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Optimality Conditions for Optimistic Bilevel Programming Problem Using Convexifactors

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Abstract In this article, we introduce two versions of nonsmooth extension of Abadie constraint qualification in terms of convexifactors and Clarke subdifferential and employ the weaker one to develop new necessary Karush–Kuhn–Tucker type optimality conditions for optimistic bilevel programming problem with convex lower-level problem, using an upper estimate of Clarke subdifferential of value function in variational analysis and the concept of convexifactor.

Keywords Bilevel programming problem · Value function · Convexifactor · Constraint qualifications · Optimality conditions

Mathematics Subject Classification (2000) 90C26 · 90C46

1 Introduction

Bilevel programming lies at the heart of modern optimization theory. The bilevel programming problem studies the two combined optimization problems, where variables of the first (or upper-level) problem are the parameters of the second (or lower-level) problem, and the optimal solution of the second problem is needed to calculate the objective function value of the first problem. Many applications of bilevel programming problems and recent developments on the subject have been discussed by Bard [1] and Dempe [2]. The most important challenge is to develop optimality conditions for the problem. Many researchers have worked in this direction, like Bard [3, 4], Dempe [5, 6], Outrata [7], Ye and Ye [8], Ye and Zhu [9], Yezza [10], Babahadda and Gadhi [11], etc.

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Constraint qualifications play an important role in deriving the Lagrange multiplier rules. Since bilevel programming problems do not satisfy the usual constraint qualifications CQs, such as Slater CQ, Mangasarian–Fromovitz CQ, in order to develop optimality conditions, we need to study those CQs, which can be applied to them. Recently, work has been done in this direction. Ye [13], in her paper, in 2004 has given necessary and sufficient conditions, for equality and inequality constrained optimization problems under the assumptions that the problem functions be either Gâteaux differentiable or locally Lipschitz. Further, she has introduced constraint qualifications in terms of the Michel–Penot subdifferential and then applied these results to bilevel programming problems. Later, in 2006, Ye [14] has introduced nonsmooth constraint qualifications for bilevel programming problems and derived Karush–Kuhn–Tucker (KKT) type necessary optimality conditions under these qualifications. Recently, Dempe, Dutta, and Mordukhovich [12] have obtained KKT type optimality conditions for optimistic bilevel programming problems under the assumption of partial calmness CQ given by Ye and Zhu [9], using tools of variational analysis.

In this paper, our aim is to develop KKT type necessary optimality conditions for bilevel programming problems with a convex lower-level problem. Since Abadie CQ [16] is weaker than most of the other CQs and, our problem is nonsmooth, therefore, we have introduced two forms of ∂^*-Abadie CQ in terms of upper convexifactors and the Clarke subdifferential. To establish the optimality conditions, we have used the weaker form. These can be regarded as nonsmooth extensions of Abadie CQ to the bilevel programming problem. Since these have been expressed in terms of a generalized subdifferential, convexifactor, they give more than general conditions for the problem. Convexifactors are important tools of nonsmooth analysis, which were introduced by Demyanov [17] and were further studied by Demyanov and Jeyakumar [18], Jeyakumar and Luc [19], Dutta and Chandra [20, 21], etc. In 2006, Li and Zhang [15] used the concept of upper and lower convexifactors for introducing constraint qualifications and obtained necessary optimality conditions for nonsmooth optimization problems involving locally Lipschitz functions. Convexifactors are subsets of many well-known subdifferentials; hence, our optimality conditions are sharper than those using Clarke, Michel–Penot subdifferentials, etc.

The paper is organized as follows. In Sect. 2, we give basic definitions of convexifactors, Clarke subdifferential, and basic subdifferential. Section 3 is devoted to the bilevel programming problem; in Sect. 4, we discuss Lipschitz continuity of the value function and introduce the above mentioned constraint qualifications. Finally, in Sect. 5, necessary conditions have been derived under ∂^*-Abadie CQ. The results developed in this section extend the corresponding ones in [12–14].

2 Tools of Analysis

In this paper, we have focused ourselves on finite dimensional spaces. We begin by defining upper and lower Dini derivatives as follows:

Let $F : \mathbb{R}^{n_1} \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real valued function and let $x \in \mathbb{R}^{n_1}$ where $F(x)$ is finite. Then the upper and lower Dini derivatives of F at x in the direction v are defined respectively by

$$(F)_d^+(x,v) := \limsup_{t\to 0^+} \frac{F(x+tv)-F(x)}{t},$$

and

$$(F)_d^-(x,v) := \liminf_{t\to 0^+} \frac{F(x+tv)-F(x)}{t}.$$

Dini derivatives may be finite as well as infinite. In particular, if F is locally Lipschitz, both the upper and lower Dini derivatives are finite.

For any set $A \subset \mathbb{R}^{n_1}$, the closure, convex hull, and the closed convex hull of A are denoted respectively by $\overline{A}(\mathrm{cl}\,A)$, $\mathrm{conv}\,A$, and $\overline{\mathrm{conv}}\,A$ $(\mathrm{cl}\,\mathrm{conv}\,A)$.

We now give the definitions of convexifactors [20].

Definition 2.1 Let $F : \mathbb{R}^{n_1} \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real valued function and let $x \in \mathbb{R}^{n_1}$ where $F(x)$ is finite.

(i) F is said to admit an upper convexifactor (UCF) $\partial^* F(x)$ at x iff $\partial^* F(x) \subseteq \mathbb{R}^{n_1}$ is a closed set and

$$(F)_d^-(x,v) \le \sup_{x^*\in\partial^* F(x)} \langle x^*, v\rangle, \quad \text{for all } v \in \mathbb{R}^{n_1},$$

(ii) F is said to admit a lower convexifactor (LCF) $\partial_* F(x)$ at x iff $\partial_* F(x) \subseteq \mathbb{R}^{n_1}$ is a closed set and

$$(F)_d^+(x,v) \ge \inf_{x^*\in\partial_* F(x)} \langle x^*, v\rangle, \quad \text{for all } v \in \mathbb{R}^{n_1},$$

(iii) F is said to admit a convexifactor (CF) $\partial_*^* F(x)$ at x iff $\partial_*^* F(x)$ is both an (UCF) and (LCF) of F at x.

(iv) F is said to admit an upper semiregular convexifactor (USRCF) $\partial^* F(x)$ at x iff $\partial^* F(x)$ is an (UCF) of F at x and

$$(F)_d^+(x,v) \le \sup_{x^*\in\partial^* F(x)} \langle x^*, v\rangle, \quad \text{for all } v \in \mathbb{R}^{n_1}.$$

In particular, if equality holds in above, then $\partial^* F(x)$ is called an upper regular convexifactor (URCF) of F at x.

(v) F is said to admit a lower semiregular convexifactor (LSRCF) $\partial_* F(x)$ at x iff $\partial_* F(x)$ is a (LCF) of F at x and

$$(F)_d^-(x,v) \ge \inf_{x^*\in\partial_* F(x)} \langle x^*, v\rangle, \quad \text{for all } v \in \mathbb{R}^{n_1}.$$

In particular, if equality holds in above, then $\partial_* F(x)$ is called a lower regular convexifactor (LRCF) of F at x.

It may be noted that convexifactors are not necessarily convex or compact [19–21]. Because of these relaxations, convexifactors can be easily applied to a large class of nonsmooth functions.

Definition 2.2 (Clarke [22]) Let $F:\mathbb{R}^{n_1}\to\mathbb{R}$ be a locally Lipschitz function on $\mathbb{R}^{n_1}$. Then Clarke subdifferential of F at $x\in\mathbb{R}^{n_1}$ is given as

$$\partial^c F(x):=\left\{\xi\in\mathbb{R}^{n_1}\,|\,F^0(x,v)\geq\langle\xi,v\rangle,\forall v\in\mathbb{R}^{n_1}\right\},$$

where $F^0(x,v)$ is Clarke generalized directional derivative of F at $x\in\mathbb{R}^{n_1}$ in direction v and is given by

$$F^0(x,v):=\limsup_{y\to x,t\to 0^+}\frac{F(y+tv)-F(y)}{t},\quad \text{where } y\in\mathbb{R}^{n_1} \text{ and } t>0.$$

$\partial^c F(x)$ is nonempty, convex, and compact set for each $x\in\mathbb{R}^{n_1}$.

