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Global Optimal Solutions of Noncyclic Mappings in Metric Spaces

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Abstract We study some minimization problems for noncyclic mappings in metric spaces. We then apply the solution to obtain some results in the theory of analytic functions.

Keywords Noncyclic mapping · Contraction mapping · Nonexpansive mapping · Fixed point

1 Introduction

Let A and B be two nonempty subsets of a metric space X, and let T be a self mapping defined on the union of A and B in such a way that T maps either of A or B into itself. In this paper, we aim to study the existence of solutions for some specific minimization problems. First, we study the problem of minimizing the distance between x and Tx when x runs through A. We then take up the problem of minimizing the distance between x and y where x and y run through A and B, respectively. Examples are provided to illustrate the applications of our results. In particular, we use our theorems to obtain some results in the theory of analytic functions.

2 Preliminaries

Let A, B be nonempty subsets of a metric space (X, d) and let T be a self-mapping on $A \cup B$. We say that T is cyclic provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. For

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cyclic mappings, the fixed point equation $Tx = x$ may not have solution. Thus it is contemplated to find an approximate solution $x \in A$ such that the error term $d(x, Tx)$ is minimum. This leads to the notion of best proximity points. A point $p \in A$ is called a best proximity point for T provided that $d(x, Tx) = \operatorname{dist}(A, B)$, where

$$\operatorname{dist}(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}.$$

The existence and convergence of best proximity points is an interesting topic in optimization theory, which recently attracted the attention of many authors; see for instance, [1–10].

Now let $T : A \cup B \to A \cup B$ be a noncyclic mapping, that is $T(A) \subseteq A$ and $T(B) \subseteq B$. In this case, we can consider the following minimization problem: Find

$$\min_{x \in A} d(x, Tx), \qquad \min_{y \in B} d(y, Ty) \quad \text{and} \quad \min_{(x,y) \in A \times B} d(x, y). \tag{1}$$

We say that $(x^*, y^*) \in A \times B$ is a solution of (1) provided that

$$Tx^* = x^*, \qquad Ty^* = y^* \quad \text{and} \quad d(x^*, y^*) = \operatorname{dist}(A, B).$$

In [11], Eldred et al. studied the existence of solution of (1) for relatively nonexpansive mappings in Banach spaces with a geometric property, called proximal normal structure. The purpose of this article is to establish some theorems for noncyclic mappings; in particular, to determine the solution of (1) in metric spaces with some appropriate geometric property.

Let A and B be nonempty subsets of a metric space (X, d). In this work, we adopt the following notations and definitions.

$$A_0 := \{x \in A : d(x, y) = \operatorname{dist}(A, B),\ \text{for some } y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = \operatorname{dist}(A, B),\ \text{for some } x \in A\}.$$

Definition 2.1 [12] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have P-property iff

$$\begin{cases} d(x_1, y_1) = \operatorname{dist}(A, B) \\ d(x_2, y_2) = \operatorname{dist}(A, B) \end{cases} \implies d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 2.1 [12] Let A, B be two nonempty closed, and convex subsets of a Hilbert space X. Then (A, B) satisfies the P-property.

Example 2.2 Let A, B be two nonempty subsets of a metric space (X, d) such that $A_0 \neq \emptyset$ and $\operatorname{dist}(A, B) = 0$. Then (A, B) has the P-property.

Example 2.3 Let A, B be two nonempty, bounded, closed, and convex subsets of a uniformly convex Banach space X. Then (A, B) has the P-property.

Proof First we prove that $A_0 \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in A, B respectively, such that $\|x_n - y_n\| \to \operatorname{dist}(A, B)$. Since A, B are bounded, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that $x_{n_k} \rightharpoonup p \in A$, and $y_{n_k} \rightharpoonup q \in B$, where "$\rightharpoonup$" stands for the weak convergence in X. But according to a well-known fact in basic functional analysis, we have

$$\|p-q\| \leq \lim_{k\to\infty} \inf \|x_{n_k} - y_{n_k}\| = \operatorname{dist}(A,B).$$

This implies that $A_0 \neq \emptyset$. Now, let $q, q' \in B$ and

$$\|p-q\| = \|p'-q'\| = \operatorname{dist}(A,B),$$

for some $p, p' \in A$. If $p - p' = q - q'$, then the conclusion follows. Assume that $p - p' \neq q - q'$. Now, by the convexity of A, B and the uniform convexity of the Banach space X, we must have

$$\operatorname{dist}(A,B) \leq \left\| \frac{p+p'}{2} - \frac{q+q'}{2} \right\| < \frac{1}{2}\big[\|p-q\| + \|p'-q'\|\big] = \operatorname{dist}(A,B),$$

which is a contradiction. Hence, (A, B) has the P-property. □

Definition 2.2 Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called relatively nonexpansive iff $d(Tx, Ty) \leq d(x, y)$, for all $(x, y) \in A \times B$. In case $A = B$, we say that T is nonexpansive.

