamma_{,}$ be rectilinear, i.e., $\gamma_{,} =]1, 2[\times\{2\}$. Instead of rectilinear crack $\gamma_{,}$ we can consider a smooth curvilinear crack $\gamma_{,}$ assuming that $\gamma_{,}$ is located outside of ω. In this case, we consider a crack extension of $\gamma_{,}$ along the curvilinear path. A formula for the derivative of the energy functional would contain an additional term, linear with respect to the solution u; see [21, 22]. which allows us to provide all the arguments of this section. Hence, a statement like Theorem 4.1 would be valid again.

Let $\Xi \subset H^2_0(0,1)$ be a bounded and weakly closed set. For any $\xi \in \Xi$, a part of the boundary of the rigid inclusion is described as the graph of the function $x_2 = \xi(x_1)$, $x_1 \in]0, 1[$. For this particular function ξ, the domain ω is denoted by ω^ξ, and its boundary is denoted by $\partial\omega^\xi$. Due to our assumptions,

$$\gamma_{,} \cap \big\{(x_1, x_2) \mid x_2 = \xi(x_1), \xi \in \Xi\big\} = \emptyset.$$

First, we provide a formulation of the equilibrium problem. For any $\xi \in \Xi$, it is necessary to find functions $u = u(\xi)$, $\rho_0 = \rho_0(\xi) \in R(\omega^\xi)$, $\sigma = \sigma(\xi)$, defined in $\Omega_{\gamma_{,}} \omega^\xi$, $\Omega_\gamma \setminus \bar{\omega}^\xi$, respectively, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_{\gamma_{,}} \setminus \bar{\omega}^\xi, \tag{35}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_{\gamma_{,}} \setminus \bar{\omega}^\xi, \tag{36}$$

$$u = \rho_0 \quad \text{in } \omega^\xi; \qquad u = 0 \quad \text{on } \Gamma, \tag{37}$$

$$-\int_{\partial\omega^\xi} \sigma n^\xi \cdot \rho = \int_{\omega^\xi} f \cdot \rho \quad \forall \rho \in R\big(\omega^\xi\big), \tag{38}$$

$$[u]\nu \ge 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \cdot [u]\nu = 0,\ \sigma_\nu \le 0,\ \sigma_\tau = 0 \text{ on } \gamma_{,}, \tag{39}$$

where n^ξ is the unit external normal vector to $\partial\omega^\xi$. A solution of the problem (35)–(39) exists. Namely, denote

$$K^\xi = \big\{v \in H^1_\Gamma(\Omega_\gamma)^2 \mid [v]\nu \ge 0 \text{ on } \gamma_{,};\ v|_{\omega^\xi} \in R\big(\omega^\xi\big)\big\}.$$

Then the solution of the problem (35)–(39) satisfies the variational inequality

$$u \in K^{\xi}, \tag{40}$$

$$\int_{\Omega_\gamma \setminus \bar{\omega}^{\xi}} \sigma(u) : \varepsilon(v-u) - \int_{\Omega_\gamma} f \cdot (v-u) \geq 0 \quad \forall v \in K^{\xi}. \tag{41}$$

In order to define a cost functional, we consider the problem perturbed with respect to (35)–(39) and introduce the perturbed crack $\gamma_,^{\delta} =]1, 2+\delta[\times \{2\}$ with a small parameter δ. Then the formulation of the perturbed problem for a given ξ is as follows. We have to find functions $u^{\delta} = u^{\delta}(\xi)$, $\rho_0^{\delta} = \rho_0^{\delta}(\xi) \in R(\omega^{\xi})$, $\sigma^{\delta} = \sigma^{\delta}(\xi)$, defined in $\Omega_{\gamma_,^{\delta}}$, ω^{ξ}, $\Omega_{\gamma_,^{\delta}} \setminus \bar{\omega}^{\xi}$, respectively, such that

$$-\operatorname{div} \sigma^{\delta} = f \quad \text{in } \Omega_{\gamma_,^{\delta}} \setminus \bar{\omega}^{\xi}, \tag{42}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_{\gamma_,^{\delta}} \setminus \bar{\omega}^{\xi}, \tag{43}$$

$$u^{\delta} = \rho_0^{\delta} \quad \text{in } \omega^{\xi}; \qquad u^{\delta} = 0 \quad \text{on } \Gamma, \tag{44}$$

$$-\int_{\partial \omega^{\xi}} \sigma^{\delta} n^{\xi} \cdot \rho = \int_{\omega^{\xi}} f \cdot \rho \quad \forall \rho \in R(\omega^{\xi}), \tag{45}$$

$$[u^{\delta}]\nu \geq 0,\ [\sigma_\nu^{\delta}] = 0,\ \sigma_\nu^{\delta} \cdot [u^{\delta}]\nu = 0,\ \sigma_\nu^{\delta} \leq 0,\ \sigma_\tau^{\delta} = 0 \text{ on } \gamma_,^{\delta}. \tag{46}$$

Here, $\Omega_{\gamma_,^{\delta}} = \Omega \setminus \bar{\gamma}_,^{\delta}$. A solution of the problem (42)–(46) exists and is unique. Thus, for any small δ, we can consider the energy functional

$$\Pi(u^{\delta}; \xi) = \frac{1}{2} \int_{\Omega_{\gamma_,^{\delta}} \setminus \bar{\omega}^{\xi}} \sigma(u^{\delta}) : \varepsilon(u^{\delta}) - \int_{\Omega_{\gamma_,^{\delta}}} f \cdot u^{\delta}.$$

Using the technique developed in [12, 19, 20], a derivative of the energy functional $\Pi(u^{\delta}; \xi)$ with respect to δ can be found, i.e.,

$$G(\xi) = \frac{d}{d\delta} \Pi(u^{\delta}; \xi)|_{\delta=0},$$

and moreover the following formula holds true:

$$G(\xi) = \int_{\Omega_{\gamma_,}} \left\{ \frac{1}{2} \sigma_{ij}(u)\varepsilon_{ij}(u)\eta_{,1} - \sigma_{ij}(u)u_{i,1}\eta_{,j} \right\} - \int_{\Omega_{\gamma_,}} (\eta f_i)_{,1} u_i, \tag{47}$$

where η is an arbitrary smooth function such that $\eta = 1$ in a neighborhood of the point $(2, 2)$, and $\eta = 0$ outside of a neighborhood of the point $(2, 2)$. Note that there is no dependence of the derivative (47) on the choice of η with the prescribed properties, and moreover, $G(\xi) \leq 0$ for all $\xi \in \Xi$. Indeed, increasing the crack length, we decrease the energy, and conversely, decreasing the crack length means increasing the energy, thus $G(\xi) \leq 0$; see details in [12, 19]. The formula (47) provides the energy release rate.

