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On Iterative Methods with Accelerated Convergence for Solving Systems of Nonlinear Equations

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Abstract We present a modified method for solving nonlinear systems of equations with order of convergence higher than other competitive methods. We generalize also the efficiency index used in the one-dimensional case to several variables. Finally, we show some numerical examples, where the theoretical results obtained in this paper are applied.

Keywords Nonlinear systems · Iterative method · Order of convergence · Computational efficiency

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1 Introduction

In this paper, we present a modified method for solving nonlinear systems of equations with higher order of convergence. We also generalize the efficiency index of Ostrowski [1] generally used in the one-dimensional case to several variables. The convergence order of the method is higher than that of other well-known competitive methods [2]. The presented study of the efficiency index is new, and the numerical results illustrate the connection between the efficiency index and the practical performance of the method.

Two are the goals of this paper. Firstly, by conveniently modifying a well-known result [3] that improves the efficiency index, we increase the local order of convergence and the efficiency index. To do this, we use a variant of Chebyshev's method [3–5] instead of the variant of Newton's method given in [3]. Secondly, we present a new way to compute the efficiency of an iterative method which we call computational efficiency index (CEI).

2 Preliminary Analysis

To approximate a simple root α of a system of nonlinear equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a vector function sufficiently differentiable in a neighborhood $I \subseteq D$ of α, we usually apply iterative methods of the form

$$x_{n+1} = \Phi(x_n), \quad n \geq 0, \tag{1}$$

starting with a given initial approximation x_0 of the root α, where Φ is a function defined on a closed subset Ω of $\mathbb{R}^m$ that maps Ω into itself. It is well-known that the choice of some methods depends on the local order of convergence and the computational cost. The classic efficiency index [1] of an iterative method is defined by

$$EI = \rho^{1/\nu}, \tag{2}$$

where ρ is the local order of convergence of the method and ν represents the number of the evaluations of the scalar component functions necessary to apply (1). Clearly, we can improve (2) by increasing the local order of convergence with minimum computational cost. Following this idea, a well-known result that improves the efficiency index and adapted to $\mathbb{R}^m$ states [3]:

If the iterative method (1) has order of convergence ρ, then the iterative method defined by

$$\begin{cases} z_n = \Phi(x_n), \\ x_{n+1} = z_n - [F'(x_n)]^{-1} F(z_n), \end{cases} \tag{3}$$

has order of convergence at least $\rho + 1$.

Notice that using an additional evaluation of the vectorial function F in (3), the order of convergence of (1) has been increased by one.

Hereafter, we consider iterative methods of the form (1) involving evaluations of the function F and its derivatives up to $\rho - 1$ order. To obtain the efficiency indices EI of (1) and (3), we compute the number of scalar functions that are used in the evaluation of the vectorial function and its derivatives. Namely, if $F = (f_1, f_2, \ldots, f_m)$, for (1), we have:

- m evaluations of $f_i(x)$, $1 \leq i \leq m$, to obtain $F(x)$;
- m^2 evaluations of functions to obtain $F'(x) = (\frac{\partial f_i(x)}{\partial x_j})_{i,j=1,2,\ldots,m}$;
- The number of evaluations given by the combinations with repetition $\binom{m+1}{2}$ for $(\frac{\partial^2 f_i(x)}{\partial x_j \partial x_k})_{j,k=1,2,\ldots,m}$, and the total number of evaluations $m\binom{m+1}{2} = \frac{m^2(m+1)}{2}$ needed to compute $F''(x)$;
- The number of evaluations given by the combinations with repetition $\binom{m+2}{3}$ for $(\frac{\partial^3 f_i(x)}{\partial x_j \partial x_k \partial x_\ell})_{j,k,\ell=1,2,\ldots,m}$, and the total number of evaluations $m\binom{m+2}{3} = \frac{m^2(m+1)(m+2)}{6}$ needed to compute $F'''(x)$, and so on.

In general, we need

$$m\binom{m+\rho-2}{\rho-1}$$

evaluations of functions to obtain $F^{(\rho-1)}(x)$. Taking into account the preceding, the efficiency indices of the iterative methods (1) and (3) are given respectively by

$$EI_{(1)} = \rho^{\frac{(m-1)!(\rho-1)!}{(m+\rho-1)!}} \quad \text{and} \quad EI_{(3)} = (\rho+1)^{\frac{(m-1)!(\rho-1)!}{m!(\rho-1)!+(m+\rho-1)!}}. \tag{4}$$

In the above expressions, it is considered that the computational cost of all the evaluations of the component functions and their partial derivatives is 1. Moreover, from (4), we have that $EI_{(3)} > EI_{(1)}$ for $\rho > 2$.

