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Strong Convergence Theorems for Nonexpansive Mappings and Ky Fan Inequalities

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Abstract We introduce a new iteration method and prove strong convergence theorems for finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of monotone and Lipschitz-type continuous Ky Fan inequality. Under certain conditions on parameters, we show that the iteration sequences generated by this method converge strongly to the common element in a real Hilbert space. Some preliminary computational experiences are reported.

Keywords Nonexpansive mapping · Fixed point · Monotone · Lipschitz-type continuous · Ky Fan inequality

1 Introduction

We consider a well-known Ky Fan inequality [1], which is very general in the sense that it includes, as special cases, the optimization problem, the variational inequality, the saddle point problem, the Nash equilibrium problem in noncooperative games and the Kakutani fixed point problem; see [2–9]. Recently, methods for solving the Ky Fan inequality have been studied extensively. One of the most popular methods is the proximal point method. This method was introduced first by Martinet in [10] for variational inequality and then was extended by Rockafellar in [11] for finding the zero point of a maximal monotone operator. Konnov in [12] further extended the proximal point method to the Ky Fan inequality with a monotone and weakly monotone bifunction, respectively. Other solution methods well developed in mathematical programming and the variational inequality, such as the gap function, extragradient,

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and bundle methods, recently have been extended to the Ky Fan inequality; see [5, 6, 13, 14].

In this paper, we are interested in the problem of finding a common element of the solution set of the Ky Fan inequality and the set of fixed points of a nonexpansive mapping. Our motivation originates from the following observations. The problem can be on one hand considered as an extension of the Ky Fan inequality when the nonexpansive mapping is the identity mapping. On the other hand, it has been significant in many practical problems. Since the Ky Fan inequality has found many direct applications in economics, transportation, and engineering, it is natural that when the feasible set of this problem results as a fixed-point solution set of a fixed-point problem, then the obtained problem can be reformulated equivalently to the problem. An important special case of the Ky Fan inequality is the variational inequality, and this problem is reduced to finding a common element of the solution set of variational inequality and the solution set of a fixed-point problem; see [15–17].

The paper is organized as follows. Section 2 recalls some concepts related to Ky Fan inequality and fixed point problems that will be used in the sequel and a new iteration scheme. Section 3 investigates the convergence theorem of the iteration sequences presented in Sect. 2 as the main results of our paper. Applications are presented in Sect. 4.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ and Proj_C be the projection of $\mathcal{H}$ onto C. When $\{x^k\}$ is a sequence in $\mathcal{H}$, then $x^k \to \bar{x}$ (resp. $x^k \rightharpoonup \bar{x}$) will denote strong (resp. weak) convergence of the sequence $\{x^k\}$ to $\bar{x}$. Let $f : C \times C \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in C$. The Ky Fan inequality consists in finding a point in

$$P(f, C) := \{x^* \in C : f(x^*, y) \geq 0 \ \forall y \in C\},$$

where $f(x, \cdot)$ is convex and subdifferentiable on C for every $x \in C$. The set of solutions of problem $P(f, C)$ is denoted by $\mathrm{Sol}(f, C)$. When $f(x, y) = \langle F(x), y - x\rangle$ with $F : C \to \mathcal{H}$, problem $P(f, C)$ amounts to the variational inequality problem (shortly, $\mathrm{VI}(F, C)$)

$$\text{Find } x^* \in C \quad \text{such that } \langle F(x^*), y - x^*\rangle \geq 0 \quad \text{for all} y \in C.$$

The bifunction f is called *strongly monotone* on C with $\beta > 0$ iff

$$f(x, y) + f(y, x) \leq -\beta\|x - y\|^2, \quad \forall x, y \in C;$$

monotone on C iff

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C;$$

pseudomonotone on C iff

$$f(x, y) \geq 0 \quad \Rightarrow \quad f(y, x) \leq 0, \quad \forall x, y \in C;$$

Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$ in the sense of Mastroeni in [8] iff

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \quad \forall x,y,z \in C.$$

When $f(x,y) = \langle F(x), y-x\rangle$ with $F : C \to \mathcal{H}$,

$$f(x,y) + f(y,z) - f(x,z) = \bigl\langle F(x) - F(y), y-z\bigr\rangle \quad \text{for all } x,y,z \in C,$$

and it is easy to see that if F is Lipschitz continuous on C with constant $L > 0$, i.e., $\|F(x) - F(y)\| \leq L\|x-y\|$ for all $x, y \in C$, then

$$\bigl|\bigl\langle F(x) - F(y), y-z\bigr\rangle\bigr| \leq L\|x-y\|\,\|y-z\| \leq \frac{L}{2}\bigl(\|x-y\|^2 + \|y-z\|^2\bigr),$$

and thus, f satisfies Lipschitz-type continuous condition with $c_1 = c_2 = \frac{L}{2}$. Furthermore, when $z = x$, this condition becomes

$$f(x,y) + f(y,x) \geq -(c_1 + c_2)\|y-x\|^2 \quad \forall x,y \in C.$$

This gives a lower bound on $f(x,y) + f(y,x)$ while the strong monotonicity gives an upper bound on $f(x,y) + f(y,x)$.

A mapping $S : C \to C$ is said to be *contractive* with $\delta \in]0,1[$ iff

$$\bigl\|S(x) - S(y)\bigr\| \leq \delta\|x-y\|, \quad \forall x,y \in C.$$

If $\delta = 1$ then S is called *nonexpansive* on C. $\mathrm{Fix}(S)$ denotes the set of fixed points of S.

In 1953, Mann [18] introduced a well-known classical iteration method to approximate a fixed point of a nonexpansive mapping S in a real Hilbert space $\mathcal{H}$. This iteration is defined as

$$x^0 \in C, \qquad x^{k+1} = \alpha_k x^k + (1-\alpha_k)S\bigl(x^k\bigr), \quad \forall k \geq 0,$$

where C is a nonempty, closed, and convex subset of $\mathcal{H}$ and $\{\alpha_k\} \subset [0,1]$. Then $\{x^k\}$ converges weakly to $x^* \in \mathrm{Fix}(S)$.