By choosing different functions $\xi \in \Xi$ and changing, therefore, the shape of the rigid inclusion ω^ξ, we can consider $G(\xi)$ as a cost functional being responsible for the crack propagation from the standpoint of the Griffith criterion. Recall that the Griffith criterion characterizes the stable and unstable behavior of cracks in terms of derivatives of energy functionals with respect to the crack length. Namely, if the derivative reaches a critical value (a given material parameter) then the crack propagates. Otherwise the crack is stable. In this section, we analyze an optimal control problem with rigid inclusion shapes ξ being control functions. For any given function ξ, we find the derivative (47) of the energy functional with respect to the crack length, and aim at maximizing this derivative over the set Ξ. Thus, we consider the optimal control problem:

$$\text{Find} \quad \xi_0 \in \Xi \quad \text{s.t.} \quad G(\xi_0) = \sup_{\xi \in \Xi} G(\xi). \tag{48}$$

Theorem 4.1 *There exists a solution of the optimal control problem* (48).

Proof Let $\xi^n \in \Xi$ be a maximizing sequence. Due to the boundedness of Ξ and classical imbedding theorems, we can assume that as $n \to \infty$

$$\begin{aligned} &\xi^n \to \xi \quad \text{weakly in } H_0^2(0,1), \quad \xi \in \Xi, \text{ as } n \to \infty, \\ &\xi^n \to \xi \quad \text{strongly in } C^1[0,1], \quad \left|\xi^n_{x_1} - \xi_{x_1}\right| < \frac{1}{n} \text{ on }]0,1[. \end{aligned}$$

For any n, we can find a solution $u(\xi^n)$ of the problem (35)–(39) satisfying the variational inequality. Put $u^n = u(\xi^n)$. Then

$$u^n \in K^{\xi^n}, \tag{49}$$

$$\int_{\Omega_\gamma \setminus \bar{\omega}^{\xi^n}} \sigma\left(u^n\right) : \varepsilon\left(v - u^n\right) - \int_{\Omega_\gamma} f \cdot \left(v - u^n\right) \geq 0 \quad \forall v \in K^{\xi^n} \tag{50}$$

with the set of admissible displacements

$$K^{\xi^n} = \left\{v \in H^1_\Gamma(\Omega_\gamma)^2 \mid [v]\nu \geq 0 \text{ on } \gamma,; v|_{\omega^{\xi^n}} \in R\left(\omega^{\xi^n}\right)\right\}.$$

From (50), it follows

$$\int_{\Omega_\gamma} \sigma\left(u^n\right) : \varepsilon\left(u^n\right) = \int_{\Omega_\gamma} f \cdot u^n, \tag{51}$$

thus, uniformly in n,

$$\left\|u^n\right\|_{H^1_\Gamma(\Omega_\gamma)^2} \leq c.$$

We can assume that as $n \to \infty$

$$u^n \to u \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2. \tag{52}$$

Like in the previous section, the transformation of the independent variables of the following form:

$$\begin{cases} y_1 = x_1, \\ y_2 = x_2 + \theta(x)(\xi^n(x_1) - \xi(x_1)), \end{cases} \tag{53}$$

can be performed in (49), (50), where $x = (x_1, x_2) \in \Omega_{\gamma,} y = (y_1, y_2) \in \Omega_\gamma$. A smooth function θ with compact support in Ω_γ is chosen in such a way that $\theta = 1$ in a neighborhood of the union $\cup\{(x_1, x_2), x_2 = \xi^n(x_1), x_1 \in]0, 1[\}$, where all functions ξ, ξ^n are extended by zero outside of $[0, 1]$.

Revisiting the arguments of Sect. 3, it turns out that it is sufficient to have $(\xi^n - \xi) \to 0$ in $C^1[0, 1]$ instead of the particular dependence on the small parameter λ which was used there. Namely, we have

$$y = x + \frac{1}{n} V_n(x), \quad x, y \in \Omega_{\gamma,}$$

$$V_n = \left(V_n^1, V_n^2\right), \quad V_n^1 = 0,\ V_n^2 = n\theta\left(\xi^n - \xi\right),$$

$$\left\| V^n \right\|_{C^1(\Omega_\gamma)^2} \le c \quad \text{for all } n.$$

The arguments of Sect. 3 then show that

$$u^n \to u(\xi) \quad \text{strongly in } H^1_\Gamma(\Omega_\gamma), \tag{54}$$

where $u(\xi)$ is the unique solution to the problem (40), (41).

Recalling (47), the formula for the derivative of the energy functional with respect to the crack length for the solutions to the problem (49), (50) can be written as

$$G\left(\xi^n\right) = \int_{\Omega_{\gamma,}} \left\{ \frac{1}{2} \sigma_{ij}\left(u^n\right) \varepsilon_{ij}\left(u^n\right) \eta_{,1} - \sigma_{ij}\left(u^n\right) u^n_{i,1} \eta_{,j} \right\} - \int_{\Omega_{\gamma,}} (\eta f_i)_{,1} u_i^n,$$

while for the solution to (40), (41), we have

$$G(\xi) = \int_{\Omega_{\gamma,}} \left\{ \frac{1}{2} \sigma_{ij}(u) \varepsilon_{ij}(u) \eta_{,1} - \sigma_{ij}(u) u_{i,1} \eta_{,j} \right\} - \int_{\Omega_{\gamma,}} (\eta f_i)_{,1} u_i.$$

Using the convergence (54), we obtain

$$G\left(\xi^n\right) \to G(\xi),$$

hence the limiting function ξ is the solution of the optimal control problem (48). Theorem 4.1 is proved. □

