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On Well-Posedness and Hausdorff Convergence of Solution Sets of Vector Optimization Problems

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Abstract In this paper, we refine and improve the results established in a 2003 paper by Deng in a number of directions. Specifically, we establish a well-posedness result for convex vector optimization problems under a condition which is weaker than that used in the paper. Among other things, we also obtain a characterization of well-posedness in terms of Hausdorff distance of associated sets.

Keywords Well-posedness · Weakly efficient solution · Hausdorff distance · Convexity

1 Introduction

Vector optimization plays an important role in many branches of applied sciences. Mathematically, the problem is to minimize (in certain sense) a vector-valued function over some feasible region. For the clarity of exposition, we assume throughout that each component of the function is continuous over some finite-dimensional Euclidean space, and the feasible region is a nonempty, closed subset of the Euclidean space. When each component is a finite convex function, and the feasible region is a convex set, we say that the problem is a *convex vector optimization* problem (CVOP for short).

This research is mainly devoted to furthering the study of well-posedness of the vector optimization problem considered by Deng [1]. It is well understood in the optimization community that well-posedness plays an important role in both sensitivity analysis and convergence analysis of a wide range of numerical optimization methods. The literature on well-posedness for scalar optimization problems is rich, and

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the reader is directed to the monograph by Dontchev and Zolezzi [2] for motivations, examples, and theories.

The study of well-posedness on vector optimization has gained momentum in recent years by various scalarization techniques. Along this line of research, the existing results can be classified into two groups: the former group is concerned with some strong forms (e.g., weak sharp minima) of well-posedness with more restrictive assumptions, and the latter group is concerned with weak forms of well-posedness with less restrictive assumptions. A sample of results for the former group includes the weak sharp minimality property for multicriteria linear and piecewise linear programming problems (see, e.g., Deng and Yang [3], Zheng and Yang [4], and references therein), and optimality conditions for weak ψ-sharp minima for a local Pareto minimizer (see, e.g., Xu and Li [5] and references therein). There are quite a number of results belonging to the latter group, including categorization of several types of well-posedness (see, e.g., Miglierina, Molho, and Rocca [6] and references therein), and work in generalized finite-dimensional normed spaces (see, e.g., Luo, Huang, and Peng [7] and references therein). Results using other techniques include classification of the relationships between well-posedness of vector optimization problems and well-posedness of vector variational inequalities (see, e.g., Crespi, Guerraggio, and Rocca [8], and references therein).

Here, we review the results directly related to the current research. In a 2003 paper by Deng [1], he initiated the study of "global" Tykhonov well-posedness for convex vector optimization problems. A key result (Theorem 3.3 of [1]) states that, for convex vector optimization problems, the "global" Tykhonov well-posedness holds whenever each associated scalar optimization problem is globally well-posed. This is significant since a number of verifiable sufficient conditions for well-posedness are well known for scalar optimization problems [9]. Additional results along these lines can also be found in [7] and references therein. We improve and refine the results in [1] in a number of directions. Under a weaker assumption, we establish the conclusions of Theorem 3.3 in [1]. We illustrate by examples that the obtained results are the sharpest possible. See Theorem 3.2 and Examples 3.1 and 2.1 for more details. Furthermore, we examine the interrelations of four properties: Hausdorff convergence of the approximate solution sets of the scalarized problem, well-posedness of the scalar problem, well-posedness of the vector problem, and Hausdorff convergence of the approximate solution sets of the vector problem. Our contributions include, among other things, a sufficient condition for epi-convergence of scalarized problems, and a characterization of well-posedness in terms of Hausdorff distance of associated sets.

2 Preliminaries

For notational clarity, we denote the aforementioned vector optimization problem as follows:

$$(\mathcal{P}) \quad \min\ F(x) \quad \text{subject to} \quad x \in X,$$

where $F(x) = (f_1(x), \dots, f_l(x)) : \mathbb{R}^m \to \mathbb{R}^l$ is continuous, and $X \subset \mathbb{R}^m$ is a nonempty and closed set.

It is well known that there are several solution concepts associated with $(\mathcal{P})$ [10].

In this paper, we consider the notion of weakly efficient set $\mathrm{WEff}(0, F, X)$ for $(\mathcal{P})$. Recall that

$$\mathrm{WEff}(0, F, X) := \left\{\overline{x} \in X \mid F(x) - F(\overline{x}) \notin -\operatorname{int}\mathbb{R}^l_+, \ \forall x \in X\right\}.$$

Throughout, we assume that $\mathrm{WEff}(0, F, X)$ is nonempty.

Also, we define the ϵ-weakly efficient solution set

$$\mathrm{WEff}(\epsilon, F, X) := \left\{\overline{x} \in X \mid F(x) + \epsilon\mathbf{1} - F(\overline{x}) \notin -\operatorname{int}\mathbb{R}^l_+, \ \forall x \in X\right\},$$

where $\mathbf{1}$ is the vector in $\mathbb{R}^l$ with all components equal to 1.