Definition 2.3 Let $T : X \to X$ be a mapping on a metric space (X, d). We say that T is expansive provided that $d(Tx, Ty) \geq d(x, y)$, for all $x, y \in X$.

Definition 2.4 A self-mapping $T : X \to X$ is said to be asymptotically regular iff $\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$, for all $x \in X$.

Example 2.4 (See Theorem 8.22 of [13]) Let D be a nonempty, bounded, and convex subset of a Banach space X. Assume that $f : D \to D$ is a nonexpansive mapping. Then the mapping $T : X \to X$ defined by $T := \frac{I+f}{2}$, where I is the identity mapping on D, is an asymptotically regular mapping.

3 The Main Results

We begin our main results with the following theorem.

Theorem 3.1 *Let (A, B) be a pair of nonempty, and closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and that (A, B) satisfies the P-property. Let $T : A \cup B \to A \cup B$ be a noncyclic mapping. Then the minimization problem (1) has a solution provided that the following conditions are satisfied*:

(i) *$T|_A$ is contraction,*
(ii) *T is relatively nonexpansive.*

Proof At first we note that $T(A_0) \subseteq A_0$. Indeed, if $x \in A_0$, then there exists $y \in B$ such that $d(x, y) = \operatorname{dist}(A, B)$. But T is relatively nonexpansive; then

$$d(Tx, Ty) \leq d(x, y) = \operatorname{dist}(A, B).$$

Hence, $Tx \in A_0$. Now let $x_0 \in A_0$. By the Banach contraction principle if $x_{n+1} = Tx_n$, then $x_n \to x^*$ where x^* is a fixed point of T in A. Since $x_0 \in A_0$, there exists $y_0 \in B$ such that $d(x_0, y_0) = \text{dist}(A, B)$. Again, since $x_1 = Tx_0 \in A_0$, there exists $y_1 \in B$ such that $d(x_1, y_1) = \text{dist}(A, B)$. Using this process, we obtain a sequence $\{y_n\}$ in B such that

$$d(x_n, y_n) = \text{dist}(A, B),$$

for all $n \in \mathbb{N} \cup \{0\}$. Since (A, B) has the P-property, we must have

$$d(x_n, x_m) = d(y_n, y_m),$$

for all $m, n \in \mathbb{N} \cup \{0\}$. This implies that $\{y_n\}$ is a Cauchy sequence, and hence there exists $y^* \in B$ such that $y_n \to y^*$. We now have

$$d\big(x^*, y^*\big) = \lim_{n\to\infty} d(x_n, y_n) = \text{dist}(A, B).$$

On the other hand, since T is relatively nonexpansive, we obtain

$$d\big(Tx^*, Ty^*\big) \le d\big(x^*, y^*\big) = \text{dist}(A, B).$$

Thus, $d(x^*, Tx^*) = d(y^*, Ty^*)$, according to the P-property of the pair (A, B). Hence, $(x^*, y^*) \in A \times B$ is a solution of (1). □

In order to explore the existence of solution of (1), we state the following theorem.

Theorem 3.2 *Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) such that A is compact and B is closed. Let $A_0 \neq \emptyset$ and (A, B) satisfy the P-property. Let $T : A \cup B \to A \cup B$ be a noncyclic mapping. Then the minimization problem* (1) *has a solution provided that the following conditions are satisfied*:

(i) *T is relatively nonexpansive*,
(ii) *$T|_A$ is expansive*,
(iii) *$T|_B$ is contractive*.

Proof Let $x_0 \in A_0$, and define $x_{n+1} = Tx_n$, for all $n \in \mathbb{N} \cup \{0\}$. An argument similar to the one in the proof of Theorem 3.1 implies that $T(A_0) \subseteq A_0$ and hence there exists $y_n \in B$ such that $d(x_n, y_n) = \text{dist}(A, B)$, for all $n \in \mathbb{N} \cup \{0\}$. Since A is compact, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} \to x^* \in A$. By using the P-property of the pair (A, B), we have $d(x_{n_k}, x_{n_s}) = d(y_{n_k}, y_{n_s})$, for all $k, s \in \mathbb{N}$. Thus, $\{y_{n_k}\}$ is a Cauchy sequence, and hence there exists $y^* \in B$ such that $y_{n_k} \to y^*$. Therefore,

$$d\big(x^*, y^*\big) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = \text{dist}(A, B).$$