Recently, Xu gave the strong convergence theorems for the following sequences in a real Hilbert space $\mathcal{H}$:

$$x^0 \in C, \qquad x^{k+1} = \alpha_k g\bigl(x^k\bigr) + (1-\alpha_k)S\bigl(x^k\bigr), \quad \forall k \geq 0,$$

where $\{\alpha_k\} \subset]0,1[$, $g : C \to C$ is contractive and $S : C \to C$ is nonexpansive. In [19], the author proved that the sequence $\{x^k\}$ converges strongly to x^*, where x^* is the unique solution of the variational inequality:

$$\bigl\langle (I-g)\bigl(x^*\bigr), x - x^*\bigr\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(S).$$

Chen et al. in [16] studied the viscosity approximation methods for a nonexpansive mapping S and an α-inverse-strongly monotone mapping $A : C \to \mathcal{H}$, i.e., $\langle A(x) - A(y), x-y\rangle \geq \alpha\|A(x) - A(y)\|^2$ for all $x, y \in C$ in a real Hilbert space $\mathcal{H}$:

$$x^0 \in C, \qquad x^{k+1} = \alpha_k g\bigl(x^k\bigr) + (1-\alpha_k)S\,\mathrm{Proj}_C\bigl(x^k - \lambda_k A\bigl(x^k\bigr)\bigr), \quad \forall k \geq 0,$$

where $\{\alpha_k\} \subset]0, 1[$, $\{\lambda_k\} \subset [a, b]$ with $0 < a < b < 2\alpha$ and Proj_C denotes the metric projection from $\mathcal{H}$ onto C. They proved that if some certain conditions on $\{\alpha_k\}$ and $\{\lambda_k\}$ are satisfied, then the sequence $\{x^k\}$ converges strongly to a common element of the set of fixed points of the nonexpansive mapping S and the set of solutions of the variational inequality for the inverse-strongly monotone mapping A. To overcome the restriction of the above methods to the class of α-inverse-strongly monotone mappings, by using the extragradient method of Korpelevich in [7], Ceng et al. in [15] could show the strong convergence result of the following method:

$$\begin{cases} x^0 \in C, \\ y^k = (1-\gamma_k)x^k + \gamma_k \mathrm{Proj}_C(x^k - \lambda_k A(x^k)), \\ z^k = (1-\alpha_k-\beta_k)x^k + \alpha_k y^k + \beta_k S\,\mathrm{Proj}_C(x^k - \lambda_k A(y^k)), \\ C_k = \{z \in C : \|z-y^k\| \le \|z-x^k\| + (3-3\gamma_k+\alpha_k)b^2\|A(x^k)\|^2\}, \\ Q_k = \{z \in C : \langle z-x^k, x^0-x^k\rangle \le 0\}, \\ x^{k+1} = \mathrm{Proj}_{C_k\cap Q_k}(x^0), \end{cases}$$

where the sequences $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, and $\{\lambda_k\}$ were chosen appropriately. The authors showed that the iterative sequences $\{x^k\}$, $\{y^k\}$, and $\{z^k\}$ converged strongly to the same point $\bar{x} = \mathrm{Proj}_{\mathrm{Sol}(F,C)\cap \mathrm{Fix}(S)}(x^0)$.

For obtaining a common element of set of solutions of problem $P(f, C)$ and the set of fixed points $\mathrm{Fix}(S)$ of a nonexpansive mapping S of a real Hilbert space $\mathcal{H}$ into itself, Takahashi and Takahashi in [20] first introduced an iterative scheme by the viscosity approximation method. The sequence $\{x^k\}$ is defined by

$$\begin{cases} x^0 \in \mathcal{H}, \\ \text{Find } u^k \in C \text{ such that } f(u^k, y) + \frac{1}{r_k}\langle y-u^k, u^k-x^k\rangle \ge 0, \quad \forall y \in C, \\ x^{k+1} = \alpha_k g(x^k) + (1-\alpha_k)S(u^k), \quad \forall k \ge 0, \end{cases}$$

where C is a nonempty, closed, and convex subset of $\mathcal{H}$ and g is a contractive mapping of $\mathcal{H}$ into itself. The authors showed that under certain conditions over $\{\alpha_k\}$ and $\{r_k\}$, sequences $\{x^k\}$ and $\{u^k\}$ converge strongly to $z = \mathrm{Proj}_{\mathrm{Sol}(f,C)\cap \mathrm{Fix}(S)}(g(z))$.

Recently, iterative methods for finding a common element of the set of solutions of Ky Fan inequality and the set of fixed points of a nonexpansive mapping in a real Hilbert space have further developed by many authors; see [21–24]. At each iteration k in all of the current algorithms, it requires solving an *approximation auxiliary Ky Fan inequality*.

Motivated by the approximation method in [15] and the iterative method in [20] via an improvement set of a hybrid extragradient method in [25], we introduce a new iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of Ky Fan inequality for monotone and Lipschitz-type continuous bifunctions. At each iteration, we only solve *two strongly convex optimization problems* instead of a regularized Ky Fan inequality. The iterative process is given by

$$\begin{cases} y^k = \operatorname{argmin}\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\}, \\ t^k = \operatorname{argmin}\{\lambda_k f(y^k, t) + \frac{1}{2}\|t - x^k\|^2 : t \in C\}, \end{cases} \tag{1}$$

and compute the next iteration point

$$x^{k+1} = \alpha_k g(x^k) + \beta_k x^k + \gamma_k(\mu S(x^k) + (1-\mu)t^k), \quad \forall k \geq 0, \tag{2}$$

where g is a contractive mapping of $\mathcal{H}$ into itself. To investigate the convergence of this scheme, we recall the following technical lemmas which will be used in the sequel.

Lemma 2.1 [25] *Let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. Let $f : C \times C \to \mathbb{R}$ be a pseudomonotone and Lipschitz-type continuous bifunction. For each $x \in C$, let $f(x, \cdot)$ be convex and subdifferentiable on C. Then, for each $x^* \in \operatorname{Sol}(f, C)$, the sequences $\{x^k\}, \{y^k\}, \{t^k\}$ generated by (1) satisfy the following inequalities*:

$$\begin{aligned} \|t^k - x^*\|^2 \leq \|x^k - x^*\|^2 - (1 - 2\lambda_k c_1)\|x^k - y^k\|^2 \\ - (1 - 2\lambda_k c_2)\|y^k - t^k\|^2, \quad \forall k \geq 0. \end{aligned}$$

Lemma 2.2 [26] *Let $\{x^k\}$ and $\{y^k\}$ be two bounded sequences in a Banach space and let $\{\beta_k\}$ be a sequence of real numbers such that $0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1$. Suppose that*

$$\begin{cases} x^{k+1} = \beta_k x^k + (1 - \beta_k) y^k, \quad \forall k \geq 0, \\ \limsup_{k\to\infty}(\|y^{k+1} - y^k\| - \|x^{k+1} - x^k\|) \leq 0. \end{cases}$$

Then

$$\lim_{k\to\infty} \|x^k - y^k\| = 0.$$

Lemma 2.3 [27] *Let T be a nonexpansive self-mapping of a nonempty, closed, and convex subset C of a real Hilbert space $\mathcal{H}$. Then $I - T$ is demiclosed; that is, whenever $\{x^k\}$ is a sequence in C weakly converging to some $\bar{x} \in C$ and the sequence $\{(I - T)(x^k)\}$ strongly converges to some $\bar{y}$, it follows that $(I - T)(\bar{x}) = \bar{y}$. Here, I is the identity operator of $\mathcal{H}$.*

Lemma 2.4 [19] *Let $\{a_k\}$ be a nonnegative real number sequence satisfying*

$$a_{k+1} \leq (1 - \alpha_k) a_k + o(\alpha_k), \quad \forall k \geq 0,$$

where $\{\alpha_k\} \subset]0, 1[$ is a real number sequence. If $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$, then $\lim_{k\to\infty} a_k = 0$.