For a given $\lambda \in \Lambda$, we define a scalarized problem $(\mathcal{P}(\lambda))$ as

$$\big(\mathcal{P}(\lambda)\big) \quad \min\ \lambda^T F(x) \quad \text{subject to} \quad x \in X,$$

where F and X are as above, and where Λ is the $l-1$ simplex (the set of all vectors in $\mathbb{R}^l_+$, with the sum of the entries totaling 1).

For $\epsilon \geq 0$, we denote the ϵ-approximate solution set to the scalarized problem $(\mathcal{P}(\lambda))$ by

$$S_\lambda(\epsilon, F, X) := \operatorname*{argmin}_{x\in X} \lambda^T F(x) = \left\{\overline{x} \in X \mid \lambda^T F(\overline{x}) \leq \inf_{x\in X} \lambda^T F(x) + \epsilon\right\}.$$

To quantify the difference between the ϵ-approximate solution sets and the ϵ-weakly efficient solution sets, we use convergence in the sense of Hausdorff distance. Given $d \in \mathbb{R}^m$ and $D \subset \mathbb{R}^m$, we define

$$\operatorname{dist}(d, D) := \inf_{d'\in D}\left\{\|d - d'\|\right\},$$

where $\|\cdot\|$ is the Euclidean distance.

Definition 2.1 For nonempty subsets C and $D \subset \mathbb{R}^m$, the Hausdorff distance between them is defined as

$$\operatorname{Haus}(C, D) := \max\left\{\sup_{c\in C} \operatorname{dist}(c, D), \sup_{d\in D} \operatorname{dist}(d, C)\right\}.$$

Definition 2.2 We say that problem $(\mathcal{P})$ is well-posed if and only if for any sequence $\{x_n\} \subset X$, the following holds:

$$\begin{aligned} &\left[\operatorname{dist}\big(F(x_n), F\big(\mathrm{WEff}(0, F, X)\big)\big) \to 0 \text{ as } n \to \infty\right] \\ &\quad \Rightarrow \left[\operatorname{dist}\big(x_n, \mathrm{WEff}(0, F, X)\big) \to 0 \text{ as } n \to \infty\right]. \end{aligned} \tag{1}$$

Note that this is slightly different from the definition from [1] because we do not require $F(\mathrm{WEff}(0, F, X))$ to be closed. The following example demonstrates that the image of the weakly efficient solution set need not always be closed.

Example 2.1 Suppose $F:\mathbb{R}^2\to\mathbb{R}^2$ to be defined by $f_1(x,y)=x^2/y$ and $f_2(x,y)=|x-1/y|$, where $X=[0,\infty[\times[1,\infty[$.

The set $\mathrm{WEff}(0,F,X)=(\{0\}\times[1,\infty[)\cup\{(x,1/x):0<x\le 1\}$ is closed. But its image is $(\{0\}\times]0,1])\cup(]0,1]\times\{0\})$, which is not closed.

Definition 2.3 For a given $\overline{\lambda}\in\Lambda$, we define $(\mathcal{P}(\overline{\lambda}))$ to be well-posed if and only if $\inf_{x\in X}\lambda^T F(x)$ is finite,

$$S_{\overline{\lambda}}(0,F,X)\neq\emptyset,$$

and for any sequence $\{x_n\}\subset X$ we have that

$$\begin{aligned}&\Big[\overline{\lambda}^T F(x_n)\to\inf_{x\in X}\overline{\lambda}^T F(x)\text{ as }n\to\infty\Big]\\ &\quad\Rightarrow\quad\big[\mathrm{dist}\big(x_n,S_{\overline{\lambda}}(0,F,X)\big)\to 0\text{ as }n\to\infty\big].\end{aligned}\tag{2}$$

We say problem $(\mathcal{P}(\lambda))$ is well-posed over Λ if and only if it is well-posed for all $\lambda\in\Lambda$.

The following proposition is a very useful characterization of epiconvergence and will be used in the next section to prove one of our main results, Proposition 3.1.

Proposition 2.1 [11] *Suppose g_n to be a sequence of real-valued functions on $\mathbb{R}^m$. We have that $g_n\xrightarrow{e}g$ if and only if at each point $x\in\mathbb{R}^m$ both of the following conditions hold*:

$$\liminf_n g_n(x_n)\ge g(x)\quad\textit{for every sequence } x_n\to x,\tag{3}$$

$$\limsup_n g_n(x_n)\le g(x)\quad\textit{for some sequence } x_n\to x.\tag{4}$$

Moreover, if it is true that $e\text{-}\limsup_n(g_n)\le g$, *then it is also true that* $\limsup_n(\inf g_n)\le\inf g$, *where we define*

$$\mathrm{epi}\Big\{e\text{-}\limsup_n g_n\Big\}=\liminf_n\big(\mathrm{epi}\{g_n\}\big).$$

The following theorem is also needed for Proposition 3.1. We use $g_n\xrightarrow{CC}g$ to denote continuous convergence of functions.