For a locally Lipschitz function F, $\partial^c F(x)$ is convexifactor of F at x [19].

Definition 2.3 (Dempe et al. [12, Definition 2.3]) Let $F:\mathbb{R}^{n_1}\to\mathbb{R}$ be Lipschitz continuous around $\bar{x}$, its basic (limiting or Mordukhovich) subdifferential at $\bar{x}$ is defined by

$$\partial F(\bar{x}):=\limsup_{x\to\bar{x}}\hat{\partial}F(x)$$

via the Painlevé–Kuratowski outer limit of the so-called Fréchet subdifferentials

$$\hat{\partial}F(x):=\left\{v\in\mathbb{R}^{n_1}\,|\,\liminf_{u\to x}\frac{F(u)-F(x)-\langle v,u-x\rangle}{\|u-x\|}\geq 0\right\}$$

of F at x.

The basic subdifferential is always nonempty and compact for every locally Lipschitz function. It reduces to the classical gradient, that is, $\partial F(\bar{x}):=\{\nabla F(\bar{x})\}$ for strictly differentiable functions and to the subdifferential of convex analysis for convex ones.

Definition 2.4 (Mordukhovich [23, Definition 1.63]) Let $S:\mathbb{R}^{n_1}\rightrightarrows\mathbb{R}^{n_2}$ be a set valued mapping and $\bar{x}\in\operatorname{dom}S$.

(i) Given $\bar{y}\in S(\bar{x})$, S is inner semicontinuous at $(\bar{x},\bar{y})$ iff for every sequence $x_k\to\bar{x}$ there is a sequence $y_k\in S(x_k)$ converging to $\bar{y}$ as $k\to\infty$.
(ii) S is inner semicompact at $\bar{x}$ with $S(\bar{x})\neq\phi$ iff for every sequence $x_k\to\bar{x}$ with $S(x_k)\neq\phi$ there is a sequence $y_k\in S(x_k)$ that contains a convergent subsequence as $k\to\infty$.

The inner semicontinuity of S at $(\bar{x},\bar{y})$ for every $\bar{y}\in S(\bar{x})$ goes back to the standard notion of inner/lower semicontinuity of S at $\bar{x}$.

In finite dimensions, the inner semicompactness holds whenever S is uniformly bounded around $\bar{x}$.

3 Bilevel Programming Problem

In this section, we study the bilevel programming problem given as follows:

$$\text{(BLPP)} \qquad \min_{x,y} F(x,y) \quad \text{s.t.} \quad G_j(x,y) \le 0, \quad j \in J,\ y \in \psi(x),$$

where, for each $x \in \mathbb{R}^{n_1}$, $\psi(x)$ is the set of optimal solutions to the following optimization problem:

$$\min_y f(x,y) \quad \text{s.t.} \quad g_i(x,y) \le 0, \quad i \in I,$$

where $F, f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $G_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $j \in J := \{1, 2, \ldots, m_2\}$, and $g_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $i \in I := \{1, 2, \ldots, m_1\}$; n_i and m_i, $i := 1, 2$ are integers with $n_i \ge 1$ and $m_i \ge 0$. $f(\cdot,\cdot)$ and $g_i(.,.)$, $i \in I$ are continuous, convex, and $\psi(x) := \arg\min_y\{f(x,y) : g(x,y) \le 0\}$.

So, the idea is that the lower level decision maker, or the follower minimizes his/her objective function based on the leader's choice x and returns the solution $y = y(x)$ to the leader, who then uses it to minimize his/her objective function. If the optimal solution of the lower-level problem is uniquely determined for all $x \in \mathbb{R}^{n_1}$, then the problem (BLPP) is well defined. However, if there are multiple solutions to the lower-level problem for a given x, then the upper-level objective becomes a set-valued map. In order to overcome this difficulty, two different solution concepts have been considered in the literature, namely the optimistic solution and the pessimistic one.

In this article, we have focused on the optimistic approach only. According to this approach, the leader assumes the cooperation of the follower in the sense that the follower will in any case take an optimal solution which is a best one from the leader's point of view. This leads to the following optimistic bilevel programming problem (OBLPP):

$$\text{(OBLPP)} \qquad \min_x \varphi_0(x), \quad x \in \mathbb{R}^{n_1}$$

$$\text{where } \varphi_0(x) := \min_y \bigl\{F(x,y) : G_j(x,y) \le 0, j \in J, y \in \psi(x)\bigr\}$$

and $\psi(x)$ is the set of optimal solutions to the lower-level problem

$$\min_y f(x,y) \quad \text{s.t.} \quad g_i(x,y) \le 0, \quad i \in I.$$

A point $\bar{x} \in \mathbb{R}^{n_1}$ is called a local optimistic solution [12] of the bilevel programming problem iff $\bar{y} \in \psi(\bar{x})$, $\bar{x} \in \mathbb{R}^{n_1}$, $F(\bar{x},\bar{y}) = \varphi_0(\bar{x})$ and there is a number $\varepsilon > 0$ such that $\varphi_0(x) \ge \varphi_0(\bar{x})$, for all $x \in \mathbb{R}^{n_1}$, $\| x - \bar{x} \| < \varepsilon$.

To obtain necessary optimality conditions for optimistic bilevel programming problem (OBLPP), we follow value function approach initiated by Outrata [7] according to which optimistic bilevel programming problem can be converted into single level mathematical programming problem with the help of the value function of the lower-level problem given by

$$V(x) := \min_y \bigl\{f(x,y) : g_i(x,y) \le 0, i \in I, y \in \mathbb{R}^{n_2}\bigr\}.$$

Then the reformulated optimistic bilevel programming problem (ROBLPP) is given as:

$$\text{(ROBLPP)} \qquad \min_{x,y} F(x,y) \quad \text{s.t.} \quad \begin{aligned} & f(x,y) - V(x) \leq 0, \\ & g_i(x,y) \leq 0, \quad i \in I, \\ & G_j(x,y) \leq 0, \quad j \in J, \\ & (x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{aligned}$$

Let $X \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ denote the feasible set for (ROBLPP), that is,

$$X := \big\{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | f(x,y) - V(x) \leq 0, \\ g_i(x,y) \leq 0, i \in I, G_j(x,y) \leq 0, j \in J\big\}.$$

However, the price to pay in this reformulation is that (ROBLPP) is nonsmooth even for smooth initial data. Formulation (ROBLPP) of (OBLPP) has been developed by Ye and Zhu [9], Babahadda and Gadhi [11], and Ye [13, 14]. The latest published results for optimistic bilevel programs are given by Dempe, Dutta, and Mordukhovich [12]. The developments in their paper are based on the value function approach. Under the assumption of the partial calmness constraint qualification [9], they have given necessary optimality conditions for bilevel programs with smooth, convex, linear, and Lipschitzian functions, describing the initial data of the (BLPP). Their results have been proved by assuming solution set map of lower-level problem to be inner semicontinuous and inner semicompact and by using advanced formulas for computing basic subgradients of value function in variational analysis. In this paper, we further develop the value function approach to prove necessary optimality conditions for (BLPP).

Remark 3.1 (Dempe et al. [12]) Note that (ROBLPP) is globally equivalent to (OBLPP), while local optimal solutions to (OBLPP) are always locally optimal to (ROBLPP). The problems (BLPP) and (OBLPP) are not equivalent with respect to local optimal solutions. But the problems (BLPP) and (ROBLPP) are fully equivalent both with respect to local and global optimal solutions.

4 Regularity Conditions and Constraint Qualifications

In this section, we shall give regularity conditions with discussion on Lipschitz continuity of value function and shall introduce constraint qualifications.

We begin with the following definitions.

