J Optim Theory Appl (2012) 152:31–50
DOI 10.1007/s10957-011-9892-5

On the Solvability and Optimal Controls of Fractional Integrodifferential Evolution Systems with Infinite Delay

JinRong Wang · Yong Zhou · Milan Medved'

Received: 27 December 2010 / Accepted: 22 June 2011 / Published online: 26 July 2011

Abstract In this paper, we study the solvability and optimal controls of a class of fractional integrodifferential evolution systems with infinite delay in Banach spaces. Firstly, a more appropriate concept for mild solutions is introduced. Secondly, existence and continuous dependence of mild solutions are investigated by utilizing the techniques of a priori estimation and extension of step by steps. Finally, existence of optimal controls for system governed by fractional integrodifferential evolution systems with infinite delay is proved.

Keywords Fractional integrodifferential evolution systems · Infinite delay · Solvability · Continuous dependence · Optimal controls

1 Introduction

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics. We can find numerous applications in viscoelasticity, electrochemistry, control, and electromagnetic. There has been a significant development in fractional

J. Wang
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China
e-mail: wjr9668@126.com

Y. Zhou (✉)
Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, China
e-mail: yzhou@xtu.edu.cn

M. Medved'
Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, Slovakia
e-mail: Milan.Medved@fmph.uniba.sk

differential equations. One can see the monographs of Kilbas et al. [1], Miller and Ross [2], Podlubny [3], Lakshmikantham et al. [4], and the survey of Agarwal et al. [5, 6]. Recently, some authors focused on fractional functional differential equations and inclusions in Banach spaces [7–16].

To study the theory of abstract fractional evolution equations involving the Caputo derivative in Banach spaces, the nature of the difficulties is how to introduce a concept of a mild solution. A pioneering work has been reported by El-Borai [17, 18]. Hernández et al. [19] pointed that some recent works [6, 8, 12, 14–16] of abstract fractional evolution equations in Banach spaces were incorrect and used another approach to treat abstract equations with fractional derivatives based on the well developed theory of resolvent operators for integral equations. Particularly, we investigated some fractional evolution equations and optimal controls [20–30], introduced an appropriate definition for mild solutions based on the well-known theory of Laplace transform and probability density functions.

Fractional order semilinear integrodifferential evolution equations with infinite delay have been studied by Ren et al. [14]. Let us mention, however, that the definition of mild solutions (see Definition 3.1, [14]) is not appropriate enough. Thus, it is necessary to revisit the work and give a more appropriate definition for mild solutions. To our knowledge, optimal control problems of system, governed by fractional evolution equations with infinite delay, has not been studied extensively.

The aim of this paper is to investigate the solvability of fractional integrodifferential evolution systems with infinite delay. Meanwhile, optimal controls for system governed by fractional integrodifferential evolution systems with infinite delay and control terms is studied. Comparing with the literature [14], an appropriate definition for mild solutions is introduced (see Sect. 3, Definition 3.1). Different techniques are used here, including a priori estimation of mild solutions and extension of the mild solutions from local interval to global interval. More details can be found in our proof. Furthermore, we discuss the continuous dependence of mild solutions and optimal controls problem, which extend the existence results for mild solutions to the existence result for optimal controls.

The rest of this paper is organized as follows. In Sect. 2, some notations and preparation results are given. In Sect. 3, the existence and uniqueness results of mild solutions for system (1) are given. In Sect. 4, continuous dependence of mild solutions is discussed. In Sect. 5, the Lagrange problem (P) of system (1) is formulated and existence result for optimal controls are presented. Finally, an example is given to illustrate the results.

2 Preliminaries

We investigate the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay:

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + f(t, x_t, \int_0^t g(t,s,x_s)\,ds) + B(t)u(t), & t \in J := [0,T], \\ x(t) = \varphi(t) \in \mathcal{B}, & -\infty < t \le 0, \end{cases} \tag{1}$$

where ${}^C D_t^q$ denotes the Caputo fractional derivative of order $q \in (0,1)$, A is the infinitesimal generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$ (see Definitions 2.1 and 2.2) on a separable reflexive Banach space X, f, and g are X-value functions specified latter, u takes value from another separable reflexive Banach space Y, B is a linear operator from Y into X. The histories $x_t\colon\,]-\infty, 0] \to X$, $x_t = x(t+s)$, belong to some abstract phase space $\mathcal{B}$, that will be introduced later.

Throughout this paper, $L_b(X,Y)$ denotes the space of bounded and linear operators from X to Y and $L_b(X)$ denotes the space of bounded and linear operators in X. Let $\{S(t), t \geq 0\}$ be a family of bounded and linear operators in X, that is, for each $t \geq 0$, $S(t) \in L_b(X)$.

Definition 2.1 The family of bounded and linear operators $\{S(t), t \geq 0\}$ is said to be a semigroup of operators in X if and only if (i) $S(0) = I$; (ii) $S(t+s) = S(t)S(s) = S(s)S(t)$ for all $t, s \geq 0$.

Definition 2.2 A semigroup $\{S(t), t \geq 0\}$ in X is called a strongly continuous semigroup if and only if $\lim_{t\to 0} S(t)x = x$ for each $x \in X$.

Definition 2.3 A linear operator A, defined by

(i) $D(A) := \{x \in X : \lim_{t\to 0} A_t x = \lim_{t\to 0} \frac{S(t)x - x}{t}$ exists$\}$;
(ii) $Ax := \lim_{t\to 0} A_t x$ for all $x \in D(A)$, is called the infinitesimal generator of the semigroup $\{S(t), t \geq 0\}$ in X.

Suppose that A be the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ in a Banach space X. Denote $M := \sup_{t\in J} \|S(t)\|_{L_b(X)}$, which is a finite number. Let $\mathcal{C}(J,X)$, be the Banach space of continuous functions from J to X with the usual supreme norm $\|x\|_{\mathcal{C}} := \sup_{t\in J}\{\|x(t)\|\}$.

We employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [31]. Let $\mathcal{B}$ be a linear space of functions mapping $]-\infty, 0]$ to X endowed with seminorm $\|\cdot\|_{\mathcal{B}}$ and satisfy the following axioms:

(S1) If $x\colon\,]-\infty, 0] \to X$, is such that $x_0 \in \mathcal{B}$, then for every $t \in J$, the following conditions hold:
  (i) x_t is in $\mathcal{B}$,
  (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + \overline{M}(t)\|x_0\|_{\mathcal{B}}$,
  where $H \geq 0$ is a constant, $K : J \to [0, +\infty[$ is continuous, $\overline{M} : [0, +\infty[\to [0, +\infty[$ is locally bounded and $H, K, \overline{M}$ are independent of x.
(S2) For the functions x in (S1), x_t is a $\mathcal{B}$-valued function in J.
(S3) The space $\mathcal{B}$ is compete.

Define $\mathcal{BC} := \{x :]-\infty, 0] \to X, x|_{]-\infty,0]} \in \mathcal{B}$ and $x|_J \in \mathcal{C}(J,X)\}$ and let $\|\cdot\|_{\mathcal{BC}}$ be the seminorm in $\mathcal{BC}$ defined by $\|x\|_{\mathcal{BC}} = \|x_0\|_{\mathcal{B}} + \sup_{s\in J}\{\|x(t)\|\}$. It is easy to see $(\mathcal{BC}, \|\cdot\|_{\mathcal{BC}})$ is a Banach space.

We also set $\mathcal{BC}^0 := \{y \in \mathcal{BC} : y_0 = 0 \in \mathcal{B}\}$ and let $\|\cdot\|_{\mathcal{BC}^0}$ be the seminorm in $\mathcal{BC}^0$ defined by $\|y\|_{\mathcal{BC}^0} := \|y_0\|_{\mathcal{B}} + \sup_{s\in J}\{\|y(t)\|\} = \sup_{s\in J}\{\|y(t)\|\}$. It is easy to see that $(\mathcal{BC}^0, \|\cdot\|_{\mathcal{BC}^0})$ is a Banach space.

Let us recall the following known definitions. For more details, see [1].