Theorem 2.1 [11] *Consider a sequence of functions $\{g_n\}$ and a function g, each from $\mathbb{R}^m$ to $\mathbb{R}$. If $g_n\xrightarrow{CC}g$, then $g_n\xrightarrow{e}g$.*

Unless otherwise stated, the notations used in this paper are standard [11].

3 Main Results

The first of our main results relates and compares the solution sets of the scalarized problem to well-posedness of the scalarized problem.

To quantify the difference between these sets, we use convergence in the sense of Hausdorff distance, which is not a universal choice.

Theorem 3.1 does not use any kind of convexity assumptions. Although Theorem 3.1 can be stated in a more general form, we prefer this format for the upcoming Corollary 3.1.

Theorem 3.1 *Suppose* $\lambda \in \Lambda$ *to be fixed and* $S_\lambda(0, F, X) \neq \emptyset$. *We have that*

$$\mathrm{Haus}\big(S_\lambda(0, F, X), S_\lambda(\epsilon, F, X)\big) \to 0 \quad \text{as } \epsilon \downarrow 0$$

if and only if $(\mathcal{P}(\lambda))$ *is well-posed.*

Proof First we show one direction, that Hausdorff convergence implies well-posedness. Suppose $\{x_n\}$ to be a sequence in X such that

$$\lambda^T F(x_n) \to \inf_{x\in X} \lambda^T F(x) \quad \text{as } n \to \infty.$$

For each n, define $\epsilon_n = \lambda^T F(x_n) - \inf_{x\in X} \lambda^T F(x)$. So $x_n \in S_\lambda(\epsilon_n, F, X)$. Consider

$$\begin{aligned}\mathrm{dist}\big(x_n, S_\lambda(0, F, X)\big) &\leq \sup_{y\in S_\lambda(\epsilon_n, F, X)} \mathrm{dist}\big(y, S_\lambda(0, F, X)\big)\\ &= \mathrm{Haus}\big(S_\lambda(0, F, X), S_\lambda(\epsilon_n, F, X)\big)\\ &\to 0 \quad \text{as } n \to \infty \text{ by hypothesis.}\end{aligned}$$

Therefore, $(\mathcal{P}(\lambda))$ is well-posed.

Now we show the other direction that well-posedness implies Hausdorff convergence. We proceed by contradiction. Suppose that

$$\mathrm{Haus}\big(S_\lambda(0, F, X), S_\lambda(\epsilon_n, F, X)\big) \not\to 0 \quad \text{for some } \epsilon_n \downarrow 0.$$

Since $S_\lambda(0, F, X) \subseteq S_\lambda(\epsilon_n, F, X)$, the Hausdorff distance can be expressed simply by

$$\sup_{x\in S_\lambda(\epsilon_n, F, X)} \big\{\mathrm{dist}\big(x, S_\lambda(0, F, X)\big)\big\},$$

so there exists a sequence $\{x_n\} \subset X$ such that each $x_n \in S_\lambda(\epsilon_n, F, X)$, and we have that

$$\mathrm{dist}\big(x_n, S_\lambda(0, F, X)\big) \geq \Delta$$

for some $\Delta > 0$ and for all sufficiently large n.

By location of x_n, we have that

$$\lambda^T F(x_n) \le \inf_{x\in X} \lambda^T F(x) + \epsilon_n,$$

$$\lim_{n\to\infty} \lambda^T F(x_n) \le \inf_{x\in X} \lambda^T F(x).$$

And since $(\mathcal{P}(\lambda))$ is well-posed, we have that $\operatorname{dist}(x_n, S_\lambda(0, F, X)) \to 0$ as $n \to \infty$. This is a contradiction. □

The following proposition is needed for the proof of Lemma A.1, which is needed for the proof of Theorem 3.2. However, Proposition 3.1 may be of independent interest, as it deals with the well-known notion of epiconvergence.

Proposition 3.1 *Suppose, for problem* $(\mathcal{P}(\lambda))$, *each component of* F *to be continuous.*

If $\lambda_n \to \overline{\lambda}$, *then* $\lambda_n^T F \xrightarrow{CC} \overline{\lambda}^T F$, *and in particular, by Theorem* 2.1 [11], *we get that* $\lambda_n^T F \xrightarrow{e} \overline{\lambda}^T F$.

Proof For any sequence of points $\{x_n\} \subset \mathbb{R}^m$ approaching some $\overline{x} \in \mathbb{R}^m$, consider

$$\begin{aligned}&\left|\lambda_n^T F(x_n) - \overline{\lambda}^T F(\overline{x})\right| \\ &\quad \le \left|\lambda_n^T F(x_n) - \lambda_n^T F(\overline{x})\right| + \left|\lambda_n^T F(\overline{x}) - \overline{\lambda}^T F(\overline{x})\right|.\end{aligned}$$

The first term goes to zero since F is continuous and $\{\lambda_n\} \subset \Lambda$ is bounded. The second term goes to zero since $\lambda_n \to \overline{\lambda}$. Thus $\lambda_n^T F(x_n) \to \overline{\lambda}^T F(\overline{x})$, and so $\lambda_n^T F \xrightarrow{CC} \overline{\lambda}^T F$. □

The following theorem expands on Theorem 3.3 [1] by removing the requirement that the set $F(\operatorname{WEff}(0, F, X))$ be closed. In this proof, we refer to the conclusion of Lemma A.1 from Appendix A.