5 Optimal Control of the Crack Shape

It follows from the previous section that for a given crack, we can choose an optimal shape of the rigid inclusion. The cost functional characterizes the derivative of the energy functional with respect to the crack length. In our considerations, we assumed that the crack be rectilinear. Similar arguments allow us to prove the same results provided that the crack is not rectilinear. For example, assume that the curve

$$x_2 = 2 + \varphi(x_1), \quad x_1 \in]1, 2[, \tag{55}$$

describes the crack shape with a function φ such that $\varphi \in H_0^2(1,2)$. Moreover, we assume that $\varphi = 0$ on the interval $]2-\mu, 2[$, $\mu = \text{const} > 0$. On the other hand, a perturbation of the crack shape can be analyzed from different points of view. In particular, it is interesting to know the influence of perturbation on the parameters of the Griffith rupture criterion. This section provides the analysis of these questions.

The formulation of the equilibrium problem corresponding to the rigid inclusion ω and the crack (55) is as follows. Denote $\gamma_, = \{(x_1, x_2) \mid x_2 = 2 + \varphi(x_1), x_1 \in]1, 2[)\}$, $\Omega_{\gamma_,} = \Omega \setminus \bar{\gamma}_,$. Assuming that $\gamma_, \cap \bar{\omega} = \emptyset$ we have to find functions $u = (u_1, u_2)$, $\rho_0 \in R(\omega)$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in $\Omega_{\gamma_,} \omega, \Omega_\gamma \setminus \bar{\omega}$, respectively, such that

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega_{\gamma_,} \setminus \bar{\omega}, \tag{56}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_{\gamma_,} \setminus \bar{\omega}, \tag{57}$$

$$u = \rho_0 \quad \text{in } \omega, \tag{58}$$

$$u = 0 \quad \text{on } \Gamma, \tag{59}$$

$$-\int_{\partial\omega} \sigma n \cdot \rho = \int_\omega f \cdot \rho \quad \forall \rho \in R(\omega), \tag{60}$$

$$[u]\nu \ge 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \cdot [u]\nu = 0,\ \sigma_\nu \le 0,\ \sigma_\tau = 0 \text{ on } \gamma_,. \tag{61}$$

The solution of this problem exists, and it can be found from a suitable variational inequality.

We can consider the problem perturbed with respect to (56)–(61) and corresponding to the crack $x_2 = 2 + \varphi^\delta(x_1), x_1 \in]1, 2+\delta[$, with

$$\varphi^\delta(x_1) = \begin{cases} \varphi(x_1), & x_1 \in]1, 2[, \\ 0, & x_1 \in]2, 2+\delta[, \end{cases}$$

where δ is a small parameter. At this step, it makes sense to assume that $\delta \ge 0$, but our analysis covers the case $\delta \le 0$. A formulation of the crack problem perturbed with respect to (56)–(61) is as follows. We have to find functions $u^\delta = (u_1^\delta, u_2^\delta)$, $\rho_0^\delta \in R(\omega)$, $\sigma^\delta = \{\sigma_{ij}^\delta\}, i, j = 1, 2$, defined in $\Omega_{\gamma_,^\delta}, \omega, \Omega_{\gamma_,^\delta} \setminus \bar{\omega}$, respectively, such that

$$-\operatorname{div} \sigma^\delta = f \quad \text{in } \Omega_{\gamma_,^\delta} \setminus \bar{\omega}, \tag{62}$$

$$\sigma^\delta - A\varepsilon(u^\delta) = 0 \quad \text{in } \Omega_{\gamma_,^\delta} \setminus \bar{\omega}, \tag{63}$$

$$u^\delta = \rho_0^\delta \quad \text{in } \omega, \tag{64}$$

$$u^\delta = 0 \quad \text{on } \Gamma, \tag{65}$$

$$-\int_{\partial\omega} \sigma^\delta n \cdot \rho = \int_\omega f \cdot \rho \quad \forall \rho \in R(\omega), \tag{66}$$

$$[u^\delta]\nu \geq 0,\ [\sigma_\nu^\delta] = 0,\ \sigma_\nu^\delta \cdot [u^\delta]\nu = 0,\ \sigma_\nu^\delta \leq 0, \sigma_\tau^\delta = 0 \text{ on } \gamma_,. \tag{67}$$

Here, $\gamma_,^\delta = \{(x_1, x_2) \mid x_2 = 2 + \varphi^\delta(x_1), x_1 \in]1, 2+\delta[\}$, $\Omega_{\gamma_,^\delta} = \Omega \setminus \bar{\gamma}_,^\delta$. The case $\delta = 0$ corresponds to the unperturbed problem, i.e., $\varphi^0 = \varphi$, $\gamma_,^0 = \gamma_,$.

We can prove the existence of the solution to problem (62)–(67). This solution can be found from the corresponding variational inequality. Namely, denote

$$K^\delta = \{v \in H_\Gamma^1(\Omega_{\gamma_,^\delta})^2 \mid [v]\nu \geq 0 \text{ on } \gamma_,^\delta;\ v|_\omega \in R(\omega)\}.$$

In this case, the functional

$$\left\{\frac{1}{2}\int_{\Omega_{\gamma_,^\delta}\setminus\bar{\omega}} \sigma(v) : \varepsilon(v) - \int_{\Omega_{\gamma_,^\delta}} f \cdot v\right\}$$

has a unique minimizer $u^\delta \in K^\delta$, i.e.,

$$u^\delta \in K^\delta, \tag{68}$$

$$\int_{\Omega_{\gamma_,^\delta}\setminus\bar{\omega}} \sigma(u^\delta) : \varepsilon(v - u^\delta) - \int_{\Omega_{\gamma_,^\delta}} f \cdot (v - u^\delta) \geq 0 \quad \forall v \in K^\delta. \tag{69}$$