Definition 2.4 The fractional integral of the order γ with the lower limit zero for a function f is defined as

$$I^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds, \quad t>0,\ \gamma>0,$$

provided the right hand be point-wise defined on $[0,\infty[$, where $\Gamma(\cdot)$ is the Gamma function, which is defined by $\Gamma(\gamma) := \int_0^{\infty} t^{\gamma-1} e^{-t}\, dt$.

Definition 2.5 The Riemann–Liouville derivative of the order γ with the lower limit zero for a function $f : [0,\infty[\to \mathbb{R}$ can be written as

$${}^{L}D^{\gamma} f(t) := \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}\, ds, \quad t>0,\ n-1<\gamma<n.$$

Definition 2.6 The Caputo derivative of order γ for a function $f : [0,\infty[\to \mathbb{R}$ can be written as

$${}^{C}D^{\gamma} f(t) := {}^{L}D^{\gamma}\left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right), \quad t>0,\ n-1<\gamma<n.$$

Remark 2.1

(i) If $f(t) \in C^n[0,\infty[$, then

$${}^{C}D^{\gamma} f(t) := \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}}\, ds = I^{n-\gamma} f^{(n)}(t),$$
$$t>0,\ n-1<\gamma<n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.4 and 2.5 are taken in Bochner's sense.

In what follows, we collect the Henry–Gronwall inequality (see Lemma 7.1.1, [32]).

Lemma 2.1 *Let $z, \omega : J \to [0,+\infty[$ be continuous functions. If ω is nondecreasing and there are constants $\kappa \ge 0$ and $q>0$ such that*

$$z(t) \le \omega(t) + \kappa \int_0^t (t-s)^{q-1} z(s)\, ds, \quad t \in J,$$

then

$$z(t) \le \omega(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\kappa\Gamma(q))^n}{\Gamma(nq)} (t-s)^{nq-1} \omega(s) \right] ds, \quad t \in J.$$

If $\omega(t) := \bar{a}$, constant on J, then the above inequality is reduce to

$$z(t) \leq \bar{a} E_q\big(\kappa \Gamma(q) t^q\big), \quad t \in J,$$

where E_q is the Mittag-Leffler function [1] *defined by*

$$E_q(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(kq+1)}, \quad y \in \mathbb{C}, \ \mathfrak{Re}(q) > 0.$$

For more generalized Henry–Gronwall inequalities see Ye et al. [33].

Lemma 2.2 *A measurable function $V: J \to X$ is Bochner integrable, if $\|V\|$ is Lebesgue integrable.*

3 Solvability of System

We make the following assumptions.

[HF]: $f : J \times \mathcal{B} \times X \to X$ satisfies:
- (i) f is measurable for $t \in J$.
- (ii) For arbitrary $\xi_1, \xi_2 \in \mathcal{B}$, $\eta_1, \eta_2 \in X$ satisfying $\|\xi_1\|_{\mathcal{B}}$, $\|\xi_2\|_{\mathcal{B}}$, $\|\eta_1\|$, $\|\eta_2\| \leq \rho$, there exists a $L_f(\rho) > 0$ such that

$$\big\|f(t, \xi_1, \eta_1) - f(t, \xi_2, \eta_2)\big\| \leq L_f(\rho)\big(\|\xi_1 - \xi_2\|_{\mathcal{B}} + \|\eta_1 - \eta_2\|\big),$$

for all $t \in J$.

- (iii) There exists a $a_f > 0$ such that

$$\big\|f(t, \xi, \eta)\big\| \leq a_f\big(1 + \|\xi\|_{\mathcal{B}} + \|\eta\|\big), \quad \text{for all } \xi \in \mathcal{B}, \eta \in X \text{ and } t \in J.$$

[HG]: $g : D := \{(t, s) \in J \times J \mid 0 \leq s \leq t\} \times \mathcal{B} \to X$ satisfies:
- (i) g is continuous for $(t, s) \in D$.
- (ii) For arbitrary $(t, s) \in D$ and $\xi_1, \xi_2 \in \mathcal{B}$ satisfying $\|\xi_1\|_{\mathcal{B}}, \|\xi_2\|_{\mathcal{B}} \leq \rho$, there exists a $L_g(\rho) > 0$ such that

$$\big\|g(t, s, \xi_1) - g(t, s, \xi_2)\big\| \leq L_g(\rho)\|\xi_1 - \xi_2\|_{\mathcal{B}}.$$

- (iii) There exists a $M_g > 0$ such that

$$\big\|g(t, s, \xi)\big\| \leq M_g\big(1 + \|\xi\|_{\mathcal{B}}\big) \quad \text{for all } \xi \in \mathcal{B}.$$

[HB]: Let Y be a separable reflexive Banach space from which the control u take the values. Operator $B \in L_\infty(J, L(Y, X))$, $\|B\|_\infty$ stands for the norm of operator B on Banach space $L_\infty\big(J, L(Y, X)\big)$.

[HU]: Multivalued maps $U(\cdot) : J \rightrightarrows 2^Y \backslash \{\emptyset\}$ has closed, convex, and bounded values. $U(\cdot)$ is graph measurable and $U(\cdot) \subseteq \Omega$ where Ω is a bounded set of Y.

Set the admissible set

$$U_{\mathrm{ad}} = \{v(\cdot) \mid J \to Y \text{ strongly measurable}, v(t) \in U(t) \text{ a.e.}\}.$$

Obviously, $U_{\mathrm{ad}} \neq \emptyset$ (Theorem 2.1, [34]) and $U_{\mathrm{ad}} \subset L^p(J, Y)(1 < p < +\infty)$ is bounded, closed, and convex. It is obvious that $Bu \in L^p(J, X)$ for all $u \in U_{\mathrm{ad}}$.

Based on Lemma 3.1 and Definition 3.1 of our earlier work [29], we can introduce the following definition.

Definition 3.1 If for every $u \in U_{\mathrm{ad}}$ there exists a $T = T(u) > 0$ and $x \in \mathcal{BC}$ such that

$$x(t) = \begin{cases} \mathcal{T}(t)\varphi(0) \\ \quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) f(s, x_s, \int_0^s g(s,\tau,x_\tau)d\tau)\,ds \\ \quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)B(s)u(s)\,ds, \quad 0 \le t \le T, \\ \varphi(t), \quad -\infty < t \le 0, \end{cases} \tag{2}$$

then system (1) is called mildly solvable with respect to u on $]-\infty, T]$, where

$$\mathcal{T}(t) = \int_0^\infty \xi_q(\theta) S(t^q\theta)\,d\theta, \qquad \mathcal{S}(t) = q\int_0^\infty \theta\xi_q(\theta)S(t^q\theta)\,d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\varpi_q(\theta^{-\frac{1}{q}}) \ge 0,$$

$$\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in]0, \infty[,$$

ξ_q is a probability density function defined on $]0, \infty[$, that is,

$$\xi_q(\theta) \ge 0, \quad \theta \in]0,\infty[\quad \text{and} \quad \int_0^\infty \xi_q(\theta)\,d\theta = 1.$$

The following properties of the operators $\mathcal{T}$ and $\mathcal{S}$ have been proved in our earlier work (see Lemmas 3.2–3.4, [29]).

Lemma 3.1 *The operators $\mathcal{T}$ and $\mathcal{S}$ have the following properties*:

(i) *For any fixed $t \ge 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are linear and bounded operators, i.e., for any $x \in X$,*

$$\|\mathcal{T}(t)x\| \le M\|x\| \quad \text{and} \quad \|\mathcal{S}(t)x\| \le \frac{qM}{\Gamma(1+q)}\|x\|.$$

(ii) *$\{\mathcal{T}(t), t \ge 0\}$ and $\{\mathcal{S}(t), t \ge 0\}$ are strongly continuous.*

(iii) *For every $t > 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are also compact operators if $T(t)$ is compact.*

In order to discuss the solvability of system (1), we need the following important a priori estimation.