Theorem 3.2 *Suppose* $(\mathcal{P})$ *to be a CVOP, and that* $S_\lambda(0, F, X) \ne \emptyset$ *for all* $\lambda \in \Lambda$. *If* $(\mathcal{P}(\lambda))$ *is well-posed over* Λ, *then* $(\mathcal{P})$ *is well-posed in the sense of* (1).

Proof Suppose $\{x_n\} \subset X$ to be a sequence with

$$\operatorname{dist}\bigl(F(x_n), F\bigl(\operatorname{WEff}(0, F, X)\bigr)\bigr) \to 0 \quad \text{as } n \to \infty. \tag{5}$$

We do not require $F(\operatorname{WEff}(0, F, X))$ to be closed like in Deng's 2003 Theorem 3.3 [1], but for each n there exists a point $a_n \in \operatorname{WEff}(0, F, X)$ such that

$$\bigl\|F(x_n) - F(a_n)\bigr\| \le \operatorname{dist}\bigl(F(x_n), F\bigl(\operatorname{WEff}(0, F, X)\bigr)\bigr) + \frac{1}{n}. \tag{6}$$

So, by statements (5) and (6),

$$\bigl\|F(x_n) - F(a_n)\bigr\| \to 0 \quad \text{as } n \to \infty. \tag{7}$$

By Theorem 2.1 in [12], which states that for $\epsilon \geq 0$ and $(\mathcal{P})$ a CVOP, $\mathrm{WEff}(\epsilon, F, X) = \bigcup_{\lambda \in \Lambda} S_\lambda(\epsilon, F, X)$, and thus we have that for each n there exists $\lambda_n \in \Lambda$ such that $a_n \in S_{\lambda_n}(0, F, X)$. By boundedness of $\{\lambda_n\}$, without loss of generality, we can say that $\lambda_n \to \overline{\lambda} \in \Lambda$.

By Lemma A.1, we have that $\overline{\lambda}^T F(a_n) \to \inf_{x\in X} \overline{\lambda}^T F(x)$ as $n \to \infty$. And then by (7), we have that $\overline{\lambda}^T F(x_n) \to \inf_{x\in X} \overline{\lambda}^T F(x)$ as $n \to \infty$.

Again, by Theorem 2.1 in [12], we have that $S_{\overline{\lambda}}(0, F, X) \subseteq \mathrm{WEff}(0, F, X)$, and so

$$\begin{aligned} \operatorname{dist}\big(x_n, \mathrm{WEff}(0, F, X)\big) &\leq \operatorname{dist}\big(x_n, S_{\overline{\lambda}}(0, F, X)\big) \\ &\to 0 \quad \text{as } n \to \infty \text{ since } \big(\mathcal{P}(\lambda)\big) \text{ is well-posed over } \Lambda. \end{aligned}$$

Therefore, $(\mathcal{P})$ is well-posed, as desired. □

Various sufficient conditions under which the assumptions of Theorem 3.2 hold are given in [1].

By Theorems 3.1 and 3.2, we get the following corollary.

Corollary 3.1 *Suppose $(\mathcal{P})$ to be a CVOP, and $S_\lambda(0, F, X)$ to be nonempty for all $\lambda \in \Lambda$. If*

$$\mathrm{Haus}\big(S_\lambda(0, F, X), S_\lambda(\epsilon, F, X)\big) \to 0 \quad \textit{as } \epsilon \downarrow 0$$

for all $\lambda \in \Lambda$, then $(\mathcal{P})$ is well-posed in the sense of (1).

The following corollary is a direct consequence of Theorem 3.2 and Corollary 2.2 in [9].

Corollary 3.2 *If for $(\mathcal{P})$, each f_i is convex quadratic function (in other words, $f_i(x) = \frac{1}{2}x^T B_i x + c_i^T x$ where each B_i is a symmetric, positive semi-definite $m \times m$ matrix, and the c_i are m-dimensional vectors), and if X is a convex polyhedral set, and each $\operatorname{argmin}_{x\in X} f_i \neq \emptyset$, then $(\mathcal{P}(\lambda))$ is well-posed over Λ, and thus $(\mathcal{P})$ is well-posed in the sense of* (1).

The following is a nontrivial vector example that shows there is no converse to Theorem 3.2 in the situation where F is convex on X but not on all of $\mathbb{R}^l$. We have that $(\mathcal{P})$ is well-posed in the sense of (1), but $(\mathcal{P}(\lambda))$ not well-posed for some $\lambda \in \Lambda$.