3 Main Result

In the following, we present theoretical results that improve the local order of convergence (≥ 3) of a given iterative method. We also show numerical examples where the previously obtained theoretical results are applied and how the efficiency index is improved.

3.1 Theoretical Results

We begin with the iterative method (1), where we evaluate $F(x_n)$, $F'(x_n)$, $F''(x_n)$, and we consider the following two-step iterative algorithm

$$\begin{cases} z_n = \Phi(x_n), \\ x_{n+1} = z_n - (I + \frac{1}{2}L_F(x_n, z_n))U(z_n), \end{cases} \tag{5}$$

where $L_F(x_n, z_n) = [F'(z_n)]^{-1}F''(x_n)U(z_n)$ and $U(z_n) = [F'(z_n)]^{-1}F(z_n)$.

Observe that in (3) an evaluation of the function F is computed while an evaluation of the function and its derivative are computed in (5). We note that in the following theorem $F^{(k)}(\alpha) \in \mathscr{L}(\mathbb{R}^m \times \overset{k}{\cdots} \times \mathbb{R}^m, \mathbb{R}^m)$, $\Gamma \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^m)$, $A_k = \frac{1}{k!}\Gamma F^{(k)}(\alpha)$ and $A_k h^k = A_k(h, h, \overset{k}{\ldots}, h) \in \mathbb{R}^m$ with $h \in \mathbb{R}^m$, where $\mathscr{L}$ denotes the set of bounded linear functions.

Theorem 3.1 *If the errors in sequences x_n and z_n are respectively $e_n = x_n - \alpha$ and $z_n - \alpha = C_0 e_n^\rho + O(e_n^{\rho+1})$, where $C_0 \in \mathscr{L}(\mathbb{R}^m \times \overset{\rho}{\cdots} \times \mathbb{R}^m, \mathbb{R}^m)$, then the local order of convergence of the iterative method defined in (5) is at least $2\rho+1$. More precisely, if $\Gamma = [F'(\alpha)]^{-1}$ exists, then*

$$e_{n+1} = -3A_3C_0^2 e_n^{2\rho+1} + O\big(e_n^{2\rho+2}\big),$$

where $A_3C_0^2 e_n^{2\rho+1} = A_3(C_0 e_n^\rho, C_0 e_n^\rho, e_n)$.

Proof We will denote $e = e_n$ and $E = z_n - \alpha$. Taking into account Taylor's series of $F(z_n)$ and $F'(z_n)$ at α, we obtain

$$F(z_n) = F(\alpha + E) = F'(\alpha)\big(E + A_2E^2 + O\big(E^3\big)\big) \tag{6}$$

and

$$F'(z_n) = F'(\alpha + E) = F'(\alpha)\big(I + 2A_2E + O\big(E^2\big)\big). \tag{7}$$

From (6) and (7) and developing $[F'(z_n)]^{-1}$ and $U(z_n)$ at α, we obtain

$$\begin{aligned} \big[F'(z_n)\big]^{-1} &= \big(I - 2A_2E + O\big(E^2\big)\big)\Gamma, \\ U(z_n) &= \big(I - 2A_2E + O\big(E^2\big)\big)\Gamma\Gamma^{-1}\big(E + A_2E^2 + O\big(E^3\big)\big) \\ &= E - A_2E^2 + O\big(E^3\big). \end{aligned} \tag{8}$$