Lemma 3.2 *Let* $\varphi(0)\in X$, [HF](iii) *and* [HG](iii) *hold. Suppose system* (1) *is mildly solvable on* $]-\infty,T]$ *with respect to* $u\in U_{\mathrm{ad}}$, *then there exists a constant* $\rho>0$ *such that*

$$\|x(t)\|\le\rho\quad \textit{for all } t\in J.$$

Proof Since system (1) is mildly solvable on $]-\infty,T]$ with respect to $u\in U_{\mathrm{ad}}$, by Definition 3.1, we can suppose x is a mild solution of system (1) with respect to u on $]-\infty,T]$, then x satisfies the form (2). Let $x(t)=y(t)+\widetilde{\varphi}(t)$ where $\widetilde{\varphi}$: $]-\infty,T]\to X$ be function given by

$$\widetilde{\varphi}(t)=\begin{cases}\varphi(t), & -\infty<t\le 0,\\ \mathcal{T}(t)\varphi(0), & t\in J.\end{cases}\tag{3}$$

It is obvious that x satisfies the form (2) if and only if

$$\begin{cases} y_0=0, \quad -\infty<t\le 0,\\ y(t)=\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)f(s,y_s+\widetilde{\varphi}_s,\int_0^s g(s,\tau,y_\tau+\widetilde{\varphi}_\tau)d\tau)\,ds,\\ \qquad +\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)B(s)u(s)\,ds, \quad t\in J.\end{cases}\tag{4}$$

For $t\in J$, directly calculation gives that

$$\begin{aligned}\|y(t)\|&\le\int_0^t(t-s)^{q-1}\left\|\mathcal{S}(t-s)f\left(s,y_s+\widetilde{\varphi}_s,\int_0^s g(s,\tau,y_\tau+\widetilde{\varphi}_\tau)\,d\tau\right)\right\|ds\\ &\quad+\int_0^t(t-s)^{q-1}\|\mathcal{S}(t-s)B(s)u(s)\|\,ds\\ &\le\frac{qM}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}a_f\big(1+\|y_s+\widetilde{\varphi}_s\|_{\mathcal{B}}+M_gT\big(1+\|y_\tau+\widetilde{\varphi}_\tau\|_{\mathcal{B}}\big)\big)\,ds\\ &\quad+\frac{qM\|B\|_\infty}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}\|u(s)\|_Y\,ds\\ &\le a+\frac{a_fqM(1+M_gT)}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}\|y_s+\widetilde{\varphi}_s\|_{\mathcal{B}}\,ds,\end{aligned}\tag{5}$$

where

$$a=\frac{a_f(1+M_gT)MT^q}{\Gamma(1+q)}+\frac{qM\|B\|_\infty}{\Gamma(1+q)}\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}T^{q-\frac{1}{p}}\|u\|_{L^p(J,Y)}.$$

Let $K_T=\max\{K(t):t\in J\}$ and $M_T=\max\{\overline{M}(t):t\in J\}$. Then

$$\begin{aligned}\|y_s+\widetilde{\varphi}_s\|_{\mathcal{B}}&\le\|y_s\|_{\mathcal{B}}+\|\widetilde{\varphi}_s\|_{\mathcal{B}}\\ &\le K(t)\sup\big\{\|y(s)\|:0\le s\le t\big\}+M(t)\|y_0\|_{\mathcal{B}}\end{aligned}$$

$$+ K(t)\sup\{\|\widetilde{\varphi}(s)\| : 0 \le s \le t\} + M(t)\|\widetilde{\varphi}_0\|_{\mathcal{B}}$$
$$\le K_T \sup\{\|y(s)\| : 0 \le s \le t\} + K_T M\|\varphi(0)\| + M_T\|\varphi\|_{\mathcal{B}}.$$

Set

$$z(t) = K_T \sup\{\|y(s)\| : 0 \le s \le t\} + K_T M\|\varphi(0)\| + M_T\|\varphi\|_{\mathcal{B}},$$

then

$$\|y_s + \widetilde{\varphi}_s\|_{\mathcal{B}} \le z(t),$$

which implies that (5) can be written as

$$\|y(t)\| \le a + \frac{a_f q M(1+M_g T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} z(s)\,ds. \tag{6}$$

Note that (6) and the definition of z, we can obtain

$$z(t) \le K_T M\|\varphi(0)\| + M_T\|\varphi\|_{\mathcal{B}} + K_T a$$
$$+ \frac{K_T a_f q M(1+M_g T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} z(s)\,ds.$$

Applying Lemma 3.2, there is a constant $\widehat{M} > 0$ such that

$$z(t) \le \widehat{M}\big(K_T M\|\varphi(0)\| + M_T\|\varphi\|_{\mathcal{B}} + K_T a\big) := \widetilde{M}, \quad t \in J.$$

Then we have

$$\|y(t)\| \le a + \frac{a_f q M(1+M_g T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \widetilde{M}\,ds,$$

which implies that

$$\|y(t)\| \le a + \frac{a_f M(1+M_g T)T^q}{\Gamma(1+q)} \widetilde{M} := M^*.$$

As a result, for $t \in J$,

$$\|x(t)\| \le \|y(t)\| + \|\mathcal{T}(t)\varphi(0)\| \le M^* + M\|\varphi(0)\| := \rho.$$

The proof is completed. □

Remark 3.1 By the definition of the seminorm in $\mathcal{BC}$, it is not difficult to see $\|x\|_{\mathcal{BC}} \le \|\varphi\|_{\mathcal{B}} + \rho := \rho^*$.

Theorem 3.1 *Suppose* [HF], [HG], [HB], *and* [HU] *hold*, $\varphi(0) \in X$. *Then for each* $u \in U_{\mathrm{ad}}$ *and for some* $p \in]1, \infty[$ *such that* $pq > 1$, *system* (1) *is mildly solvable on* $]-\infty, T]$ *with respect to* u, *and the mild solution is unique.*

Proof Let $\mathcal{BC}|_{T_1} := \{x :]-\infty, 0] \to X, x|_{]-\infty,0]} \in \mathcal{B}$ and $x|_{[0,T_1]} \in \mathcal{C}([0, T_1], X)\}$ and

$$\mathcal{S}(1, T_1) := \Big\{x \in \mathcal{BC}|_{T_1} \;\Big|\; \max_{s\in[0,T_1]} \big\|x(s) - \varphi(0)\big\| \le 1,\ x(s) = \varphi(s) \text{ for } -\infty < s \le 0\Big\}.$$

Then $\mathcal{S}(1, T_1) \subseteq \mathcal{BC}|_{T_1}$ is a closed convex subset of $\mathcal{BC}|_{T_1}$. According to [HF](i) and [HG](i), it is easy to obtain that $f(s, x_s, \int_0^s g(s, \tau, x_\tau)\,d\tau)$ is a measurable function for $s \in [0, t]$, $t \in [0, T_1]$. Let $x \in \mathcal{S}(1, T_1)$, there exists a constant $\rho^* := \|\varphi(0)\| + 1 + \|\varphi\|_{\mathcal{B}} > 0$ such that $\|x\|_{\mathcal{BC}|_{T_1}} \le \rho^*$. Using [HF](iii) and [HG](iii), we have

$$\begin{aligned} &\left\| f\left(s, x_s, \int_0^s g(s, \tau, x_\tau)\,d\tau\right)\right\| \\ &\quad\le a_f\big(1 + \|x_s\|_{\mathcal{B}} + M_g T\big(1 + \|x_\tau\|_{\mathcal{B}}\big)\big) \\ &\quad\le a_f\big(1 + \rho^* + M_g T\big(1 + \rho^*\big)\big) \equiv K^*, \quad \text{for } t \in [0, T_1]. \end{aligned} \tag{7}$$

In light of Lemma 3.1(i) and (7), we obtain that

$$\int_0^t (t-s)^{q-1}\left\|\mathcal{S}(t-s) f\left(s, x_s, \int_0^s g(s, \tau, x_\tau)\,d\tau\right)\right\| ds \le \frac{MK^*T_1^q}{\Gamma(1+q)}.$$

Thus, $(t-s)^{q-1}\mathcal{S}(t-s)f(s, x_s, \int_0^s g(s, \tau, x_\tau)\,d\tau)$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in [0, T_1]$ due to Lemma 2.2.