Example 3.1 Suppose $F(x_1, x_2) = (\frac{x_1^2}{x_2}, x_1) : \mathbb{R}^2 \to \mathbb{R}^2$ and that the feasible region $X = [0, \infty[\times [1, \infty[\subset \mathbb{R}^2$.

The first component of this function is convex on X; a proof is provided in Appendix B.

The weakly efficient solution set of F on X is the set $\{0\} \times [1, \infty[$, but its image is simply the origin.

Also, note that for $\lambda = (1, 0)$, it is not true that $(\mathcal{P}(\lambda))$ is well-posed. Consider the sequence of points $\{(1, n)\} \subset X$. It is true that

$$\lambda^T F(1, n) = \frac{1}{n} \to 0 = \inf_{x \in X} \lambda^T F(x).$$

However, $\operatorname{dist}((1, n), S_\lambda(0, F, X)) = 1$ for all n.

It is clear that $(\mathcal{P})$ is well-posed in the sense of (1).

In the upcoming Corollary 3.3, we compare well-posedness of the scalar problem and Hausdorff convergence of the ϵ-weakly efficient solution sets, using the condition that

$$\operatorname{Haus}\big(F\big(\operatorname{WEff}(0, F, X)\big), F\big(\operatorname{WEff}(\epsilon, F, X)\big)\big) \to 0 \quad \text{as } \epsilon \downarrow 0. \tag{8}$$

In the case where F is a scalar function it is clear that having $(\mathcal{P})$ be well-posed is equivalent to having condition (8) imply Hausdorff convergence of the solution sets.

The following example justifies that it is reasonable in Corollary 3.3 to assume that condition (8) holds.

Example 3.2 Suppose $F : \mathbb{R} \to \mathbb{R}^2$ to be defined by $F(x) = (x^2, (x-1)^2)$, and that $X = \mathbb{R}$. We see the following:

$$\begin{aligned}
&\operatorname{WEff}(0, F, X) = [0, 1],\\
&\operatorname{WEff}(\epsilon, F, X) = [-\sqrt{\epsilon}, 1 + \sqrt{\epsilon}],\\
&\operatorname{Haus}\big(F\big(\operatorname{WEff}(0, F, X)\big), F\big(\operatorname{WEff}(\epsilon, F, X)\big)\big) \to 0 \quad \text{as } \epsilon \downarrow 0.
\end{aligned}$$

Note that $F(\operatorname{WEff}(0, F, X))$ is an arc in the first quadrant of $\mathbb{R}^2$, and that $F(\operatorname{WEff}(\epsilon, F, X))$ is a slightly longer arc.

Corollary 3.3 *Suppose $(\mathcal{P})$ to be a CVOP and*

$$\operatorname{Haus}\big(F\big(\operatorname{WEff}(0, F, X)\big), F\big(\operatorname{WEff}(\epsilon, F, X)\big)\big) \to 0 \quad \textit{as } \epsilon \downarrow 0.$$

If $(\mathcal{P}(\lambda))$ is well-posed over Λ, then

$$\operatorname{Haus}\big(\operatorname{WEff}(0, F, X), \operatorname{WEff}(\epsilon, F, X)\big) \to 0 \quad \textit{as } \epsilon \downarrow 0.$$

Proof First, we must show that $(\mathcal{P})$ being well-posed in the sense of (1) is equivalent to the following:

$$\begin{aligned}
&\big[\operatorname{Haus}\big(F\big(\operatorname{WEff}(\epsilon, F, X)\big), F\big(\operatorname{WEff}(0, F, X)\big)\big) \to 0 \text{ as } \epsilon \downarrow 0\big]\\
&\quad \Rightarrow \quad \big[\operatorname{Haus}\big(\operatorname{WEff}(\epsilon, F, X), \operatorname{WEff}(0, F, X)\big) \to 0 \text{ as } \epsilon \downarrow 0\big].
\end{aligned}$$

Suppose not, so $\operatorname{Haus}(F(\operatorname{WEff}(\epsilon, F, X)), F(\operatorname{WEff}(0, F, X))) \to 0$ as $\epsilon \downarrow 0$; however, we have that $\operatorname{Haus}(\operatorname{WEff}(\epsilon, F, X), \operatorname{WEff}(0, F, X)) \not\to 0$. So, there exists a real sequence $\{\epsilon_n\}$ decreasing to zero and a sequence $\{x_n\} \subset X$ such that each $x_n \in \mathrm{WEff}(\epsilon_n, F, X)$, but

$$\mathrm{dist}\big(x_n, \mathrm{WEff}(0, F, X)\big) \geq \Delta$$

for some $\Delta > 0$ and for all large n. Therefore, since $(\mathcal{P})$ is well-posed, we have that

$$\mathrm{dist}\big(F(x_n), F\big(\mathrm{WEff}(0, F, X)\big)\big) \geq \hat{\Delta}$$

for some $\hat{\Delta} > 0$ and for all large n. However, consider

$$\begin{aligned}
&\mathrm{dist}\big(F(x_n), F\big(\mathrm{WEff}(0, F, X)\big)\big) \\
&\quad \leq \sup_{x \in \mathrm{WEff}(\epsilon_n, F, X)} \mathrm{dist}\big(F(x), F\big(\mathrm{WEff}(0, F, X)\big)\big) \\
&\quad = \mathrm{Haus}\big(F\big(\mathrm{WEff}(\epsilon_n, F, X)\big), F\big(\mathrm{WEff}(0, F, X)\big)\big) \\
&\quad \to 0
\end{aligned}$$

by hypothesis.