Since

$$\begin{aligned} F(x_n) &= F'(\alpha)\big(e + A_2\,e^2 + A_3e^3 + O\big(e^4\big)\big), \\ F'(x_n) &= F'(\alpha)\big(I + 2A_2e + 3A_3e^2 + O\big(e^3\big)\big), \\ F''(x_n) &= F'(\alpha)\big(2A_2 + 6A_3e + O\big(e^2\big)\big), \end{aligned}$$

where I is the identity operator and

$$\begin{aligned} L_F(x_n, z_n) &= \big[F'(z_n)\big]^{-1}F''(x_n)U(z_n) \\ &= \big(I - 2A_2E + O\big(E^2\big)\big)\big(2A_2 + 6A_3e + O\big(e^2\big)\big)\big(E - A_2E^2 + O\big(E^3\big)\big) \\ &= 2A_2E + 6A_3eE + O\big(e^2E\big), \end{aligned}$$

where $L_F(x_n, z_n) \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^m)$, we have

$$\begin{aligned} L_F(x_n, z_n)U(z_n) &= \big(2A_2E + 6A_3eE + O\big(e^2E\big)\big)\big(E - A_2E^2 + O\big(E^3\big)\big) \\ &= 2A_2E^2 + 6A_3eE^2 + O\big(e^2E^2\big). \end{aligned} \tag{9}$$

From (8) and (9), by subtracting α from the two sides of the definition of the iterative function defined in (5), we get the following vectorial difference equation

$$\begin{aligned} e_{n+1} &= E - U(z_n) - \frac{1}{2}L_f(x_n, z_n)U(z_n) \\ &= E - \big(E - A_2E^2 + O\big(E^3\big)\big) - \frac{1}{2}\big(2A_2E^2 + 6A_3eE^2 + O\big(e^2E^2\big)\big) \\ &= -3A_3eE^2 + O\big(e^2E^2\big). \end{aligned}$$

Replacing E by $E = C_0e^\rho + O(e^{\rho+1})$, the proof is complete. □

Particularizing Theorem 3.1 for quadratic functions, the following result is obtained.

Corollary 3.1 *If F is a quadratic function, then the local order of convergence of the iterative method defined in (5) is 3ρ. Moreover, if $\Gamma = [F'(\alpha)]^{-1}$ exists, then*

$$e_{n+1} = 2A_2^2C_0^3e_n^{3\rho} + O\big(e_n^{3(\rho+1)}\big),$$

where $A_2^2C_0^3e_n^{3\rho} = (A_2C_0e_n^\rho)^2C_0e_n^\rho$.

Proof Since $A_k = 0$, for $k \geq 3$, and considering in (6) terms until the third order, we obtain

$$\begin{aligned} \big[F'(z_n)\big]^{-1} &= \big(I - 2A_2E + 4A_2^2E^2 + O\big(E^3\big)\big)\Gamma^{-1}, \\ U(z_n) &= E - A_2E^2 + 2A_2^2E^3 + O\big(E^4\big), \\ L_F(x_n, z_n) &= \big[F'(z_n)\big]^{-1}2A_2U(z_n) \\ &= 2A_2E - 6A_2^2E^2 + 16A_2^3E^3 + O\big(E^4\big). \end{aligned}$$

Therefore,

$$\begin{aligned} e_{n+1} &= E - \left(I + \frac{1}{2}L_F(x_n, z_n)\right)U(z_n) \\ &= 2A_2^2E^3 + O\big(E^4\big), \end{aligned}$$

and replacing E by $E = C_0e^\rho + O(e^{\rho+1})$ in the last expression, it follows that the local order of convergence is $\geq 3\rho$. □

Note that the efficiency index of method (5) is:

$$EI_{(5)} = (2\rho + 1)^{\frac{(m-1)!(\rho-1)!}{(m+1)!(\rho-1)!+(m+\rho-1)!}}. \tag{10}$$

In Fig. 1, the efficiency index (10) of the iterative method defined in (5) is presented when the number of equations is $m = 4, 5, \ldots, 12$ and the local order of convergence is $\rho = 3, 4, 5$. Notice that this index is greater than the ones obtained for the methods (1) and (3) (see (4)).

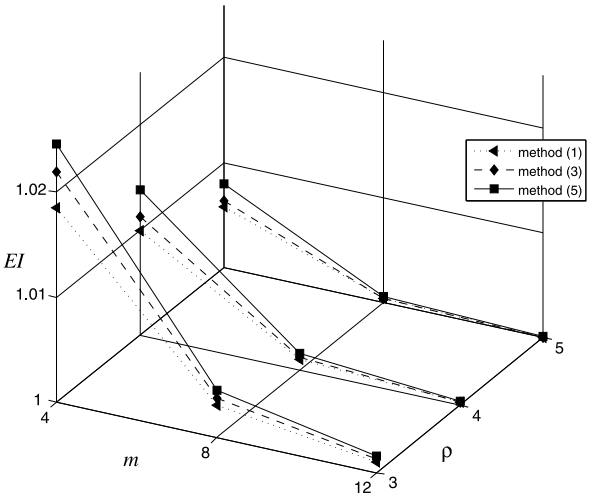

Fig. 1 Efficiency indices of iterative methods (1), (3) and (5)