On the other hand, by Lemma 3.1(i), [HB], [HU] and $pq > 1$, we have

$$\begin{aligned} &\int_0^t (t-s)^{q-1}\big\|\mathcal{S}(t-s)B(s)u(s)\big\|\,ds \\ &\quad\le \frac{qM\|B\|_\infty}{\Gamma(1+q)} \int_0^t (t-s)^{q-1}\big\|u(s)\big\|_Y\,ds \\ &\quad\le \frac{qM\|B\|_\infty}{\Gamma(1+q)} \left(\int_0^t (t-s)^{\frac{p}{p-1}(q-1)}\,ds\right)^{\frac{p-1}{p}} \left(\int_0^t \big\|u(s)\big\|_Y^p\,ds\right)^{\frac{1}{p}} \\ &\quad\le \frac{qM\|B\|_\infty}{\Gamma(1+q)} \left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}} T^{q-\frac{1}{p}}\|u\|_{L^p(J,Y)}. \end{aligned} \tag{8}$$

Thus, $(t-s)^{q-1}\mathcal{S}(t-s)B(s)u(s)$ is also Bochner integrable with respect to $s \in [0, t]$ for all $t \in [0, T_1]$ due to Lemma 2.2 again.

Now we can define $P\colon \mathcal{S}(1, T_1) \to \mathcal{BC}|_{T_1}$ by

$$(Px)(t) = \begin{cases} \mathcal{T}(t)\varphi(0) \\ \quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)f(s, x_s, \int_0^s g(s, \tau, x_\tau)\,d\tau)\,ds \\ \quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)B(s)u(s)\,ds, \quad 0 < t \le T_1, \\ \varphi(t), \quad -\infty < t \le 0. \end{cases} \tag{9}$$

Let $x(t)=y(t)+\widetilde{\varphi}(t)$ where $\widetilde{\varphi}\colon\ ]-\infty,T]\to X$ be the function given by (3). Then x satisfies (2), if and only if $y_0=0$ and

$$y(t)=\int_0^t (t-s)^{q-1}\mathcal{S}(t-s)f\left(s,y_s+\widetilde{\varphi}_s,\int_0^s g(s,\tau,y_\tau+\widetilde{\varphi}_\tau)\,d\tau\right)ds \\ +\int_0^t (t-s)^{q-1}\mathcal{S}(t-s)B(s)u(s)\,ds,\quad t\in J.$$

Define

$$\mathcal{BC}|_{T_1}^0:=\{y\in\mathcal{BC}|_{T_1}:y_0=0\in\mathcal{B}\}$$

and let $\|\cdot\|_{\mathcal{BC}|_{T_1}^0}$ be the seminorm in $\mathcal{BC}|_{T_1}^0$ defined by

$$\|y\|_{\mathcal{BC}|_{T_1}^0}:=\|y_0\|_{\mathcal{B}}+\sup_{s\in[0,T_1]}\{\|y(t)\|\}=\sup_{s\in[0,T_1]}\{\|y(t)\|\}.$$

Then $(\mathcal{BC}|_{T_1}^0,\|\cdot\|_{\mathcal{BC}|_{T_1}^0})$ is a Banach space.

Set

$$\mathcal{S}^0(1,T_1):=\left\{y\in\mathcal{BC}|_{T_1}^0\ \Big|\ \max_{s\in[0,T_1]}\|y(s)\|\le 1,\ y(s)=0\text{ for }-\infty<s\le 0\right\}.$$

Then $\mathcal{S}^0(1,T_1)\subseteq\mathcal{BC}|_{T_1}^0$ is a closed convex subset of $\mathcal{BC}|_{T_1}^0$.

Define $P^0\colon\mathcal{S}^0(1,T_1)\to\mathcal{BC}|_{T_1}^0$ by

$$(P^0y)(t)=\begin{cases}\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)f(s,y_s+\widetilde{\varphi}_s,\int_0^s g(s,\tau,y_\tau+\widetilde{\varphi}_\tau)\,d\tau)\,ds\\ \quad+\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)B(s)u(s)\,ds,\quad 0<t\le T_1,\\ 0,\quad -\infty<t\le 0.\end{cases}\tag{10}$$

Next, we verify that P^0 is a contraction mapping on $\mathcal{S}^0(1,T_1)$ with chosen $T_1>0$. For $t\in[0,T_1]$, it is not difficult to see

$$\begin{aligned}\|(P^0y)(t)\|&\le\int_0^t(t-s)^{q-1}\left\|\mathcal{S}(t-s)f\left(s,y_s+\widetilde{\varphi}_s,\int_0^s g(s,\tau,y_\tau+\widetilde{\varphi}_\tau)\,d\tau\right)\right\|ds\\&\quad+\int_0^t(t-s)^{q-1}\|\mathcal{S}(t-s)B(s)u(s)\|\,ds\\&\le\frac{MK^*}{\Gamma(1+q)}t^q+\frac{qM\|B\|_\infty\|u\|_{L^p(J,Y)}}{\Gamma(1+q)}\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}t^{q-\frac{1}{p}}.\end{aligned}\tag{11}$$

Let

$$T_{11}=\left[\frac{\Gamma(1+q)}{M(K^*+q\|B\|_\infty\|u\|_{L^p(J,Y)})(\frac{p-1}{pq-1})^{\frac{p-1}{p}}}\right]^{\frac{p}{pq-1}},$$

then for all $t \le T_{11}$, it comes from (11) that

$$\left\| \left(P^0 y\right)(t) \right\| \le 1.$$

On the other hand, for $-\infty < t \le 0$, $(P^0 y)(t) = 0$.

Hence, $P^0(\mathcal{S}^0(1, T_1)) \subseteq \mathcal{S}^0(1, T_1)$.

For each $t \in [0, T_1]$, $y, \widehat{y} \in \mathcal{S}^0(1, T_1)$ and $\|y\|_{\mathcal{BC}|^0_{T_1}}, \|\widehat{y}\|_{\mathcal{BC}|^0_{T_1}} \le \rho^*$. For $t \in [0, T_1]$, using Lemma 3.1(i), [HF](ii), and [HG](ii), we also obtain

$$\begin{aligned}
&\left\| \left(P^0 y\right)(t) - \left(P^0 \widehat{y}\right)(t) \right\| \\
&\quad \le \int_0^t (t-s)^{q-1} \left\| \mathcal{S}(t-s)\left[f\left(s, y_s + \widetilde{\varphi}_s, \int_0^s g(s,\tau, y_\tau + \widetilde{\varphi}_\tau)\, d\tau\right) \right.\right. \\
&\qquad \left.\left. - f\left(s, \widehat{y}_s + \widetilde{\varphi}_s, \int_0^s g(s,\tau, \widehat{y}_\tau + \widetilde{\varphi}_\tau)\, d\tau\right)\right] \right\| ds \\
&\quad \le \frac{qML_f(\rho^*)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|y_s - \widehat{y}_s\|_{\mathcal{B}}\, ds \\
&\qquad + \frac{qML_f(\rho^*)L_g(\rho^*)T}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|y_\tau - \widehat{y}_\tau\|_{\mathcal{B}}\, ds \\
&\quad \le \frac{qML_f(\rho^*)(1+L_g(\rho^*)T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|y_s - \widehat{y}_s\|_{\mathcal{B}}\, ds, \\
&\quad \le \frac{qML_f(\rho^*)(1+L_g(\rho^*)T)K_T}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \sup_{s\in J} \left\| y(s) - \widehat{y}(s) \right\| ds,
\end{aligned}$$

which implies that

$$\sup_{t\in J} \left\| \left(P^0 y\right)(t) - \left(P^0 \widehat{y}\right)(t) \right\| \le \frac{ML_f(\rho^*)(1+L_g(\rho^*)T)K_T}{\Gamma(1+q)} t^q \sup_{s\in J} \left\| y(s) - \widehat{y}(s) \right\|.$$

Thus,

$$\left\| P^0 y - P^0 \widehat{y} \right\|_{\mathcal{BC}|^0_{T_1}} \le \frac{ML_f(\rho^*)(1+L_g(\rho^*)T)K_T}{\Gamma(1+q)} t^q \|y - \widehat{y}\|_{\mathcal{BC}|^0_{T_1}}.$$

Let

$$T_{12} = \frac{1}{2}\left[\frac{\Gamma(1+q)}{ML_f(\rho^*)(1+L_g(\rho^*)T)K_T}\right]^{\frac{1}{q}}, \quad T_1 = \min\{T_{11}, T_{12}\},$$

then P^0 is a contraction mapping on $\mathcal{S}^0(1, T_1)$. It follows from the contraction mapping principle that P^0 has a unique fixed point $y \in \mathcal{S}^0(1, T_1)$. Therefore, $x(t) = y(t) + \widetilde{\varphi}(t)$ is just the unique mild solution of system (1) with respect to u on $(-\infty, T_1]$.