The rest of the proof follows from Theorem 3.2. □

4 Concluding Remarks

In this paper, we have established a number of new results for global Tykhonov well-posedness of $(\mathcal{P})$. Specifically, among other things, we show in Theorem 3.2 that $(\mathcal{P})$ is well-posed if $(\mathcal{P})$ is a CVOP and $(\mathcal{P}(\lambda))$ is well-posed for all $\lambda \in \Lambda$. Theorem 3.2 improves Theorem 3.3 of [1] by removing the closedness assumption of $F(\mathrm{WEff}(0, F, X))$. As demonstrated in [1], the verification of the closedness of $F(\mathrm{WEff}(0, F, X))$ is a nontrivial issue. Therefore, the improvement is useful and it allows us to study well-posedness of a vector optimization problem via that of its associated scalar optimization problems. Furthermore, we display an example of non-closedness of $F(\mathrm{WEff}(0, F, X))$ (see Example 2.1). For scalar optimization problem $(\mathcal{P}(\lambda))$, in Theorem 3.1, we provide a new characterization of well-posedness in terms of the convergence of Hausdorff distances of its ϵ-approximate solutions. Finally, we work out a number of consequences of Theorems 3.1 and 3.2 (see Corollaries 3.2 and 3.3).

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Appendix A

The following is a technical lemma.

Lemma A.1 *Suppose* $F : X \subset \mathbb{R}^m \to \mathbb{R}^l$, $(\mathcal{P})$ *to be a CVOP*, $\mathrm{argmin} f_i \neq \emptyset$ *for each* i, *and* $\lambda_n, \bar{\lambda} \in \Lambda$.

If $\lambda_n \to \overline{\lambda}$ *and* $\{a_n\} \subset X$ *such that each* $a_n \in S_{\lambda_n}(0, F, X)$*, then* $\overline{\lambda}^T F(a_n) \to \inf_{x\in X} \overline{\lambda}^T F(x)$ *as* $n \to \infty$.

Proof By Proposition 3.1, since $\lambda_n \to \overline{\lambda}$, we have that $\lambda_n^T F \xrightarrow{e} \overline{\lambda}^T F$. Thus e-$\limsup_n \lambda_n^T F =$ e-$\lim_n \lambda_n^T F = \overline{\lambda}^T F$. Now, we can apply the second part of Proposition 2.1 to get that then $\limsup_n(\inf \lambda_n^T F) \le \inf \overline{\lambda}^T F$, and thus we can say for any $\epsilon > 0$,

$$\inf_{x\in X} \lambda_n^T F(x) \le \inf_{x\in X} \overline{\lambda}^T F(x) + \epsilon \tag{9}$$

for sufficiently large n.

Denote the ith component of $\overline{\lambda}$ by $\overline{\lambda}_i$, and denote the ith component of λ_n by $\lambda_i(n)$. Define $I = \{i \in [1,l] : \overline{\lambda}_i \ne 0\}$, the set indices of the components of $\overline{\lambda}$ which are nonzero, and define $I' = [1,l]\backslash I$ as the complement of I, the set indices of the components of $\overline{\lambda}$ which are zero. We use $[1,l]$ to denote the set of all integers between 1 and l, inclusive.

Because each $\operatorname{argmin}_{x\in X} f_i(x)$ is nonempty, each f_i is bounded below on the set X for $i \in [1,l]$, so we can assume that each f_i is bounded below on $\{x \in \mathbb{R}^m : \operatorname{dist}(x, X) \le \delta\}$ for sufficiently small δ. Thus, we have that each collection $\{f_i(a_n)\}_{n=1}^{\infty}$ is bounded below in particular.

Since each f_i is bounded below and there are only finitely many i, there exists $\alpha \in \mathbb{R}$ such that

$$f_i(x) \ge \alpha, \quad \forall i \in [1,l],\ \forall x \in X. \tag{10}$$

In particular, if we wish, we can define α as $\min_{i\in[1,l]}(\inf_{x\in X} f_i(x))$. So, now we know that $f_i(a_n) \ge \alpha$ for all $i \in [1,l]$ and for all n.