Given a nonempty subset S of $\mathbb{R}^{n_1}$, the negative polar cone of S is defined by

$$S^- := \big\{v \in \mathbb{R}^{n_1} | \langle x, v\rangle \leq 0, \forall x \in S\big\}.$$

S^- is always closed and convex.

We now give definitions of two tangent cones that will be useful in the sequel. Let $x \in \overline{S}$, then the adjacent cone $A(S,x)$ and the contingent cone $T(S,x)$ to S at x are defined, respectively, by

$$A(S,x) := \{v \in \mathbb{R}^{n_1} | \forall t_n \downarrow 0 \exists v_n \to v \text{ such that } x + t_n v_n \in S\}.$$
$$T(S,x) := \{v \in \mathbb{R}^{n_1} | \exists t_n \downarrow 0 \text{ and } v_n \to v \text{ such that } x + t_n v_n \in S\}.$$

Let $(\bar{x}, \bar{y}) \in X$ be feasible for (ROBLPP), we assume that g_i, $i \in I(\bar{x}, \bar{y})$, G_j, $j \in J(\bar{x}, \bar{y})$ admit UCFs $\partial^* g_i(\bar{x}, \bar{y})$ and $\partial^* G_j(\bar{x}, \bar{y})$, respectively, at $(\bar{x}, \bar{y})$, where $I(\bar{x}, \bar{y}) := \{i \in I | g_i(\bar{x}, \bar{y}) := 0\}$ and $J(\bar{x}, \bar{y}) := \{j \in J | G_j(\bar{x}, \bar{y}) := 0\}$.

Here, $f(\cdot,\cdot)$ and $g_i(\cdot,\cdot)$, $i \in I$ are convex functions, therefore, it is easy to check that $V(\cdot)$ is a convex function and so we have used the same symbol for Clarke subdifferential and the subdifferential of f and V since they coincide for convex functions.

We know that when objective function and constraints of the lower-level problem are convex the value function is convex. We now give two examples to show that there are situations where it may happen that the convexity of value function depends on the convexity of objective function of lower-level problem but not on the convexity of the constraint function.

(i) Let

$$f(x,y) := x^2 + y^2 - 1,$$
$$g(x,y) := x^2 y \le 0,$$
$$V(x) := \begin{cases} -1, & x = 0, \\ x^2 - 1, & x \neq 0. \end{cases}$$

Here, V is convex because f is convex though g is nonconvex.

(ii) Let

$$f(x,y) := x^2 y + y,$$
$$g_1(x,y) := y \le 0, \qquad g_2(x,y) := -y - 1 \le 0,$$
$$V(x) := \begin{cases} -1, & x = 0, \\ -(x^2 + 1), & x \neq 0. \end{cases}$$

Here, V is nonconvex because f is nonconvex though g_1 and g_2 are convex.

To proceed further, we need to formulate the notion of lower-level regularity and upper-level regularity followed by the discussion on Lipschitz continuity of value function.

Given a point $(\bar{x}, \bar{y})$ satisfying the lower level inequality constraints $g_i(\bar{x}, \bar{y}) \le 0$, $i \in I$ with the index set $I(\bar{x}, \bar{y})$. We say that $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is lower-level regular if the following implication holds in terms of the Clarke subdifferential.

$$\left[\sum_{i \in I(\bar{x},\bar{y})} \lambda_i v_i := 0, \lambda_i \ge 0\right] \quad \Rightarrow \quad \left[\lambda_i := 0 \text{ for all } i \in I(\bar{x}, \bar{y})\right]$$

whenever $(u_i, v_i) \in \partial g_i(\bar{x}, \bar{y})$ with some $u_i \in \mathbb{R}^{n_1}$ as $i \in I(\bar{x}, \bar{y})$.

We can also define lower-level regularity in terms of an upper convexifactor of $g_i(\cdot,\cdot), i \in I$ at $(\bar{x},\bar{y})$ by replacing the Clarke subdifferential in the above implication by the convex hull of convexifactor of $g_i(\cdot,\cdot), i \in I$ at $(\bar{x},\bar{y})$.
Similarly, given $(\bar{x},\bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying the upper-level inequality constraints $G_j(\bar{x},\bar{y}) \le 0,\ j \in J$ with the index set $J(\bar{x},\bar{y})$. We say that $(\bar{x},\bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is upper-level regular if

$$\Bigl[(0,0) \in \sum_{j \in J(\bar{x},\bar{y})} \lambda_j \partial^c G_j(\bar{x},\bar{y}), \lambda_j \ge 0\Bigr] \quad \Rightarrow \quad \bigl[\lambda_j := 0 \text{ whenever } j \in J(\bar{x},\bar{y})\bigr].$$

It may be noted that these regularity conditions developed in [23, Sect. 4.3] and used in [12] in terms of basic subdifferential can be regarded as nonsmooth counter part of the classical Mangasarian–Fromovitz constraint qualification for the lower-level and upper-level problems, respectively.

Using Corollary 4.39 of [23], the constraint mapping $K(x) := \{y \in \mathbb{R}^{n_2} | g_i(\bar{x},\bar{y}) \le 0, i \in I(\bar{x},\bar{y})\}$ is Lipschitz-like around $(\bar{x},\bar{y})$ under the lower-level regularity of $(\bar{x},\bar{y})$. Let f be Lipschitz around $(\bar{x},\bar{y})$ for every $\bar{y} \in \psi(\bar{x})$.

In addition to above, if

(i) ψ is inner semicompact at $\bar{x}$, then, Theorem 5.2(ii) [24] ensures the Lipschitz continuity of value function V around $\bar{x}$.
(ii) ψ is inner semicontinuous at $(\bar{x},\bar{y})$ then, Theorem 5.2(i) [24] ensures the Lipschitz continuity of value function V around $\bar{x}$.

Remark 4.1 In this paper, value function V is convex as argued earlier, and hence locally Lipschitz. The assumptions of inner semicontinuity or inner semicompactness are not necessary for convexity of the value function.

It has been observed by Ye and Zhu [9] that the usual CQs fail to hold for (BLPP) and in this regard they have suggested partial calmness CQ for (BLPP). For a detailed discussion on partial calmness CQ, one can see [9, 12]. We do have situations where partial calmness CQ may not hold. Example 3.7 [12] illustrates this fact.

We now introduce two forms of ∂^*-Abadie constraint qualification using the concept of convexifactors and Clarke subdifferential. These CQs generalize those of [13, 14].

∂^*-Abadie CQ

∂^*-ACQ

$$\Bigl(\bigcup_{i \in I(\bar{x},\bar{y})} \operatorname{conv} \partial^* g_i(\bar{x},\bar{y}) \cup \bigcup_{j \in J(\bar{x},\bar{y})} \operatorname{conv} \partial^* G_j(\bar{x},\bar{y}) \cup \partial f(\bar{x},\bar{y}) - \partial V(\bar{x}) \times \{0\}\Bigr)^- \subseteq T\bigl(X,(\bar{x},\bar{y})\bigr)$$

∂^*-ACQ′

$$\Bigl(\bigcup_{i \in I(\bar{x},\bar{y})} \operatorname{conv} \partial^* g_i(\bar{x},\bar{y}) \cup \bigcup_{j \in J(\bar{x},\bar{y})} \operatorname{conv} \partial^* G_j(\bar{x},\bar{y}) \cup \partial f(\bar{x},\bar{y}) - \partial V(\bar{x}) \times \{0\}\Bigr)^- \subseteq A\bigl(X,(\bar{x},\bar{y})\bigr).$$

Since $A(X,(\bar{x},\bar{y})) \subseteq T(X,(\bar{x},\bar{y}))$, by definition of the ∂^*-Abadie CQ, we have the following proposition.

Proposition 4.1 *∂^*-ACQ$'$ $\Rightarrow$ ∂^*-ACQ.*

5 Optimality Conditions

In this section, we shall prove Karush–Kuhn–Tucker (KKT) type necessary optimality conditions under ∂^*-ACQ. We begin with the following two lemmas, which will be used in the derivation of our optimality conditions.