3 Convergence Results

Now, we prove the main convergence theorem.

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. Let $f : C \times C \to \mathbb{R}$ be a monotone, continuous, and Lipschitz-type continuous bifunction, $g : C \to C$ be a contractive mapping with constant $\delta \in]0,1[$, S be a nonexpansive mapping of C into itself, and $\mathrm{Fix}(S) \cap \mathrm{Sol}(f,C) \neq \emptyset$. Suppose that $x^0 \in C$, $\mu \in]0,1[$, positive sequences $\{\lambda_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$ satisfy the following restrictions*:

$$\begin{cases} \lim_{k\to\infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = \infty, \\ 0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1, \\ \lim_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0, \{\lambda_k\} \subset [a,b] \subset \left]0, \frac{1}{L}\right[, \quad \text{where } L = \max\{2c_1, 2c_2\}, \\ \alpha_k + \beta_k + \gamma_k = 1, \\ \alpha_k(2 - \alpha_k - 2\beta_k\delta - 2\gamma_k) \in]0,1[. \end{cases} \tag{3}$$

Then the sequences $\{x^k\}$, $\{y^k\}$, and $\{t^k\}$ generated by (1) and (2) converge strongly to the same point $x^* \in \mathrm{Fix}(S) \cap \mathrm{Sol}(f,C)$, which is the unique solution of the following variational inequality:

$$\left\langle (I - g)\big(x^*\big), x - x^* \right\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(S) \cap \mathrm{Sol}(f,C).$$

The proof of this theorem is divided into several steps.

Step 1 Claim that $\{x^k\}$ is bounded.

Proof of Step 1 By Lemma 2.1 and $x^{k+1} = \alpha_k g(x^k) + \beta_k x^k + \gamma_k(\mu S(x^k) + (1-\mu)t^k)$, we have

$$\begin{aligned} &\|x^{k+1} - x^*\| \\ &\quad = \big\|\alpha_k\big(g\big(x^k\big) - x^*\big) + \beta_k\big(x^k - x^*\big) + \gamma_k\big(\mu S\big(x^k\big) + (1-\mu)t^k - x^*\big)\big\| \\ &\quad \leq \alpha_k\big\|g\big(x^k\big) - x^*\big\| + \beta_k\big\|x^k - x^*\big\| + \gamma_k\big\|\mu S\big(x^k\big) + (1-\mu)t^k - x^*\big\| \\ &\quad \leq \alpha_k\big\|g\big(x^k\big) - x^*\big\| + \beta_k\big\|x^k - x^*\big\| + \gamma_k\big(\mu\big\|S\big(x^k\big) - x^*\big\| + (1-\mu)\big\|t^k - x^*\big\|\big) \\ &\quad \leq \alpha_k\big\|g\big(x^k\big) - x^*\big\| + \beta_k\big\|x^k - x^*\big\| + \gamma_k\big(\mu\big\|x^k - x^*\big\| + (1-\mu)\big\|t^k - x^*\big\|\big) \\ &\quad \leq \alpha_k\big\|g\big(x^k\big) - x^*\big\| + \beta_k\big\|x^k - x^*\big\| + \gamma_k\big\|x^k - x^*\big\| \\ &\quad \leq \alpha_k\big\|g\big(x^k\big) - g\big(x^*\big)\big\| + \alpha_k\big\|g\big(x^*\big) - x^*\big\| + \beta_k\big\|x^k - x^*\big\| + \gamma_k\big\|x^k - x^*\big\| \\ &\quad \leq \alpha_k\delta\big\|x^k - x^*\big\| + \alpha_k\big\|g\big(x^*\big) - x^*\big\| + (1-\alpha_k)\big\|x^k - x^*\big\| \\ &\quad = \big(1 - (1-\delta)\alpha_k\big)\big\|x^k - x^*\big\| + (1-\delta)\alpha_k\left\|\frac{g(x^*) - x^*}{1-\delta}\right\| \end{aligned}$$

$$\le \max\left\{\|x^k - x^*\|, \left\|\frac{g(x^*) - x^*}{1-\delta}\right\|\right\}$$
$$\le \cdots$$
$$\le \max\left\{\|x^0 - x^*\|, \left\|\frac{g(x^*) - x^*}{1-\delta}\right\|\right\}.$$

Then

$$\|x^{k+1} - x^*\|^2 \le \max\left\{\|x^0 - x^*\|, \left\|\frac{g(x^*) - x^*}{1-\delta}\right\|\right\}, \quad \forall k \ge 0.$$

So, $\{x^k\}$ is bounded. Therefore, by Lemma 2.1, the sequences $\{y^k\}$ and $\{t^k\}$ are bounded. □

Step 2 Claim that $\lim_{k\to\infty} \|t^k - x^k\| = 0$.

Proof of Step 2 Since $f(x, \cdot)$ is convex on C for each $x \in C$, we see that

$$t^k = \operatorname{argmin}\left\{\frac{1}{2}\|t - x^k\|^2 + \lambda_k f\left(y^k, t\right) : t \in C\right\}$$

if and only if

$$0 \in \partial_2\left(\lambda_k f\left(y^k, t\right) + \frac{1}{2}\|t - x^k\|^2\right)\left(t^k\right) + N_C\left(t^k\right), \tag{4}$$

where $N_C(x)$ is the (outward) normal cone of C at $x \in C$. Thus, since $f(y^k, \cdot)$ is subdifferentiable on C, by the well-known Moreau–Rockafellar theorem [11], there exists $w \in \partial_2 f(y^k, t^k)$ such that

$$f\left(y^k, t\right) - f\left(y^k, t^k\right) \ge \left\langle w, t - t^k\right\rangle, \quad \forall t \in C.$$