Since we can establish a one-to-one mapping between the sets K^δ and K, the derivative $G(\varphi) = \frac{\Pi(\Omega_{\gamma_,^\delta}; u^\delta)}{d\delta}|_{\delta=0}$ of the energy functional

$$\Pi(\Omega_{\gamma_,^\delta}; u^\delta) = \frac{1}{2}\int_{\Omega_{\gamma_,^\delta}\setminus\bar{\omega}} \sigma(u^\delta) : \varepsilon(u^\delta) - \int_{\Omega_{\gamma_,^\delta}} f \cdot u^\delta$$

with respect to δ, as $\delta = 0$, exists, and the following formula holds:

$$G(\varphi) = \int_{\Omega_{\gamma_,}\setminus\bar{\omega}} \left\{\frac{1}{2}\sigma_{ij}(u)\varepsilon_{ij}(u)\eta_{,1} - \sigma_{ij}(u)u_{i,1}\eta_{,j}\right\} - \int_{\Omega_{\gamma_,}} (\eta f_i)_{,1} u_i, \tag{70}$$

where η is a smooth function equal to 1 near the crack tip $(2,2)$ and equal to zero outside of a neighborhood of the point $(2,2)$. As before, the derivative (70) does not depend on η.

Now we assume that $\Theta \subset H_0^2(1,2)$ be a bounded and weakly closed set, and for any $\varphi \in \Theta$ we have $\varphi = 0$ on the interval $]2-\mu, [)$, $\mu > 0$. For any $\varphi \in \Theta$, it is possible to prove the existence of the solutions to the equilibrium problem (62)–(67) and, therefore, to find the derivative (70). Clearly, $G(\varphi) \leq 0$. Consider the optimal control problem: Find $\varphi \in \Theta$ such that φ solves

$$\sup_{\varphi \in \Theta} G(\varphi). \tag{71}$$

To prove the existence of a solution to the problem (71), we provide additional arguments. First of all, we analyze the behavior of the solution of the problem (56)–(61), as $\lambda \to 0$, provided that the family of cracks $x_2 = 2 + \lambda\varphi(x_1)$, $x_1 \in]1, 2[$, be considered for a fixed $\varphi \in \Theta$, and λ be a small parameter converging to zero.

Let Ω^λ be the domain corresponding to the crack $x_2 = 2 + \lambda\varphi(x_1)$, i.e. $\gamma_{,\lambda} = \{(x_1, x_2) \mid x_2 = 2 + \lambda\varphi(x_1), x_1 \in]1, 2[\}$, $\Omega^\lambda = \Omega \setminus \bar{\gamma_{,\lambda}}$. Denote by

$$\nu^\lambda = \frac{(-\lambda\varphi', 1)}{\sqrt{1 + (\lambda\varphi')^2}}$$

a unit normal vector to $\gamma_{,\lambda}$ and consider the set of admissible displacements

$$K_\lambda = \{v \in H^1_\Gamma(\Omega^\lambda)^2 \mid [v]\nu^\lambda \geq 0 \text{ on } \gamma_{,\lambda};\, v|_\omega \in R(\omega)\}.$$

We introduce the solution of the problem corresponding to the crack $\gamma_{,\lambda}$ and the rigid inclusion ω, i.e.,

$$u^\lambda \in K_\lambda, \int_{\Omega^\lambda \setminus \bar{\omega}} \sigma(u^\lambda) : \varepsilon(v - u^\lambda) - \int_{\Omega^\lambda} f \cdot (v - u^\lambda) \geq 0 \quad \forall v \in K_\lambda. \tag{72}$$

Analogously, as $\lambda = 0$, we can consider the solution of the problem corresponding to a rectilinear crack, i.e.,

$$u \in K_0, \int_{\Omega^0 \setminus \bar{\omega}} \sigma(u) : \varepsilon(v - u) - \int_{\Omega^0} f \cdot (v - u) \geq 0 \quad \forall v \in K_0 \tag{73}$$

with a convex and closed set

$$K_0 = \{v \in H^1_\Gamma(\Omega^0)^2 \mid [v]\nu^0 \geq 0 \text{ on } \gamma_{,0};\, v|_\omega \in R(\omega)\}.$$

It is easy to set up a one-to-one correspondence between the domains Ω^λ and Ω^0. Indeed, we introduce a transformation of the independent variables $x = \Psi(\lambda, y)$,

$$x_1 = y_1, \qquad x_2 = y_2 - \lambda\xi(y)\varphi(y_1), \quad x \in \Omega^0,\ y \in \Omega^\lambda, \tag{74}$$

with $\xi \in C^\infty_0(\Omega)$, $\xi = 1$ in a neighborhood of the curve $\gamma_{,0}$. In order to make the transformation (74) correct, we extend the function φ by zero outside of interval $]1, 2[$. For a small λ, the transformation (74) is one-to-one, thus $y = \Phi(\lambda, x)$.

Denote $u_\lambda(x) = u^\lambda(y)$, $y \in \Omega^\lambda$, $x \in \Omega^0$. Notice that $u_\lambda|_\omega \in R(\omega)$ due to the inclusion $u^\lambda|_\omega \in R(\omega)$. We can prove the following statement concerning a behavior of the solution u_λ.

Lemma 5.1 *Let u be a solution of the problem* (73). *Then as* $\lambda \to 0$,

$$u_\lambda \to u \quad \textit{strongly in } H^1_\Gamma(\Omega^0)^2.$$

Proof The main difficulty in the proof consists in the absence of the one-to-one correspondence between K_λ and K_0. Note that $\nu^\lambda(y) = \nu^\lambda(y_1) = \nu^\lambda(x_1) = \nu^\lambda(x)$. Denote

$$K_{0\lambda} = \{v \in H^1_\Gamma(\Omega^0)^2 \mid [v(x)]\nu^\lambda(x) \geq 0 \text{ on } \gamma_{,0};\, v|_\omega \in R(\omega)\}.$$