Lemma 5.1 (Li and Zhang [15]) *Let S_1 and S_2 be two nonempty subsets of $\mathbb{R}^{n_1}$. Then*

(i) $\operatorname{conv}(S_1+S_2) := \operatorname{conv} S_1 + \operatorname{conv} S_2$
(ii) $\operatorname{cl}(\operatorname{cl} S_1 + \operatorname{cl} S_2) := \operatorname{cl}(S_1 + \operatorname{cl} S_2) := \operatorname{cl}(S_1+S_2)$.

Lemma 5.2 *Let B be a nonempty, convex, and compact set and A be a convex cone. If*

$$\sup_{v\in B}\langle v,d\rangle \geq 0, \quad \textit{for all } d\in A^-,$$

then $0\in B-A$.

Theorem 5.1 *Let $(\bar{x},\bar{y})$ be an optimal solution of (OBLPP). Assume that F be locally Lipschitz and admit bounded (USRCF) $\partial^* F(\bar{x},\bar{y})$ at $(\bar{x},\bar{y})$. Furthermore, we suppose that g_i, $i\in I(\bar{x},\bar{y})$, G_j, $j\in J(\bar{x},\bar{y})$ admit (UCFs) $\partial^* g_i(\bar{x},\bar{y})$, $i\in I(\bar{x},\bar{y})$, $\partial^* G_j(\bar{x},\bar{y})$, $j\in J(\bar{x},\bar{y})$, respectively at $(\bar{x},\bar{y})$ and ∂^*-ACQ holds at $(\bar{x},\bar{y})$.*

(a) *Suppose that the argminimum map ψ be inner semicompact at $\bar{x}$, that for each vector $y\in\psi(\bar{x})$, $(\bar{x},y)$ be lower-level regular. Then, there exist scalars $\lambda\geq 0$, $\mu_i\geq 0$, $i\in I(\bar{x},\bar{y})$, $\tau_j\geq 0$, $j\in J(\bar{x},\bar{y})$, $\lambda_i\geq 0$, $i\in I$ and also $y^*\in\psi(\bar{x})$ such that the following conditions hold*:

$$\text{(i) } (0,0)\in \operatorname{cl}\Bigg[\operatorname{conv}\partial^* F(\bar{x},\bar{y}) - \Bigg\{\lambda\big(\partial f(\bar{x},\bar{y}) - \partial_x f(\bar{x},y^*)\times\{0\}\big) - \lambda\Bigg(\sum_{i\in I}\lambda_i\partial_x g_i(\bar{x},y^*)\times\{0\}\Bigg) + \sum_{i\in I(\bar{x},\bar{y})}\mu_i\operatorname{conv}\partial^* g_i(\bar{x},\bar{y}) + \sum_{j\in J(\bar{x},\bar{y})}\tau_j\operatorname{conv}\partial^* G_j(\bar{x},\bar{y})\Bigg\}\Bigg],$$

$$\text{(ii) } 0\in\partial_y f(\bar{x},y^*) + \sum_{i\in I}\lambda_i\partial_y g_i(\bar{x},y^*),$$

$$\text{(iii) } \lambda_i g_i(\bar{x},y^*) := 0, \quad \lambda_i\geq 0, \quad i\in I.$$

(b) *Suppose that the argminimum map* ψ *be inner semicontinuous at* $(\bar{x}, \bar{y})$, *that the pair* $(\bar{x}, \bar{y})$ *be lower-level regular. Then there exist scalars* $\lambda \geq 0$, $\mu_i \geq 0$, $i \in I(\bar{x}, \bar{y})$, $\tau_j \geq 0$, $j \in J(\bar{x}, \bar{y})$, $\lambda_i \geq 0$, $i \in I$ *such that* (i), (ii), *and* (iii) *hold with* y^* *replaced by* $\bar{y}$.

Here, $\partial, \partial_x,$ *and* ∂_y *stand*, *respectively*, *for the full and partial subdifferentials of convex analysis*.

Proof Since $(\bar{x}, \bar{y})$ is an optimal solution of (OBLPP), hence by Remark 3.1, an optimal solution of (ROBLPP).

Let $(v_1, v_2) \in T(X, (\bar{x}, \bar{y}))$, from the definition of tangent cone, it follows that $\exists t_n \downarrow 0$ and $(v_{n_1}, v_{n_2}) \to (v_1, v_2)$ such that $(\bar{x}, \bar{y}) + t_n(v_{n_1}, v_{n_2}) \in X$ for all n.

Since $(\bar{x}, \bar{y})$ is minimum of F over X, therefore, we have

$$\frac{F((\bar{x}, \bar{y}) + t_n(v_{n_1}, v_{n_2})) - F(\bar{x}, \bar{y})}{t_n} \geq 0, \quad \text{for sufficiently large } n. \tag{1}$$

Now

$$\begin{aligned} &\frac{F((\bar{x}, \bar{y}) + t_n(v_{n_1}, v_{n_2})) - F(\bar{x}, \bar{y})}{t_n} \\ &\quad = \frac{F((\bar{x}, \bar{y}) + t_n(v_{n_1}, v_{n_2})) - F((\bar{x}, \bar{y}) + t_n(v_1, v_2))}{t_n} \\ &\qquad + \frac{F((\bar{x}, \bar{y}) + t_n(v_1, v_2)) - F(\bar{x}, \bar{y})}{t_n}. \end{aligned} \tag{2}$$

Since F is locally Lipschitz, therefore,

$$\frac{F((\bar{x}, \bar{y}) + t_n(v_{n_1}, v_{n_2})) - F((\bar{x}, \bar{y}) + t_n(v_1, v_2))}{t_n} \to 0 \quad \text{as } n \to \infty.$$

Taking Limit supremum on both the sides of equation (2) and noting above and (1), we get

$$\limsup_{t_n \to 0^+} \frac{F((\bar{x}, \bar{y}) + t_n(v_1, v_2)) - F(\bar{x}, \bar{y})}{t_n} = (F)_d^+\big((\bar{x}, \bar{y}), (v_1, v_2)\big) \geq 0.$$

That is, we have

$$(F)_d^+\big((\bar{x}, \bar{y}), (v_1, v_2)\big) \geq 0, \quad \text{for all } (v_1, v_2) \in T\big(X, (\bar{x}, \bar{y})\big).$$

By upper semiregularity of $\partial^* F(\bar{x}, \bar{y})$ at $(\bar{x}, \bar{y})$, it follows that

$$\sup_{\eta \in \partial^* F(\bar{x}, \bar{y})} \big\langle \eta, (v_1, v_2) \big\rangle \geq 0, \quad \text{for all } (v_1, v_2) \in T\big(X, (\bar{x}, \bar{y})\big), \tag{3}$$

Since ∂^*-ACQ holds at $(\bar{x}, \bar{y})$, we have

$$\sup_{\eta \in \operatorname{conv} \partial^* F(\bar{x}, \bar{y})} \big\langle \eta, (v_1, v_2) \big\rangle \geq 0, \quad \text{for all } (v_1, v_2) \in A^- \tag{4}$$

where A^- is the negative polar cone of A and A is defined by

$$A := \Big(\bigcup_{i\in I(\bar{x},\bar{y})} \operatorname{conv} \partial^* g_i(\bar{x},\bar{y}) \cup \bigcup_{j\in J(\bar{x},\bar{y})} \operatorname{conv} \partial^* G_j(\bar{x},\bar{y}) \cup \big(\partial f(\bar{x},\bar{y}) - \partial V(\bar{x}) \times \{0\}\big)\Big).$$

Now, using Lemma 5.2, we get

$$(0,0) \in \overline{\operatorname{conv}}\big(\operatorname{conv} \partial^* F(\bar{x},\bar{y}) - A\big).$$

That is,

$$(0,0) \in \overline{\operatorname{conv}}\Big(\operatorname{conv} \partial^* F(\bar{x},\bar{y}) - \Big\{ \bigcup_{i\in I(\bar{x},\bar{y})} \operatorname{conv} \partial^* g_i(\bar{x},\bar{y}) \cup \bigcup_{j\in J(\bar{x},\bar{y})} \operatorname{conv} \partial^* G_j(\bar{x},\bar{y}) \cup \big(\partial f(\bar{x},\bar{y}) - \partial V(\bar{x}) \times \{0\}\big)\Big\}\Big)$$

which implies that there exists a sequence

$$(x_n, y_n) \in \operatorname{conv}\Big(\operatorname{conv} \partial^* F(\bar{x},\bar{y}) - \Big\{ \bigcup_{i\in I(\bar{x},\bar{y})} \operatorname{conv} \partial^* g_i(\bar{x},\bar{y}) \cup \bigcup_{j\in J(\bar{x},\bar{y})} \operatorname{conv} \partial^* G_j(\bar{x},\bar{y}) \cup \big(\partial f(\bar{x},\bar{y}) - \partial V(\bar{x}) \times \{0\}\big)\Big\}\Big)$$

such that $(x_n, y_n) \to (0,0)$.