Let $T_{21} = T_1 + T_{11}$, $T_{22} = T_1 + T_{12}$, $\Delta T = \min\{T_{21} - T_1, T_{12}\} > 0$. Similarly, one can verify that system (1) has an unique mild solutions on $(-\infty, \Delta T]$. Repeating the above procedures in each interval $[\Delta T, 2\Delta T]$, $[2\Delta T, 3\Delta T]$, …, and using the methods of steps, we immediately obtain the global existence of mild solutions for the system (1). □

4 Continuous Dependence

In this section, we discuss the continuous dependence of mild solutions for system (1).

Theorem 4.1 *Suppose $\varphi^i(0) \in \Pi$, where Π be a bounded subset of X, $\varphi^i \in \mathcal{B}$ and $u^i \in U_{\mathrm{ad}}$, $i = 1, 2$. Let*

$$x^i(t) = \begin{cases} \mathcal{T}(t)\varphi^i(0) \\ \quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) f(s, x_s^i, \int_0^s g(s,\tau,x_\tau^i)\,d\tau)\,ds \\ \quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)B(s)u^i(s)\,ds, \quad 0 \le t \le T, \\ \varphi^i(t), \quad -\infty < t \le 0. \end{cases}$$

Then there exists a constant $C^* > 0$ such that

$$\begin{cases} \|x^1(t) - x^2(t)\| \le C^*(\|\varphi^1(0) - \varphi^2(0)\| + \|\varphi^1 - \varphi^2\|_{\mathcal{B}} \\ \qquad\qquad + \|u^1 - u^2\|_{L^p(J,Y)}), & t \in J, \\ \|x^1(t) - x^2(t)\| = \|\varphi^1(t) - \varphi^2(t)\|, & -\infty < t \le 0. \end{cases}$$

Proof Let $x^i(t) = y^i(t) + \widetilde{\varphi}^i(t)$ where $\widetilde{\varphi}^i \colon]-\infty, T] \to X$, $i = 1, 2$, be function defined by

$$\widetilde{\varphi}^i(t) = \begin{cases} \varphi^i(t), & -\infty < t \le 0, \\ \mathcal{T}(t)\varphi^i(0), & t \in J, \end{cases} \tag{12}$$

where

$$\begin{cases} y_0^i = 0, \quad -\infty < t \le 0, \\ y^i(t) = \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) f(s, x_s^i + \widetilde{\varphi}_s^i, \int_0^s g(s,\tau,x_\tau^i + \widetilde{\varphi}_\tau^i)\,d\tau)\,ds \\ \qquad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)B(s)u^i(s)\,ds, \quad 0 \le t \le T. \end{cases}$$

By Lemma 3.2 and [HG](iii), one can check that there exists a constant $\rho > 0$ such that

$$\|y_s^i + \widetilde{\varphi}_s^i\|_{\mathcal{B}} \le \rho \quad \text{and} \quad \left\| \int_0^s g\big(s,\tau,y_\tau^i + \widetilde{\varphi}_\tau^i\big)\,d\tau \right\| \le \rho.$$

For $t \in J$, by virtue of Lemma 3.1(i), [HF](ii), [HG](ii), [HB], [HU], and Hölder inequality, we have

$$
\begin{aligned}
&\left\| y^1(t) - y^2(t) \right\| \\
&\quad \le \int_0^t (t-s)^{q-1} \left\| \mathcal{S}(t-s) \left[f\left(s, y_s^1 + \widetilde{\varphi}_s^1, \int_0^s g\big(s, \tau, y_\tau^1 + \widetilde{\varphi}_\tau^1\big)\, d\tau \right) \right.\right. \\
&\qquad \left.\left. - f\left(s, y_s^2 + \widetilde{\varphi}_s^2, \int_0^s g\big(s, \tau, y_\tau^2 + \widetilde{\varphi}_\tau^2\big)\, d\tau \right) \right] \right\| ds \\
&\qquad + \int_0^t (t-s)^{q-1} \left\| \mathcal{S}(t-s) \big[B(s) u^1(s) - B(s) u^2(s) \big] \right\| ds \\
&\quad \le \frac{q M L_f(\rho)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \big(\| y_s^1 - y_s^2 \|_{\mathcal{B}} + \| \widetilde{\varphi}_s^1 - \widetilde{\varphi}_s^2 \|_{\mathcal{B}} \big)\, ds \\
&\qquad + \frac{q M L_f(\rho) L_g(\rho) T}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \big(\| y_\tau^1 - y_\tau^2 \|_{\mathcal{B}} + \| \widetilde{\varphi}_\tau^1 - \widetilde{\varphi}_\tau^2 \|_{\mathcal{B}} \big)\, ds \\
&\qquad + \frac{q M \|B\|_\infty}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left\| u^1(s) - u^2(s) \right\|_Y ds \\
&\quad \le \frac{q M \|B\|_\infty}{\Gamma(1+q)} \left(\int_0^t (t-s)^{\frac{p}{p-1}(q-1)}\, ds \right)^{\frac{p-1}{p}} \left(\int_0^t \left\| u^1(s) - u^2(s) \right\|_Y^p ds \right)^{\frac{1}{p}} \\
&\qquad + \frac{q M L_f(\rho)(1 + L_g(\rho) T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \big(\| y_s^1 - y_s^2 \|_{\mathcal{B}} + \| \widetilde{\varphi}_s^1 - \widetilde{\varphi}_s^2 \|_{\mathcal{B}} \big)\, ds \\
&\quad \le \frac{q M \|B\|_\infty}{\Gamma(1+q)} \left(\frac{p-1}{pq-1} \right)^{\frac{p-1}{p}} T^{q - \frac{1}{p}} \left\| u^1 - u^2 \right\|_{L^p(J,Y)} \\
&\qquad + \frac{q M L_f(\rho)(1 + L_g(\rho) T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \big(\| y_s^1 - y_s^2 \|_{\mathcal{B}} + \| \widetilde{\varphi}_s^1 - \widetilde{\varphi}_s^2 \|_{\mathcal{B}} \big)\, ds.
\end{aligned}
$$

It is easy to obtain

$$
\begin{aligned}
&\| y_s^1 - y_s^2 \|_{\mathcal{B}} + \| \widetilde{\varphi}_s^1 - \widetilde{\varphi}_s^2 \|_{\mathcal{B}} \\
&\quad \le K(t) \sup\big\{ \left\| y^1(s) - y^2(s) \right\| : 0 \le s \le t \big\} + M(t) \left\| y_0^1 - y_0^2 \right\|_{\mathcal{B}} \\
&\qquad + K(t) \sup\big\{ \left\| \widetilde{\varphi}^1(s) - \widetilde{\varphi}^2(s) \right\| : 0 \le s \le t \big\} + M(t) \left\| \widetilde{\varphi}_0^1 - \widetilde{\varphi}_0^2 \right\|_{\mathcal{B}} \\
&\quad \le V(t),
\end{aligned}
$$

where

$$
\begin{aligned}
V(t) := {} & K_T \sup\big\{ \left\| y^1(s) - y^2(s) \right\| : 0 \le s \le t \big\} + K_T M \left\| \varphi^1(0) - \varphi^2(0) \right\| \\
& + M_T \left\| \varphi^1 - \varphi^2 \right\|_{\mathcal{B}}.
\end{aligned}
$$