Substituting $t = x^*$ into this inequality to obtain

$$f\left(y^k, x^*\right) - f\left(y^k, t^k\right) \ge \left\langle w, x^* - t^k\right\rangle. \tag{5}$$

On the other hand, it follows from (4) that $0 = \lambda_k w + t^k - x^k + \bar{\eta}$, where $w \in \partial_2 f(y^k, t^k)$ and $\bar{\eta} \in N_C(t^k)$. By the definition of the normal cone N_C we have, from this relation that

$$\left\langle t^k - x^k, t - t^k\right\rangle \ge \lambda_k\left\langle w, t^k - t\right\rangle, \quad \forall t \in C. \tag{6}$$

Set

$$\eta^k = \mu S\left(x^k\right) + (1-\mu)t^k \quad \text{and} \quad x^{k+1} = \beta_k x^k + (1-\beta_k)z^k. \tag{7}$$

For each $k \ge 0$, we have $z^k = \frac{\alpha_k g(x^k) + \gamma_k \eta^k}{1-\beta_k}$, and hence

$$
\begin{aligned}
z^{k+1} - z^k &= \frac{\alpha_{k+1} g(x^{k+1}) + \gamma_{k+1}\eta^{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k g(x^k) + \gamma_k \eta^k}{1-\beta_k} \\
&= \frac{\alpha_{k+1}(g(x^{k+1}) - g(x^k))}{1-\beta_{k+1}} + \frac{\gamma_{k+1}(\eta^{k+1} - \eta^k)}{1-\beta_{k+1}} \\
&\quad + \left(\frac{\alpha_{k+1} g(x^k)}{1-\beta_{k+1}} + \frac{\gamma_{k+1}\eta^k}{1-\beta_{k+1}} - \frac{\alpha_k g(x^k) + \gamma_k \eta^k}{1-\beta_k}\right) \\
&= \frac{\alpha_{k+1}(g(x^{k+1}) - g(x^k))}{1-\beta_{k+1}} + \frac{\gamma_{k+1}(\eta^{k+1} - \eta^k)}{1-\beta_{k+1}} \\
&\quad + \left(\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}\right)\bigl(g\bigl(x^k\bigr) - \eta^k\bigr).
\end{aligned} \tag{8}
$$

Since $f(x,\cdot)$ is convex on C for all $x \in C$, we have

$$
f\bigl(y^k, t^{k+1}\bigr) - f\bigl(y^k, t^k\bigr) \geq \bigl\langle w, t^{k+1} - t^k \bigr\rangle,
$$

where $w \in \partial_2 f(y^k, t^k)$. Substituting $t = t^{k+1}$ into (6), then we have

$$
\begin{aligned}
\bigl\langle t^k - x^k, t^{k+1} - t^k \bigr\rangle &\geq \lambda_k \bigl\langle w, t^k - t^{k+1} \bigr\rangle \\
&\geq \lambda_k \bigl(f\bigl(y^k, t^k\bigr) - f\bigl(y^k, t^{k+1}\bigr)\bigr).
\end{aligned} \tag{9}
$$

By the similar way, we also have

$$
\bigl\langle t^{k+1} - x^{k+1}, t^k - t^{k+1} \bigr\rangle \geq \lambda_{k+1}\bigl(f\bigl(y^{k+1}, t^{k+1}\bigr) - f\bigl(y^{k+1}, t^k\bigr)\bigr). \tag{10}
$$

Using (9), (10), and f is Lipschitz-type continuous and monotone, we get

$$
\begin{aligned}
&\frac{1}{2}\bigl\|x^{k+1} - x^k\bigr\|^2 - \frac{1}{2}\bigl\|t^{k+1} - t^k\bigr\|^2 \\
&\quad \geq \bigl\langle t^{k+1} - t^k, t^k - x^k - t^{k+1} + x^{k+1} \bigr\rangle \\
&\quad \geq \lambda_k \bigl(f\bigl(y^k, t^k\bigr) - f\bigl(y^k, t^{k+1}\bigr)\bigr) \\
&\qquad + \lambda_{k+1}\bigl(f\bigl(y^{k+1}, t^{k+1}\bigr) - f\bigl(y^{k+1}, t^k\bigr)\bigr) \\
&\quad \geq \lambda_k \bigl(-f\bigl(t^k, t^{k+1}\bigr) - c_1\bigl\|y^k - t^k\bigr\|^2 - c_2\bigl\|t^k - t^{k+1}\bigr\|^2\bigr) \\
&\qquad + \lambda_{k+1}\bigl(-f\bigl(t^{k+1}, t^k\bigr) - c_1\bigl\|y^{k+1} - t^{k+1}\bigr\|^2 - c_2\bigl\|t^k - t^{k+1}\bigr\|^2\bigr) \\
&\quad \geq (\lambda_{k+1} - \lambda_k) f\bigl(t^k, t^{k+1}\bigr) \\
&\quad \geq -|\lambda_{k+1} - \lambda_k| \, \bigl|f\bigl(t^k, t^{k+1}\bigr)\bigr|.
\end{aligned}
$$

Hence,

$$
\bigl\|t^{k+1} - t^k\bigr\|^2 \leq \bigl\|x^{k+1} - x^k\bigr\|^2 + 2|\lambda_{k+1} - \lambda_k| \bigl|f\bigl(t^k, t^{k+1}\bigr)\bigr|.
$$

Therefore, we have

$$\begin{aligned}\|\eta^{k+1}-\eta^k\|^2 &= \big\|\big(\mu S(x^{k+1})+(1-\mu)t^{k+1}\big)-\big(\mu S(x^k)+(1-\mu)t^k\big)\big\|^2\\ &= \big\|\mu\big(S(x^{k+1})-S(x^k)\big)+(1-\mu)\big(t^{k+1}-t^k\big)\big\|^2\\ &\le \mu\big\|S(x^{k+1})-S(x^k)\big\|^2+(1-\mu)\big\|t^{k+1}-t^k\big\|^2\\ &\le \mu\big\|x^{k+1}-x^k\big\|^2+(1-\mu)\big\|t^{k+1}-t^k\big\|^2\\ &\le \mu\big\|x^{k+1}-x^k\big\|^2+(1-\mu)\big(\big\|x^{k+1}-x^k\big\|^2\\ &\quad +2|\lambda_{k+1}-\lambda_k|\big|f\big(t^k,t^{k+1}\big)\big|\big)\\ &\le \big\|x^{k+1}-x^k\big\|^2+2(1-\mu)|\lambda_{k+1}-\lambda_k|\big|f\big(t^k,t^{k+1}\big)\big|.\end{aligned}$$