In such a case, the transformation (74) maps K_λ on $K_{0\lambda}$. To simplify the formulae, for $v, w \in H^1_\Gamma(\Omega^0)^2$, we introduce the bilinear form

$$B_\lambda(v, w) = \int_{\Omega^0} \sigma(v) : \varepsilon(w) J(\lambda), \tag{75}$$

where $J(\lambda) = |\frac{\partial \Psi(\lambda, y)}{\partial y}|^{-1}$ is the Jacobian of the transformation inverse to (74). Observe that we should integrate over $\Omega \setminus \bar{\omega}$ in (75) provided that $v|_\omega \in R(\omega)$ or $w|_\omega \in R(\omega)$. Clearly, $J(\lambda) = (1 - \lambda\varphi\xi_{y_2})^{-1} > 0$ for a small λ. By changing the independent variables in (72), we arrive at the relation

$$u_\lambda \in K_{0\lambda} : B_\lambda(u_\lambda, \bar{u}_\lambda - u_\lambda) + \int_{\Omega^0} F(\lambda, u^2_{\lambda x}, \bar{u}_{\lambda x} u_{\lambda x}, \lambda(\varphi\xi)_y) J(\lambda)$$
$$\geq \int_{\Omega^0} f_\lambda(\bar{u}_\lambda - u_\lambda) J(\lambda) \quad \forall \bar{u}_\lambda \in K_{0\lambda}, \tag{76}$$

where $f_\lambda(x) = f(\Phi(\lambda, x))$, and the following formulae for the first derivatives

$$u^\lambda_{y_1} = u_{\lambda x_1} - \lambda u_{\lambda x_2}(\xi\varphi)_{y_1}, \qquad u^\lambda_{y_2} = u_{\lambda x_2}(1 - \lambda\xi_{y_2}\varphi)$$

are used, $u^\lambda(y) = u_\lambda(x)$, $y \in \Omega^\lambda$, $x \in \Omega^0$. The function F depends linearly on $u^2_{\lambda x}$, $\bar{u}_{\lambda x} u_{\lambda x}$. In particular, as $\lambda \to 0$,

$$\int_{\Omega^0} F(\lambda, u^2_{\lambda x}, \bar{u}_{\lambda x} u_{\lambda x}, \lambda(\varphi\xi)_y) J(\lambda) \to 0 \tag{77}$$

provided that u_λ, $\bar{u}_\lambda$ are bounded in $H^1_\Gamma(\Omega^0)^2$ uniformly in λ. Choosing $\bar{u}_\lambda \equiv 0$ in the inequality (76), it follows

$$\|u_\lambda\|_{H^1_\Gamma(\Omega^0)^2} \leq c_2,$$

uniformly in λ. Hence, we can assume that, as $\lambda \to 0$,

$$u_\lambda \to u_0 \quad \text{weakly in } H^1_\Gamma(\Omega^0)^2, u_0|_\omega \in R(\omega). \tag{78}$$

Note that if we have any given function $w = (w_1, w_2) \in K_0$, then the function $w_\lambda = w + h_\lambda$ can be constructed with $h_\lambda = (0, \lambda\xi\varphi_{x_1} w_1)$ such that $w_\lambda \in K_{0\lambda}$ and $w_\lambda|_\omega = w|_\omega \in R(\omega)$. Moreover,

$$w_\lambda \to w \quad \text{strongly in } H^1_\Gamma(\Omega^0)^2. \tag{79}$$

Hence, for the function $u_0 \in K_0$ from (78), we can find a sequence $\bar{u}_\lambda \in K_{0\lambda}$ such that

$$\bar{u}_\lambda = u_0 + v_\lambda, \quad v_\lambda \to 0 \text{ strongly in } H^1_\Gamma(\Omega^0)^2 \tag{80}$$

with the property $v_\lambda|_\omega = 0$. Taking into account the inequality (76), we have

$$B_\lambda(u_\lambda - u_0, u_\lambda - u_0) \le B_\lambda(u_0, u_0 - u_\lambda) + B_\lambda(u_0, v_\lambda) + B_\lambda(u_\lambda - u_0, v_\lambda) + \int_{\Omega^0} f_\lambda(u_\lambda - u_0 - v_\lambda) J(\lambda) + \int_{\Omega^0} F\big(\lambda, u_{\lambda x}^2, u_{\lambda x}(u_{0x} + v_{\lambda x}), \lambda(\varphi\xi)_y\big) J(\lambda).$$

Thus, by (77), (78), (80), we derive

$$\|u_\lambda - u_0\|_{H^1_\Gamma(\Omega^0)^2} \to 0, \quad \lambda \to 0. \tag{81}$$

Moreover, the convergence (81) and the convergence (79) allow us to pass to the limit as in (76) which implies

$$u_0 \in K_0, B_0(u_0, \bar{u} - u_0) \ge \int_{\Omega^0} f \cdot (\bar{u} - u_0) \quad \forall \bar{u} \in K_0.$$

Consequently, $u_0 = u$ is the solution of the variational inequality (73). Lemma 5.1 is proved. □

For any fixed λ in accordance with (70), it is possible to find the derivative of the energy functional with respect to the crack length. To this end, we consider the problem perturbed with respect to (72), like (62)–(67). It gives the following formula for the derivative:

$$G(\lambda\varphi) = \int_{\Omega_{\gamma,\lambda} \setminus \bar{\omega}} \left\{ \frac{1}{2} \sigma_{ij}\big(u^\lambda\big) \varepsilon_{ij}\big(u^\lambda\big) \eta_{,1} - \sigma_{ij}\big(u^\lambda\big) u_{i,1}^\lambda \eta_{,j} \right\} - \int_{\Omega_{\gamma,\lambda}} (\eta f_i)_{,1} u_i^\lambda, \tag{82}$$

where $\gamma_{,\lambda} = \{(x_1, x_2) \mid x_2 = 2 + \lambda\varphi(x_1), x_1 \in]1, 2[\}$, $\Omega_{\gamma^\lambda_,} = \Omega \setminus \bar{\gamma_{,\lambda}}$. The function η is smooth, finite, and is equal to 1 near the crack tip $(2, 2)$. Also, we have a formula for the derivative of the energy functional with respect to the crack length for the case $\lambda = 0$, i.e.,

$$G(0) = \int_{\Omega_{\gamma,0} \setminus \bar{\omega}} \left\{ \frac{1}{2} \sigma_{ij}(u) \varepsilon_{ij}(u) \eta_{,1} - \sigma_{ij}(u) u_{i,1} \eta_{,j} \right\} - \int_{\Omega_{\gamma,0}} (\eta f_i)_{,1} u_i. \tag{83}$$

We can change the independent variables in (82) in accordance with (74). By Lemma 5.1, we derive

$$G(\lambda\varphi) \to G(0), \quad \lambda \to 0. \tag{84}$$

Thus, the continuity of the derivative of the energy functional with respect to the crack shape (i.e. as $\lambda \to 0$) is established.