Thus, by that fact, and since every component of λ_i is nonnegative, we have that

$$\sum_{i\in I'} \lambda_i(n) f_i(a_n) \ge \alpha \sum_{i\in I'} \lambda_i(n). \tag{11}$$

So, by (11) and location of a_n,

$$\begin{aligned}
\alpha \sum_{i\in I'} \lambda_i(n) + \sum_{i\in I} \lambda_i(n) f_i(a_n) &\le \sum_{i\in I'} \lambda_i(n) f_i(a_n) + \sum_{i\in I} \lambda_i(n) f_i(a_n) \\
&= \sum_{i\in I\cup I'} \lambda_i(n) f_i(a_n) \\
&= \lambda_n^T F(a_n) \\
&= \inf_{x\in X} \lambda_n^T F(x).
\end{aligned} \tag{12}$$

Recall that $\lambda_i(n) \to \overline{\lambda}_i > 0$ as $n \to \infty$ for all $i \in I$, and that $\lambda_i(n) \to \overline{\lambda}_i = 0$ as $n \to \infty$ for all $i \in I'$. So,

$$\alpha \sum_{i\in I'} \lambda_i(n) \to 0 \quad \text{as } n \to \infty. \tag{13}$$

Therefore, by (12), (13), and (9), we get

$$\sum_{i\in I}\lambda_i(n)f_i(a_n)\le \inf_{x\in X}\lambda_n^T F(x)\le \inf_{x\in X}\overline{\lambda}^T F(x)+\epsilon \tag{14}$$

for some $\epsilon>0$ and for sufficiently large n.

Since $\inf_{x\in X}\overline{\lambda}^T F(x)$ is a constant, by (14) we see that there exists a β such that

$$\sum_{i\in I}\lambda_i(n)f_i(a_n)\le\beta \quad \text{for all sufficiently large } n. \tag{15}$$

Now we shall show that there is a γ such that $f_i(a_n)\le\gamma$ for all $i\in I$ and for sufficiently large n.

If there were no such γ, then for some $j\in I$, it would not be true that $f_j(a_n)$ is bounded above (recall that it must be bounded below). So without loss of generality, $f_j(a_n)\to\infty$ as $n\to\infty$. But (15) implies that either $\lambda_j(n)\to 0$, which cannot happen because $j\in I$, or it would imply that $\sum_{i\in I\setminus\{j\}}\lambda_i f_i(a_n)\to-\infty$, which cannot happen because the λ_i are nonnegative and the $f_i(a_n)$ are bounded below. Therefore, for all $i\in I$, $\{f_i(a_n)\}_{n=1}^{\infty}$ is bounded above by some γ. Since it is also bounded below, it is bounded. Because $\lambda_i(n)\to\overline{\lambda}_i$, we have that $\forall\eta>0$, there exists an $N\in\mathbb{N}$ such that for all $n\ge N$, we have

$$\left|\sum_{i\in I}\bigl(\overline{\lambda}_i-\lambda_i(n)\bigr)f_i(a_n)\right|\le\frac{\eta}{2}. \tag{16}$$

Now for all $i\in I'$, we know that $\lambda_i(n)\to 0$ as $n\to\infty$. We also know that $\{f_i(a_n)\}_{i=1}^{l}$ is bounded below. So, for every $\eta>0$, there exists $\hat{N}\in\mathbb{N}$ such that

$$-\frac{\eta}{2}\le\sum_{i\in I'}\lambda_i(n)f_i(a_n),\qquad \forall n\ge\hat{N}. \tag{17}$$

Therefore, for all sufficiently large n, we have

$$0\le\overline{\lambda}^T F(a_n)-\inf_{x\in X}\overline{\lambda}^T F(x) \tag{18}$$

$$=\sum_{i\in I\cup I'}\overline{\lambda}_i f_i(a_n)-\inf_{x\in X}\overline{\lambda}^T F(x) \tag{19}$$

$$=\sum_{i\in I}\overline{\lambda}_i f_i(a_n)+\sum_{i\in I'}\overline{\lambda}_i f_i(a_n)-\inf_{x\in X}\overline{\lambda}^T F(x) \tag{20}$$

$$=\sum_{i\in I}\overline{\lambda}_i f_i(a_n)-\inf_{x\in X}\overline{\lambda}^T F(x)+\sum_{i\in I}\lambda_i(n)f_i(a_n)-\sum_{i\in I}\lambda_i(n)f_i(a_n) \tag{21}$$

$$=\sum_{i\in I}\lambda_i(n)f_i(a_n)-\inf_{x\in X}\overline{\lambda}^T F(x)+\sum_{i\in I}\bigl(\overline{\lambda}_i-\lambda_i(n)\bigr)f_i(a_n) \tag{22}$$

$$= \sum_{i\in I\cup I'} \lambda_i(n) f_i(a_n) - \sum_{i\in I'} \lambda_i(n) f_i(a_n) - \inf_{x\in X} \overline{\lambda}^T F(x)$$
$$+ \sum_{i\in I} \big(\overline{\lambda}_i - \lambda_i(n)\big) f_i(a_n) \tag{23}$$
$$\le \sum_{i\in I\cup I'} \lambda_i(n) f_i(a_n) - \sum_{i\in I'} \lambda_i(n) f_i(a_n) - \inf_{x\in X} \overline{\lambda}^T F(x) + \frac{\eta}{2} \tag{24}$$
$$\le \sum_{i\in I\cup I'} \lambda_i(n) f_i(a_n) + \eta/2 - \inf_{x\in X} \overline{\lambda}^T F(x) + \eta/2 \tag{25}$$
$$\le \sum_{i\in I\cup I'} \lambda_i(n) f_i(a_n) - \inf_{x\in X} \overline{\lambda}^T F(x) + \eta \tag{26}$$
$$= \lambda_n^T F(a_n) - \inf_{x\in X} \overline{\lambda}^T F(x) + \eta. \tag{27}$$