Combining this with (8), we obtain

$$\begin{aligned}&\big\|z^{k+1}-z^k\big\|^2\\ &\quad=\left\|\frac{\alpha_{k+1}(g(x^{k+1})-g(x^k))}{1-\beta_{k+1}}+\frac{\gamma_{k+1}(\eta^{k+1}-\eta^k)}{1-\beta_{k+1}}\right\|^2\\ &\qquad+\left|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\right|^2\big\|g\big(x^k\big)-\eta^k\big\|^2+M_k\\ &\quad\le\frac{\alpha_{k+1}\|g(x^{k+1})-g(x^k)\|^2}{1-\beta_{k+1}}+\frac{\gamma_{k+1}\|\eta^{k+1}-\eta^k\|^2}{1-\beta_{k+1}}\\ &\qquad+\left|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\right|^2\big\|g\big(x^k\big)-\eta^k\big\|^2+M_k\\ &\quad\le\frac{\alpha_{k+1}\|x^{k+1}-x^k\|^2}{1-\beta_{k+1}}+\frac{\gamma_{k+1}\|\eta^{k+1}-\eta^k\|^2}{1-\beta_{k+1}}\\ &\qquad+\left|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\right|^2\big\|g\big(x^k\big)-\eta^k\big\|^2+M_k\\ &\quad\le\frac{\alpha_{k+1}\|x^{k+1}-x^k\|^2}{1-\beta_{k+1}}\\ &\qquad+\frac{\gamma_{k+1}(\|x^{k+1}-x^k\|^2+2(1-\mu)|\lambda_{k+1}-\lambda_k|\,|f(t^k,t^{k+1})|)}{1-\beta_{k+1}}\\ &\qquad+\left|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\right|^2\big\|g\big(x^k\big)-\eta^k\big\|^2+M_k\\ &\quad=\big\|x^{k+1}-x^k\big\|^2+\frac{2\gamma_{k+1}(1-\mu)|\lambda_{k+1}-\lambda_k|\,|f(t^k,t^{k+1})|}{1-\beta_{k+1}}\\ &\qquad+\left|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\right|^2\big\|g\big(x^k\big)-\eta^k\big\|^2+M_k,\end{aligned}\tag{11}$$

where M_k is defined by

$$M_k = \left(\frac{2\alpha_{k+1}}{1-\beta_{k+1}} - \frac{2\alpha_k}{1-\beta_k}\right) \times \left\langle \frac{\alpha_{k+1}(g(x^{k+1}) - g(x^k))}{1-\beta_{k+1}} + \frac{\gamma_{k+1}(\eta^{k+1} - \eta^k)}{1-\beta_{k+1}}, g\left(x^k\right) - \eta^k \right\rangle.$$

Since Step 2, $\lim_{k\to\infty} \lambda_k = 0, 0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1$ and $\lim_{k\to\infty} \|\lambda_{k+1} - \lambda_k\| = 0$, we have $\lim_{k\to\infty} M_k = 0$ and

$$\limsup_{k\to\infty}\left(\left\|z^{k+1} - z^k\right\|^2 - \left\|x^{k+1} - x^k\right\|^2\right) \le 0.$$

So,

$$\limsup_{k\to\infty}\left(\left\|z^{k+1} - z^k\right\| - \left\|x^{k+1} - x^k\right\|\right) \le 0.$$

By Lemma 2.2, we have $\lim_{k\to\infty} \|z^k - x^k\| = 0$ and hence

$$\lim_{k\to\infty} \frac{1}{1-\beta_k}\left\|x^{k+1} - x^k\right\| = 0.$$

Note that $0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1$, we have

$$\lim_{k\to\infty} \left\|x^{k+1} - x^k\right\| = 0. \tag{12}$$

Since

$$\begin{aligned}
&\left\|x^{k+1} - x^*\right\|^2 \\
&\quad = \left\|\alpha_k\left(g\left(x^k\right) - x^*\right) + \beta_k\left(x^k - x^*\right) + \gamma_k\left(\mu S\left(x^k\right) + (1-\mu)t^k - x^*\right)\right\|^2 \\
&\quad \le \alpha_k\left\|g\left(x^k\right) - x^*\right\|^2 + \beta_k\left\|x^k - x^*\right\|^2 + \gamma_k\left\|\mu S\left(x^k\right) + (1-\mu)t^k - x^*\right\|^2 \\
&\quad \le \alpha_k\left\|g\left(x^k\right) - x^*\right\|^2 + \beta_k\left\|x^k - x^*\right\|^2 \\
&\qquad + \gamma_k\left(\mu\left\|S\left(x^k\right) - x^*\right\|^2 + (1-\mu)\left\|t^k - x^*\right\|^2\right) \\
&\quad \le \alpha_k\left\|g\left(x^k\right) - x^*\right\|^2 + \beta_k\left\|x^k - x^*\right\|^2 + \gamma_k\left(\mu\left\|x^k - x^*\right\|^2 + (1-\mu)\left\|t^k - x^*\right\|^2\right)
\end{aligned}$$

and Step 1, we have

$$\begin{aligned}
&\left\|x^{k+1} - x^*\right\|^2 \\
&\quad \le \alpha_k\left\|g\left(x^k\right) - x^*\right\|^2 + \beta_k\left\|x^k - x^*\right\|^2 + \gamma_k\left(\mu\left\|x^k - x^*\right\|^2 + (1-\mu)\left\|t^k - x^*\right\|^2\right) \\
&\quad \le \alpha_k\left\|g\left(x^k\right) - x^*\right\|^2 + \beta_k\left\|x^k - x^*\right\|^2 + \gamma_k\mu\left\|x^k - x^*\right\|^2 + (1-\mu)\gamma_k\left(\left\|x^k - x^*\right\|^2\right. \\
&\qquad \left. - (1-2\lambda_k c_1)\left\|x^k - y^k\right\|^2 - (1-2\lambda_k c_2)\left\|y^k - t^k\right\|^2\right)
\end{aligned}$$

$$= \alpha_k \| g(x^k) - x^* \|^2 + \| x^k - x^* \|^2 - (1-\mu)\gamma_k (1 - 2\lambda_k c_1) \| x^k - y^k \|^2$$
$$- (1-\mu)\gamma_k (1 - 2\lambda_k c_2) \| y^k - t^k \|^2.$$