Now we are ready to prove a solution existence of the problem (71).

Theorem 5.1 *There exists a solution of the optimal control problem* (71).

Proof Let $\varphi^n \in \Theta$ be a maximizing sequence for the problem (71). Since Θ is bounded in the space $H_0^2(1,2)$, we can assume that, as $n \to 0$,

$$\varphi^n \to \varphi \quad \text{weakly in } H_0^2(1,2), \qquad (\varphi^n)' \to \varphi' \quad \text{in } C[1,2]. \tag{85}$$

For any fixed n, we can find the solution of the problem (56)–(61) satisfying the variational inequality

$$u^n \in K_{\varphi^n}, \int_{\Omega_{\gamma_,^n}\setminus\bar{\omega}} \sigma(u^n) : \varepsilon(v - u^n) - \int_{\Omega_{\gamma_,^n}} f \cdot (v - u^n) \geq 0 \quad \forall v \in K_{\varphi^n} \tag{86}$$

with the set of admissible displacements

$$K_{\varphi^n} = \{v \in H_\Gamma^1(\Omega_{\gamma_,^n})^2 \mid [v]\nu^n \geq 0 \text{ on } \gamma_,^n; v|_\omega \in R(\omega)\}$$

and $\gamma_,^n = \{(x_1, x_2) \mid x_2 = 2 + \varphi^n(x_1), x_1 \in]1,2[\}$, $\Omega_{\gamma_,^n} = \Omega \setminus \bar{\gamma}_,^n$.

Consider the transformation of the independent variables

$$x_1 = y_1, \qquad x_2 = y_2 + \xi(y)\big(\varphi(y_1) - \varphi^n(y_1)\big), \tag{87}$$

where $y \in \Omega_{\gamma_,^n}$, $x \in \Omega_{\gamma_,}$, and the crack $\gamma_,$ corresponds to the limit curve φ from (85). All functions φ^n and φ in (87) are extended by zero outside of $(1,2)$.

Now we can find the derivative of the energy functional with respect to the crack length for a given n in the problem (86). Again, for any n, we consider a problem perturbed with respect to (86), which is similar to (62)–(67), and find the derivative of the energy functional with respect to δ as $\delta = 0$. We have

$$G(\varphi^n) = \int_{\Omega_{\gamma_,^n}\setminus\bar{\omega}} \left\{\frac{1}{2}\sigma_{ij}(u^n)\varepsilon_{ij}(u^n)\eta_{,1} - \sigma_{ij}(u^n)u_{i,1}^n\eta_{,j}\right\} - \int_{\Omega_{\gamma_,^n}} (\eta f_i)_{,1} u_i^n. \tag{88}$$

In addition to this, for the limit function φ we also have the formula for the derivative of the energy functional

$$G(\varphi) = \int_{\Omega_{\gamma_,}\setminus\bar{\omega}} \left\{\frac{1}{2}\sigma_{ij}(u)\varepsilon_{ij}(u)\eta_{,1} - \sigma_{ij}(u)u_{i,1}\eta_{,j}\right\} - \int_{\Omega_{\gamma_,}} (\eta f_i)_{,1} u_i, \tag{89}$$

where u is the solution of the problem

$$u \in K_\varphi, \int_{\Omega_{\gamma_,^0}\setminus\bar{\omega}} \sigma(u) : \varepsilon(v - u) - \int_{\Omega_{\gamma_,^0}} f \cdot (v - u) \geq 0 \quad \forall v \in K_\varphi. \tag{90}$$

Similarly to Lemma 5.1, it is possible to prove that

$$u_n \to u \quad \text{strongly in } H_\Gamma^1(\Omega_{\gamma_,})^2, \tag{91}$$

where $u_n(x) = u^n(y)$, $y \in \Omega_{\gamma_,^n}$, $x \in \Omega_{\gamma_,^0}$. We can change the independent variables in (88) in accordance with (87). Like in (84) the convergences (85), (91) allow us to pass to the limit in (88). It provides

$$G(\varphi^n) \to G(\varphi).$$

Since $\varphi \in \Theta$, $u = u(\varphi)$, the limit function φ solves the optimal control problem (71). Theorem 5.1 is proved. □

6 Optimal Control of Elastic Inclusion Shape

In this section, we analyze an optimal control problem for the elastic inclusion shapes in order to maximize the derivative of the energy functional with respect to the crack length. The geometry of the problem coincides with that of Sect. 2; see Fig. 1. We keep the main notations used. The crack shape is given by $\gamma_, =]1, 2[\times\{2\}$. Assume that a part of the boundary $\partial\omega$ be described by a function $x_2 = \xi(x_1)$, $x_1 \in]0, 1[$, $\xi \in H_0^1(0, 1)$, and denote by ω^ξ the domain ω corresponding to the function ξ. Unlike in the previous section, the part ω^ξ is assumed to be elastic, hence the equilibrium equations and Hooke's law are fulfilled in Ω_γ. In what follows, we consider the elasticity tensor:

$$A^\xi(x) = \begin{cases} A^1(x), & x \in \Omega_\gamma \setminus \bar{\omega}^\xi, \\ A^2(x), & x \in \omega^\xi, \end{cases}$$

where $A^1 = \{a^1_{ijkl}\}$, $A^2 = \{a^2_{ijkl}\}$ are elasticity tensors with the usual properties of the symmetry and positive definiteness, $A^1, A^2 \in C^1(\bar{\Omega})$.