Thus,

$$
V(t) \le a' + K_T c' \int_0^t (t-s)^{q-1} V(s)\, ds,
$$

where

$$a' = K_T M \|\varphi^1(0) - \varphi^2(0)\| + M_T \|\widetilde{\varphi}^1 - \widetilde{\varphi}^2\|_{\mathcal{B}} + K_T b' \|u^1 - u^2\|_{L^p(J,Y)},$$

$$b' = \frac{qM\|B\|_\infty}{\Gamma(1+q)} \left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}} T^{q-\frac{1}{p}},$$

$$c' = \frac{qML_f(\rho)(1+L_g(\rho)T)}{\Gamma(1+q)}.$$

By Lemma 2.1 again, there exists a constant $\widehat{M} > 0$ such that

$$V(t) \le \widehat{M} a' \quad \text{for all } t \in J.$$

Thus, for all $t \in J$,

$$\begin{aligned}
&\|y^1(t) - y^2(t)\| \\
&\quad \le b'\|u^1 - u^2\|_{L^p(J,Y)} + c'\widehat{M}\frac{T^q}{q}a' \\
&\quad \le c'\widehat{M}\frac{T^q}{q}K_T M\|\varphi^1(0) - \varphi^2(0)\| \\
&\qquad + c'\widehat{M}\frac{T^q}{q}M_T\|\widetilde{\varphi}^1 - \widetilde{\varphi}^2\|_{\mathcal{B}} + \left(c'\widehat{M}\frac{T^q}{q}K_T + 1\right)b'\|u^1 - u^2\|_{L^p(J,Y)},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x^1(t) - x^2(t)\| \\
&\quad \le \left(c'\widehat{M}\frac{T^q}{q}K_T + 1\right)M\|\varphi^1(0) - \varphi^2(0)\| + c'\widehat{M}\frac{T^q}{q}M_T\|\widetilde{\varphi}^1 - \widetilde{\varphi}^2\|_{\mathcal{B}} \\
&\qquad + \left(c'\widehat{M}\frac{T^q}{q}K_T + 1\right)b'\|u^1 - u^2\|_{L^p(J,Y)}.
\end{aligned}$$

Let

$$C^* := \max\left\{\left(c'\widehat{M}\frac{T^q}{q}K_T + 1\right)M, c'\widehat{M}\frac{T^q}{q}M_T, \left(c'\widehat{M}\frac{T^q}{q}K_T + 1\right)b'\right\} > 0.$$

Then, one can obtain

$$\begin{aligned}
&\|x^1(t) - x^2(t)\| \\
&\quad \le C^*\big(\|\varphi^1(0) - \varphi^2(0)\| + \|\varphi^1 - \varphi^2\|_{\mathcal{B}} + \|u^1 - u^2\|_{L^p(J,Y)}\big), \quad \text{for } t \in J.
\end{aligned}$$

Note that,

$$\|x^1(t) - x^2(t)\| \le \|\varphi^1(t) - \varphi^2(t)\|, \quad \text{for } -\infty < t \le 0.$$

This completes the proof. □

5 Existence of Optimal Controls

In the following, we consider the Lagrange problem (P):

Find a control $u^0 \in U_{\mathrm{ad}}$ such that

$$\mathcal{J}\left(u^0\right) \le \mathcal{J}(u), \quad \text{for all } u \in U_{\mathrm{ad}}$$

where

$$\mathcal{J}(u) := \int_0^{\mathrm{T}} \mathcal{L}\left(t, x_t^u, x^u(t), u(t)\right) dt,$$

and x^u denotes the mild solution of system (1) corresponding to the control $u \in U_{\mathrm{ad}}$.

For the existence of solution for problem (P), we shall introduce the following assumption:

[HL]: (i) The functional $\mathcal{L}: J \times \mathcal{B} \times X \times Y \longrightarrow R \cup \{\infty\}$ is Borel measurable.
(ii) $\mathcal{L}(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $\mathcal{B} \times X \times Y$ for almost all $t \in J$.
(iii) $\mathcal{L}(t, x, y, \cdot)$ is convex on Y for each $x \in \mathcal{B}$, $y \in X$ and almost all $t \in J$.
(iv) There exist constants $d, e \ge 0$, $j > 0$, μ is nonnegative and $\mu \in L^1(J, R)$ such that

$$\mathcal{L}(t, x, y, u) \ge \mu(t) + d\|x\|_{\mathcal{B}} + e\|y\| + j\|u\|_Y^p.$$

Now, we can give the following result on existence of optimal controls for problem (P).

Theorem 5.1 *Let assumptions of Theorem 3.1 and* [HL] *hold. Suppose that B be a strongly continuous operator. Then Lagrange problem* (P) *admits at least one optimal pair, that is there exists an admissible control $u^0 \in U_{\mathrm{ad}}$ such that*

$$\mathcal{J}\left(u^0\right) = \int_0^{\mathrm{T}} \mathcal{L}\left(t, x_t^0, x^0(t), u^0(t)\right) dt \le \mathcal{J}(u), \quad \textit{for all } u \in U_{\mathrm{ad}}.$$

Proof If $\inf\{\mathcal{J}(u) \mid u \in U_{\mathrm{ad}}\} = +\infty$, there is nothing to prove. Without loss of generality, we assume that $\inf\{\mathcal{J}(u) \mid u \in U_{\mathrm{ad}}\} = \epsilon < +\infty$. Using [HL], we have $\epsilon > -\infty$. By definition of infimum there exists a minimizing sequence feasible pair $\{(x^m, u^m)\} \subset \mathcal{A}_{\mathrm{ad}} \equiv \{(x, u) \mid x$ is a mild solution of system (1) corresponding to $u \in U_{\mathrm{ad}}\}$, such that $\mathcal{J}(x^m, u^m) \to \epsilon$ as $m \to +\infty$. Since $\{u^m\} \subseteq U_{\mathrm{ad}}$, $m = 1, 2, \ldots$, $\{u^m\}$ is a bounded subset of the separable reflexive Banach space $L^p(J, Y)$, there exists a subsequence, relabeled as $\{u^m\}$, and $u^0 \in L^p(J, Y)$ such that $u^m \xrightarrow{w} u^0$ in $L^p(J, Y)$. Since U_{ad} is closed and convex, thanks to Marzur lemma, $u^0 \in U_{\mathrm{ad}}$.

Let $\{x^m\} \subset \mathcal{BC}$ denote the corresponding sequence of solutions of the integral equation

$$x^m(t) = \begin{cases} \mathcal{T}(t)\varphi(0) \\ \quad + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s, x_s^m, \int_0^s g(s, \tau, x_\tau^m)\, d\tau)\, ds \\ \quad + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) B(s) u^m(s)\, ds, \quad 0 \le t \le T, \\ \varphi(t), \quad -\infty < t \le 0. \end{cases}$$