We now prove that $Tx^* = x^*$ and $Ty^* = y^*$. Since T is relatively nonexpansive,

$$d\big(T^2x^*, T^2y^*\big) = d\big(Tx^*, Ty^*\big) = \text{dist}(A, B).$$

Again, by the P-property of (A, B), we obtain

$$d\big(x^*, Tx^*\big) = d\big(y^*, Ty^*\big) \quad \text{and} \quad d\big(Tx^*, T^2x^*\big) = d\big(Ty^*, T^2y^*\big).$$

Let $Ty^* \neq T^2y^*$. Since $T|_A$ is expansive and $T|_B$ is contractive, we must have

$$d\big(Ty^*, T^2y^*\big) < d\big(y^*, Ty^*\big) = d\big(x^*, Tx^*\big) \le d\big(Tx^*, T^2x^*\big) \\ = d\big(Ty^*, T^2y^*\big),$$

which is a contradiction. This implies that $Ty^* = T^2y^*$, from which it follows that $x^* = Tx^*$, $y^* = Ty^*$. □

The next theorem establishes the existence of solution of (1) under some other conditions.

Theorem 3.3 *Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) such that A is compact and B is closed. Let $A_0 \neq \emptyset$ and (A, B) satisfy the P-property and let $T : A \cup B \to A \cup B$ be a noncyclic mapping. Then the minimization problem (1) has a solution provided that the following conditions are satisfied*:

(i) *T is relatively nonexpansive,*
(ii) *$T|_A$ is continuous and asymptotically regular.*

Proof Let $\{x_n\}, \{y_n\}, \{x_{n_k}\}, \{y_{n_k}\}, x^*$ and y^* be as in Theorem 3.2. Then we have $x_{n_k} \to x^* \in A$, $y_{n_k} \to y^* \in B$, and $d(x^*, y^*) = \mathrm{dist}(A, B)$. Since $T|_A$ is continuous, $x_{n_k+1} = T(x_{n_k}) \to Tx^*$. On the other hand, by the asymptotic regularity of $T|_A$, we obtain

$$d\big(x^*, Tx^*\big) = \lim_{k\to\infty} d\big(x_{n_k}, T(x_{n_k})\big) \\ = \lim_{k\to\infty} d\big(T^{n_k}(x_0), T^{n_k+1}(x_0)\big) \le \limsup_{n\to\infty} d\big(T^n(x_0), T^{n+1}(x_0)\big) = 0.$$

This implies that $Tx^* = x^*$. Since T is relatively nonexpansive, we conclude that

$$d\big(Tx^*, Ty^*\big) \le d\big(x^*, y^*\big) = \mathrm{dist}(A, B).$$

Using the P-property of (A, B), we conclude that $d(x^*, Tx^*) = d(y^*, Ty^*)$. Hence, $Ty^* = y^*$. □

Let us illustrate the above theorems with the following examples.

Example 3.1 Let $X := \mathbb{R}$ with the usual metric, and $A := [-2, 0]$, $B := \mathbb{N}$. We note that (A, B) has the P-property, by the fact that A_0 and B_0 have exactly one element. Define $T : A \cup B \to A \cup B$ as follows:

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in A, \\ \frac{x+1}{2}, & \text{if } x \in \mathbb{N}_o, \\ x - 1, & \text{if } x \in \mathbb{N}_e, \end{cases}$$

where $\mathbb{N}_o, \mathbb{N}_e$ are the set of odd natural numbers and even natural numbers, respectively. It is easy to see that T is noncyclic on $A \cup B$ and $T(A_0) \subseteq A_0$. Moreover, $T|_A$

is contraction. We show that T is relatively nonexpansive on $A \cup B$. Let $x \in A$ and $y \in B$. If $y \in \mathbb{N}_o$, then

$$d(Tx, Ty) = \left|\frac{x}{2} - \frac{y+1}{2}\right| = \frac{1}{2}(y-x) + \frac{1}{2} \leq (y-x) = d(x,y),$$

by the choice of x, y. If $y \in \mathbb{N}_e$, then

$$|Tx - Ty| = \left|\frac{x}{2} - (y-1)\right| = y - \frac{x}{2} - 1 \leq |y-x|.$$

Therefore, all conditions of Theorem 3.1 hold, and hence (1) has a solution. It is clear that $(x^*, y^*) = (0, 1)$ is the solution of (1).

Example 3.2 Suppose that $X := \mathbb{R}^2$ with the metric

$$d\big((x,y),(x',y')\big) = \max\big\{|x-x'|, |y-y'|\big\}.$$

Let $A := \{(x,0) : -1 \leq x \leq 1\}$, $B := \{(0,y) : 0 \leq y \leq 1\}$ and define $T : A \cup B \to A \cup B$ by

$$T(x,0) = (-x, 0) \quad \text{and} \quad T(0,y) = \left(0, \frac{y}{2}\right).$$

Since $\operatorname{dist}(A, B) = 0$, the pair (A, B) has the P-property. It is easy to check that all the conditions of Theorem 3.2 hold. Thus, the minimization problem (1) has a solution, and this solution is $(x^*, y^*) = (0, 0)$.