Let Ξ be bounded and weakly closed set in $H_0^1(0, 1)$. For any $\xi \in \Xi$, we can formulate the equilibrium problem for the elastic body Ω_γ with different (generally speaking) elasticity tensors A^2 and A^1 in ω^ξ and $\Omega_\gamma \setminus \bar{\omega}^\xi$, respectively. We have to find functions $u = u(\xi)$, $\sigma = \sigma(\xi)$, defined in $\Omega_{\gamma_,}$ such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_{\gamma_,}, \tag{92}$$

$$\sigma - A^\xi\varepsilon(u) = 0 \quad \text{in } \Omega_{\gamma_,}, \tag{93}$$

$$u = 0 \quad \text{on } \Gamma, \tag{94}$$

$$[u]\nu \geq 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \cdot [u]\nu = 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0 \text{ on } \gamma_,. \tag{95}$$

For the variational formulation, we introduce the set of admissible displacements

$$M = \big\{v \in H^1_\Gamma(\Omega_\gamma)^2 \mid [v]\nu \geq 0 \text{ on } \gamma_,\big\}.$$

Then there exists a unique solution $u = u(\xi)$ of the variational inequality

$$u \in M, \tag{96}$$

$$\int_{\Omega_\gamma} \sigma(u) : \varepsilon(v - u) - \int_{\Omega_\gamma} f \cdot (v - u) \geq 0 \quad \forall v \in M. \tag{97}$$

Problem formulations (92)–(95) and (96)–(97) are equivalent. We can consider the problem perturbed with respect to (92)–(95). Let $\gamma_,^\delta =]1, 2 + \delta[\ \times\{2\}$ be a perturbed crack with a small parameter δ, $\Omega_{\gamma_,^\delta} = \Omega \setminus \bar{\gamma}_,^\delta$. For any small δ, we can find a solution of the problem similar to (92)–(95), i.e., we have to find functions $u^\delta = u^\delta(\xi)$, $\sigma^\delta = \sigma^\delta(\xi)$, defined in $\Omega_{\gamma_,^\delta}$, such that

$$-\operatorname{div}\sigma^\delta = f \quad \text{in } \Omega_{\gamma_,^\delta}, \tag{98}$$

$$\sigma - A^\xi \varepsilon(u) = 0 \quad \text{in } \Omega_{\gamma_,^\delta}, \tag{99}$$

$$u^\delta = 0 \quad \text{on } \Gamma, \tag{100}$$

$$\big[u^\delta\big]\nu \geq 0,\ \big[\sigma_\nu^\delta\big] = 0,\ \sigma_\nu^\delta \cdot \big[u^\delta\big]\nu = 0,\ \sigma_\nu^\delta \leq 0,\ \sigma_\tau^\delta = 0 \text{ on } \gamma_,^\delta. \tag{101}$$

This problem is solvable, and its solution u^δ satisfies the variational inequality

$$u^\delta \in M^\delta, \tag{102}$$

$$\int_{\Omega_{\gamma_,^\delta}} \sigma\big(u^\delta\big) : \varepsilon\big(v - u^\delta\big) - \int_{\Omega_{\gamma_,^\delta}} f \cdot \big(v - u^\delta\big) \geq 0 \quad \forall v \in M^\delta, \tag{103}$$

where the set M^δ of admissible displacements is defined as follows:

$$M^\delta = \big\{v \in H_\Gamma^1(\Omega_\gamma) \mid [v]\nu \geq 0 \text{ on } \gamma_,^\delta\big\}.$$

The energy functional in the problem (102)–(103) can be written as

$$\pi\big(u^\delta;\xi\big) = \frac{1}{2}\int_{\Omega_{\gamma_,^\delta}} \sigma\big(u^\delta\big) : \varepsilon\big(u^\delta\big) - \int_{\Omega_{\gamma_,^\delta}} f \cdot u^\delta,$$

and its derivative $E(\xi)$ with respect to the parameter δ can be found by the formula

$$\begin{aligned} E(\xi) &= \frac{d\pi(u^\delta;\xi)}{d\delta}\Big|_{\delta=0} \\ &= \int_{\Omega_{\gamma_,}} \left\{\frac{1}{2}\varepsilon_{kl}(u)\varepsilon_{ij}(u)\big(a_{ijkl}^1\eta\big)_{,1} - \sigma_{ij}(u)u_{i,1}\eta_{,j}\right\} - \int_{\Omega_{\gamma_,}} (\eta f_i)_{,1} u_i. \end{aligned} \tag{104}$$

Here, the smooth function η is equal to 1 in a neighborhood of the point $(2,2)$, and $\eta = 0$ outside of a neighborhood of the point $(2,2)$. Like in the formula (47), there is no dependence of $E(\xi)$ on the choice of η with the prescribed properties, and $E(\xi) \leq 0\ \forall \xi \in \Xi$. Since we have to integrate in (104) over the neighborhood of the point $(2,2)$, in (104) the tensor A^1 is considered. Thus, we arrive at the following optimal control problem:

$$\sup_{\xi \in \Xi} E(\xi). \tag{105}$$

Theorem 6.1 *There exists a solution of the optimal control problem* (105).

Proof Let $\xi^n \in \Xi$ be a maximizing sequence. Since Ξ is bounded, it can be assumed that as $n \to \infty$

$$\begin{aligned} &\xi^n \to \xi \quad \text{weakly in } H_0^1(0,1), \xi \in \Xi, \\ &\left|\xi^n - \xi\right| < \frac{1}{n} \quad \text{on } (0,1). \end{aligned} \tag{106}$$

For any n, we can find a solution of the problem (96), (97). Namely, there exists a unique solution $u^n = u(\xi^n)$ of the variational inequality

$$u^n \in M, \tag{107}$$

$$\int_{\Omega_\gamma} \sigma^n(u^n) : \varepsilon(v - u^n) - \int_{\Omega_\gamma} f \cdot (v - u^n) \geq 0 \quad \forall v \in M. \tag{108}$$

Here,

$$\sigma_{ij}^n(u^n) = a_{ijkl}^{\xi^n} \varepsilon_{kl}(u^n), \quad i, j = 1, 2,$$

and $a_{ijkl}^{\xi^n}$ corresponds to the inclusion shape ω^{ξ^n}, i.e.,

$$A^{\xi^n}(x) = \begin{cases} A^1(x), & x \in \Omega_\gamma \setminus \bar{\omega}^{\xi^n}, \\ A^2(x), & x \in \omega^{\xi^n}. \end{cases}$$