For Lemma 3.2 and Remark 3.1, there exists a $\rho > 0$ such that

$$\|x^m\|_{\mathcal{BC}} \le \rho, \quad m = 0, 1, 2, \dots.$$

Let $x^m(t) = y^m(t) + \widetilde{\varphi}(t)$ where $y^m \in \mathcal{BC}^0$ and $\widetilde{\varphi}\colon\,]-\infty, T] \to X$ be the function given by (3). For $t \in J$, we have

$$\begin{aligned}
&\|y^m(t) - y^0(t)\| \\
&\le \int_0^t (t-s)^{q-1} \bigg\| \mathcal{S}(t-s)\bigg[f\bigg(s, y_s^m + \widetilde{\varphi}_s, \int_0^s g\big(s,\tau, y_\tau^m + \widetilde{\varphi}_\tau\big)\, d\tau\bigg) \\
&\quad - f\bigg(s, y_s^0 + \widetilde{\varphi}_s, \int_0^s g\big(s,\tau, y_\tau^0 + \widetilde{\varphi}_\tau\big)\, d\tau\bigg)\bigg]\bigg\|\, ds \\
&\quad + \int_0^t (t-s)^{q-1} \big\| \mathcal{S}(t-s)\big[B(s)u^m(s) - B(s)u^0(s)\big]\big\|\, ds \\
&\le \frac{qML_f(\rho)(1+L_g(\rho)T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|y_s^m - y_s^0\|_{\mathcal{B}}\, ds \\
&\quad + \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|B(s)u^m(s) - B(s)u^0(s)\|\, ds \\
&\le \frac{qM}{\Gamma(1+q)} \bigg(\frac{p-1}{pq-1}\bigg)^{\frac{p-1}{p}} t^{(q-\frac{1}{p})} \bigg(\int_0^t \|B(s)u^m(s) - B(s)u^0(s)\|^p\, ds\bigg)^{\frac{1}{p}} \\
&\quad + \frac{qML_f(\rho)(1+L_g(\rho)T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \sup_{s\in J} \|y^m(s) - y^0(s)\|_{\mathcal{B}}\, ds,
\end{aligned}$$

which implies that there exists a constant $M^* > 0$ such that

$$\sup_{t\in J} \|y^m(t) - y^0(t)\| \le M^* \|Bu^m - Bu^0\|_{L^p(J,Y)}, \quad \text{for } t \in J. \tag{13}$$

Since B is strongly continuous, we have

$$\|Bu^m - Bu^0\|_{L^p(J,Y)} \xrightarrow{s} 0 \quad \text{as } m \to \infty. \tag{14}$$

Then we have

$$\|y^m - y^0\|_{\mathcal{BC}^0} \xrightarrow{s} 0 \quad \text{as } m \to \infty,$$

which is equivalent to

$$\|x^m - x^0\|_{\mathcal{BC}} \xrightarrow{s} 0 \quad \text{as } m \to \infty.$$

This yields that

$$x^m \xrightarrow{s} x^0 \quad \text{in } \mathcal{BC} \text{ as } m \to \infty.$$

Note that [HL] implies the assumptions of Balder (see Theorem 2.1, [35]). Hence, by Balder's theorem, we can conclude that $(x_t \times x, u) \longrightarrow \int_0^{\mathrm{T}} \mathcal{L}(t, x_t, x(t), u(t))\, dt$

is sequentially lower semicontinuous in the weak topology of $L^p(J,Y) \subset L^1(J,Y)$, and strong topology of $L^1(J,\mathcal{B}\times X)$. Hence, $\mathcal{J}$ is weakly lower semicontinuous on $L^p(J,Y)$, and since by [HL](iv), $\mathcal{J} > -\infty$, $\mathcal{J}$ attains its infimum at $u^0 \in U_{\mathrm{ad}}$, that is,

$$\epsilon = \lim_{m\to\infty} \int_0^{\mathrm{T}} \mathcal{L}\big(t, x_t^m, x^m(t), u^m(t)\big)\,dt \ge \int_0^{\mathrm{T}} \mathcal{L}\big(t, x_t^0, x^0(t), u^0(t)\big)\,dt = \mathcal{J}\big(u^0\big) \ge \epsilon.$$

This completes the proof. □

6 An Example

At last, an example is given to illustrate our theory. Consider the following problem:

$$\begin{cases} {}^{C}D_t^q x(t,y) - \frac{\partial^2}{\partial y^2} x(t,y) \\ \quad = \mu\big(t, \int_{-\infty}^t \mu_1(s-t)x(s,y)\,ds, \int_0^t \int_{-\infty}^0 \mu_2(s,y,\tau-s)x(\tau,y)\,d\tau\,ds\big), \\ \qquad + \int_{[0,1]} \mathcal{K}(y,s)u(s,t)\,ds, \quad q \in (\frac{1}{2},1),\ y\in[0,1],\ t\in J, \\ x(t,0) = x(t,1) = 0, \quad t \ge 0, \\ x(t,y) = \varphi(t,y), \quad -\infty < t \le 0,\ y\in[0,1], \end{cases} \tag{15}$$

where φ is continuous and satisfies certain smoothness conditions, $u \in L^2(J\times[0,1])$, and $\mathcal{K}:[0,1]\times[0,1]\to\mathbb{R}$ is continuous. Moreover, we assume that:

(h1) $\mu_1(s) \ge 0$ is continuous in $]-\infty,0]$ and $\int_{-\infty}^0 \mu_1^2(s)\,ds < \infty$.

(h2) μ is continuous in $J\times[0,1]\times[0,1]$ and there exists a $L_\mu > 0$ such that

$$\big\|\mu(t,v_1,w_1) - \mu(t,v_2,w_2)\big\| \le L_\mu\big(\delta\|v_1-v_2\| + \|w_1-w_2\|\big), \quad \text{for all } t\in J.$$

where $\delta = (\frac{-1}{2\nu}\int_{-\infty}^0 \mu_1^2(s)\,ds)^{-\frac{1}{2}}$.

(h2′) μ is continuous in $J\times[0,1]\times[0,1]$ and there exists a $a_\mu > 0$ such that

$$\big\|\mu(t,v,w)\big\| \le a_\mu\big(\delta\|v\| + \|w\|\big), \quad \text{for all } t\in J.$$

(h3) $\mu_2(t,y,s) \ge 0$ is continuous in $J\times[0,1]\times]-\infty,0]$ and $\int_{-\infty}^0 \mu_2(t,y,s)\,ds = \beta(t,y) < \infty$ and $a_g = \max\{\beta(t,y) : t\in J, y\in[0,1]\}$.

Let $X = Y = L^2(0,1)$ be endowed with the usual norm $\|\cdot\|_{L^2}$, and $D(A) := W^{2,2}(0,1)\cap W_0^{1,2}(0,1)$, and $Ax := -\frac{\partial^2 x}{\partial y^2}$ for $x\in D(A)$. Then A can generate a strongly continuous semigroup $\{S(t), t\ge 0\}$ in X. The controls are functions $u : Sx([0,1])\to\mathbb{R}$, such that $u\in L^2(Sx([0,1]))$. This claim is that $t\to u(\cdot,t)$ going from J into Y is measurable. Set $U(t) := \{u\in Y : \|u\|_Y \le \varpi\}$, where $\varpi \in L^2(J,\mathbb{R}^+)$. We restrict the admissible controls U_{ad} to be all $u\in L^2(Sx([0,1]))$ such that $\|u(\cdot,t)\|_{L^2([0,1])} \le \varpi(t)$, a.e.

Let $\nu < 0$, defined the phase space

$$\mathcal{B} := \Big\{\phi \in C\big(]-\infty,0], X\big) : \lim_{s\to-\infty} e^{\nu s}\phi(s) \text{ exists in } X\Big\},$$

and let

$$\|\phi\|_{\mathcal{B}} := \sup_{-\infty<s\leq 0}\left\{e^{\nu s}\left\|\phi(s)\right\|\right\}.$$

Then $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space which satisfies (S1)–(S3) with $H=1$, $K(t)=\max\{1, e^{-\nu t}\}$, $\bar{M}(t)=e^{-\nu t}$.

For $(t,\phi)\in[0,1]\times\mathcal{B}$, where $\phi(s)(y)=\varphi(s,y)$, $(s,y)\in\,]-\infty,0]\times[0,1]$, let

$$x(t)(y)=x(t,y),$$

$$g(t,\phi)(y)=\int_{-\infty}^{0}\mu_2(t,y,s)\phi(s)(y)\,ds,$$

$$f\left(t,\phi,\int_0^t g(s,\phi)\,ds\right)(y)=\mu\left(t,\int_{-\infty}^{0}\mu_1(s)\phi(s)(y)\,ds,\int_0^t g(s,\phi)(y)\,ds\right),$$

$$B(t)u(t)(y)=\int_{[0,1]}\mathcal{K}(y,s)u(s,t)\,ds.$$

Then the system (1) can be abstracted as the problem (15).