On the other hand, if we are interested in studying the efficiency, in the sense defined by the authors, of the iterative methods based on the first order divided differences, we can see [6], where the following k-step iterative method with memory is presented:

$$x_n^{(k)} = x_n^{(k-1)} - [x_{n-1}, x_n; F]^{-1} F\big(x_n^{(k-1)}\big), \quad k \geq 1,$$

which has first been proposed in [7], where a semilocal analysis and sharp error bounds are given. Some preliminary results of the efficiency index of the methods from [7] can be found in [8].

3.2 Numerical Examples

In the following numerical examples, we compare the order and the efficiency of the well-known Super-Halley method (SHM), whose iterative function is [2]:

$$\Phi(x) = x - \left(I + \frac{1}{2}L_F(x)\right)\big(I - L_F(x)\big)^{-1}U(x), \tag{11}$$

with $L_F(x) = [F'(x)]^{-1}F''(x)U(x)$ and $U(x) = [F'(x)]^{-1}F(x)$, and its corresponding modification (MSHM) given by (5). To do this, we present two numerical examples where integral equations are involved.

Example 3.1 (Hammerstein's equation) We consider the following mixed Hammerstein equation [9]:

$$x(s) = 1 + \frac{1}{5}\int_0^1 G(s,t)x(t)^3\,dt, \quad s \in [0,1],$$

Table 1 Abscissas and weights of the Gauss–Legendre formula with $m = 8$

j	t_j	ϖ_j
1	0.0198...	0.0506...
2	0.1016...	0.1111...
3	0.2372...	0.1568...
4	0.4082...	0.1813...
5	0.5917...	0.1813...
6	0.7627...	0.1568...
7	0.8983...	0.1111...
8	0.9801...	0.0506...

where $x \in C[0,1]$, $t \in [0,1]$, and the kernel G is

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

Firstly, we write the integral equation as $F(x) = 0$, where $F : C[0,1] \to C[0,1]$ and

$$F(x)(s) = x(s) - 1 - \frac{1}{5}\int_0^1 G(s,t)x(t)^3\,dt, \quad s \in [0,1].$$

Then, we transform it into a finite-dimensional problem by using the following Gauss–Legendre formula:

$$\int_0^1 f(t)\,dt \approx \sum_{j=1}^{m} \varpi_j f(t_j), \tag{12}$$

where the abscissas t_j and the weights ϖ_j are known and presented in Table 1 for $m = 8$.

Denoting the approximation of $x(t_i)$ by x_i $(i = 1, 2, \ldots, 8)$, we obtain the system of nonlinear equations

$$x_i - 1 - \frac{1}{5}\sum_{j=1}^{8} a_{ij}x_j^3 = 0, \quad \text{where } a_{ij} = \begin{cases} \varpi_j t_j(1-t_i) & \text{if } j \le i, \\ \varpi_j t_i(1-t_j) & \text{if } j < i, \end{cases} \quad i = 1, 2, \ldots, 8. \tag{13}$$

Now, system (13) is written in matrix form as

$$F(\overline{x}) = \overline{x} - \overline{1} - \frac{1}{5}A\hat{x},$$

where $F : \mathbb{R}^8 \longrightarrow \mathbb{R}^8$, $\overline{x} = (x_1, x_2, \ldots, x_8)^T$, $\overline{1} = (1, 1, \ldots, 1)^T$, $A = (a_{ij})$ and $\hat{x} = (x_1^3, x_2^3, \ldots, x_8^3)^T$.