Then

$$\begin{aligned}
&(1-\mu)\gamma_k(1-2\lambda_k c_1)\|x^k - y^k\|^2 \\
&\quad \le \alpha_k \| g(x^k) - x^* \|^2 + \| x^k - x^* \|^2 - \| x^{k+1} - x^* \|^2 \\
&\quad = \alpha_k \| g(x^k) - x^* \|^2 + \left(\| x^k - x^* \| - \| x^{k+1} - x^* \| \right)\left(\| x^k - x^* \| + \| x^{k+1} - x^* \| \right) \\
&\quad \le \alpha_k \| g(x^k) - x^* \|^2 + \| x^k - x^{k+1} \| \left(\| x^k - x^* \| + \| x^{k+1} - x^* \| \right),
\end{aligned}$$

for every $k = 0, 1, \ldots$. By Step 2, $\mu \in]0, 1[$, $\alpha_k + \beta_k + \gamma_k = 1$, $\lim_{k\to\infty} \alpha_k = 0$, (12), and $0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1$, and we have

$$\lim_{k\to\infty} \| x^k - y^k \| = 0. \tag{13}$$

By the similar way, we also have

$$\lim_{k\to\infty} \| y^k - t^k \| = 0. \tag{14}$$

Using $\|x^k - t^k\| \le \|x^k - y^k\| + \|y^k - t^k\|$, (13) and (14), we have

$$\lim_{k\to\infty} \| x^k - t^k \| = 0. \qquad \square$$

Step 3 Claim that

$$\lim_{k\to\infty} \| x^k - S(x^k) \| = 0.$$

Proof of Step 3 From $x^{k+1} = \alpha_k g(x^k) + \beta_k x^k + \gamma_k (\mu S(x^k) + (1-\mu) t^k)$, we have

$$\begin{aligned}
x^{k+1} - x^k &= \alpha_k g(x^k) + \beta_k x^k + \gamma_k \left(\mu S(x^k) + (1-\mu) t^k \right) - x^k \\
&= \alpha_k \left(g(x^k) - x^k \right) + \mu \gamma_k \left(S(x^k) - x^k \right) + (1-\mu)\gamma_k \left(t^k - x^k \right)
\end{aligned}$$

and hence

$$\mu \gamma_k \| S(x^k) - x^k \| \le \| x^{k+1} - x^k \| + \alpha_k \| g(x^k) - x^k \| + (1-\mu)\gamma_k \| t^k - x^k \|.$$

Using this, $\lim_{k\to\infty} \alpha_k = 0, \alpha_k + \beta_k + \gamma_k = 1, 0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1$, Step 2, Step 3, and (12), we have

$$\lim_{k\to\infty} \| x^k - S(x^k) \| = 0. \qquad \square$$

Step 4 Claim that

$$\limsup_{k\to\infty}\langle x^* - g(x^*), \eta^k - x^*\rangle \geq 0,$$

where η^k is defined by (7).

Proof of Step 4 Since $\{\eta^k\}$ is bounded, there exists a subsequence $\{\eta^{k_i}\}$ of $\{\eta^k\}$ such that

$$\limsup_{k\to\infty}\langle x^* - g(x^*), \eta^k - x^*\rangle = \lim_{i\to\infty}\langle x^* - g(x^*), \eta^{k_i} - x^*\rangle.$$

By Step 1, the sequence $\{\eta^{k_i}\}$ is bounded, and hence there exists a subsequence $\{\eta^{k_{i_j}}\}$ of $\{\eta^{k_i}\}$ which converges weakly to $\bar{\eta}$. Without loss of generality, we suppose that the sequence $\{\eta^{k_i}\}$ converges weakly to $\bar{\eta}$ such that

$$\limsup_{k\to\infty}\langle x^* - g(x^*), \eta^k - x^*\rangle = \lim_{i\to\infty}\langle x^* - g(x^*), \eta^{k_i} - x^*\rangle. \tag{15}$$

Using Step 2, Step 3, and $\eta^k = \mu S(x^k) + (1-\mu)t^k$, we also have

$$\lim_{k\to\infty}\|x^k - \eta^k\| = 0.$$

Since Lemma 2.3, $\{\eta^{k_i}\}$ converges weakly to $\bar{\eta}$ and Step 3, we get

$$S(\bar{\eta}) = \bar{\eta} \quad \Leftrightarrow \quad \bar{\eta} \in \mathrm{Fix}(S). \tag{16}$$

Now, we show that $\bar{\eta} \in \mathrm{Sol}(f, C)$. By Step 2, we have

$$x^{k_i} \rightharpoonup \bar{\eta}, \qquad y^{k_i} \rightharpoonup \bar{\eta}.$$

Since y^k is the unique solution of the strongly convex problem

$$\min\left\{\frac{1}{2}\|y - x^k\|^2 + f(x^k, y) : \ y \in C\right\},$$

we have

$$0 \in \partial_2\left(\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\right)(y^k) + N_C(y^k).$$

This follows that

$$0 = \lambda_k w + y^k - x^k + w^k,$$

where $w \in \partial_2 f(x^k, y^k)$ and $w^k \in N_C(y^k)$. By the definition of the normal cone N_C, we have

$$\langle y^k - x^k, y - y^k\rangle \geq \lambda_k\langle w, y^k - y\rangle, \quad \forall y \in C. \tag{17}$$

On the other hand, since $f(x^k, \cdot)$ is subdifferentiable on C, by the well-known Moreau–Rockafellar theorem, there exists $w \in \partial_2 f(x^k, y^k)$ such that

$$f(x^k, y) - f(x^k, y^k) \geq \langle w, y - y^k\rangle, \quad \forall y \in C.$$

Combining this with (17), we have

$$\lambda_k\big(f\big(x^k, y\big) - f\big(x^k, y^k\big)\big) \geq \big\langle y^k - x^k, y^k - y\big\rangle, \quad \forall y \in C.$$

Hence,

$$\lambda_{k_j}\big(f\big(x^{k_j}, y\big) - f\big(x^{k_j}, y^{k_j}\big)\big) \geq \big\langle y^{k_j} - x^{k_j}, y^{k_j} - y\big\rangle, \quad \forall y \in C.$$

Then, using $\{\lambda_k\} \subset [a, b] \subset \left]0, \frac{1}{L}\right[$ and continuity of f, we have

$$f(\bar{\eta}, y) \geq 0, \quad \forall y \in C.$$

Combining this and (16), we obtain

$$\eta^{k_i} \rightharpoonup \bar{\eta} \in \mathrm{Fix}(S) \cap \mathrm{Sol}(f, C).$$

By (15) and the definition of x^*, we have

$$\limsup_{k\to\infty}\big\langle x^* - g\big(x^*\big), \eta^k - x^*\big\rangle = \big\langle x^* - g\big(x^*\big), \bar{\eta} - x^*\big\rangle \geq 0.$$

□

Step 5 Claim that the sequences $\{x^k\}$, $\{y^k\}$, and $\{t^k\}$ converge strongly to x^*.