Now, consider the following cost function:

$$\mathcal{J}(u):=\int_0^{\mathrm{T}}\mathcal{L}\big(t,x_t^u,x^u(t),u(t)\big)\,dt,$$

where $\mathcal{L}: J\times C^{1,0}(]-\infty,0]\times[0,1])\times L^2(J\times[0,1])\to\mathbb{R}\cup\{+\infty\}$ for $x\in C^{1,0}(]-\infty,T]\times[0,1])$ and $u\in L^2([0,1]\times J)$,

$$\mathcal{L}\big(t,x_t^u,x^u(t),u(t)\big)(y)$$
$$:=\int_{[0,1]}\int_{-\infty}^{0}\left|x^u(t+s,y)\right|^2 ds\,dy+\int_{[0,1]}\left|x^u(t,y)\right|^2 dy+\int_{[0,1]}\left|u(y,t)\right|^2 dy.$$

It is easy to see that all the assumptions in Theorem 5.1 are satisfied; the problem (15) has at least one optimal pair.

7 Conclusions

This paper contains the existence, uniqueness, and continuous dependence of mild solutions for the fractional integrodifferential evolution systems with infinite delay in Banach spaces by utilizing the techniques of a priori estimation, extension of step by steps via the Banach fixed point principle. Also, we discussed the existence of optimal control problems of the fractional integrodifferential evolution controlled systems with infinite delay. The result shows that the priori estimation, extension of step by steps via the Banach fixed point principle can effectively be used in existence and control problems to obtain sufficient conditions. Here, it is proved that, under some hypotheses, the Lagrange problem admits at least one optimal pair.

Acknowledgements The authors thank the referees for their careful reading of the manuscript and insightful comments, which helped to improve the quality of the paper. We would also like to acknowledge the valuable comments and suggestions from the editors, which vastly contributed to improve the presentation of the paper. Finally, the first author acknowledges the support by the Tianyuan Special Funds of the National Natural Science Foundation of China (No. 11026102) and Key Projects of Science and Technology Research in the Ministry of Education (211169); the second author acknowledges the support by the National Natural Science Foundation of China (No. 10971173); and the third author acknowledges the support by the Slovak Research and Development Agency under the contract No. APVV-0134-10.

References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and applications of fractional differential equations. In: North-Holland Mathematics Studies, vol. 204. Elsevier Science, Amsterdam (2006)
2. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
3. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
4. Lakshmikantham, V., Leela, S., Devi, J.V.: Theory of Fractional Dynamic Systems. Scientific Publishers Cambridge, Cambridge (2009)
5. Agarwal, R.P., Benchohra, M., Hamani, S.: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973–1033 (2010)
6. Agarwal, R.P., Belmekki, M., Benchohra, M.: A survey on semilinear differential equations and inclusions involving Riemann–Liouville fractional derivative. Adv. Differ. Equ. **2009**, e1–e47 (2009)
7. Agarwal, R.P., Zhou, Y., He, Y.: Existence of fractional neutral functional differential equations. Comput. Math. Appl. **59**, 1095–1100 (2010)
8. Belmekki, M., Benchohra, M.: Existence results for fractional order semilinear functional differential with nondense domain. Nonlinear Anal. TMA **72**, 925–932 (2010)
9. Benchohra, M., Henderson, J., Ntouyas, S.K., Ouahab, A.: Existence results for fractional order functional differential equations with infinite delay. J. Math. Anal. Appl. **338**, 1340–1350 (2008)
10. Benchohra, M., Henderson, J., Ntouyas, S.K., Ouahab, A.: Existence results for fractional functional differential inclusions with infinite delay and application to control theory. Fract. Calc. Appl. Anal. **11**, 35–56 (2008)
11. Henderson, J., Ouahab, A.: Fractional functional differential inclusions with finite delay. Nonlinear Anal. TMA **70**, 2091–2105 (2009)
12. Hu, L., Ren, Y., Sakthivel, R.: Existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delays. Semigroup Forum **79**, 507–514 (2009)
13. Mophou, G.M., N'Guérékata, G.M.: Existence of mild solutions of some semilinear neutral fractional functional evolution equations with infinite delay. Appl. Math. Comput. **216**, 61–69 (2010)
14. Ren, Y., Qin, Y., Sakthivel, R.: Existence results for fractional order semilinear integro-differential evolution equations with infinite delay. Integral Equ. Oper. Theory **67**, 33–49 (2010)
15. Chang, Y.K., Kavitha, V., Arjunan, M.M.: Existence and uniqueness of mild solutions to a semilinear integrodifferential equation of fractional order. Nonlinear Anal. TMA **71**, 5551–5559 (2009)
16. Jaradat, O.K., Al-Omari, A., Momani, S.: Existence of the mild solution for fractional semilinear initial value problems. Nonlinear Anal. TMA **69**, 3153–3159 (2008)
17. El-Borai, M.M.: Some probability densities and fundamental solutions of fractional evolution equations. Chaos Solitons Fractals **14**, 433–440 (2002)
18. El-Borai, M.M.: The fundamental solutions for fractional evolution equations of parabolic type. J. Appl. Math. Stoch. Anal. **3**, 197–211 (2004)
19. Hernández, E., O'Regan, D., Balachandran, K.: On recent developments in the theory of abstract differential equations with fractional derivatives. Nonlinear Anal. TMA **73**, 3462–3471 (2010)
20. Wang, J., Zhou, Y.: Study of an approximation process of time optimal control for fractional evolution systems in Banach spaces. Adv. Differ. Equ. **2011**, e1–e16 (2011)
21. Wang, J., Yang, Y., Wei, W.: Nonlocal impulsive problems for fractional differential equations with time-varying generating operators in Banach spaces. Opusc. Math. **30**, 361–381 (2010)
22. Wang, J., Wei, W., Yang, Y.: On some impulsive fractional differential equations in Banach spaces. Opusc. Math. **30**, 507–525 (2010)

23. Wang, J., Wei, W., Yang, Y.: Fractional nonlocal integrodifferential equations and its optimal control in Banach spaces. J. SIAM **14**, 79–91 (2010)
24. Wang, J., Zhou, Y., Wei, W.: A class of fractional delay nonlinear integrodifferential controlled systems in Banach spaces. Commun. Nonlinear Sci. Numer. Simul. **16**, 4049–4059 (2011)
25. Wang, J., Zhou, Y.: A class of fractional evolution equations and optimal controls. Nonlinear Anal., Real World Appl. **12**, 262–272 (2011)
26. Zhou, Y.: Existence and uniqueness of fractional functional differential equations with unbounded delay. Int. J. Dyn. Differ. Equ. **1**, 239–244 (2008)
27. Zhou, Y., Jiao, F., Li, J.: Existence and uniqueness for fractional neutral differential equations with infinite delay. Nonlinear Anal. TMA **71**, 3249–3256 (2009)
28. Zhou, Y., Jiao, F.: Existence of extremal solutions for discontinuous fractional functional differential equations. Int. J. Dyn. Differ. Equ. **2**, 237–252 (2009)
29. Zhou, Y., Jiao, F.: Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. **59**, 1063–1077 (2010)
30. Zhou, Y., Jiao, F.: Nonlocal Cauchy problem for fractional evolution equations. Nonlinear Anal., Real World Appl. **11**, 4465–4475 (2010)
31. Hale, J., Kato, J.: Phase spaces for retarded equations with infinite delay. Funkc. Ekvacioj **21**, 11–41 (1978)
32. Henry, D.: Geometric Theory of Semilinear Parabolic Equations. LNM, vol. 840. Springer, Berlin (1981)
33. Ye, H., Gao, J., Ding, Y.: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. **328**, 1075–1081 (2007)
34. Zeidler, E.: Nonlinear Functional Analysis and Its Application II/A. Springer, New York (1990)
35. Balder, E.: Necessary and sufficient conditions for L_1-strong-weak lower semicontinuity of integral functional. Nonlinear Anal., Real World Appl. **11**, 1399–1404 (1987)