So, we have for large n that

$$0 \le \overline{\lambda}^T F(a_n) - \inf_{x\in X} \overline{\lambda}^T F(x) \le \lambda_n^T F(a_n) - \inf_{x\in X} \overline{\lambda}^T F(x) + \eta,$$

and now we can say that, for large n,

$$\begin{aligned} \inf_{x\in X} \overline{\lambda}^T F(x) &\le \overline{\lambda}^T F(a_n) \quad \text{by definition of infimum,} \\ &\le \lambda_n^T F(a_n) + \eta \quad \text{by above,} \\ &= \inf_{x\in X} \lambda_n^T F(x) + \eta \quad \text{by location of } a_n, \\ &\le \inf_{x\in X} \overline{\lambda}^T F(x) + \eta + \epsilon \quad \text{by (9).} \end{aligned}$$

Thus, since η and ϵ are arbitrary, we have proved our claim that

$$\overline{\lambda}^T F(a_n) \to \inf_{x\in X} \overline{\lambda}^T F(x).$$

This completes the proof. □

Appendix B

Proposition B.1 *The function* $f(x_1, x_2) = \frac{x_1^2}{x_2}$ *is convex on the feasible region* $X = [0, \infty[\times[1, \infty[$.

Proof Consider (x_1, x_2) and $(z_1, z_2) \in X$, and $\lambda \in [0, 1]$. We will show convexity with the "poorly written backwards method," whereby we begin with the statement which we wish to prove, and write equivalent statements until we arrive at a statement that is clearly true.

$$f\big(\lambda\vec{x}+(1-\lambda)\vec{z}\big)\le\lambda f(\vec{x})+(1-\lambda)f(\vec{z})$$
$$\Leftrightarrow\quad \frac{(\lambda x_1+(1-\lambda)z_1)^2}{\lambda x_2+(1-\lambda)z_2}\le\frac{\lambda x_1^2}{x_2}+\frac{(1-\lambda)z_1^2}{z_2}$$
$$\Leftrightarrow\quad \big(\lambda x_1+(1-\lambda)z_1\big)^2x_2z_2\le\big(\lambda x_1^2z_2+(1-\lambda)z_1^2x_2\big)\big(\lambda x_2+(1-\lambda)z_2\big).$$

That last line is true since the denominators are all positive by the definition of X.

$$\Leftrightarrow\quad \lambda^2x_1^2x_2z_2+2\lambda(1-\lambda)x_1x_2z_1z_2+(1-\lambda)^2x_2z_1^2z_2$$
$$\le\lambda^2x_1^2x_2z_2+(1-\lambda)\lambda x_1^2z_2^2+(1-\lambda)\lambda x_2^2z_1^2+(1-\lambda)^2x_2z_1^2z_2$$
$$\Leftrightarrow\quad 2\lambda(1-\lambda)x_1x_2z_1z_2\le\lambda(1-\lambda)\big[x_2^2z_1^2+x_1^2z_2^2\big].$$

This statement is clearly true in the case where $\lambda=0$ or $\lambda=1$. Thus, the only case that remains is when $\lambda\in(0,1)$.

$$\Leftrightarrow\quad 2x_1x_2z_1z_2\le x_2^2z_1^2+x_1^2z_2^2$$
$$\Leftrightarrow\quad x_1x_2z_1z_2+x_1x_2z_1z_2-x_2^2z_1^2-x_1^2z_2^2\le 0$$
$$\Leftrightarrow\quad x_1x_2z_1z_2-x_1^2z_2^2+x_1x_2z_1z_2-x_2^2z_1^2\le 0$$
$$\Leftrightarrow\quad x_1z_2[x_2z_1-x_1z_2]+x_2z_1[x_1z_2-x_2z_1]\le 0$$
$$\Leftrightarrow\quad -x_1z_2[x_1z_2-x_2z_1]+x_2z_1[x_1z_2-x_2z_1]\le 0$$
$$\Leftrightarrow\quad (-x_1z_2+x_2z_1)(x_1z_2-x_2z_1)\le 0$$
$$\Leftrightarrow\quad -(x_1z_2-x_2z_1)(x_1z_2-x_2z_1)\le 0$$
$$\Leftrightarrow\quad -(x_1z_2-x_2z_1)^2\le 0$$

which is clearly true. Thus, our original statement that f was convex on X is true, as desired. □

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