Applying Lemma 5.1(i) in the above condition, we get

$$(x_n, y_n) \in \operatorname{conv} \partial^* F(\bar{x},\bar{y}) + \operatorname{conv}\Big\{ -\Big(\bigcup_{i\in I(\bar{x},\bar{y})} \operatorname{conv} \partial^* g_i(\bar{x},\bar{y}) \cup \bigcup_{j\in J(\bar{x},\bar{y})} \operatorname{conv} \partial^* G_j(\bar{x},\bar{y}) \cup \big(\partial f(\bar{x},\bar{y}) - \partial V(\bar{x}) \times \{0\}\big)\Big)\Big\}.$$

Using the convex hull property of a subset S of $\mathbb{R}^{n_1}$, $\operatorname{conv}(-S) := -\operatorname{conv} S$, we get

$$(x_n, y_n) \in \operatorname{conv} \partial^* F(\bar{x},\bar{y}) - \operatorname{conv}\Big\{ \bigcup_{i\in I(\bar{x},\bar{y})} \operatorname{conv} \partial^* g_i(\bar{x},\bar{y}) \cup \bigcup_{j\in J(\bar{x},\bar{y})} \operatorname{conv} \partial^* G_j(\bar{x},\bar{y}) \cup \big(\partial f(\bar{x},\bar{y}) - \partial V(\bar{x}) \times \{0\}\big)\Big\}.$$

Since convexifactors are in general nonconvex sets, therefore, there exist scalars $\lambda \geq 0$, $\mu_i \geq 0$, $i \in I(\bar{x}, \bar{y})$, $\tau_j \geq 0$, $j \in J(\bar{x}, \bar{y})$ such that

$$(x_n, y_n) \in \operatorname{conv} \partial^* F(\bar{x}, \bar{y}) - \Bigg[\sum_{i \in I(\bar{x}, \bar{y})} \mu_i \operatorname{conv} \partial^* g_i(\bar{x}, \bar{y}) + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \operatorname{conv} \partial^* G_j(\bar{x}, \bar{y}) + \lambda \big(\partial f(\bar{x}, \bar{y}) - \partial V(\bar{x}) \times \{0\} \big) \Bigg]$$

with

$$\lambda + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j := 1. \tag{5}$$

Thus,

$$(0, 0) \in \operatorname{cl} \Bigg[\operatorname{conv} \partial^* F(\bar{x}, \bar{y}) - \Bigg\{ \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \operatorname{conv} \partial^* g_i(\bar{x}, \bar{y}) + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \operatorname{conv} \partial^* G_j(\bar{x}, \bar{y}) + \lambda \big(\partial f(\bar{x}, \bar{y}) - \partial V(\bar{x}) \times \{0\} \big) \Bigg\} \Bigg]. \tag{6}$$

Furthermore, we use the following relationship for convex, continuous function $f(x, y)$ that holds, e.g., by [23, Corollary 3.44]

$$\partial f(x, y) \subset \partial_x f(x, y) \times \partial_y f(x, y). \tag{7}$$

Now, we have to determine an efficient estimate of subdifferential $\partial V(x)$ of the value function. Applying Theorem 8 [25] (its inner semicompact counterpart) and using property (7), we get the following upper estimate of Clarke subdifferential of the value function at $\bar{x}$:

$$\partial V(\bar{x}) := \Bigg[\bigcup_{y \in \psi(\bar{x})} \Bigg\{ \bigcup_{(\lambda_1, \lambda_2, \ldots, \lambda_{m_1}) \in \Lambda(\bar{x}, y)} \bigg(\partial_x f(\bar{x}, y) + \sum_{i \in I} \lambda_i \partial_x g_i(\bar{x}, y) \bigg) \Bigg\} \Bigg] \tag{8}$$

where $\Lambda(\bar{x}, y)$ is defined by

$$\Lambda(\bar{x}, y) := \Bigg\{ (\lambda_1, \ldots, \lambda_{m_1}) \in \mathbb{R}^{m_1} | 0 \in \partial_y f(\bar{x}, y) + \sum_{i \in I} \lambda_i \partial_y g_i(\bar{x}, y), \; \lambda_i \geq 0, \; \lambda_i g_i(\bar{x}, y) := 0, i \in I \Bigg\}. \tag{9}$$

Combining (6), (8), and (9), we arrive at the necessary conditions (i), (ii), and (iii).

(b) To derive optimality conditions under the inner semicontinuity assumption on ψ, we shall argue in the same way as above and shall get the following instead of (8):

$$\partial V(\bar{x}) := \bigcup_{(\lambda_1,\lambda_2,\ldots,\lambda_{m_1})\in\Lambda(\bar{x},\bar{y})} \left(\partial_x f(\bar{x},\bar{y}) + \sum_{i\in I}\lambda_i \partial_x g_i(\bar{x},\bar{y})\right)$$

where

$$\Lambda(\bar{x},\bar{y}) := \Bigg\{(\lambda_1,\ldots,\lambda_{m_1}) \in \mathbb{R}^{m_1} | 0 \in \partial_y f(\bar{x},\bar{y}) + \sum_{i\in I}\lambda_i \partial_y g_i(\bar{x},\bar{y}),$$
$$\lambda_i \geq 0,\ \lambda_i g_i(\bar{x},\ \bar{y}) := 0, i \in I\Bigg\}.$$

Proceeding on the same lines, as in (a), we arrive at the necessary conditions (i), (ii), and (iii) with the replacement of y^* by $\bar{y}$. □

Remark 5.1 The notion of (USRCF) has been introduced by Dutta and Chandra [20] where they have shown that for a locally Lipschitz function, most known subdifferentials such as subdifferential of Clarke, Michel–Penot, Mordukhovich, etc. are (USRCFs). We now provide an example [19, 20] which shows that these well known subdifferentials may often contain convex hull of an (USRCF), and hence optimality conditions in terms of (USRCFs) and upper convexifactors provide sharp results.

Example 5.1 (See [19, 20]) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$F(x, y) := |x| - |y|.$$

The convexifactor of F at $(0, 0)$ is given by

$$\partial^* F(0,0) = \{(-1,1),(1,-1)\},$$

Clarke and Michel–Penot subdifferentials are given by

$$\partial^c F(0,0) = \partial^\diamond F(0,0) = \operatorname{conv}\{(1,-1),(-1,1),(1,1),(-1,-1)\}.$$

It has been shown in [23, 25], Mordukhovich (basic) subdifferential of F at $(0, 0)$ is given by

$$\partial F(0,0) = \{(v,-1) | -1 \leq v \leq 1\} \cup \{(v,1) | -1 \leq v \leq 1\}$$

It follows that $\operatorname{conv} \partial^* F(0,0) \subset \partial F(0,0) \subset \partial^c F(0,0) = \partial^\diamond F(0,0)$.

Remark 5.2 It is interesting to compare the above results with those in [12]. We observe that our optimality conditions are close to those of Theorem 4.1 [12] and also generalizes them in view of Remark 5.1.