From (108), it follows

$$\int_{\Omega_\gamma} \sigma^n(u^n) : \varepsilon(u^n) = \int_{\Omega_\gamma} f \cdot u^n;$$

thus. we have the uniform in n estimate

$$\|u^n\|_{H_\Gamma^1(\Omega_\gamma)^2} \leq c.$$

Assume that as $n \to \infty$

$$u^n \to u_0 \quad \text{weakly in } H_\Gamma^1(\Omega_\gamma)^2. \tag{109}$$

Now we consider the variational inequality corresponding to the limiting function ξ from (106), i.e., we define a solution $u^0 = u^0(\xi), \sigma(u^0) = A^\xi \varepsilon(u^0)$ to the variational inequality

$$u^0 \in M, \tag{110}$$

$$\int_{\Omega_\gamma} \sigma(u^0) : \varepsilon(v - u^0) - \int_{\Omega_\gamma} f \cdot (v - u^0) \geq 0 \quad \forall v \in M. \tag{111}$$

We aim at proving that u^n converges to u^0 strongly in $H_\Gamma^1(\Omega_\gamma)^2$. We take $v = u^0$ in (108), and take $v = u^n$ in (111) as test functions. This implies

$$\int_{\Omega_\gamma} a_{ijkl}^{\xi^n} \varepsilon_{kl}(u^n) \varepsilon_{ij}(u^0 - u^n) - \int_{\Omega_\gamma} f_i(u_i^0 - u_i^n) \geq 0, \tag{112}$$

$$\int_{\Omega_\gamma} a^{\xi}_{ijkl}\varepsilon_{kl}\big(u^0\big)\varepsilon_{ij}\big(u^n-u^0\big)-\int_{\Omega_\gamma} f_i\big(u^n_i-u^0_i\big)\geq 0. \tag{113}$$

Here, the tensors $a^{\xi^n}_{ijkl}, a^{\xi}_{ijkl}$ correspond to the elastic inclusion shapes $\omega^{\xi^n}, \omega^{\xi}$, respectively. Summing the inequalities (112), (113), we find

$$\int_{\Omega_\gamma} a^{\xi^n}_{ijkl}\varepsilon_{kl}\big(u^n\big)\varepsilon_{ij}\big(u^0-u^n\big)+\int_{\Omega_\gamma} a^{\xi}_{ijkl}\varepsilon_{kl}\big(u^0\big)\varepsilon_{ij}\big(u^n-u^0\big)$$
$$\pm\int_{\Omega_\gamma} a^{\xi^n}_{ijkl}\varepsilon_{kl}\big(u^0\big)\varepsilon_{ij}\big(u^0-u^n\big)\geq 0.$$

Hence, we obtain

$$\int_{\Omega_\gamma} a^{\xi^n}_{ijkl}\varepsilon_{kl}\big(u^n-u^0\big)\varepsilon_{ij}\big(u^n-u^0\big)$$
$$\leq\int_{\Omega_\gamma}\big(a^{\xi}_{ijkl}-a^{\xi^n}_{ijkl}\big)\varepsilon_{kl}\big(u^0\big)\varepsilon_{ij}\big(u^n-u^0\big). \tag{114}$$

Since, by (106), as $n\to\infty$

$$\operatorname{meas}\big\{x\in\Omega_\gamma : \big|A^{\xi}(x)-A^{\xi^n}(x)\big|>0\big\}\to 0,$$

from (114) it follows

$$\big\|u^n-u^0\big\|^2_{H^1_\Gamma(\Omega_\gamma)^2}\leq\big\|u^n-u^0\big\|_{H^1_\Gamma(\Omega_\gamma)^2}\cdot g(n)$$

with $g(n)\to 0$ as $n\to\infty$ and, therefore,

$$u^n\to u^0 \quad \text{strongly in } H^1_\Gamma(\Omega_\gamma)^2. \tag{115}$$

By (115), we can pass to the limit in (107), (108) as $n\to\infty$. The limiting variational inequality coincides with (110)–(111). Accounting (109), we see $u^0=u_0$, i.e., $u_0=u(\xi)$.

Due to (104), for any n the formula for the derivative of the energy functional with respect to the crack length in the problem (107)–(108) has the form

$$E\big(\xi^n\big)=\int_{\Omega_{\gamma,}}\left\{\frac{1}{2}\varepsilon_{ij}\big(u^n\big)\varepsilon_{kl}\big(u^n\big)\big(a^1_{ijkl}\eta\big)_{,1}-\sigma_{ij}(u)u^n_{i,1}\eta_{,j}\right\}-\int_{\Omega_{\gamma,}}(\eta f_i)_{,1}u^n_i.$$

By (115), we have

$$E\big(\xi^n\big)\to E(\xi),$$

where

$$E(\xi)=\int_{\Omega_{\gamma,}}\left\{\frac{1}{2}\varepsilon_{ij}\big(u^0\big)\varepsilon_{kl}\big(u^0\big)\big(a^1_{ijkl}\eta\big)_{,1}-\sigma_{ij}\big(u^0\big)u^0_{i,1}\eta_{,j}\right\}-\int_{\Omega_{\gamma,}}(\eta f_i)_{,1}u^0_i,$$

and, consequently, the limiting function $\xi\in\Xi$ solves the optimal control problem (105). Theorem 6.1 is proved. □

7 Conclusions and Further Remarks

We have shown in this paper that the energy release rate associated to an incipient crack depends continuously on the shape and the material properties of inclusions embedded into the domain, as long as the crack-tip stays away from these inclusions. We have emphasized that the nonpenetration of mutual crack surfaces is essential from a mechanical point of view. We have used the speed-method in order to formulate the domain variations of the respective inclusions and have shown the existence of solutions to crack-optimization problem. However, we have not been able to derive optimality conditions so far. It is to be expected that conical derivatives are to be considered in this context. We will consider, in a forthcoming publication, second-order derivatives of the energy functional, in order to study stability properties of cracks with respect to shape-variations of material inclusions. Finally, the shape-regularity properties of the energy-release-rate will be subject to further research. As for numerical methods and examples, we have also to refer to a forthcoming publication.

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