Observe that $F'(\overline{x})$ is the linear operator defined by

$$F'(\overline{x})\overline{y} = \overline{y} - \frac{3}{5}A\,\mathrm{diag}\{x_1^2, x_2^2, \ldots, x_8^2\}\overline{y}$$

Table 2 Initial approximations $\overline{x}_0$, k iteration number, $\bar{\rho}$ (COC), NFE for SHM and MSHM methods and the Hammerstein's equation

	SHM			MSHM		
$\overline{x}_0$	k	$\bar{\rho}$	NFE	k	$\bar{\rho}$	NFE
$\overline{1}$	5	3.00002	1800	3	6.42547	1296
$\overline{3}$	6	2.99999	2160	4	7.00028	1728
Iterations	11			7		
TNFE			3960			3024

and $F''(\overline{x})$ is the bilinear operator defined by

$$F''(\overline{x})\overline{yz} = -\frac{6}{5}A(\overline{x} \otimes \overline{y} \otimes \overline{z}),$$

where $\overline{y} = (y_1, y_2, \ldots, y_8)^T$, $\overline{z} = (z_1, z_2, \ldots, z_8)^T$ and $\otimes$ denotes the componentwise product on $\mathbb{R}^8$.

The solution $\overline{x}^* = (x_1^*, x_2^*, \ldots, x_8^*)^T$ is

$$x_1^* = x_8^* = 1.0020\ldots, \qquad x_2^* = x_7^* = 1.0099\ldots,$$
$$x_3^* = x_6^* = 1.0197\ldots, \qquad x_4^* = x_5^* = 1.0264\ldots.$$

Table 2 shows the initial approximations $\overline{x}_0$, the k iteration number necessary for accomplishing the stopping criterion $\|\overline{x}_n - \overline{x}^*\| < 10^{-250}$, the Computational Order of Convergence (COC) computed by the formula [10]

$$\bar{\rho} = \ln \frac{\|\overline{x}_{n+1} - \alpha\|}{\|\overline{x}_n - \alpha\|} \Big/ \ln \frac{\|\overline{x}_n - \alpha\|}{\|\overline{x}_{n-1} - \alpha\|}, \quad n \in \mathbb{N}, \tag{14}$$

and the Number of Function Evaluation (NFE) for SHM (11) and MSHM (5). Note that in the last two rows we can see the total number of iterations and the total number of functions evaluations (TNFE) for nonlinear system (13) with two different initial approximations $\overline{x}_0 = \overline{1}$ and $\overline{x}_0 = \overline{3}$. Table 2 shows the speed of convergence of MSHM and how the total number of function evaluations decreases with respect to SHM.

If we choose $\overline{x}_0 = \overline{1}$, we obtain the numerical solution $\overline{x}^* = (x_1^*, x_2^*, \ldots, x_8^*)^T$ after five iterations by SHM and after three iterations by MSHM. If we choose $\overline{x}_0 = \overline{3}$, we obtain the numerical solution $\overline{x}^*$ after six iterations by SHM and after four iterations by MSHM.

We remark that in system (13) the number of scalar function evaluations is 360 for SHM and 432 for MSHM. The efficiency indices of SHM and MSHM are respectively $3^{\frac{1}{360}} \approx 1.0031$ and $7^{\frac{1}{432}} \approx 1.0045$.

Example 3.2 (Chandrasekhar's equation) Now, we consider a quadratic integral equations related with Chandrasekhar's work [11] of type

$$x(s) = f(s) + \lambda x(s) \int_0^1 \kappa(s,t)x(t)\,dt, \tag{15}$$

which arise in the study of the radiative transfer theory, the transport of neutrons and the kinetic theory of the gases. It is studied in [12] and, under certain conditions for the kernel, in [13] and [14].

We consider the max-norm, the kernel $\kappa(s,t)$ as a continuous function with $s,t \in [0,1]$ and such that $0 < \kappa(s,t) < 1$ and $\kappa(s,t) + \kappa(t,s) = 1$. Moreover, we suppose that $f(s) \in C[0,1]$ is a given function and λ is a real number.

Notice that finding a solution of (15) is equivalent to solving the equation $F(x) = 0$, where $F : C[0,1] \to C[0,1]$ and

$$F(x)(s) = x(s) - f(s) - \lambda x(s) \int_0^1 \kappa(s,t)x(t)\,dt, \quad x \in C[0,1], s \in [0,1].$$

In particular, we consider

$$F(x)(s) = x(s) - 1 - \frac{x(s)}{4} \int_0^1 \frac{s}{s+t} x(t)\,dt, \quad x \in C[0,1], s \in [0,1]. \tag{16}$$