Remark 5.3 If we assume all the functions $F, G_j, j \in J, f, g_i, i \in I$ to be differentiable in above theorem, then our optimality conditions under inner semicontinuity assumption become close to those of Theorem 4.2 [12], as for differentiable function F, $\partial^* F(\bar{x},\bar{y}) := \{\nabla F(\bar{x},\bar{y})\}$ and $\partial F(\bar{x},\bar{y}) := \{\nabla F(\bar{x},\bar{y})\}$.

Here, we would like to point out that if our lower-level problem is general, that is, functions involving are not necessarily convex, then the value function will be Lipschitz continuous by Theorem 5.2(ii) [24], if ψ is assumed to be inner semicompact at $\bar{x}$ and by Theorem 5.2(i) [24], if ψ is assumed to be inner semicontinuous at $(\bar{x}, \bar{y})$.

Then, in Theorem 5.1, after (6), we shall proceed in the following way.

Now, we have to determine an efficient estimate of Clarke subdifferential $\partial^c V(x)$ of the value function. Applying Theorem 3.57(ii) [23] in Theorem 8 [25] (its inner semicompact counterpart), we get the following upper estimate of Clarke subdifferential of the value function at $\bar{x}$:

$$\partial^c V(\bar{x}) \subset \operatorname{cl}\operatorname{conv}\Bigg[\bigcup_{y\in\psi(\bar{x})}\Bigg\{\bigcup_{(\lambda_1,\lambda_2,\ldots,\lambda_{m_1})} u\in\mathbb{R}^{n_1} \mid (u,0)\in\partial^c f(\bar{x},y)+\sum_{i\in I}\lambda_i\partial^c g_i(\bar{x},y),\ \lambda_i\geq 0,\ \lambda_i g_i(\bar{x},y):=0,\ i\in I\Bigg\}\Bigg]. \tag{7}$$

Now, before proceeding further we have to evaluate

$$\operatorname{cl}\operatorname{conv}\Bigg[\bigcup_{y\in\psi(\bar{x})}\Bigg\{\bigcup_{(\lambda_1,\lambda_2,\ldots,\lambda_{m_1})} u\in\mathbb{R}^{n_1} \mid (u,0)\in\partial^c f(\bar{x},y)+\sum_{i\in I}\lambda_i\partial^c g_i(\bar{x},y),\ \lambda_i\geq 0,\ \lambda_i g_i(\bar{x},y):=0,\ i\in I\Bigg\}\Bigg].$$

Let

$$\xi\in\operatorname{cl}\operatorname{conv}\Bigg[\bigcup_{y\in\psi(\bar{x})}\Bigg\{\bigcup_{(\lambda_1,\lambda_2,\ldots,\lambda_{m_1})} u\in\mathbb{R}^{n_1} \mid (u,0)\in\partial^c f(\bar{x},y)+\sum_{i\in I}\lambda_i\partial^c g_i(\bar{x},y),\ \lambda_i\geq 0,\ \lambda_i g_i(\bar{x},y):=0,\ i\in I\Bigg\}\Bigg]. \tag{8}$$

Then there exists a sequence

$$\xi_n\in\operatorname{conv}\Bigg[\bigcup_{y\in\psi(\bar{x})}\Bigg\{\bigcup_{(\lambda_1,\lambda_2,\ldots,\lambda_{m_1})} u\in\mathbb{R}^{n_1} \mid (u,0)\in\partial^c f(\bar{x},y)+\sum_{i\in I}\lambda_i\partial^c g_i(\bar{x},y),\ \lambda_i\geq 0,\ \lambda_i g_i(\bar{x},y):=0,\ i\in I\Bigg\}\Bigg]$$

such that $\xi_n\to\xi$ as $n\to\infty$.

Since $\partial^c f(\bar{x},y)\subset\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and $\partial^c g_i(\bar{x},y)\subset\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $i\in I$, by the classical Carathéodory theorem, there exist $\gamma_n^s\geq 0$, $\sum_{s=1}^{n_1+1}\gamma_n^s:=1$, $y_n^s\in\psi(\bar{x})$, $\lambda_{in}^s\geq 0$, $s:=1,2,\ldots,n_1+1$, $i\in I$, such that the following hold:

$$(\xi_n, 0) \in \sum_{s=1}^{n_1+1} \gamma_n^s \Big(\partial^c f\big(\bar{x}, y_n^s\big) + \sum_{i \in I} \lambda_{in}^s \partial^c g_i\big(\bar{x}, y_n^s\big) \Big),$$

$$\lambda_{in}^s g_i\big(\bar{x}, y_n^s\big) := 0, \quad i \in I.$$

We may assume that $\gamma_n^s \to \gamma^s$, $y_n^s \to y^s$, $\lambda_{in}^s \to \lambda_i^s$ as $n \to \infty$. Since g_i, $i \in I$, are continuous functions and $\partial^c f$ and $\partial^c g_i$, $i \in I$ are closed, therefore, as $n \to \infty$, we have

$$\lambda_i^s g_i\big(\bar{x}, y^s\big) := 0, \quad i \in I \tag{9}$$

and

$$(\xi, 0) \in \sum_{s=1}^{n_1+1} \gamma^s \Big(\partial^c f\big(\bar{x}, y^s\big) + \sum_{i \in I} \lambda_i^s \partial^c g_i\big(\bar{x}, y^s\big) \Big).$$

Using the above and (8) and then (7), we get

$$\partial^c V(\bar{x}) \subset \sum_{s=1}^{n_1+1} \gamma^s \Big(\partial^c f\big(\bar{x}, y^s\big) + \sum_{i \in I} \lambda_i^s \partial^c g_i\big(\bar{x}, y^s\big) \Big). \tag{10}$$

Combining (6) and (10), we arrive at the following necessary condition:

$$\begin{aligned}(0,0) \in \mathrm{cl}\Bigg[&\mathrm{conv}\, \partial^* F(\bar{x}, \bar{y}) - \Bigg\{ \lambda \Bigg(\partial^c f(\bar{x}, \bar{y}) - \sum_{s=1}^{n_1+1} \gamma^s \partial^c f\big(\bar{x}, y^s\big) \\ &- \sum_{s=1}^{n_1+1} \gamma^s \sum_{i \in I} \lambda_i^s \partial^c g_i\big(\bar{x}, y^s\big) \Bigg) \\ &+ \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \,\mathrm{conv}\, \partial^* g_i(\bar{x}, \bar{y}) + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \,\mathrm{conv}\, \partial^* G_j(\bar{x}, \bar{y}) \Bigg\} \Bigg]. \end{aligned} \tag{11}$$

Hence, (9) and (11) are necessary optimality conditions obtained in (a).

(b) To derive optimality conditions under the inner semicontinuity assumption on ψ, we shall argue in the same way as above and shall get the following instead of (7):

$$\begin{aligned}\partial^c V(\bar{x}) \subset \mathrm{cl}\,\mathrm{conv}\Bigg[\Bigg\{ \bigcup_{(\lambda_1, \lambda_2, \ldots, \lambda_{m_1})} u \in \mathbb{R}^{n_1} \,|\, (u, 0) \in \partial^c f(\bar{x}, \bar{y}) + \sum_{i \in I} \lambda_i \partial^c g_i(\bar{x}, \bar{y}), \\ \lambda_i \geq 0, \lambda_i g_i(\bar{x}, \bar{y}) := 0, i \in I \Bigg\} \Bigg]. \end{aligned}$$

Now, proceeding on the same lines, as in (a), we arrive at the necessary optimality conditions (9) and (11) with the replacement of y^s by $\bar{y}$. □

We now give an example to illustrate Theorem 5.1.