Proof of Step 5 Since $\eta^k = \mu S(x^k) + (1-\mu)t^k$ and Lemma 2.1, we have

$$\begin{aligned}
\big\|\eta^k - x^*\big\|^2 &= \big\|\mu\big(S\big(x^k\big) - x^*\big) + (1-\mu)\big(t^k - x^*\big)\big\|^2 \\
&\leq \mu\big\|S\big(x^k\big) - x^*\big\|^2 + (1-\mu)\big\|t^k - x^*\big\|^2 \\
&\leq \mu\big\|x^k - x^*\big\|^2 + (1-\mu)\big(\big\|x^k - x^*\big\|^2 - (1-2\lambda_k c_1)\big\|x^k - y^k\big\|^2 \\
&\quad - (1-2\lambda_k c_2)\big\|y^k - t^k\big\|^2\big) \\
&\leq \big\|x^k - x^*\big\|^2.
\end{aligned}$$

Using this and $x^{k+1} = \alpha_k g(x^k) + \beta_k x^k + \gamma_k \eta^k$, we have

$$\begin{aligned}
&\big\|x^{k+1} - x^*\big\|^2 \\
&\quad = \big\|\alpha_k\big(g\big(x^k\big) - x^*\big) + \beta_k\big(x^k - x^*\big) + \gamma_k\big(\eta^k - x^*\big)\big\|^2 \\
&\quad \leq \alpha_k^2\big\|g\big(x^k\big) - x^*\big\|^2 + \beta_k^2\big\|x^k - x^*\big\|^2 + \gamma_k^2\big\|x^k - x^*\big\|^2 \\
&\qquad + 2\alpha_k\beta_k\big\langle g\big(x^k\big) - x^*, x^k - x^*\big\rangle + 2\beta_k\gamma_k\big\|x^k - x^*\big\|^2 \\
&\qquad + 2\gamma_k\alpha_k\big\langle g\big(x^k\big) - x^*, \eta^k - x^*\big\rangle \\
&\quad = \alpha_k^2\big\|g\big(x^k\big) - x^*\big\|^2 + (1-\alpha_k)^2\big\|x^k - x^*\big\|^2 + 2\alpha_k\beta_k\big\langle g\big(x^k\big) - g\big(x^*\big), x^k - x^*\big\rangle \\
&\qquad + 2\alpha_k\beta_k\big\langle g\big(x^*\big) - x^*, x^k - x^*\big\rangle + 2\gamma_k\alpha_k\big\langle g\big(x^k\big) - g\big(x^*\big), \eta^k - x^*\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\gamma_k\alpha_k\langle g(x^*) - x^*, \eta^k - x^*\rangle \\
&\le \alpha_k^2\|g(x^k) - x^*\|^2 + (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k\beta_k\delta\|x^k - x^*\|^2 \\
&\quad + 2\alpha_k\beta_k\langle g(x^*) - x^*, x^k - x^*\rangle + 2\gamma_k\alpha_k\|x^k - x^*\|^2 \\
&\quad + 2\gamma_k\alpha_k\langle g(x^*) - x^*, \eta^k - x^*\rangle \\
&= \big((1-\alpha_k)^2 + 2\alpha_k\beta_k\delta + 2\gamma_k\alpha_k\big)\|x^k - x^*\|^2 + \alpha_k^2\|g(x^k) - x^*\|^2 \\
&\quad + 2\alpha_k\beta_k\langle g(x^*) - x^*, x^k - x^*\rangle + 2\gamma_k\alpha_k\langle g(x^*) - x^*, \eta^k - x^*\rangle \\
&\le \big((1-\alpha_k)^2 + 2\alpha_k\beta_k\delta + 2\gamma_k\alpha_k\big)\|x^k - x^*\|^2 + \alpha_k^2\|g(x^k) - x^*\|^2 \\
&\quad + 2\alpha_k\beta_k \max\{0, \langle g(x^*) - x^*, x^k - x^*\rangle\} \\
&\quad + 2\gamma_k\alpha_k \max\{0, \langle g(x^*) - x^*, \eta^k - x^*\rangle\} \\
&= (1 - A_k)\|x^k - x^*\|^2 + B_k,
\end{aligned}
$$

where A_k and B_k are defined by

$$
\begin{cases}
A_k = 2\alpha_k - \alpha_k^2 - 2\alpha_k\beta_k\delta - 2\gamma_k\alpha_k, \\
B_k = \alpha_k^2\|g(x^k) - x^*\|^2 + 2\alpha_k\beta_k \max\{0, \langle g(x^*) - x^*, x^k - x^*\rangle\} \\
\qquad + 2\gamma_k\alpha_k \max\{0, \langle g(x^*) - x^*, \eta^k - x^*\rangle\}.
\end{cases}
$$

Since $\lim_{k\to\infty}\alpha_k = 0$, $\sum_{k=1}^{\infty}\alpha_k = \infty$, Step 2, and Step 4, we have

$$
\limsup_{k\to\infty}\langle x^* - g(x^*), x^k - x^*\rangle \ge 0,
$$

and hence

$$
B_k = o(A_k), \qquad \lim_{k\to\infty} A_k = 0, \qquad \sum_{k=1}^{\infty} A_k = \infty.
$$

By Lemma 2.4, we obtain that the sequence $\{x^k\}$ converges strongly to x^*. It follows from Step 3 that the sequences $\{y^k\}$ and $\{t^k\}$ also converge strongly to the unique solution x^*. □

4 Applications and Numerical Results

Let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ and F be a function from C into $\mathcal{H}$. In this section, we consider the variational inequality $\mathrm{VI}(F, C)$. The set of solutions of $\mathrm{VI}(F, C)$ is denoted by $\mathrm{Sol}(F, C)$. Recall that the function F is called

- *Strongly monotone* on C with $\beta > 0$ iff

$$
\langle F(x) - F(y), x - y\rangle \ge \beta\|x - y\|^2, \quad \forall x, y \in C.
$$

- *Monotone* on C iff

$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in C.$$

- *Pseudomonotone* on C iff

$$\langle F(y), x - y\rangle \geq 0 \quad \Rightarrow \quad \langle F(x), x - y\rangle \geq 0, \quad \forall x, y \in C.$$

- *Lipschitz continuous* on C with constants $L > 0$ (shortly, L-Lipschitz continuous) iff

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

Since

$$\begin{aligned} y^k &= \operatorname{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : \ y \in C\right\} \\ &= \operatorname{argmin}\left\{\lambda_k\langle F(x^k), y - x^k\rangle + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\} \\ &= \operatorname{Proj}_C(x^k - \lambda_k F(x^k)), \end{aligned}$$

equation (1), and Theorem 3.1, the convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping S and the solution set $\operatorname{Sol}(F, C)$ is presented as follows.