To approximate numerically a solution of $F(x) = 0$, where F is given in (16), we approach the integral that appears in (16) by the Gauss–Legendre quadrature formula given in (12) and Table 1. If we denote by x_i the approximations of $x(t_i)$, $i = 1, 2, \ldots, 8$, we obtain the following nonlinear system:

$$x_i = 1 + x_i \sum_{j=1}^{8} a_{ij} x_j, \quad \text{where } a_{ij} = \frac{t_i \varpi_j}{4(t_i + t_j)}, \ i = 1, 2, \ldots, 8. \tag{17}$$

Now, if we denote $\overline{x} = (x_1, \ldots, x_8)^T$, $\overline{1} = (1, \ldots, 1)^T$ and $A = (a_{ij})$, then we write (17) in the matrix form:

$$F(\overline{x}) = \overline{x} - \overline{1} - \overline{x} \odot A\overline{x},$$

where $\odot$ denotes the scalar product. Moreover, $F'(\overline{x})$ is then the linear operator given by

$$F'(\overline{x})\overline{y} = \overline{y} - (\overline{x} \odot A\overline{y} + \overline{y} \odot A\overline{x}),$$

and $F''(\overline{x})$ is the bilinear operator defined by

$$F''(\overline{x})\overline{yz} = -(\overline{z} \odot A\overline{y} + \overline{y} \odot A\overline{z}).$$

The solution $\overline{x}^* = (x_1^*, x_2^*, \ldots, x_8^*)^T$ of the nonlinear system (17) is

$$x_1^* = 1.0217\ldots, \quad x_2^* = 1.0731\ldots, \quad x_3^* = 1.1257\ldots, \quad x_4^* = 1.1697\ldots,$$
$$x_5^* = 1.2030\ldots, \quad x_6^* = 1.2264\ldots, \quad x_7^* = 1.2415\ldots, \quad x_8^* = 1.2494\ldots.$$

In Table 3, the stopping criterion $\|\overline{x}^* - \overline{x}_n\| < 10^{-750}$ is used.

Notice that in system (17) the number of scalar function evaluations is again 360 for SHM and 432 for MSHM, exactly as in system (13). But the efficiency indices are $4^{\frac{1}{360}} \approx 1.0039$ for SHM and $12^{\frac{1}{432}} \approx 1.0058$ for MSHM.

Table 3 Initial approximations $\overline{x}_0$, k iteration number, $\bar{\rho}$ (COC), NFE for SHM and MSHM methods and Chandrasekhar's equations

	SHM			MSHM		
$\overline{x}_0$	k	$\bar{\rho}$	NFE	k	$\bar{\rho}$	NFE
$\overline{0}$	5	4.0077	1800	4	12.009	1728
$\overline{3}$	6	4.0030	2160	4	12.011	1728
Iterations	11			8		
TNFE			3960			3456

4 On the Computational Efficiency Index

In this section, the traditional way to present the computational efficiency index of iterative methods (see [1, 3, 10]) is revisited and adapted for systems of nonlinear equations. It will be presented in general form for SHM and MSHM, and some comparisons will be given. Finally, the preceding is particularized to the two integral equations presented previously.

When dealing with a system of nonlinear equations, the total operational cost is the sum of the evaluations of functions (the function and the derivatives involved) and the operational cost of doing a step of the iterative method. Therefore, for nonlinear systems with m equations and m unknowns, we suggest the following definition of the computational efficiency index (CEI) of an iterative method of order of convergence ρ as

$$CEI(\mu, m) = \rho^{1/\mathcal{C}(\mu,m)}, \tag{18}$$

where $\mathcal{C}(\mu, m)$ is the computational cost given by

$$\mathcal{C}(\mu, m) = A(m)\mu + P(m), \tag{19}$$

$A(m)$ represents the number of evaluations of the scalar functions, $P(m)$ is the number of products to do an iteration and μ is a ratio between products and evaluations required to express the value of $\mathcal{C}(\mu, m)$ in terms of products. Notice that if $\mu = 1$ and $P(m) = 0$, (18) is reduced to (2). When the computation of (19) is particularized for a system, we can count the number of products that appear in the evaluation of the corresponding scalar functions, and the notation $A(m)\mu = \widetilde{P}(m)$ is used.

According to the above, an estimation of the factor μ is claimed. To do this, we express the cost of the evaluation of the elementary functions in terms of products [15, 16], which depends on the machine, the software and the used arithmetics. In Table 4, an estimation of the cost of the elementary functions in product units is shown, where running time of one product is measured in milliseconds.