Example 5.2 Consider the problem

$$\min_{x,y}\ F(x,y) := \begin{cases} |y|^2, & x \le 0, y < 0, \\ 1, & x < 0, y = 0, \\ |x| + |y|, & \text{otherwise.} \end{cases} \quad \text{s.t.} \quad G(x,y) \le 0, \quad y \in \psi(x),$$

$$\text{where } G(x,y) := \begin{cases} 2, & x > 0,\ y \in \mathbb{R}, \\ -\sqrt{-x} + y, & x \le 0,\ y \in \mathbb{R}, \end{cases}$$

and, for each $x \in \mathbb{R}$, $\psi(x)$ is the set of optimal solutions to the following optimization problem:

$$\min_{y}\ f(x,y) := x^2 + |y-1| + 2(y-1) \quad \text{s.t.} \quad g(x,y) := y^2 - y \le 0$$

where $F, G, f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The corresponding value function for lower-level problem is given by

$$V(x) := \begin{cases} x^2, \\ x^2 - 1, \end{cases} \quad \text{for all } x \in \mathbb{R}$$

and the set $\psi(x)$ of optimal solutions to the lower-level problem is given by

$$\psi(x) := \{0, 1\}, \quad \text{for all } x \in \mathbb{R}.$$

We have

$$F(x,0) := \begin{cases} 1, & x < 0, \\ x, & x \ge 0, \end{cases} \qquad F(x,1) := |x| + 1.$$

Hence, $(0,0)$ is an optimal solution of the problem.

It can be seen that F admits (USRCF) $\partial^* F(0,0) = \{(x^*, y^*) | -1 \le x^* \le 1, -1 \le y^* \le 0\}$, G and g admit (UCFs) $\partial^* G(0,0) = \{(x^*, y^*) | x^* \ge 0, y^* > 0\}$, $\partial^* g(0,0) = \{(0, y^*) | y^* < 0\}$, respectively, at $(0,0)$. $\partial f(0,0) = \{0\} \times [1,3]$, $\partial_y f(0,0) = [1,3]$, $\partial_x f(0,0) = \{0\}$, $\partial V(0) \times \{0\} = \{(0,0)\}$, $\partial g(0,0) = \{(0,-1)\}$, $\partial_x g(0,0) = \{0\}$, $\partial_y g(0,0) = \{-1\}$, $\partial g(0,1) = \{(0,1)\}$.

∂^*-ACQ holds at $(0,0)$ where $T(X,(0,0)) = \{(x,0) : x \le 0\}$ and $X \subset \mathbb{R} \times \mathbb{R}$ is given by

$$X := \big\{(x,0) | x \le 0\big\} \cup \big\{(x,1) | x \le 0\big\}.$$

$(0,0)$ and $(0,1)$ are lower-level regular.

Then there exist scalars $\lambda = \frac{1}{2}$, $\lambda_1 = 1$, $\mu = \frac{1}{2}$, $\tau = 1$ and $y^* = 0 \in \psi(0)$ such that

$$(0,0) \in \text{cl}\Big[\text{conv}\, \partial^* F(0,0) - \Big\{\lambda\big(\partial f(0,0) - \partial_x f(0,0) \times \{0\} - \lambda_1 \partial_x g(0,0) \times \{0\}\big)$$
$$+ \mu\, \text{conv}\, \partial^* g(0,0) + \tau\, \text{conv}\, \partial^* G(0,0)\Big\}\Big],$$

$$0 \in \partial_y f(0,0) + \lambda_1 \partial_y g(0,0),$$

$$\lambda_1 g(0,0) = 0.$$

We now show with the help of an example that lower-level regularity assumption in the above theorem cannot be relaxed.

Example 5.3 Suppose that in the above example we take

$$g_1(x,y) := -y \le 0, \qquad g_2(x,y) := y - 1 \le 0.$$

Then

$$\begin{aligned}
&\partial g_1(0,0) = \{(0,-1)\}, && \partial_x g_1(0,0) = \{0\}, && \partial_y g_1(0,0) = \{-1\},\\
&\partial g_1(0,1) = \{(0,-1)\}, && \partial_x g_1(0,1) = \{0\}, && \partial_y g_1(0,1) = \{-1\},\\
&\partial g_2(0,0) = \{(0,1)\}, && \partial_x g_2(0,0) = \{0\}, && \partial_y g_2(0,0) = \{1\},\\
&\partial g_2(0,1) = \{(0,1)\}, && \partial_x g_2(0,1) = \{0\}, && \partial_y g_2(0,1) = \{1\},\\
&\partial^* g_1(0,0) = \{(0,-1)\}, && \partial^* g_2(0,0) = \{(0,1)\}.
\end{aligned}$$

Here, $(0,0)$ and $(0,1)$ are not lower-level regular.

We can see that there exist scalars $\lambda = 1$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu_1 = 1$, $\mu_2 = \frac{1}{2}$, $\tau = \frac{1}{2}$, and $y^* = 0 \in \psi(0)$ such that

$$\begin{aligned}
(0,0) \in \operatorname{cl}\Big[&\operatorname{conv}\partial^* F(0,0) - \Big\{\lambda\big(\partial f(0,0) - \partial_x f(0,0) \times \{0\}\big)\\
&- \lambda\big(\lambda_1 \partial_x g_1(0,0) \times \{0\} + \lambda_2 \partial_x g_2(0,0) \times \{0\}\big)\\
&+ \mu_1 \operatorname{conv}\partial^* g_1(0,0) + \mu_2 \operatorname{conv}\partial^* g_2(0,0) + \tau \operatorname{conv}\partial^* G(0,0)\Big\}\Big],
\end{aligned}$$

$$0 \in \partial_y f(0,0) + \lambda_1 \partial_y g_1(0,0) + \lambda_2 \partial_y g_2(0,0),$$

$$\lambda_1 g_1(0,0) = 0 \quad \text{but} \quad \lambda_2 g_2(0,0) \ne 0.$$

The following example illustrates that we cannot relax the assumption of the Lipschitz condition on the objective function of the upper-level problem.

Example 5.4 Suppose that in Example 5.2 we take

$$F(x,y) := \begin{cases} \sqrt{x} + y, & x > 0,\ y > 0,\\ x^2, & x > 0,\ y \le 0,\\ \sqrt{y}, & x \le 0,\ y > 0,\\ x^2 - |x| + 1, & x < 0,\ y = 0,\\ \sqrt{-x} - y, & x \le 0,\ y < 0,\\ 0, & x = 0,\ y = 0, \end{cases}$$

F is not locally Lipschitz at $(0,0)$.

F attains its optimal value at $(0,0)$.

$$\partial^* F(0,0) = \{(x^*, y^*) | x^* \le -1, y^* \ge -1\} \quad \text{is an (USRCF) of } F \text{ at } (0,0).$$

We can see that for all $\lambda \geq 0$, $1 \leq \lambda_1 \leq 3$, $\mu \geq 0$, $\tau \geq 0$, there exists $y^* = 0 \in \psi(0)$ such that

$$\begin{aligned}
&(0,0) \notin \mathrm{cl}\bigl[\mathrm{conv}\,\partial^* F(0,0) - \bigl\{\lambda(\partial f(0,0) - \partial_x f(0,0) \times \{0\} - \lambda_1 \partial_x g(0,0) \times \{0\}) \\
&\qquad\qquad + \mu\,\mathrm{conv}\,\partial^* g(0,0) + \tau\,\mathrm{conv}\,\partial^* G(0,0)\bigr\}\bigr], \\
&0 \in \partial_y f(0,0) + \lambda_1 \partial_y g(0,0), \\
&\lambda_1 g(0,0) = 0.
\end{aligned}$$

The next example shows that the (SR) assumption on (UCFs) in Theorem 5.1 cannot be relaxed.

Example 5.5 In Example 5.2, if we take

$$F(x,y) := \begin{cases} |x| \sin\log|x| + |y|, & x > 0, y > 0; x > 0, y < 0; \\ & x < 0, y > 0; x < 0, y < 0, \\ 0, & \text{otherwise.} \end{cases}$$

F attains its minimum at $(0,0)$.