Theorem 4.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $F : C \to \mathcal{H}$ be monotone and L-Lipschitz continuous, $g : C \to C$ be a contractive mapping, S be a nonexpansive mapping of C into itself, and* $\operatorname{Fix}(S) \cap \operatorname{Sol}(F, C) \neq \emptyset$. *Suppose that $\mu \in]0, 1[$, positive sequences $\{\lambda_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$ satisfy the following restrictions*:

$$\begin{cases} \lim_{k\to\infty} \alpha_k = 0, \ \sum_{k=0}^{\infty} \alpha_k = \infty, \\ 0 < \liminf_{k\to\infty} \beta_k < \limsup_{k\to\infty} \beta_k < 1, \\ \lim_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0, \quad \{\lambda_k\} \subset [a, b] \text{ for some } a, b \in \left]0, \frac{1}{L}\right[, \\ \alpha_k + \beta_k + \gamma_k = 1, \\ \alpha_k(2 - \alpha_k - 2\beta_k\delta - 2\gamma_k) \in]0, 1[. \end{cases}$$

Then the sequences $\{x^k\}$, $\{y^k\}$, and $\{t^k\}$ generated by

$$\begin{cases} x^0 \in C, \\ y^k = \operatorname{Proj}_C(x^k - \lambda_k F(x^k)), \\ t^k = \operatorname{Proj}_C(x^k - \lambda_k F(y^k)), \\ x^{k+1} = \alpha_k g(x^k) + \beta_k x^k + \gamma_k(\mu S(x^k) + (1 - \mu)t^k), \quad \forall k \geq 0, \end{cases}$$

converge strongly to the same point $x^* \in \mathrm{Fix}(S) \cap \mathrm{Sol}(F, C)$, *which is the unique solution of the following variational inequality*:

$$\big\langle (I-g)\big(x^*\big), x - x^*\big\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(S) \cap \mathrm{Sol}(F, C).$$

Now, we consider a special case of problem $P(f, C)$, the nonexpansive mapping S is the identity mapping. Then iterative schemes (1) and (2) are to find a solution of Ky Fan inequality $P(f, C)$. The iterative process is given by

$$\begin{cases} x^0 \in C, \\ y^k = \operatorname{argmin}\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : \ y \in C\}, \\ t^k = \operatorname{argmin}\{\lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : \ y \in C\}, \\ x^{k+1} = \alpha_k g(x^k) + \beta_k x^k + \gamma_k(\mu x^k + (1-\mu)t^k), \quad \forall k \geq 0, \end{cases} \tag{18}$$

where g is δ-contractive and the parameters satisfy (3). By Theorem 3.1, the sequence $\{x^k\}$ converges to the unique solution x^* of the following variational inequality:

$$\big\langle (I-g)\big(x^*\big), x - x^*\big\rangle \geq 0, \quad \forall x \in \mathrm{Sol}(f, C).$$

It is easy to see that if $x^k = t^k$ then x^k is a solution of $P(f, C)$. So, we can say that x^k is an ϵ-solution to $P(f, C)$ if $\|t^k - x^k\| \leq \epsilon$. To illustrate this scheme, we consider to numerical examples in $\mathbb{R}^5$. The set C is a polyhedral convex set given by

$$C = \begin{cases} x \in \mathbb{R}^5_+, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 \leq 10, \\ 2x_1 + x_2 - x_3 + x_4 + 3x_5 \leq 15, \\ x_1 + x_2 + x_3 + x_4 + 0.5x_5 \geq 4, \end{cases}$$

and the bifunction f is defined by

$$f(x, y) = \langle Ax + By + q, x - y\rangle,$$

where the matrices A, B, q (randomly generated) are

$$A = \begin{pmatrix} 3 & 1.5 & 0 & 0 & 0 \\ 1.5 & 2.5 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 3 & 2.5 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1.5 & 0 & 0 & 0 \\ 1.5 & 3.5 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2.5 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix},$$

$$q = (2, 4, 6, 8, 1)^{\mathrm{T}}.$$

Then A is symmetric positive semidefinite and f is Lipschitz-type continuous on C with $L = 2c_1 = 2c_2 = \|A - B\| = 3.7653$. Since the eigenvalues of the matrix $B - A$

are $-3.5, -0.5, -1, -1, 0$, we get that $B - A$ is negative semidefinite. Therefore, f is monotone on C. With

$$g(x) = \frac{1}{2}x, \qquad \alpha_k = \frac{1}{k+2}, \qquad \beta_k = \frac{1}{2}, \qquad \gamma_k = \frac{k}{2(k+2)},$$
$$\lambda_k = \frac{k+20}{10(k+10)}, \quad \forall k \geq 0,$$

$x^0 = (1, 2, 1, 1, 1)^{\mathrm{T}}$ and $\epsilon = 10^{-6}$, the conditions (3) are satisfied and we obtain the following iterates:

Iter (k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k
0	1	2	1	1	1
1	0.6695	1.5337	0.7686	0.7481	0.6672
2	0.7092	1.3673	0.8069	0.8058	0.7217
3	0.9045	1.0437	0.9009	0.8992	0.5033
4	0.9338	0.9751	0.9298	0.9278	0.4670
5	0.9428	0.9540	0.9387	0.9366	0.4559
6	0.9455	0.9475	0.9414	0.9393	0.4524
7	0.9464	0.9456	0.9422	0.9402	0.4514
8	0.9466	0.9449	0.9425	0.9404	0.4511
9	0.9467	0.9448	0.9426	0.9405	0.4510
10	0.9467	0.9447	0.9426	0.9405	0.4510
11	0.9467	0.9447	0.9426	0.9405	0.4510

The approximate solution obtained after 11 iterations is

$$x^{11} = (0.9467, 0.9447, 0.9426, 0.9405, 0.4510)^{\mathrm{T}}.$$

We perform the iterative scheme (18) in Matlab R2008a running on a PC Desktop Intel(R) Core(TM)2 Duo CPU T5750@ 2.00 GHz 1.32 GB, 2 Gb RAM.

5 Conclusion

This paper presented an iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of monotone and Lipschitz-type continuous Ky Fan inequality. To solve the problem, most of current algorithms are based on solving strongly auxiliary equilibrium problems. The fundamental difference here is that, at each main iteration in the proposed algorithms, we only solve strongly convex problems. Moreover, under certain parameters, we show that the iterative sequences converge strongly to the unique solution of a strong variational inequality problem in a real Hilbert space.

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