In general, notice that the value of μ in (19) is of several units for polynomial functions, while it is increased considerably for transcendental and trigonometric functions, since it will be easily of several tens or hundreds.

Now, we illustrate the above-mentioned with the two examples previously given.

Example 4.1 (Hammerstein's equation) Note that we have $\tilde{P}_1(m) = 4m^2 + 2m$ for SHM and $\tilde{P}_2(m) = 7m^2 + 4m$ for MSHM. In Table 5, the computational costs is shown in terms of products expressed as functions of m, where $\tilde{P}_j(m) + P_j(m) = G_j(m)$, $j = 1, 2$.

Table 4 Estimation of computational cost of elementary functions computed with Maple 13 and using an Intel(R) Core(TM)2 Duo CPU P8800 (32-bit machine) Microsoft Windows 7 Professional, where $x = \sqrt{3} - 1$ and $y = \sqrt{5}$

Digits	$x * y$	x/y	$\sqrt{x}$	$\exp(x)$	$\ln(x)$	$\sin(x)$	$\cos(x)$	$\arctan(x)$
250	0.0052 ms	1	15	46	35	74	74	135
750	0.0257 ms	1	7	40	13	71	72	87

Table 5 Values of $\tilde{P}(m)$ and $P(m)$ for SHM and MSHM corresponding to the Hammerstein's equation

	SHM	MSHM
ρ	$\rho_1 = 3$	$\rho_2 = 7$
$\tilde{P}_j(m)$	$4m^2 + 2m$	$7m^2 + 4m$
$P_j(m)$	$\frac{m}{3}(2m^2 + 12m + 4)$	$m^3 + 7m^2 + 4m$
$G_j(m)$	$\frac{m}{3}(2m^2 + 24m + 10)$	$m(m^2 + 14m + 8)$

Table 6 Values of $\tilde{P}(m)$ and $P(m)$ for SHM and MSHM corresponding to the Chandrasekhar's equation

	SHM	MSHM
ρ	$\rho_1 = 4$	$\rho_2 = 12$
$\tilde{P}_j(m)$	$2m^2 + m$	$4m^2 + 2m$
$P_j(m)$	$\frac{m}{3}(2m^2 + 15m - 2)$	$m^3 + 10m^2 - m$
$G_j(m)$	$\frac{m}{3}(2m^2 + 21m + 1)$	$m(m^2 + 14m + 1)$

If we denote the computational efficiency indices of SHM and MSHM, respectively, by $\widetilde{CEI}_1(m)$ and $\widetilde{CEI}_2(m)$, then from (18) we have $\widetilde{CEI}_j(m) = \rho_j^{1/G_j(m)}$, $j = 1, 2$, and

$$R_1 = \frac{\log \widetilde{CEI}_2(m)}{\log \widetilde{CEI}_1(m)} = \frac{\log 49}{\log 27} \frac{m^2 + 12m + 5}{m^2 + 14m + 8} \geq 1 \quad \text{for } m \geq 2.97.$$

Therefore, for systems with $m \geq 3$, it follows that $\widetilde{CEI}_2 > \widetilde{CEI}_1$.

Example 4.2 (Chandrasekhar's equation) Notice that we have $\tilde{P}_1(m) = 2m^2 + m$ for SHM and $\tilde{P}_2(m) = 4m^2 + 2m$ for MSHM. In Table 6, the computational costs is shown in terms of products expressed as functions of m, where $\tilde{P}_j(m) + P_j(m) = G_j(m)$, $j = 1, 2$.

Following the same notation as in the previous example, we have $\widetilde{CEI}_j(m) = \rho_j^{1/G_j(m)}$, $j = 1, 2$, and

$$R_2 = \frac{\log \widetilde{CEI}_2(m)}{\log \widetilde{CEI}_1(m)} = \frac{\log 12}{\log 64} \frac{2m^2 + 21m + 1}{m^2 + 14m + 1} \geq 1 \quad \text{for } m \geq 0.27.$$

Therefore, for systems with $m \geq 2$, it follows that $\widetilde{CEI}_2 > \widetilde{CEI}_1$.

5 Conclusions

The main result of this paper is the presentation of a modified method for solving nonlinear systems of equations whose order of convergence is higher than that of other well-known competitive methods. Moreover, we analyze a generalization of the efficiency index used in the scalar case to several variables. Numerical examples that illustrate the theoretical results presented in this paper are also given.

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