The bounded set, $\partial^* F(0,0) = \{(-1,1), (-1,-1)\}$ is (CF) but not an (USRCF) of F at $(0,0)$. We can see that for all $\lambda \geq 0$, $1 \leq \lambda_1 \leq 3$, $\mu \geq 0$, $\tau \geq 0$, there exists $y^* = 0 \in \psi(0)$ such that

$$\begin{aligned}
&(0,0) \notin \mathrm{cl}\bigl[\mathrm{conv}\,\partial^* F(0,0) - \bigl\{\lambda\bigl(\partial f(0,0) - \partial_x f(0,0) \times \{0\} - \lambda_1 \partial_x g(0,0) \times \{0\}\bigr) \\
&\qquad\qquad + \mu\,\mathrm{conv}\,\partial^* g(0,0) + \tau\,\mathrm{conv}\,\partial^* G(0,0)\bigr\}\bigr], \\
&0 \in \partial_y f(0,0) + \lambda_1 \partial_y g(0,0), \\
&\lambda_1 g(0,0) = 0.
\end{aligned}$$

We end this section by providing an example which shows that the boundedness assumption on (USRCF) of F in Theorem 5.1 cannot be relaxed.

Example 5.6 Suppose that in Example 5.2 we take

$$F(x,y) := \begin{cases} |x| + |y|, & x \geq 0, y \in \mathbb{R} \sim (x = 0,\ y < 0), \\ |x| + |y|, & x < 0,\ y > 0, \\ -|x| - |y|, & x \leq 0,\ y < 0, \\ 100x^2 + |x|, & x < 0,\ y = 0, \end{cases}$$

and

$$G(x,y) := \begin{cases} -\sqrt{x} - 1 - y, & x \geq 0,\ y < 0, \\ \sqrt{-x} + y, & x < 0,\ y < 0, \\ -\sqrt{x} - 1 - y, & x > 0,\ y \geq 0, \\ y - 2y^2, & x = 0,\ y \geq 0, \\ \sqrt{-x} + y, & x < 0,\ y \geq 0, \end{cases}$$

F attains its optimal value at $(0,0)$.

$$\partial^* F(0,0) = \{(x^*, y^*) | x^* > 0, y^* > 0\} \quad \text{is an (USRCF) of } F \text{ at } (0,0).$$

$$\partial^* G(0,0) = \{(x^*, y^*) | x^* \le 0, y^* > 0\} \quad \text{is an (UCF) of } G \text{ at } (0,0).$$

We can see that for all $\lambda \ge 0$, $1 \le \lambda_1 \le 3$, $\mu \ge 0$, $\tau \ge 0$, there exists $y^* = 0 \in \psi(0)$ such that

$$\begin{aligned}
&(0,0) \notin \operatorname{cl}\big[\operatorname{conv} \partial^* F(0,0) - \big\{\lambda\big(\partial f(0,0) - \partial_x f(0,0) \times \{0\} - \lambda_1 \partial_x g(0,0) \times \{0\}\big) \\
&\qquad + \mu \operatorname{conv} \partial^* g(0,0) + \tau \operatorname{conv} \partial^* G(0,0)\big\}\big], \\
&0 \in \partial_y f(0,0) + \lambda_1 \partial_y g(0,0), \\
&\lambda_1 g(0,0) = 0
\end{aligned}$$

but in view of Lemma 5.1(ii), there exists scalars $\lambda = 1$, $\lambda_1 = 1$, $\mu = \frac{3}{2}$, $\tau = 1$ and $y^* = 0 \in \psi(0)$ such that

$$\begin{aligned}
&(0,0) \in \operatorname{cl}\big[\overline{\operatorname{conv}} \partial^* F(0,0) \\
&\qquad + \big(\overline{-\big\{\lambda(\partial f(0,0) - \partial_x f(0,0) \times \{0\} - \lambda_1 \partial_x g(0,0) \times \{0\}) + \mu \operatorname{conv} \partial^* g(0,0) + \tau \operatorname{conv} \partial^* G(0,0)\big\}}\big)\big].
\end{aligned}$$

6 Conclusion

It is known that the bilevel programming problem does not satisfy most of the well-known constraint qualifications such as the Slater CQ, the Mangasarian–Fromovitz CQ. So, in search of the CQ applicable to the bilevel programming problem, we have found Abadie CQ which is weaker than most of the other CQs, the suitable one. Since the convexifactor is a weaker generalization of the idea of subdifferentials and is a closed set and not necessarily convex or compact unlike most existing subdifferentials in the literature, we have introduced two versions of a nonsmooth extension of the Abadie CQ in terms of convexifactors and the Clarke subdifferential. We have employed the weaker version to establish the necessary optimality conditions using an upper estimate of the Clarke subdifferential of value function in variational analysis and the concept of the convexifactor.

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References

1. Bard, J.F.: Practical Bilevel Optimization: Algorithms and Applications. Kluwer Academic, Dordrecht (1998)
2. Dempe, S.: Foundations of Bilevel Programming. Kluwer Academic, Dordrecht (2002)
3. Bard, J.F.: Optimality conditions for the bilevel programming problem. Nav. Res. Logist. Q. **31**, 13–26 (1984)

4. Bard, J.F.: Some properties of the bilevel programming problem. J. Optim. Theory Appl. **68**, 371–378 (1991)
5. Dempe, S.: A necessary and sufficient optimality condition for bilevel programming problem. Optimization **25**, 341–354 (1992)
6. Dempe, S.: First order necessary optimality conditions for general bilevel programming problems. J. Optim. Theory Appl. **95**, 735–739 (1997)
7. Outrata, J.V.: On necessary optimality conditions for Stackelberg problems. J. Optim. Theory Appl. **76**, 306–320 (1993)
8. Ye, J.J., Ye, X.Y.: Necessary optimality conditions for optimization problems with variational inequality constraints. Math. Oper. Res. **22**, 977–997 (1997)
9. Ye, J.J., Zhu, D.L.: Optimality conditions for bilevel programming problems. Optimization **33**, 9–27 (1995)
10. Yezza, A.: First order necessary optimality conditions for general bilevel programming problems. J. Optim. Theory Appl. **89**, 189–219 (1996)
11. Babahadda, H., Gadhi, N.: Necessary optimality conditions for bilevel optimization problems using convexificators. J. Glob. Optim. **34**, 535–549 (2006)
12. Dempe, S., Dutta, J., Mordukhovich, B.S.: New necessary optimality conditions in optimistic bilevel programming. Optimization **56**, 577–604 (2007)
13. Ye, J.J.: Nondifferentiable multiplier rules for optimization and bilevel optimization problems. SIAM J. Optim. **15**, 252–274 (2004)
14. Ye, J.J.: Constraint qualifications and KKT conditions for bilevel programming problems. Math. Oper. Res. **31**, 811–824 (2006)
15. Li, X.F., Zhang, J.Z.: Necessary optimality conditions in terms of convexificators in Lipschitz optimization. J. Optim. Theory Appl. **131**, 429–452 (2006)
16. Abadie, J.M.: On the Kuhn–Tucker theorem. In: Abadie, J. (ed.) Nonlinear Programming, pp. 21–36. Wiley, New York (1967)
17. Demyanov, V.F.: Convexification and concavification of positively homogeneous function by the same family of linear functions. Report 3.208,802 Universita di Pisa (1994)
18. Demyanov, V.F., Jeyakumar, V.: Hunting for a smaller convex subdifferential. J. Glob. Optim. **10**, 305–326 (1997)
19. Jeyakumar, V., Luc, D.T.: Nonsmooth calculus, maximality and monotonicity of convexificators. J. Optim. Theory Appl. **101**, 599–621 (1999)
20. Dutta, J., Chandra, S.: Convexifactors, generalized convexity and optimality conditions. J. Optim. Theory Appl. **113**, 41–65 (2002)
21. Dutta, J., Chandra, S.: Convexifactors, generalized convexity and vector optimization. Optimization **53**, 77–94 (2004)
22. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley Interscience, New York (1983)
23. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory Grundlehren Series (Fundamental Principles of Mathematical Sciences), vol. 330. Springer, Berlin (2006)
24. Mordukhovich, B.S., Nam, N.M.: Variational stability and marginal functions via generalized differentiation. Math. Oper. Res. **30**, 800–816 (2005)
25. Mordukhovich, B.S., Nam, N.M., Yen, N.D.: Subgradients of marginal functions in parametric mathematical programming. Math. Program., Ser. B **116**, 369–396 (2009)