Example 3.3 Let $X := \mathbb{R}$ with the usual metric and let $A := [-\frac{\pi}{2}, 0]$, and $B := [2, 3]$. Then (A, B) has the P-property. Consider the noncyclic mapping $T : A \cup B \to A \cup B$ with

$$T(x) = \begin{cases} \frac{x+\sin(x)}{2}, & \text{if } x \in A, \\ \frac{x+2}{2}, & \text{if } x \in B. \end{cases}$$

Then $T|_A$ is continuous, and since the function $f(x) := \sin(x)$ is nonexpansive on A, $T|_A$ is asymptotically regular by Example 2.4. Also, T is relatively nonexpansive. Indeed, if $x \in A$, $y \in B$, then

$$|Tx - Ty| = \left|\frac{x+\sin(x)}{2} - \frac{y+2}{2}\right| = \frac{1}{2}(y-x) + \frac{1}{2}\big(2 - \sin(x)\big) \leq (y-x),$$

by the choice of x, y. Hence, by Theorem 3.3, (1) has a solution, and obviously this solution is $(x^*, y^*) = (0, 2)$.

4 Application to Complex Function Theory

Theorem 4.1 *Let A and B be nonempty, compact, and convex subsets of a domain D of the complex plane. Let $f(z)$ and $g(z)$ be functions in D such that $f(z)$ is analytic in D. Suppose that*

(a) *$f(A) \subseteq A$ and $g(B) \subseteq B$,*

(b) $|f'(z)| < 1$, *for all* $z \in A$,
(c) $|f(z_1) - g(z_2)| \le |z_1 - z_2|$, *for* $z_1 \in A$ *and* $z_2 \in B$.

Then there exist $z_1^* \in A$ *and* $z_2^* \in B$ *such that*

$$z_1^* = f(z_1^*), \qquad z_2^* = g(z_2^*) \quad \textit{and} \quad |z_1^* - z_2^*| = \operatorname{dist}(A, B).$$

Proof By Example 2.3, $A_0 \neq \emptyset$ and (A, B) has the P-property. Since $|f'(z)|$ is continuous on the compact set A, it attains its maximum at some point, say $z_1^* \in A$. Let $k = |f'(z_1^*)|$. Then $k < 1$. Hence, for all $z \in A$, we have $|f'(z)| \le k < 1$. Now for all $z, w \in A$, we have

$$|f(z) - f(w)| = \left| \int_w^z f'(\xi)\, d\xi \right| \le k|z - w|.$$

So, $f(z)$ is a contraction on A. Now, if we define $T : A \cup B \to A \cup B$ with

$$T(z) = \begin{cases} f(z), & \text{if } z \in A, \\ g(z), & \text{if } z \in B, \end{cases}$$

then the result follows by invoking Theorem 3.1. □

Remark 4.1 It is interesting to note that if in Theorem 4.1, $|f'(z)| \le 1$, for all $z \in A$, then the conclusion is valid.

Proof Define the noncyclic mapping $T : A \cup B \to A \cup B$ with

$$T(z) = \begin{cases} \frac{z + f(z)}{2}, & \text{if } z \in A, \\ g(z), & \text{if } z \in B. \end{cases}$$

Since $|f'(z)| \le 1$ for all $z \in A$, a similar argument as in Theorem 4.1 implies that f is nonexpansive on A. Thus $T|_A$ is asymptotically regular and continuous. Now, it follows from Theorem 3.3 that there exists $(z_1^*, z_2^*) \in A \times B$ such that $T(z_1^*) = z_1^*$, $T(z_2^*) = z_2^*$ and $|z_1^* - z_2^*| = \operatorname{dist}(A, B)$. That is,

$$z_1^* = f(z_1^*), \qquad z_2^* = g(z_2^*) \quad \text{and} \quad |z_1^* - z_2^*| = \operatorname{dist}(A, B).$$ □

The following result for analytic functions is a special case of the preceding argument. This will improve Corollary 3.2 of [14].

Corollary 4.1 *Let A be a nonempty, compact and convex subset of a domain D of the complex plane. Let $f(z)$ be an analytic function in D. Suppose that*

(a) $f(A) \subseteq A$,
(b) $|f'(z)| \le 1$, *for all* $z \in A$.

Then the fixed point equation $f(z) = z$ *has at least one solution in* A.

5 Concluding Remarks

In [11], the authors solved the minimization problem (1) by introducing a geometric property on pairs of subsets in a Banach space X, namely, the proximal normal structure. In the current paper, we replaced the Banach space X by a metric space, and used the P-property already introduced in [12]. Finally, in the last section we have supplied an application to complex function theory.

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